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Wigner distribution function of a Lorentz–Gauss vortex beam: alternative approach

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Abstract The Wigner distribution function of a Lorentz–Gauss vortex beam with one topological charge is considered. The topics have been the object of a recent publication. However, we present an alternative compact expressionthat could complement the analysis developed in the aforementioned publication. The deduced expression can be usefully exploited to fully account for the Wigner dynamics of the Lorentz–Gauss vortex beam as far as the paraxial propagation through first-order optical systems is concerned.

1 Introduction

We propose an analytic compact expression for the Wigner distribution function (WDF) of a Lorentz–Gauss vortex beam with one topological charge. The topic has been the subject of a recent publication [1]. Here, the authors, resorting to the approximation of a Lorentzian function by a finite sum of (even-order) Hermite–Gaussian functions, as originally proposed in [2], and to the Collins integral [3, 4], worked out analytic expressions for the WDFs of the Lorentz–Gauss vortex beam at the input plane and after paraxial propagation through a separable optical system accountable by a 2×2 ray matrix. We will deduce a compact closed-form expression for the WDF of the Lorentz–Gauss vortex beam at the input plane. It involves known functions, like the erfc function, which is supported by almost any software. Also, due to the transfer properties of the WDF, it is enough to fully account for the Wigner dynamics of the beam under paraxial propagation through first-order optical systems. The analysis here presented complements that developed in [1] and can usefully be exploited for further investigations.

A definite interest for the Lorentz and Lorentz–Gauss beams clearly emerges from the literature. In fact, after the primary explicit introduction of the Lorentz beams [5] (see also [6, 7]) on the basis of the experimental observations reported in [8, 9], mainly regarding the radiation emitted by single-mode diode lasers [10], various properties of such beams as well as of their Gaussian-modulated version have been widely investigated (see, for instance, [11–15]). In particular, the WDF of Lorentz and Lorentz–Gauss beams has been considered in [16, 17] as that of a super-Lorentz–Gauss beam in [17, 18].

Lorentz–Gauss vortex beams can be produced, for instance, by letting the radiation emitted by a single-mode diode laser to pass through a spiral phase plate, by which one can suitably modulate the wavefront phase of the beam. Carrying on orbital angular momentum, Lorentz–Gauss vortex beams could be appropriate, for example, for optical micro-manipulation, quantum information processing and phase contrast light microscopy [19]. Indeed, several properties of the Lorentz–Gauss vortex beams have been examined; the paraxial propagation through optical systems has been discussed in [20] and, in particular, that through a fractional Fourier transformer in [21]. Also, the propagation through a turbulent atmosphere has been object of analysis in [22], whereas the non-paraxial propagation in uniaxial crystals whose optical axis is orthogonal to the propagation axis has been considered in [23].

As is well known, the WDF is a basic tool for the hybrid, i.e., space/spatial frequency, representation of optical signals [24–30]. The "quantum-like" pairs of Fourier

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conjugate space/spatial-frequency variables span the 4D *wave-optical phase space*. In it, following the replacement of light rays by optical wave functions as descriptors of optical disturbances, light signals with the relevant propagation "dynamics" are described by suitable phase-space distribution functions, among which the WDF is definitely the most important one. It represents the wave-optical tool closest to the geometric-optical concept of light ray, due to its localization properties and dynamical behavior, which under paraxial propagation through real (first-order) optical systems is ruled by the same transfer law of ray optics. The WDF retains in fact the information about the optical signal as conveyed by the relative wave-optical description while obeying the simple rules of evolution according to the corresponding ray-optical approach. It then accommodates for both the formal and conceptual simplicity of geometrical optics and the completeness of wave optics.

In Sect. 2, we deduce a closed-form expression for the WDF of a Lorentz–Gauss vortex beam with one topological charge. The analysis partly grounds on the results presented in [17], where the author proposed a closed-form expression for the WDF of a Lorentz–Gauss beam. It also exploits the symmetric correspondences between the x- and y-coordinate dependences in the input field. Then, in Sect. 3 we resort to the aforementioned transfer law obeyed by the WDF under paraxial propagation through real optical systems. As is well known, it strictly relates to the evolution of the "classical-like" pair of optical variables, namely the light-ray coordinates of geometrical optics, which evolution is in turn ruled by the system ray matrix. Thus, the WDF retains its formal expression under propagation, the arguments being basically transformed as the ray variables. For exemplificative purposes, we will show the Wigner charts of the Lorentz–Gauss vortex beam on some 2D planes at the initial z-position z_i and at some subsequent z-position under free propagation. In general, the expression of the WDF at the initial plane retains its crucial role in determining the WDF under paraxial propagation also through complex optical systems, although by means of a convolution-based evolution law. Therefore, as far as the paraxial propagation is concerned, the expression of the WDF at the initial plane z_i is all one needs to fully account for the Wigner dynamics at subsequent $z > z_i$. Concluding notes are given in Sect. 4.

