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Two-mode photon-added entangled coherent-squeezed states: their entanglement and nonclassical properties

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Abstract Recently, we introduced and generated new types of two-mode entangled states named "entangled coherent-squeezed states". In these states, two common states for quantum information processing, coherent state, and squeezed states have been used. Now, based on the generated entangled states, we introduce "two-mode photon-added entangled coherent-squeezed states". These states are obtained via the iterated action of two creation operators on the two modes of the "entangled coherent-squeezed states". Next, we study the amount of entanglement of the introduced states using concurrence criterion. In the continuation, some of the nonclassical features such as photon-statistics, second-order correlation function, and quadrature squeezing are considered. In addition, we study the influence of photon-addition of two modes on the mentioned properties of the introduced states. We will observe that the entanglement of the introduced photon-added entangled states increases more rapidly as photon-addition of two modes increases. Moreover, some of the nonclassical features for the first mode of the introduced states such as sub-Poissonian photon-statistics and squeezing in p appear and disappear by photon-addition of two modes, respectively.

1 Introduction

Quantum entanglement can be mentioned as the main concept and most important basis in quantum information science such as cryptography [1], quantum teleportation [2], quantum computation [3], and so on. This notion (entanglement or entangled states) which is related to special superposition of several different states was first used by Schrödinger in 1935. He called the wave function of two particles as an entangled state if it cannot be written as the product of the wave functions of two particles [4]. In another word, it can be said that the concept of entanglement is referred to the inseparability feature of a multi-mode system to multiplication of its single-mode subsystems.

In the last two decades, various types of entangled quantum states have been proposed and studied which, for example, can be mentioned entangled coherent states [5–7], entangled squeezed states [8–11], and entangled cat states [12, 13]. Soon, after the introduction of these entangled states, the corresponding photon-added and photon-subtracted entangled states attracted a great deal of attention and some of their nonclassical properties have been considered [14–24]. Moreover, many studies have been done to generate such entangled states theoretically and experimentally [25–33].

Recently, we introduced and generated new types of entangled states namely "entangled coherent-squeezed states", and studied some of their nonclassical properties [34]. It is worth noting that to generate the entangled states, we used a theoretical scheme based on the interaction between a Λ-type three-level atom and a two-mode quantized field under certain conditions (For more information, see [34]). Now, the increasing importance to achieve new entangled states with different nonclassical features motivated us to introduce "two-mode photon-added entangled coherent-squeezed states" and study some of their nonclassicalities.

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The remainder of paper organizes as follows: in the next section, we have a brief review on the "entangled coherent-squeezed states". Next, by making use of these states, we introduce the "two-mode photon-added entangled coherent-squeezed states" in Sect. 3. In Sect. 4, we study the amount of entanglement of the introduced photon-added entangled states by evaluating concurrence. In the continuation, we consider some of the nonclassical features of the introduced photon-added entangled states such as quantum statistics, second-order correlation function, and quadrature squeezing, as well as the influence of two-mode photon-addition on the mentioned nonclassical properties in Sect. 5. Finally, we end our paper by presenting a summary in Sect. 6.

2 Entangled coherent-squeezed states: a brief review

The special form of the "entangled coherent-squeezed states" has recently been introduced and generated in the following forms [34]:

$$|\psi(\alpha,\xi)\rangle_i = M_i(\alpha,\xi)\Big(|\alpha\rangle\otimes|-\xi\rangle_i + |-\alpha\rangle\otimes|\xi\rangle_i\Big), \quad i=\mathrm{e},\mathrm{o}, \tag{1}$$

where $M_{\mathrm{e}}(\alpha,\xi)$ and $M_{\mathrm{o}}(\alpha,\xi)$ are normalization constants. The states $|\alpha\rangle$ and $|\xi\rangle_i$ $(i=\mathrm{e},\mathrm{o})$ are, respectively, coherent state and squeezed states that will be explained in what follows.

The coherent state $|\alpha\rangle$ that known as the minimum uncertainty state can be generated by displacing the vacuum state $|\alpha\rangle = D(\alpha)|0\rangle$ where $D(\alpha)=e^{\alpha a^\dagger-\alpha^* a}$ is displacement operator (a and $a^\dagger$ are the bosonic annihilation and creation operators, respectively, and $\alpha = re^{i\theta}$ is a complex number). Moreover, the coherent state can be obtained by solving the eigen-value equation $a|\alpha\rangle=\alpha|\alpha\rangle$ as [35]:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{p=0}^{\infty}\frac{\alpha^p}{\sqrt{p!}}|p\rangle. \tag{2}$$

Two types of squeezed states $|\xi\rangle_{\mathrm{e}}$ and $|\xi\rangle_{\mathrm{o}}$ that for which one of the quadrature fluctuation becomes less than that of the vacuum state can be, respectively, obtained by squeezing the vacuum state $|\xi\rangle_{\mathrm{e}} = S(\xi)|0\rangle$ and one-photon state of the field $|\xi\rangle_{\mathrm{o}} = S(\xi)|1\rangle$ in the following forms [36]:

$$|\xi\rangle_{\mathrm{e}} = N_{\mathrm{e}}(\xi)\sum_{q=0}^{\infty}(-1)^q\frac{\sqrt{(2q)!}}{2^q q!}(e^{i\theta}\tanh r)^q|2q\rangle, \tag{3}$$

$$|\xi\rangle_{\mathrm{o}} = N_{\mathrm{o}}(\xi)\sum_{q=0}^{\infty}(-1)^q\frac{2^q q!}{\sqrt{(2q+1)!}}(e^{i\theta}\tanh r)^q|2q+1\rangle, \tag{4}$$

where $S(\xi)=e^{\frac{\xi^*}{2}a^2-\frac{\xi}{2}a^{\dagger 2}}$ is squeezing operator. The operators a and $a^\dagger$ have the same definitions of coherent state and $\xi = re^{i\theta}$ is a complex parameter, too.