2 The Wigner distribution function of a Lorentz–Gauss vortex beam: a closed-form expression

A Lorentz–Gaussian vortex beam with one topological charge can be represented by the wave function

$$\varphi_0(x, y) = \mathfrak{C} \frac{x + iy}{(1 + \frac{x^2}{w_x^2})(1 + \frac{y^2}{w_y^2})} e^{-\frac{x^2 + y^2}{2w_G^2}}, \quad (1)$$

at the initial plane conventionally taken at $z_i = 0$. Here, x and y address Cartesian coordinates in that plane as well as in any (transverse) plane along the z-direction of propagation. The arbitrary constant $\mathfrak{C}$ serves to give φ_0 the desired dimension. Thus, if one wished to work with a dimensionless quantity, $\mathfrak{C}$ could be fixed to have the dimensions of length^{-1} as conveyed, for instance, by $1/\sqrt{w_x w_y}$. This choice will be implicitly taken throughout the paper. If on the other hand a normalized wave function is desired, for the explicit expression of $\mathfrak{C}$ one can benefit from the fact that

$$\|\varphi_0\|^2 = |\mathfrak{C}|^2 \frac{\pi}{2} w_x w_y e^{\frac{w_x^2 + w_y^2}{2w_G^2}} \left[N(w_x, w_y, w_G) + N(w_y, w_x, w_G) \right],$$

where

$$N(w_x, w_y, w_G) = \sqrt{w_x^3 w_y} W_{-\frac{5}{4}, -\frac{1}{4}} \left(\frac{w_x^2}{w_G^2} \right) W_{-\frac{3}{4}, -\frac{3}{4}} \left(\frac{w_y^2}{w_G^2} \right),$$

and $W_{\kappa, \mu}$ denotes Whittaker's second function [31][1]. Finally, the parameters w_x, w_y and w_G relate to the widths of the Lorentzian and Gaussian parts of the wave function; in fact, the two limits of purely Lorentzian or purely Gaussian vortex beams are recovered for $w_G \to \infty$ and $w_x, w_y \to \infty$, respectively.

As is well known, for any pair of functions $\psi(x, y)$, $\phi(x, y)$, the relevant WDF is conveyed by the 2D Fourier-like integral

$$\mathcal{W}_{\psi, \phi}(x, y, k_x, k_y) = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi^*(x - x', y - y') \phi$$
$$(x + x', y + y') e^{-2ik_x x' - 2ik_y y'} \mathrm{d}x' \mathrm{d}y', \quad (2)$$

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[1] We recall that Whittaker's first and second functions, $M_{\kappa, \mu}(s)$ and $W_{\kappa, \mu}(s)$, are solutions of the second-order differential equation

$$\frac{d^2}{ds^2} f(s) + \left(\frac{-1}{4} + \frac{\kappa}{s} + \frac{\frac{1}{4} - \mu^2}{s^2} \right) f(s) = 0,$$

whose standard solutions are in fact

$$M_{\kappa, \mu}(s) = e^{-\frac{s}{2}} s^{\frac{1}{2} + \mu} \, _1F_1 \left(\frac{1}{2} + \mu - \kappa; 1 + 2\mu; s \right),$$

$$W_{\kappa, \mu}(s) = e^{-\frac{s}{2}} s^{\frac{1}{2} + \mu} \, U \left(\frac{1}{2} + \mu - \kappa; 1 + 2\mu; s \right),$$

except that $M_{\kappa, \mu}(s)$ does not exist for $2\mu = -1, -2, \dots$ Here, $_1F_1(a, c; s) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{s^n}{n!}$ and $U(a; c; s) = \frac{\pi}{\sin(\pi c)}$ $\frac{_1F_1(a, c; s)}{\Gamma(c) \Gamma(1 + a - c)} - s^{1 - c} \frac{_1F_1(a + 1 - c, 2 - c; s)}{\Gamma(a) \Gamma(2 - c)}$ denote Kummer's functions of the first and second kind, respectively, Γ being the gamma function.

where k_x and k_y denote the Fourier conjugate variables to x and y, i.e., the spatial frequencies. The above is generally addressed to as the cross-WDF of ψ and ϕ, turning into the (self-)WDF $\mathcal{W}_\psi$ when $\psi = \phi$. The dependence on the propagation variable z is implied in (2). However, due to the well-known evolution law obeyed by the WDF under paraxial propagation, the knowledge of the WDF at the "initial" plane $z_i = 0$ amounts to the knowledge of the WDF at any subsequent plane $z > 0$. Indeed, we will evaluate the WDF of the initial wave function (1); later, we will consider its evolution under paraxial propagation through a first-order optical system.

In view of evaluating the WDF of φ_0, we rewrite it as

$$\varphi_0(x, y) \equiv \varphi_1(x, y) + i\varphi_2(x, y); \tag{3}$$

the identification of φ_1 and φ_2 will be fixed later (see Eq. (5)).

The WDF will then be composed of the self- and cross-terms according to

$$\begin{aligned} \mathcal{W}_{\varphi_0}(x, y, k_x, k_y) =& \mathcal{W}_{\varphi_1}(x, y, k_x, k_y) + \mathcal{W}_{\varphi_2}(x, y, k_x, k_y) \\ &+ i\mathcal{W}_{\varphi_1,\varphi_2}(x, y, k_x, k_y) - i\mathcal{W}_{\varphi_2,\varphi_1}(x, y, k_x, k_y), \end{aligned} \tag{4}$$

with, of course, $\mathcal{W}_{\varphi_1,\varphi_2} = \mathcal{W}_{\varphi_2,\varphi_1}^*$.

For each component in (3), the dependence on x and y is factorizable. Therefore, the self-WDFs $\mathcal{W}_{\varphi_1}$ and $\mathcal{W}_{\varphi_2}$ turn out to be the product of the WDFs of the x- and y-depending functions.