Due to the fact that only the even and odd number states appear, respectively, in Eqs. (3) and (4), these squeezed states are, respectively, called even and odd squeezed states and labeled by e and o.

The normalization coefficients $N_{\mathrm{e}}(\xi)$ and $N_{\mathrm{o}}(\xi)$ can be easily obtained as:

$$N_{\mathrm{e}}^{-2}(\xi) = \sum_{q=0}^{\infty}\frac{(2q)!}{2^{2q}q!^2}(\tanh r)^{2q}, \tag{5}$$

$$N_{\mathrm{o}}^{-2}(\xi) = \sum_{q=0}^{\infty}\frac{2^{2q}q!^2}{(2q+1)!}(\tanh r)^{2q}. \tag{6}$$

It is worth noting that the even (odd) squeezed states $|\xi\rangle_{\mathrm{e}}$ and $|-\xi\rangle_{\mathrm{e}}$ ($|\xi\rangle_{\mathrm{o}}$ and $|-\xi\rangle_{\mathrm{o}}$) in Eq. (1) can be obtained by setting $\theta=0$ and $\theta=\pi$ in Eqs. (3) and (4), respectively. In the same way, the coherent states $|\pm\alpha\rangle$ are obtained from Eq. (2), too.

Now, using the coherent state in Eq. (2) and squeezed states in Eqs. (3) and (4), we can derive the number bases representation of the introduced "entangled coherent-squeezed states" in Eq. (1) in the following forms:

$$|\psi(\alpha,\xi)\rangle_{\mathrm{e}} = M_{\mathrm{e}}(\alpha,\xi)N_{\mathrm{e}}(\xi)e^{-\frac{r^2}{2}}\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}((-1)^p+(-1)^q) \times\frac{r^p}{\sqrt{p!}}\frac{\sqrt{(2q)!}}{2^q q!}(\tanh r)^q|p\rangle|2q\rangle, \tag{7}$$

$$|\psi(\alpha,\xi)\rangle_{\mathrm{o}} = M_{\mathrm{o}}(\alpha,\xi)N_{\mathrm{o}}(\xi)e^{-\frac{r^2}{2}}\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}((-1)^p+(-1)^q) \times\frac{r^p}{\sqrt{p!}}\frac{2^q q!}{\sqrt{(2q+1)!}}(\tanh r)^q|p\rangle|2q+1\rangle. \tag{8}$$

The normalization factors $M_{\mathrm{e}}(\alpha,\xi)$ and $M_{\mathrm{o}}(\alpha,\xi)$ are given by:

$$M_{\mathrm{e}}^{-2}(\alpha,\xi) = N_{\mathrm{e}}^2(\xi)e^{-r^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(2+2(-1)^{p+q}) \times\frac{r^{2p}}{p!}\frac{(2q)!}{2^{2q}q!^2}(\tanh r)^{2q}, \tag{9}$$

$$M_{\mathrm{o}}^{-2}(\alpha,\xi)=N_{\mathrm{o}}^{2}(\xi)e^{-r^{2}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(2+2(-1)^{p+q})\times\frac{r^{2p}}{p!}\frac{2^{2q}q!^{2}}{(2q+1)!}(\tanh r)^{2q}, \tag{10}$$

where $N_{\mathrm{e}}(\xi)$ and $N_{\mathrm{o}}(\xi)$ have been obtained in Eqs. (5) and (6), respectively.

It is worth mentioning that here, $|\alpha|$ and $|\xi|$ have been supposed the same and equal to r.

3 Two-mode photon-added entangled coherent-squeezed states

Now, we introduce the "two-mode photon-added entangled coherent-squeezed states". These states can be obtained via the iterated actions of two creation operators $a^{\dagger}$ and $b^{\dagger}$, respectively, on the first and second modes of the "entangled coherent-squeezed states" in Eq. (1) in the following forms:

$$|\psi(\alpha,\xi,m,n)\rangle_{i}=M_{i}(\alpha,\xi,m,n)(a^{\dagger})^{m}(b^{\dagger})^{n}\Big(|\alpha\rangle\otimes|-\xi\rangle_{i}+|-\alpha\rangle\otimes|\xi\rangle_{i}\Big),\quad i=\mathrm{e},\mathrm{o}. \tag{11}$$

Using the even and odd "entangled coherent-squeezed states" in Eqs. (7) and (8), we can get the number basis representations of the introduced photon-added entangled states in Eq. (11) as:

$$|\psi(\alpha,\xi,m,n)\rangle_{\mathrm{e}}=M_{\mathrm{e}}(\alpha,\xi,m,n)N_{\mathrm{e}}(\xi)e^{-\frac{r^{2}}{2}}\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}((-1)^{p}+(-1)^{q})\frac{\sqrt{(p+m)!}}{p!}\frac{\sqrt{(2q+n)!}}{2^{q}q!}\times r^{p}(\tanh r)^{q}|p+m\rangle|2q+n\rangle, \tag{12}$$

$$|\psi(\alpha,\xi,m,n)\rangle_{\mathrm{o}}=M_{\mathrm{o}}(\alpha,\xi,m,n)N_{\mathrm{o}}(\xi)e^{-\frac{r^{2}}{2}}\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}((-1)^{p}+(-1)^{q})\frac{\sqrt{(p+m)!}}{p!}\frac{2^{q}q!\sqrt{(2q+n+1)!}}{(2q+1)!}\times r^{p}(\tanh r)^{q}|p+m\rangle|2q+n+1\rangle, \tag{13}$$