Specifically, we have

$$\begin{aligned} \varphi_1(x, y) =& \Phi_1(x)\Psi_1(y), \\ \varphi_2(x, y) =& \Theta_2(x)\Xi_2(y), \end{aligned} \tag{5}$$

and

$$\begin{aligned} \mathcal{W}_{\varphi_1}(x, y, k_x, k_y) =& \mathcal{W}_{\Phi_1}(x, k_x)\mathcal{W}_{\Psi_1}(y, k_y), \\ \mathcal{W}_{\varphi_2}(x, y, k_x, k_y) =& \mathcal{W}_{\Theta_2}(x, k_x)\mathcal{W}_{\Xi_2}(y, k_y). \end{aligned} \tag{6}$$

It is easy to realize that $\Psi_1(y)$ and $\Theta_2(x)$ are both Lorentz–Gauss functions, namely both are like

$$\Psi(s) = \frac{1}{1 + \frac{s^2}{w^2}} e^{-\frac{s^2}{2w_G^2}}, \quad s = x, y, \ w = w_x, w_y. \tag{7}$$

Correspondingly, $\Phi_1(x)$ and $\Xi_2(y)$ have the same form as well, being both conveyed by the product of a Lorentzian function and an Hermite–Gaussian function, namely both are like

$$\Phi(s) = \mathfrak{C}\frac{1}{(1 + \frac{s^2}{w^2})} s e^{-\frac{s^2}{2w_G^2}} = \mathfrak{C}\frac{w_G/2}{(1 + \frac{s^2}{w^2})} H_1\left(\frac{s}{w_G}\right) e^{-\frac{s^2}{2w_G^2}}. \tag{8}$$

The dependence $s e^{-\frac{s^2}{2w_G^2}}$ can be understood as an Hermite–Gaussian function by virtue of the fact that the Hermite polynomial of order 1 is just $H_1(s) = 2s$ [31].

It is therefore evident that

$$\mathcal{W}_{\Theta_2} = \mathcal{W}_{\Psi_1} \quad \text{and} \quad \mathcal{W}_{\Phi_1} = \mathcal{W}_{\Xi_2} \tag{9}$$

under the interchanging $x \longleftrightarrow y$, $k_x \longleftrightarrow k_y$ and $w_x \longleftrightarrow w_y$.

Then, under the same interchanging of variables and parameters, we also have that

$$\mathcal{W}_{\varphi_1} = \mathcal{W}_{\varphi_2}. \tag{10}$$

Let us examine now the cross-term. It factorizes as well into the product of two cross-WDFs as

$$\mathcal{W}_{\varphi_1,\varphi_2}(x, y, k_x, k_y) = \mathcal{W}_{\Phi_1,\Theta_2}(x, k_x)\mathcal{W}_{\Psi_1,\Xi_2}(y, k_y), \tag{11}$$

with clearly

$$\mathcal{W}_{\Phi_1,\Theta_2} = \mathcal{W}_{\Psi_1,\Xi_2}^*, \tag{12}$$

under the interchanging $x \longleftrightarrow y$, $k_x \longleftrightarrow k_y$ and $w_x \longleftrightarrow w_y$.

In conclusion, we are left with the evaluation of $\mathcal{W}_{\Phi_1}$ or equivalently of $\mathcal{W}_{\Xi_2}$, and of $\mathcal{W}_{\Phi_1,\Theta_2}$ or equivalently of $\mathcal{W}_{\Psi_1,\Xi_2}$. This amounts to the evaluation of $\mathcal{W}_\Phi$ and $\mathcal{W}_{\Phi,\Psi}$. For completeness' sake we will review the steps leading to the result presented in [17], concerning the WDF of a Lorentz–Gauss function, i.e., $\mathcal{W}_\Psi$; as we will see, its expression will be central to the evaluation of the other WDFs of concern here.

2.1 WDF of the Lorentz–Gauss function

The WDF of Ψ reads as

$$\begin{aligned} \mathcal{W}_\Psi(s, k) = &\frac{1}{\pi} e^{-\frac{s^2}{w_G^2}} \\ &\int_{-\infty}^{+\infty} \frac{1}{[1 + (s - s')^2/w^2][1 + (s + s')^2/w^2]} e^{-\frac{s'^2}{w_G^2}} e^{-2iks'} ds'. \end{aligned} \tag{13}$$

The product of the two shifted Lorentzian functions entering the above integral can be properly managed to yield

$$\frac{1}{[1 + (s - s')^2/w^2][1 + (s + s')^2/w^2]} = \frac{w^4}{4iws}\left[\frac{1}{A^2 + s'^2} - \frac{1}{B^2 + s'^2}\right]$$

where

$$A = w - is, \quad B = w + is = A^*.$$

Accordingly, expression (13) turns into

$$\begin{aligned} \mathcal{W}_\Psi(s, k) = &\frac{1}{4i\pi s/w^3} e^{-\frac{s^2}{w_G^2}} \\ &\int_{-\infty}^{+\infty} \left[\frac{1}{A^2 + s'^2} - \frac{1}{B^2 + s'^2}\right] e^{-\frac{s'^2}{w_G^2}} e^{-2iks'} ds', \end{aligned} \tag{14}$$

showing the difference of two integrals of the same form, i.e.,

$$\mathcal{I}(s,k) = \int_{-\infty}^{+\infty} \frac{1}{\mathcal{A}^2 + y^2} e^{-\frac{y^2}{w_G^2}} e^{-2iky} dy, \tag{15}$$

where $\mathcal{A}$ denotes A or B. The evaluation of $\mathcal{I}(s,k)$ leads to the expression [17]

$$\mathcal{I}(s,k) = \frac{\pi}{2\mathcal{A}} e^{-k^2 w_G^2} [\mathcal{E}(s,k) + \mathcal{E}(s,-k)], \tag{16}$$

where

$$\mathcal{E}(s,k) = e^{\mathcal{X}(s,k)^2} \text{erfc}(\mathcal{X}(s,k)), \tag{17}$$

with erfc denoting the complementary error function [31], and

$$\mathcal{X}(s,k) = \frac{1}{w_G}(\mathcal{A} + k w_G^2), \quad \mathcal{A} = w \pm is. \tag{18}$$