where the normalization constants $M_{\mathrm{e}}(\alpha,\xi,m,n)$ and $M_{\mathrm{o}}(\alpha,\xi,m,n)$ are easily obtained as:

$$M_{\mathrm{e}}^{-2}(\alpha,\xi,m,n)=N_{\mathrm{e}}^{2}(\xi)e^{-r^{2}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(2+2(-1)^{p+q})\times\frac{(p+m)!}{p!^{2}}\frac{(2q+n)!}{2^{2q}q!^{2}}r^{2p}(\tanh r)^{2q}, \tag{14}$$

$$M_{\mathrm{o}}^{-2}(\alpha,\xi,m,n)=N_{\mathrm{o}}^{2}(\xi)e^{-r^{2}}\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(2+2(-1)^{p+q})\frac{(p+m)!}{p!^{2}}\frac{2^{2q}q!^{2}(2q+n+1)!}{(2q+1)!^{2}}r^{2p}(\tanh r)^{2q}. \tag{15}$$

In these relations, the coefficients $N_{\mathrm{e}}(\xi)$ and $N_{\mathrm{o}}(\xi)$ have been, respectively, determined in Eqs. (5) and (6).

4 Entanglement of the introduced states

In this section, we want to study the amount of entanglement of the introduced "two-mode photon-added entangled coherent-squeezed states" by evaluating "concurrence".

This criterion that known as one of the appropriate measures to study the entanglement degree of the introduced states is defined for discrete-variable entangled states in terms of the Pauli matrix σ_y as [38]:

$$C=|\langle\Psi|\sigma_{y}\otimes\sigma_{y}|\Psi^{*}\rangle|, \tag{16}$$

where $|\Psi^{*}\rangle$ stands for the complex conjugate of $|\Psi\rangle$. Notice that the maximum value of the concurrence is 1 for a maximum entangled state.

To calculate the concurrence of the introduced entangled states, we consider a two-mode entangled state in the following general form:

$$|\Psi\rangle=\frac{1}{N}\Big(\mu|\eta\rangle\otimes|\gamma\rangle+\nu|\xi\rangle\otimes|\delta\rangle\Big), \tag{17}$$

where μ and ν are complex numbers. The first (second) mode of such entangled state has been shown with $|\eta\rangle$ and $|\xi\rangle$ ($|\gamma\rangle$ and $|\delta\rangle$).

For such state, using the transformation of the continuous-variables-type components to the discrete orthogonal basis and the Schmidt decomposition, the concurrence of the entangled state (17) is determined by [39–42]:

$$C=\frac{2|\mu||\nu|}{N^{2}}\sqrt{(1-|P^{(1)}|^{2})(1-|P^{(2)}|^{2})},\qquad P^{(1)}=\langle\eta|\xi\rangle,P^{(2)}=\langle\delta|\gamma\rangle. \tag{18}$$

Now, to calculate the concurrence of the "two-mode photon-added entangled coherent-squeezed states" in Eq. (11), we first rewrite the introduced "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) in the explicit form. For this purpose, we use the photon-added coherent states introduced first by Agarwal and Tara [37] as:

$$|\pm\alpha,m\rangle=d(\alpha,m)(a^{\dagger})^{m}|\pm\alpha\rangle=d(\alpha,m)e^{-\frac{|\alpha|^{2}}{2}}\times\sum_{p=0}^{\infty}\frac{\sqrt{(p+m)!}}{p!}(\pm\alpha)^{p}|p+m\rangle, \tag{19}$$

where the coefficient $d(\alpha, m)$ is given by:

$$d^{-2}(\alpha, m) = e^{-|\alpha|^2} \sum_{p=0}^{\infty} \frac{(p+m)!}{p!^2} |\alpha|^{2p}. \tag{20}$$

In the similar way, we can define the even and odd photon-added squeezed states, respectively, in the following forms:

$$\begin{aligned} |\pm\xi, n\rangle_{\mathrm{e}} &= d_{\mathrm{e}}(\xi, n)(b^\dagger)^n |\pm\xi\rangle_{\mathrm{e}} \\ &= d_{\mathrm{e}}(\xi, n) N_{\mathrm{e}}(\xi) \sum_{q=0}^{\infty} (\mp 1)^q \frac{\sqrt{(2q+n)!}}{2^q q!} \\ &\quad \times (\tanh r)^q |2q+n\rangle, \end{aligned} \tag{21}$$

$$\begin{aligned} |\pm\xi, n\rangle_{\mathrm{o}} &= d_{\mathrm{o}}(\xi, n)(b^\dagger)^n |\pm\xi\rangle_{\mathrm{o}} \\ &= d_{\mathrm{o}}(\xi, n) N_{\mathrm{o}}(\xi) \sum_{q=0}^{\infty} (\mp 1)^q \frac{2^q q! \sqrt{(2q+n+1)!}}{(2q+1)!} \\ &\quad \times (\tanh r)^q |2q+n+1\rangle. \end{aligned} \tag{22}$$

The coefficients $d_{\mathrm{e}}(\xi, n)$ and $d_{\mathrm{o}}(\xi, n)$ are determined by the normalization condition as:

$$d_{\mathrm{e}}^{-2}(\xi, n) = N_{\mathrm{e}}^2(\xi) \sum_{q=0}^{\infty} \frac{(2q+n)!}{2^{2q} q!^2} (\tanh r)^{2q}, \tag{23}$$

$$d_{\mathrm{o}}^{-2}(\xi, n) = N_{\mathrm{o}}^2(\xi) \sum_{q=0}^{\infty} \frac{2^{2q} q!^2 (2q+n+1)!}{(2q+1)!^2} (\tanh r)^{2q}. \tag{24}$$