Using $\mathcal{I}(s,k)$ in (14), we end up with the expression [17]

$$\mathcal{W}_\Psi(s,k) = \frac{1}{4s/w^2} e^{-\frac{s^2}{w_G^2} - k^2 w_G^2} \mathcal{L}(s,k), \tag{19}$$

for the WDF of the Lorentz–Gauss function (7). Here,

$$\mathcal{L}(s,k) = \text{Im}\left\{ \frac{1}{1 - is/w} [\mathcal{E}(s,k) + \mathcal{E}(s,-k)] \right\} \tag{20}$$

where Im signifies imaginary part and the function $\mathcal{X}(s,k)$ is taken for $\mathcal{A} = A = w - is$, namely as

$$\mathcal{X}(s,k) = \frac{1}{w_G}\left(w - is + k w_G^2 \right) = \beta\left(1 - i\frac{s}{w} \right) + k w_G, \quad \beta \equiv \frac{w}{w_G}. \tag{21}$$

It is worth noting that

$$\begin{aligned} \mathcal{E}(s,k) &= e^{\mathcal{X}(s,k)^2} - \frac{2}{\sqrt{\pi}} \mathcal{X}(s,k) \, {}_1F_1\left(1; \frac{3}{2}; \mathcal{X}(s,k)^2 \right) \\ &= \sqrt{\frac{2}{\pi}} e^{\frac{\mathcal{X}(s,k)^2}{2}} D_{-1}\left(\sqrt{2}\mathcal{X}(s,k) \right), \end{aligned} \tag{22}$$

${}_1F_1$ representing the Kummer's function and D_{-1} the parabolic cylinder function [31]. Evidently, $\mathcal{E}^*(s,k) = \mathcal{E}(-s,k)$.

Clearly, with the replacements $s = y$, $k = k_y$, $w = w_y$ or $s = x$, $k = k_x$, $w = w_x$, $\mathcal{W}_\Psi(s,k)$ conveys $\mathcal{W}_{\Psi_1}$ and $\mathcal{W}_{\Theta_2}$, respectively.

Let us briefly comment on the obtained expression (19) for $\mathcal{W}_\Psi$.

It is immediate to recognize the WDF of the Gaussian part in Ψ, being well known that the WDF of the

Gaussian $e^{-\frac{s^2}{2w_G^2}}$ is $\propto e^{-\frac{s^2}{w_G^2} - k^2 w_G^2}$. So, one can presume that the remaining part of $\mathcal{W}_\Psi$, i.e., $\mathcal{L}(s,k)/s$, be related to the WDF of the Lorentzian dependence. Indeed, the parameter $\beta = \frac{w}{w_G}$, introduced in the expression (21) for $\mathcal{X}(s,k)$, fixes the valence of the Lorentzian and Gaussian dependence in Ψ and hence in φ_0. Evidently, $\beta \gg 1$ amounts to $w \gg w_G$, thus yielding the "Gaussian" limit, where Ψ is mainly dominated by the Gaussian dependence. In fact, in the limit $\beta \to \infty$ (i.e., $w \to \infty$, w_G finite), $\Psi(s) \to e^{-\frac{s^2}{2w_G^2}}$. Conversely, $\beta \ll 1$ amounts to $w_G \gg w$, thus yielding the "Lorentzian" limit, where Ψ is mainly dominated by the Lorentzian dependence; in the limit $\beta \to 0$ (i.e., $w_G \to \infty$, w finite), $\Psi(s) \to \frac{1}{1 + \frac{s^2}{w^2}}$.

In practice, resorting to the asymptotic representation of the erfc function, i.e., [31]

$$\text{erfc}(z) \approx \frac{e^{-z^2}}{\sqrt{\pi} z}, \quad |z| \to \infty, \ |\arg(z)| < \frac{3}{4}\pi,$$

we can easily deduce from (19) the expression for the WDF of a Lorentzian function in the limit $\beta \to 0$ (i.e., $w_G \to \infty$, w finite), which explicitly reads as [17]

$$\begin{aligned} \mathcal{W}_\Psi(s,k) \underset{w_G \to \infty}{\to} \mathcal{W}_L(s,k) = &\frac{1}{2/w} \frac{1}{(1 + s^2/w^2)} e^{-2|k|w} \\ &\times [\cos(2ks) + 2|k|w\text{sinc}(2ks)], \end{aligned} \tag{23}$$

on account of that

$$\mathcal{E}(s,k) + \mathcal{E}(s,-k) \sim 2e^{k^2 w_G^2} e^{-2|k|w(1 - is/w)}; \quad w_G \to \infty, w \text{ finite}. \tag{24}$$

As expected for a WDF, $\mathcal{W}_L$ combines in some way both the space and Fourier domain dependence, $\frac{1}{1 + \frac{s^2}{w_L^2}}$ and $e^{-|k|w_L}$ of Ψ.

Likewise, the WDF of a Gaussian function is recovered from $\mathcal{W}_\Psi(s,k)$ in the limit $\beta \to \infty$ (i.e., $w \to \infty$, w_G finite),

$$\mathcal{W}_\Psi(s,k) \underset{w \to \infty}{\to} \mathcal{W}_G(s,k) = \frac{w_G}{\sqrt{\pi}} e^{-\frac{s^2}{w_G^2} - k^2 w_G^2}, \tag{25}$$

being

$$\mathcal{E}(s,k) + \mathcal{E}(s,-k) \sim \frac{2}{\sqrt{\pi}\beta(1 - is/w)}, \quad w \to \infty, w_G \text{ finite}. \tag{26}$$

Finally, we recall from [17] that, notwithstanding the factor s^{-1}, $\mathcal{W}_\Psi(s,k)$ has no singularity at $s = 0$, as confirmed by the Taylor series expansion of $\mathcal{W}_\Psi(s,k)$ at $s = 0$, through which we obtain for $\mathcal{W}_\Psi(0,k)$ the finite value

$$\mathcal{W}_\Psi(0,k) = \frac{1}{4/w} e^{-k^2 w_G^2} \left\{ \frac{4\beta}{\sqrt{\pi}} + \mathcal{E}(0,k)[1 - 2\beta\mathcal{X}(0,k)] \right.$$
$$\left. + \mathcal{E}(0,-k)[1 - 2\beta\mathcal{X}(0,-k)] \right\}.$$

Plots of $\mathcal{W}_\Psi(s,k)$ are shown in Sect. 3.

2.2 WDF of the Lorentzian modulated by the Hermite–Gaussian function

For the evaluation of the WDF of the function $\Phi(s)$, we should deal with the integral

$$\mathcal{W}_\Phi(s,k) = \frac{|\mathfrak{C}|^2}{\pi} e^{-\frac{s^2}{w_G^2}}$$
$$\int_{-\infty}^{+\infty} \frac{(s-s')(s+s')}{[1+(s-s')^2/w^2][1+(s+s')^2/w^2]} e^{-\frac{s'^2}{w_G^2}} e^{-2iks'} ds'.$$

Since $(s-s')(s+s') = s^2 - s'^2$, it can be recast as

$$\mathcal{W}_\Phi(s,k) = |\mathfrak{C}|^2 \left[s^2 + \frac{d^2}{dk^2} \right] \mathcal{W}_\Psi(s,k). \tag{27}$$

The evaluation of the derivative entering the above expression is very simple; one obtains, in fact,

$$\frac{d}{dk} \mathcal{W}_\Psi(s,k) = \frac{1}{2s/w^3} e^{-\frac{s^2}{w^2} - k^2 w_G^2} \operatorname{Im}[\mathcal{E}(s,k) - \mathcal{E}(s,-k)],$$
$$\frac{d^2}{dk^2} \mathcal{W}_\Psi(s,k) = \frac{1}{s/w^4} e^{-\frac{s^2}{w^2} - k^2 w_G^2} \operatorname{Im}\left\{ \left(1 - i\frac{s}{w}\right) [\mathcal{E}(s,k) + \mathcal{E}(s,-k)] \right\}, \tag{28}$$

according to which $\mathcal{W}_\Phi(s,k)$ turns out to be

$$\mathcal{W}_\Phi(s,k) = \frac{|\mathfrak{C}|^2}{4s/w^4} e^{-\frac{s^2}{w^2} - k^2 w_G^2} \mathcal{H}(s,k), \tag{29}$$

with

$$\mathcal{H}(s,k) = \operatorname{Im}\left\{ \frac{1 - 2is/w}{1 - is/w} [\mathcal{E}(s,k) + \mathcal{E}(s,-k)] \right\}. \tag{30}$$

An alternative expression for $\mathcal{H}(s,k)$, which reproduces the same structure of $\mathcal{L}(s,k)$, is

$$\mathcal{H}(s,k) = \frac{w_G^2}{w^2} \operatorname{Im}\left\{ \frac{1}{1 - is/w} [\mathcal{J}(s,k) + \mathcal{J}(s,-k)] \right\}, \tag{31}$$

with

$$\mathcal{J}(s,k) = \left(\frac{s^2}{w_G^2} + w_G^2 k^2 \right) \mathcal{E}(s,k)$$
$$- \frac{1}{\sqrt{\pi}} e^{\frac{\mathcal{X}(s,k)^2}{2}} \mathcal{X}(s,-k) D_{-2}\left(\sqrt{2}\mathcal{X}(s,k) \right). \tag{32}$$

It grounds on the expression of $\mathcal{E}(s,k)$ in terms of the parabolic cylinder function (Eq. (22)) and the recurrence relation for $D_{-2}(z)$, which reads as $D_{-2}(z) = D_0(z) - zD_{-1}(z) = e^{-\frac{z^2}{4}} - zD_{-1}(z)$. It will be useful when looking for the "Gaussian" limit for $\mathcal{W}_\Phi(s,k)$.

Again, with the replacements $s = x$, $k = k_x$, $w = w_x$ or $s = y$, $k = k_y$, $w = w_y$ in (29), we obtain $\mathcal{W}_{\Phi_1}$ or $\mathcal{W}_{\Xi_2}$, respectively.

Some comments are in order. Like for $\mathcal{W}_\Psi$, the factor s^{-1} does not yield a singularity for $\mathcal{W}_\Phi$, which in fact at $s = 0$ takes on the finite value

$$\mathcal{W}_\Phi(0,k) = |\mathfrak{C}|^2 \frac{1}{4/w^3} e^{-k^2 w_G^2} \left[\frac{4\beta}{\sqrt{\pi}} - \mathcal{E}(0,k)[1 + 2\beta\mathcal{X}(0,k)] \right.$$
$$\left. - \mathcal{E}(0,-k)[1 + 2\beta\mathcal{X}(0,-k)] \right].$$

The Lorentzian limit ($w_G \to \infty$, w finite, i.e., $\beta \to 0$) conveys the expression for the WDF of a Lorentz vortex beam. It is obtained as

$$\mathcal{W}_{LV}(s,k) = |\mathfrak{C}|^2 \frac{1}{2/w^3} \frac{1}{(1+s^2/w^2)} e^{-2|k|w}$$
$$\left[2|k|w(1 + 2s^2/w^2)\operatorname{sinc}(2ks) - \cos(2ks) \right],$$

following from (29) for $\beta \to 0$, which implies the relation (24). Note that, according to (27), we also have that $\mathcal{W}_{LV}(s,k) = |\mathfrak{C}|^2 (s^2 + \frac{1}{4}\frac{d^2}{dk^2})\mathcal{W}_L(s,k)$.