By making use of the introduced photon-added coherent states in Eq. (19) and photon-added squeezed states in Eqs. (21) and (22), we can write the explicit form of the "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) in the following forms:

$$\begin{aligned} |\psi(\alpha, \xi, m, n)\rangle_i &= M_i(\alpha, \xi, m, n) d^{-1}(\alpha, m) d_i^{-1}(\xi, n) \\ &\times \Big(|\alpha, m\rangle \otimes |-\xi, n\rangle_i + |-\alpha, m\rangle \otimes |\xi, n\rangle_i \Big), \quad i = \mathrm{e}, \mathrm{o}. \end{aligned} \tag{25}$$

Now, we can obtain the concurrence of the introduced "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) by comparing Eq. (25) with Eq. (17) and using Eq. (18) as:

$$\begin{aligned} C_i(\alpha, \xi, m, n) &= \frac{2M_i^2(\alpha, \xi, m, n)}{d^2(\alpha, m) d_i^2(\xi, n)} \\ &\quad \times \sqrt{\Big(1 - |P^{(1)}(\alpha, m)|^2\Big)\Big(1 - |P_i^{(2)}(\xi, n)|^2\Big)}, \\ &\quad i = \mathrm{e}, \mathrm{o}. \end{aligned} \tag{26}$$

Using the Eqs. (19) and (21), we can, respectively, calculate the quantities $P^{(1)}(\alpha, m)$ and $P_{\mathrm{e}}^{(2)}(\xi, n)$ of the even "two-mode photon-added entangled coherent-squeezed states" in Eq. (25) in the following forms:

$$\begin{aligned} P^{(1)}(\alpha, m) &= \langle -\alpha, m | \alpha, m \rangle \\ &= d^2(\alpha, m) e^{-|\alpha|^2} \sum_{p=0}^{\infty} (-1)^p \frac{(p+m)!}{p!^2} |\alpha|^{2p}, \end{aligned} \tag{27}$$

$$\begin{aligned} P_{\mathrm{e}}^{(2)}(\xi, n) &= {}_{\mathrm{e}}\langle -\xi, n | \xi, n \rangle_{\mathrm{e}} \\ &= d_{\mathrm{e}}^2(\xi, n) N_{\mathrm{e}}^2(\xi) \sum_{q=0}^{\infty} (-1)^q \frac{(2q+n)!}{2^{2q} q!^2} (\tanh r)^{2q}. \end{aligned} \tag{28}$$

For the odd "two-mode photon-added entangled coherent-squeezed states", the quantity $P^{(1)}(\alpha, m)$ is the same with that of the even photon-added entangled states which has been obtained in Eq. (27). Similarly, using Eq. (22), the quantity $P_{\mathrm{o}}^{(2)}(\xi, n)$ of the odd "two-mode photon-added entangled coherent-squeezed states" can be easily obtained as:

$$\begin{aligned} P_{\mathrm{o}}^{(2)}(\xi, n) &= {}_{\mathrm{o}}\langle -\xi, n | \xi, n \rangle_{\mathrm{o}} \\ &= d_{\mathrm{o}}^2(\xi, n) N_{\mathrm{o}}^2(\xi) \sum_{q=0}^{\infty} (-1)^q \frac{2^{2q} q!^2 (2q+n+1)!}{(2q+1)!^2} \\ &\quad \times (\tanh r)^{2q+1}. \end{aligned} \tag{29}$$

Now, we plot the concurrence of the introduced "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) versus r ($|\alpha|$ and $|\xi|$) for the different values of photon-additions of two-mode, m and n, in Fig. 1.

It can be observed that the concurrence of the even and odd "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) shows similar behaviours.

The concurrence of the introduced photon-added entangled states starts from zero, increases monotonically and tends to 1 in the region of $r < 2$, and remains fixed. Clearly, it can be found that by photon-addition and also increasing the number of photon-addition of two modes, m and n, the concurrence increases more rapidly. Moreover, it is worth noting that the photon-addition of the second mode (squeezed states $|\pm\xi\rangle_{\mathrm{e,o}}$) of the introduced photon-added entangled states has more impact on the increasing rate of concurrence.

5 Other nonclassical properties of the introduced states

In this section, some of the nonclassical properties of the introduced "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) are studied. For this purpose,

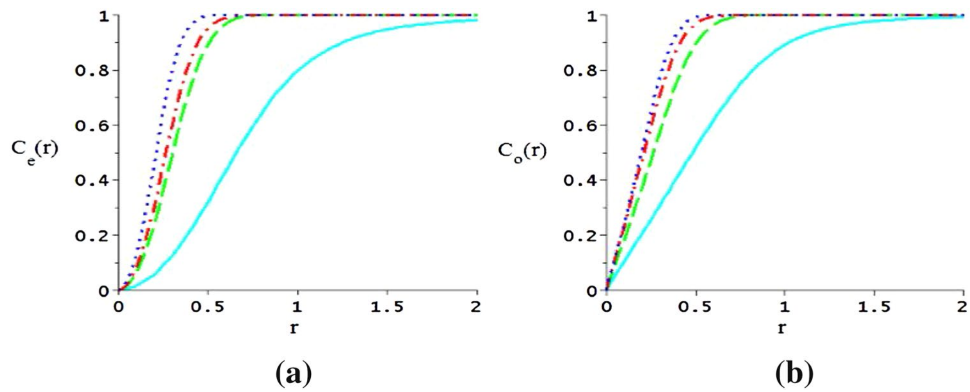

Fig. 1 The concurrence $C(r)$ of the **a** even and **b** odd "two-mode photon-added entangled coherent-squeezed states" $|\psi(\alpha,\xi,m,n)\rangle_{\mathrm{e,o}}$ versus r ($|\alpha|$ and $|\xi|$) for the different values of photon-additions $m, n = 0$ (*solid*), $m, n = 2$ (*dashed*), $m = 4, n = 2$ (*dash-dot*), and $m = 2, n = 4$ (*dot*)