Correspondingly, evaluating the limit of $\mathcal{W}_\Phi(s,k)$ for $\beta \to \infty$ (i.e., $w \to \infty$, w_G finite), we recover the expression of the WDG for the Hermite–Gaussian function of order 1, obtaining in fact

$$\mathcal{W}_\Phi(s,k) \underset{w \to \infty}{\to} \mathcal{W}_{HG}(s,k) = -|\mathfrak{C}|^2 \frac{w_G^3}{2\sqrt{\pi}} e^{-\frac{s^2}{w_G^2} - k^2 w_G^2} \left[1 - 2\left(\frac{s^2}{w_G^2} + k^2 w_G^2 \right) \right], \tag{33}$$

on the basis of the expression (31) for $\mathcal{H}(s,k)$ and the asymptotic representation of the parabolic cylinder function $D_\nu(z) \approx z^\nu e^{-\frac{z^2}{4}}$ for $|z| \to \infty$ and $|\arg(z)| < \frac{3\pi}{4}$, appropriate in our case. The result is consistent with (27), giving indeed $\mathcal{W}_\Phi(s,k) \underset{w \to \infty}{\to} |\mathfrak{C}|^2 [s^2 + \frac{d^2}{dk^2}]\mathcal{W}_G(s,k)$. It conforms also to the general expression for the WDF of the (standard) Hermite–Gaussian function $H_n(s/w_G)e^{-\frac{s^2}{2w_G^2}}$, which reads as

$$\mathcal{W}(s,k) = (-1)^n \frac{2^n n!}{\sqrt{\pi}} w_G e^{-\frac{s^2}{w_G^2} - k^2 w_G^2} L_n\left[2\left(\frac{s^2}{w_G^2} + k^2 w_G^2 \right) \right], \tag{34}$$

where L_n denotes the Laguerre polynomial of order n; in particular, $L_1 = 1 - x$.

Plots of $\mathcal{W}_\Phi(s,k)$ are shown in Sect. 3.

The occurrence of the Lorentz–Hermite–Gauss function (in particular of order 1) naturally arises from the present approach to the evaluation of the Wigner integral for φ_0.

An analysis of the behavior of the Lorentz–Hermite–Gaussian functions vs. s reveals that for $w < w_G$ the functions retain the basic features of the relevant Hermite–Gaussian functions but are squeezed around the origin. This could suggest using of the Lorentz–Hermite–Gaussian beams in applications where Hermite–Gaussian-beam features are required but across a more limited space range. On the other hand, mathematically they can help the evaluation of the WDF of beams where vortices of arbitrary topological charge ≥ 1 are embedded, for instance, in a Lorentz–Gaussian beam. In such a case, the factor $(x + iy)$ entering φ_0 would be replaced by $(x + iy)^n$. The nice relation

$$(x + iy)^n = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} i^j H_{n-j}(x) H_n(y), \qquad (35)$$

would make easier the evaluation of the WDF of a Lorentz–Gaussian beam with a vortex of topological charge n. In fact, since the WDF of Hermite–Gaussian beams is well known (Eq. (34)), one can conveniently resort to the property that the WDF of a product of functions is conveyed by the convolution of the WDFs of the single functions in the spatial-frequency domain (Sect. 3). Even better, if one wishes to consider a pair of vortices embedded in the Lorentz–Gauss beam, one with topological charge $\varepsilon_1 n$ ($\varepsilon_1 = \pm 1$) located at $x = -x_0$, $y = 0$ and another with topological charge $\varepsilon_2 m$ ($\varepsilon_2 = \pm 1$) at $x = x_0$, $y = 0$. In such a case, the factor $(x + x_0 + i\varepsilon_1 y)^n (x - x_0 + i\varepsilon_2 y)^m$ would replace the factor $(x + iy)$ in φ_0. Relation (35) along with the aforementioned property of the WDF would be useful for the "elegant" evaluation of the relevant WDF as well.

2.3 Cross-WDF of the Lorentz–Gauss and Lorentz–Hermite–Gauss functions

The cross-WDFs $\mathcal{W}_{\Phi_1,\Theta_2}$ and $\mathcal{W}_{\Psi_1,\Xi_2}$ signify the cross-WDF of the functions $\Phi(s)$ and $\Psi(s)$, and its complex conjugate. We should deal then with the integral

$$\mathcal{W}_{\Phi,\Psi}(s,k) = \frac{\mathfrak{C}}{\pi} e^{-\frac{s^2}{w_G^2}} \int_{-\infty}^{+\infty} \frac{(s-s')}{[1+(s-s')^2/w^2][1+(s+s')^2/w^2]} e^{-\frac{s'^2}{w_G^2}} e^{-2iks'} \mathrm{d}s'$$
$$= \mathfrak{C}\left(s - \frac{i}{2}\frac{d}{dk}\right)\mathcal{W}_{\Psi}(s,k). \qquad (36)$$

Using the first of (28), we readily obtain

$$\mathcal{W}_{\Psi,\Phi}(s,k) = \frac{\mathfrak{C}}{4s/w^3} e^{-\frac{s^2}{w_G^2} - k^2 w_G^2}[(s/w)\mathcal{L}(s,k) - i\mathcal{C}(s,k)], \qquad (37)$$

with

$$\mathcal{C}(s,k) = \mathrm{Im}[\mathcal{E}(s,k) - \mathcal{E}(s,-k)]. \qquad (38)$$