we consider a few important nonclassical features such as "quantum statistics", "second-order correlation function", and "quadrature squeezing". It should be noted that all necessary expectation values in this section will be obtained using the expanded forms of the introduced states in Eqs. (12) and (13).

5.1 Quantum statistics

The quantum statistics of any quantum state is usually investigated by studying the two criteria of that state as sub-Poissonian photon-statistics and oscillatory behaviour of photon distribution.

5.1.1 Sub-Poissonian photon-statistics

First, we study the sub-Poissonian photon-statistics of the introduced photon-added entangled states in Eq. (11). To achieve this goal, we calculate the Mandel parameter [43] for both modes which is defined as [43]:

$$Q_i = \frac{\langle n_i^2\rangle - \langle n_i\rangle^2}{\langle n_i\rangle} - 1, \qquad i = a, b, \tag{30}$$

where $n_a = a^\dagger a$ and $n_b = b^\dagger b$ are the number operators corresponding to the first and second modes of the introduced entangled states, respectively.

The positivity and negativity of the Mandel parameter indicate, respectively, to super-Poissonian and sub-Poissonian photon-statistics. Moreover, this parameter becomes zero for states with Poissonian photon-statistics such as coherent state. It should be noted that one of the nonclassical features of each quantum state is specified by negative values of Mandel parameter [44].

By performing the necessary calculations, the Mandel parameter of the first and second modes of the even (odd) "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) is plotted versus r ($|\alpha|$ and $|\xi|$) for different values of photon-additions of two modes, m and n, in Fig. 2 (Fig. 3).

Clearly, the Mandel parameter of the first mode $Q_a(r)$ of the introduced even and odd "two-mode photon-added entangled coherent-squeezed states" in Figs. 2a and 3a shows the same behaviours. It can be observed that the first mode of the even and odd "entangled coherent-squeezed states" ($m, n = 0$) has super-Poissonian photon-statistics over all of the regions of r. While the first mode of the even and odd "two-mode photon-added entangled coherent-squeezed states" becomes sub-Poissonian for all chosen values of photon-additions ($m, n \neq 0$) over all of the regions of r. Moreover, it can be found from Figs. 2a and 3a that the sub-Poissonian photon-statistics of the first mode of the introduced even and odd photon-added entangled states increases by increasing the photon-addition of the first mode (m) of the "two-mode photon-added entangled coherent-squeezed states".

In the case of the second mode of the introduced entangled states, the Mandel parameter $Q_b(r)$ shows no sub-Poissonian photon-statistics for the even "entangled coherent-squeezed state" (no photon-additions $m, n = 0$) in Fig. 2b while for the odd "entangled coherent-squeezed state" (no photon-additions $m, n = 0$) sub-Poissonian photon-statistics is seen over the region of small r ($|\alpha|$ and $|\xi|$) in Fig. 3b. By photon-addition of two modes ($m, n \neq 0$), the sub-Poissonian photon-statistics is observed for the second mode of both states (even and odd) "two-mode photon-added entangled coherent-squeezed states" in Figs. 2b and 3b over the region of very small r ($|\alpha|$ and $|\xi|$).

It is worth noting that the region of sub-Poissonian photon-statistics of the second mode of the even "two-mode photon-added entangled coherent-squeezed states" becomes more slightly wider with increasing the photon-addition of the second mode (n) of the introduced entangled states (Fig. 2b). While, for the second mode of the odd "two-mode photon-added entangled coherent-squeezed states", the region of sub-Poissonian photon-statistics becomes slightly narrower with increasing n (Fig. 3b).

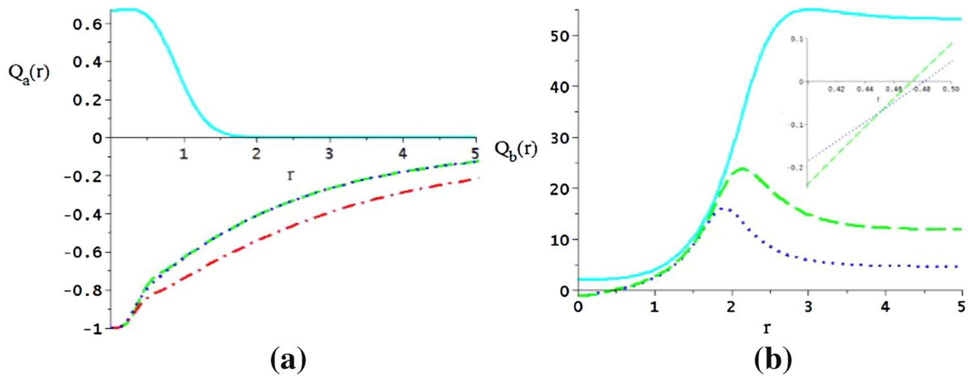

Fig. 2 Mandel parameter $Q_e(r)$ of the **a** first mode and **b** second mode of the even "two-mode photon-added entangled coherent-squeezed states" $|\psi(\alpha,\xi,m,n)\rangle_e$ versus r ($|\alpha|$ and $|\xi|$) for the different values of photon-additions $m,n=0$ (*solid*), $m,n=2$ (*dashed*), $m=4,n=2$ (*dash-dot*), and $m=2,n=4$ (*dot*) (the $m,n=2$ and $m=2,n=4$ *curves* in **a** and the $m,n=2$ and $m=4,n=2$ curves in **b** coincide with each other)