The alternative expression for $\mathcal{C}(s,k)$ can be worked out as

$$\mathcal{C}(s,k) = -\frac{w_G}{w}\mathrm{Im}\left\{\frac{1}{1-is/w}[\mathcal{Y}(s,k) - \mathcal{Y}(s,-k)]\right\},$$

with

$$\mathcal{Y}(s,k) = w_G k \mathcal{E}(s,\kappa) + \sqrt{\frac{1}{\pi}} e^{\frac{1}{2}\mathcal{X}(s,\kappa)^2} D_{-2}\left(\sqrt{2}\mathcal{X}(s,\kappa)\right).$$

At $s = 0$, the function $\mathcal{W}_{\Psi,\Phi}$ takes on the value

$$\mathcal{W}_{\Psi,\Phi}(0,k) = \mathfrak{C}\frac{i\beta}{2/w^2} e^{-k^2 w_G^2}[\mathcal{X}(0,k)\mathcal{E}(0,k) - \mathcal{X}(0,-k)\mathcal{E}(0,-k)].$$

It is worth noting that, conforming to the reality of the involved functions, the complex conjugate $\mathcal{W}^*_{\Phi,\Psi}$, needed to finalize the expression (11) for the cross-WDF, is simply obtained also by the replacement $k \to -k$, as even conveyed by (36).

Within the present context, the cross-WDF $\mathcal{W}_{\Psi,\Phi}$ has no interest itself, but only when used to finalize the expression for $\mathcal{W}_{\varphi_0}$, and hence its evolution will not be considered. Therefore, we show in Fig. 1 the (s / w, kw)-contour plots of its real and imaginary parts, and its squared amplitude for two values of β. As in the following plots, the space variable is scaled by the Lorentzian width parameter w, and accordingly, the corresponding dimensionless spatial frequency is kw.

2.4 WDF of the Lorentz–Gauss vortex beam

Let us combine together the above results in order to specify the expression for $\mathcal{W}_{\varphi_0}(x, k_x, y, k_y)$.

Firstly, on account of the expressions (19) and (29) and of the relation (10), we see that the self-terms convey the expression

$$\mathcal{W}_{\varphi_1}(x,y,k_x,k_y) + \mathcal{W}_{\varphi_2}(x,y,k_x,k_y) = \frac{|\mathfrak{C}|^2}{(4x/w_x^3)(4y/w_y^3)} e^{-\frac{x^2+y^2}{w_G^2} - w_G^2(k_x^2+k_y^2)}$$
$$\left\{\frac{w_x}{w_y}\mathcal{H}(x,k_x)\mathcal{L}(y,k_y) + \frac{w_y}{w_x}\mathcal{L}(x,k_x)\mathcal{H}(y,k_y)\right\} \qquad (39)$$

Then, as to the cross-terms, we note that

$$i[\mathcal{W}_{\varphi_1,\varphi_2}(x,y,k_x,k_y) - \mathcal{W}_{\varphi_2,\varphi_1}(x,y,k_x,k_y)] = -2\mathrm{Im}(\mathcal{W}_{\varphi_1,\varphi_2}).$$

Hence, on the basis of the relations (11) and (12) and the expression (37), it is seen to be

$$-2\mathrm{Im}(\mathcal{W}_{\varphi_1,\varphi_2}) = \frac{2|\mathfrak{C}|^2}{(4x/w_x^3)(4y/w_y^3)} e^{-\frac{x^2+y^2}{w_G^2} - w_G^2(k_x^2+k_y^2)}.$$
$$\left\{\frac{x}{w_x}\mathcal{L}(x,k_x)\mathcal{C}(y,k_y) - \frac{y}{w_y}\mathcal{L}(y,k_y)\mathcal{C}(x,k_x)\right\} \qquad (40)$$

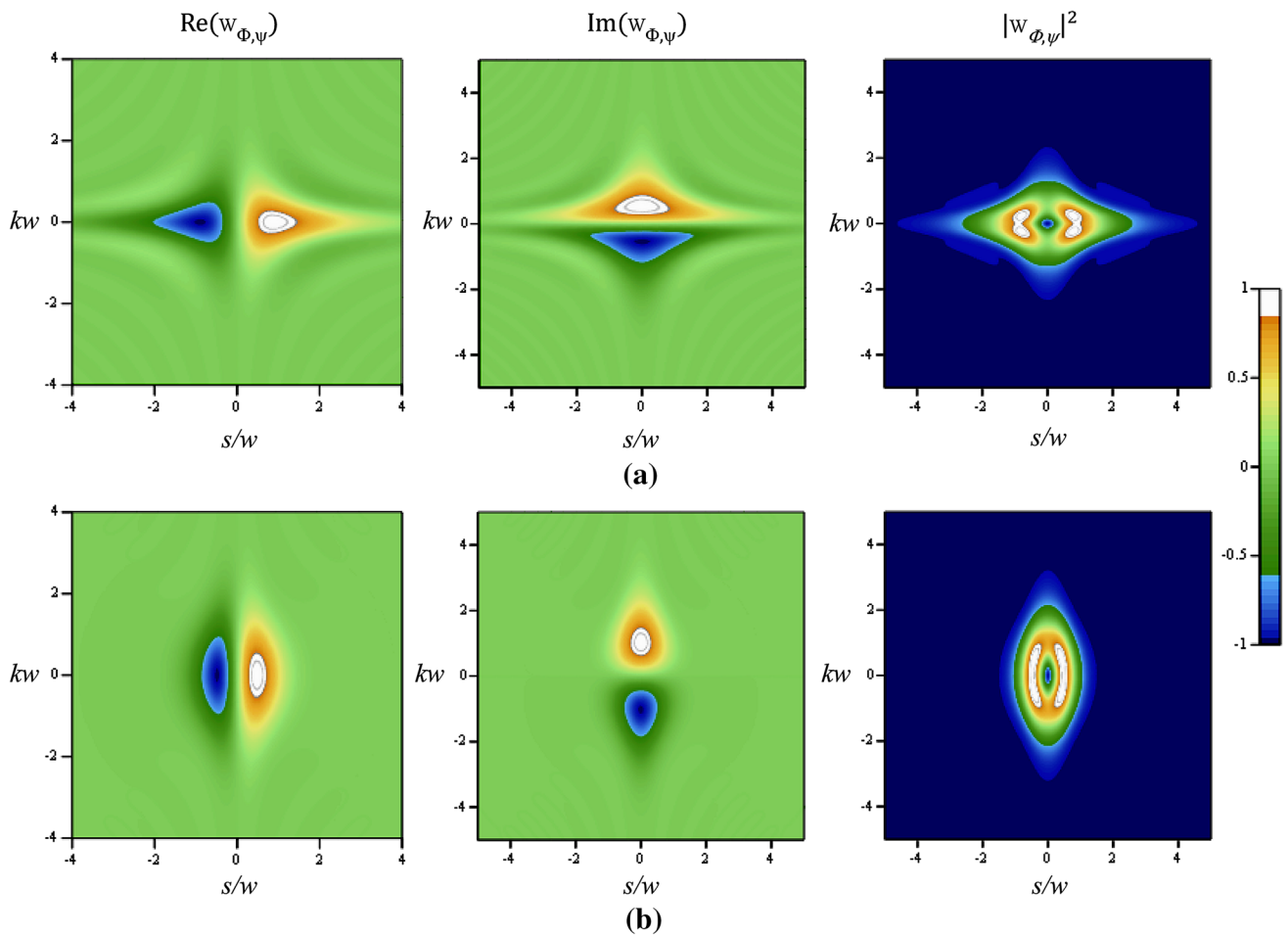Finally, using (39) and (40) in (4), we end up with