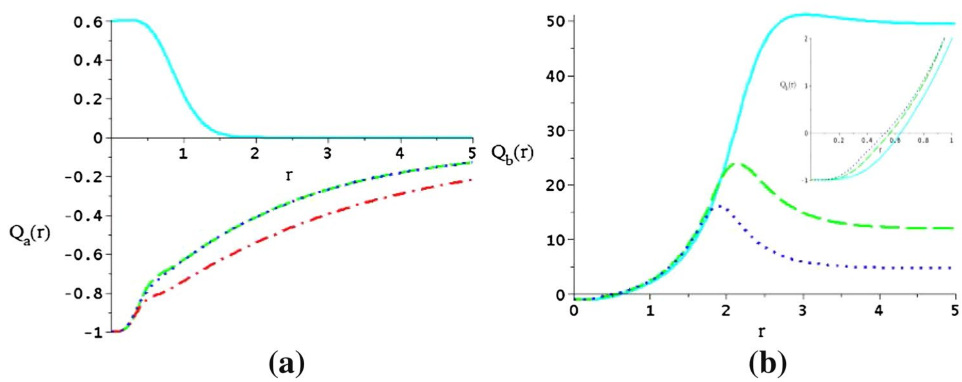

Fig. 3 Mandel parameter $Q_o(r)$ of the **a** first mode and **b** second mode of the odd "two-mode photon-added entangled coherent-squeezed states" $|\psi(\alpha,\xi,m,n)\rangle_o$ versus r ($|\alpha|$ and $|\xi|$) for the different values of photon-additions m and n (all of the parameters and explanations are the same as Fig. 2)

5.1.2 Photon number distribution

As it is pointed out before, the photon number distribution is another criterion to study the quantum statistics which its oscillatory behaviour can show the nonclassical property of each quantum state.

The photon number distribution $P(j,\ k)$ which is defined as the probability of finding j and k photons, respectively, in the first and second modes of the introduced "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) is given by:

$$P_{\mathrm{i}}(j,k)=\left|\langle j,k|\psi(\alpha,\xi,m,n)\rangle_{\mathrm{i}}\right|^2,\quad i=\mathrm{e},\mathrm{o}. \tag{31}$$

By calculating $P_i(j,k)$, we plotted the three-dimensional photon number distribution of the introduced even (odd) photon-added entangled states in Eq. (11) versus the photon numbers of the first mode j and second mode k in Fig. 4 (Fig. 5) by fixing $r=1$.

Clearly, an oscillatory behaviour can be observed in the photon number distribution of the introduced photon-added entangled states which indicates to the nonclassical feature of these states.

5.2 Two-mode correlation function

In this subsection, the second-order correlation function of the introduced "two-mode photon-added entangled coherent-squeezed states" in Eq. (11) is considered. This feature which has been defined as [45]:

$$g_{ab}^{(2)}(0)=\frac{\langle a^\dagger ab^\dagger b\rangle}{\langle a^\dagger a\rangle\langle b^\dagger b\rangle}, \tag{32}$$

becomes less than 1 for a nonclassical state [46]. By plotting the second-order correlation function of the introduced photon-added entangled states versus r ($|\alpha|$ and $|\xi|$) in Fig. 6, this nonclassical feature is observed for none of the values of photon-addition.

5.3 Quadrature squeezing

As the last proposed nonclassical feature, we study the quantum fluctuations of the quadratures of the introduced photon-added entangled states. For this purpose, the two Hermitian operators $x = (a + a^\dagger)/\sqrt{2}$ and $p = (a - a^\dagger)/i\sqrt{2}$ have been investigated which satisfy the commutation relation $[x, p] = i$.

Using the common definitions of the variances of these operators, position $(\Delta x)^2$ and momentum $(\Delta p)^2$, the quadrature squeezing is defined as $s_w = 2(\Delta w)^2 - 1, (w = x, p)$ or equivalently:

$$s_{x(p)} = 2\langle a^\dagger a\rangle \pm \langle a^2\rangle \pm \langle a^{\dagger 2}\rangle \mp \langle a\rangle^2 \mp \langle a^\dagger\rangle^2 - 2\langle a\rangle\langle a^\dagger\rangle. \tag{33}$$

A squeezed state in x (p) satisfies the inequality $-1 < s_{x(p)} < 0$.

To observe the squeezing in x or p corresponding to the two modes of the introduced photon-added entangled states in Eq. (11), we have plotted the x- and p- squeezing

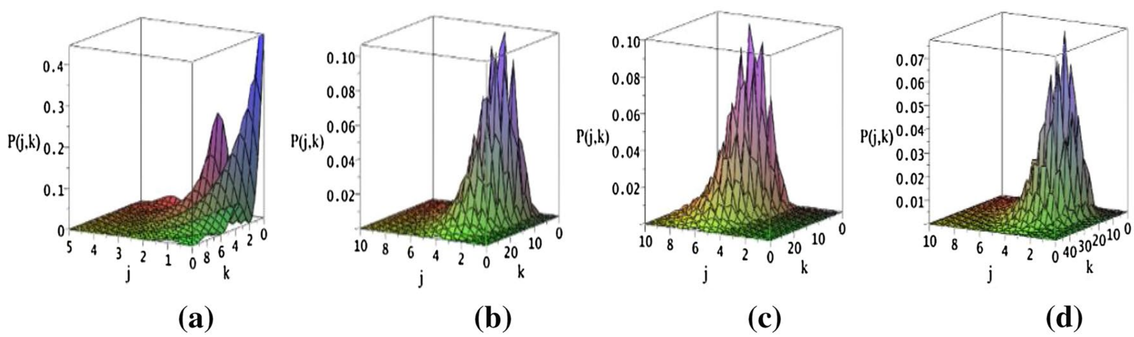

Fig. 4 Photon number distribution $P_e(j,k)$ (the probability of finding j photons in the first mode and k photons in the second mode) of the even "two-mode photon-added entangled coherent-squeezed states" $|\psi(\alpha,\xi,m,n)\rangle_e$ with $r = 1$ for the different values of photon-additions **a** $m, n = 0$, **b** $m, n = 2$, **c** $m = 4, n = 2$, and **d** $m = 2, n = 4$

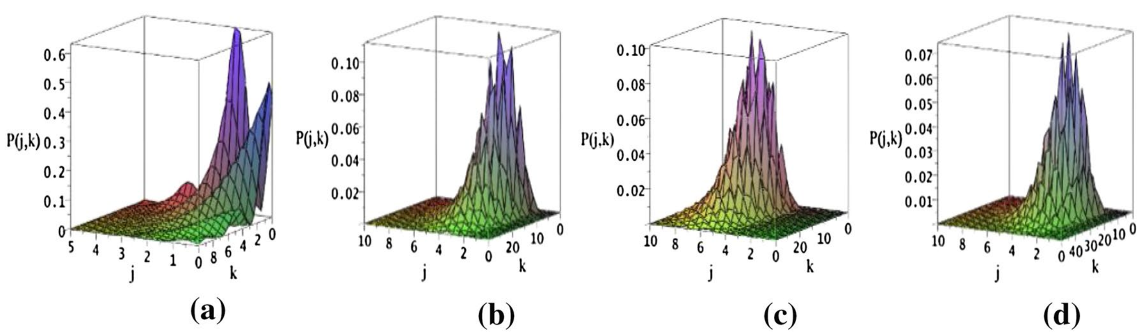

Fig. 5 Photon number distribution $P_o(j,k)$ of the odd "two-mode photon-added entangled coherent-squeezed states" $|\psi(\alpha,\xi,m,n)\rangle_o$ for the different photon-additions m and n with $r = 1$ (all of the parameters are the same as Fig. 4)

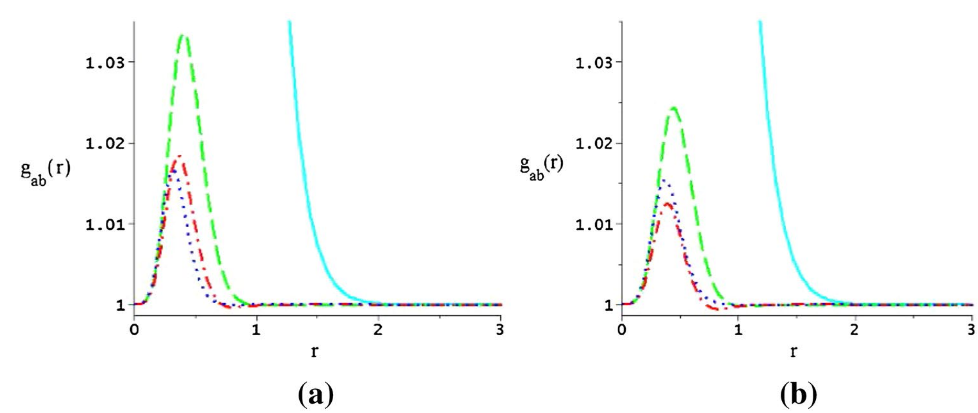

Fig. 6 Second-order correlation function $g_{ab}^{(2)}(0)$ of the **a** even and **b** odd "two-mode photon-added entangled coherent-squeezed states" $|\psi(\alpha,\xi,m,n)\rangle_{e,o}$ versus r ($|\alpha|$ and $|\xi|$) for the different values of photon-additions $m, n = 0$ (*solid*), $m, n = 2$ (*dashed*), $m = 4, n = 2$ (*dash-dot*), and $m = 2, n = 4$ (*dot*)

parameters of the even (odd) "two-mode photon-added entangled coherent-squeezed states" versus r ($|\alpha|$ and $|\xi|$) in Fig. 7 (Fig. 8). It is worth noting that in Figs. 7b and 8b, the s_p and s_x curves coincide with each other.

As it can be observed in Figs. 7a and 8a, squeezing occurs only in the p component for the first mode of the even and odd "entangled coherent-squeezed states" (no photon-additions $m, n = 0$) in Eq. (11) in the region of small r ($|\alpha|$ and $|\xi|$). Moreover, it can be seen that the squeezing in p-component is disappeared with photon-addition.

6 Summary and conclusion

In this paper, we started with a brief review on new types of entangled states named "entangled coherent-squeezed states" which have recently been introduced and generated by us. Next, using these entangled states, we introduced "two-mode photon-added entangled coherent-squeezed states". These states can be obtained via the iterated action of two creation operators $a^\dagger$ and $b^\dagger$ corresponding to the first and second modes of the "entangled coherent-squeezed states", respectively. In the continuation, we studied the amount of the entanglement by evaluating concurrence and some of the nonclassical features of the introduced photon-added entangled states such as quantum statistics, second-order correlation function, and quadrature squeezing, too. Moreover, we considered the influence of two-mode photon-addition on the mentioned properties.

In conclusion, the outcome results are briefly summarized as follows:

- The concurrence (that showing the amount of entanglement) of the even and odd "two-mode photon-added entangled coherent-squeezed states" starts from zero, increases suddenly in the region of small r ($r < 2$), and finally gets its maximum value 1 for all of the considered values of photon-additions m and n. Moreover, the rate of increasing of the entanglement increased with two-mode photon-addition.
- The sub-Poissonian photon-statistics appears for the first mode of the even and odd "two-mode photon-added entangled coherent-squeezed states" for all con-

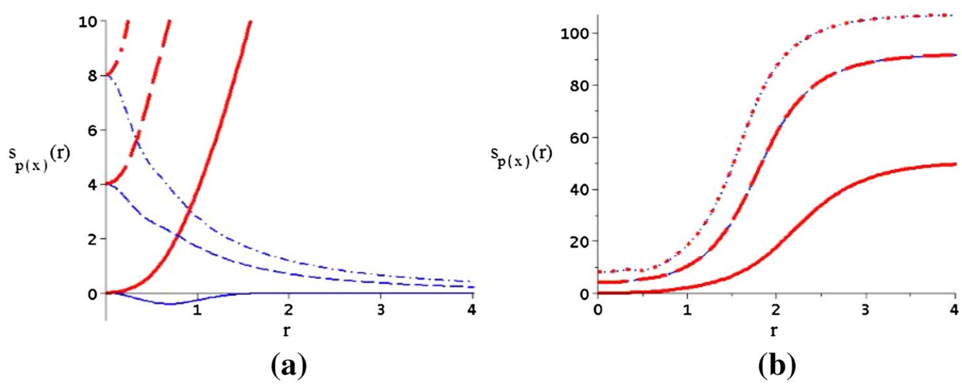

Fig. 7 Squeezing parameters $s_p(r)$ (*thin*) and $s_x(r)$ (*thick*) of **a** the first mode and **b** the second mode of the even "two-mode photon-added entangled coherent-squeezed states" $|\psi(\alpha, \xi, m, n)\rangle_e$ versus r ($|\alpha|$ and $|\xi|$) for the different values of photon-additions $m, n = 0$ (*solid*), $m, n = 2$ (*dashed*), $m = 4, n = 2$ (*dash-dot*), and $m = 2, n = 4$ (*dot*) (in **b**, the s_p and s_x curves coincide with each other. Moreover, the $m, n = 2$ and $m = 2, n = 4$ curves in **a** and the $m, n = 2$ and $m = 4, n = 2$ *curves* in **b** coincide with each other)

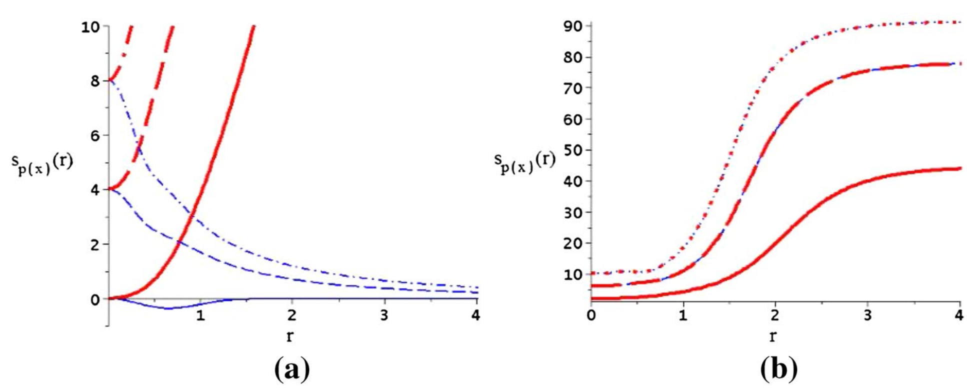

Fig. 8 Squeezing parameters $s_p(r)$ (*thin*) and $s_x(r)$ (*thick*) of **a** the first mode and **b** the second mode of the odd "two-mode photon-added entangled coherent-squeezed states" $|\psi(\alpha, \xi, m, n)\rangle_o$ versus r ($|\alpha|$ and $|\xi|$) for the different photon-additions m and n (all of the parameters and explanations are the same as Fig. 7)

sidered values of photon-additions m and n over all of the regions of r. Furthermore, the sub-Poissonian characteristics of the first mode of these entangled states increase with increasing the photon-addition of the first mode (m). The second mode of the odd "entangled coherent-squeezed state" (no photon-additions $m, n = 0$) and also the second mode of the even and odd "two-mode photon-added entangled coherent-squeezed states" show sub-Poissonian photon-statistics for all considered values of photon-additions m and n in the region of very small r ($|\alpha|$ and $|\xi|$). For the second mode of the even (odd) "two-mode photon-added entangled coherent-squeezed states", the region of sub-Poissonian photon-statistics becomes more slightly wider (slightly narrower) with increasing the photon-addition of the second mode (n).

- The oscillatory behaviour is clearly observed in the photon number distribution of both even and odd "two-mode photon-added entangled coherent-squeezed states".
- Studying the second-order correlation function does not show any nonclassical feature for the even and odd "two-mode photon-added entangled coherent-squeezed states" for all considered values of photon-additions m and n.
- Since the squeezed states (which are the second mode of the introduced entangled states) have no quadrature squeezing, this nonclassical feature is observed only in the p-component for the first mode (which is the coherent state) of the even and odd "entangled coherent-squeezed states" (no photon-additions $m, n = 0$) in the region of small r ($|\alpha|$ and $|\xi|$).

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