Fig. 1 $(s/w, kw)$-contour plots of the real part, the imaginary part and the squared amplitude of the cross-WDF $\mathcal{W}_{\Phi,\Psi}$ for **(a)** $\beta = 0.2$ and **(b)** $\beta = 1$

$$\mathcal{W}_{\varphi_0}(x, y, k_x, k_y) = \frac{|\mathcal{C}|^2}{(4x/w_x^3)(4y/w_y^3)} e^{-\frac{x^2+y^2}{w_G^2} - w_G^2(k_x^2+k_y^2)}.$$

$$\{ \mathcal{V}(y, k_y, w_y, x, w_x)\mathcal{L}(x, k_x) + \mathcal{V}(x, k_x, w_x, -y, w_y)\mathcal{L}(y, k_y) \}. \tag{41}$$

where

$$\mathcal{V}(y, k_y, x, w_y, w_x) = \frac{w_y}{w_x}\mathcal{H}(y, k_y) + \frac{2x}{w_x}\mathcal{C}(y, k_y), \tag{42}$$

with $\mathcal{H}$ given by (30). The validity of the above formula has been checked by a direct evaluation of the 2D Wigner integral.

Expression (41) represents the main result of this note, offering indeed a compact closed-form expression for the WDF of the Lorentz–Gauss vortex beam (1), in terms of known functions, like the erfc function.

We finally note that the alternative expression (31) for $\mathcal{H}$ would yield the equivalent expression for $\mathcal{W}_{\varphi_0}$ as

$$\mathcal{W}_{\varphi_0}(x, y, k_x, k_y) = |\mathcal{C}|^2 \frac{w_G^2}{(4x/w_x^2)(4y/w_y^2)} e^{-\frac{x^2+y^2}{w_G^2} - w_G^2(k_x^2+k_y^2)}.$$

$$\cdot \{ \mathcal{P}(x, y, k_y)\mathcal{L}(x, k_x) + \mathcal{P}(-y, x, k_x)\mathcal{L}(y, k_y) \}, \tag{43}$$

where

$$\mathcal{P}(x, y, k_y) = \mathcal{H}(y, k_y) + \frac{2x}{w_G}\mathcal{C}(y, k_y). \tag{44}$$

It looks more symmetric than (41). However, the latter is perhaps more suited for numerical implementation, resorting only to the erfc function; typically, software easily deal with the erfc function (even with complex arguments) while rather hardly manage the parabolic cylinder function (or, equivalently the Kummer's function), especially when complex arguments are involved.

3 Wigner charts for Lorentz–Gauss vortex beams

Expression (41) allows us to determine the WDF of the Lorentz–Gaussian vortex beam, represented at the input plane by (1), under paraxial propagation through any optical system.

As is well known, in fact, the WDF obeys the transfer law [11, 25, 26]

$$\mathcal{W}(\mathbf{v}, z_o) = \mathcal{W}(\overline{\mathbf{M}}^{-1}\mathbf{v}, z_i), \tag{45}$$

when the corresponding wavefield paraxially propagates through a real optical system. Here, the array $\mathbf{v}$ conveniently gathers the WDF variables, namely the Fourier conjugate pairs (x, k_x) and (y, k_y) as $\mathbf{v}^\top = (x, y, k_x, k_y)$; z_i and z_o identify the z-locations respectively of the input and output planes, between which the optical system is understood to operate, transporting the wave function on the plane at z_i to the wave function on the plane at z_o. Finally, $\overline{\mathbf{M}} = \begin{pmatrix} \mathbf{A} & \mathbf{B}/k \\ k\mathbf{C} & \mathbf{D} \end{pmatrix}$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are 2×2 matrices, is the 4×4 (symplectic) ray transfer matrix $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$, up to the scaling by the field wavenumber k. According to the geometrical optics formalism, the ray matrix $\mathbf{M}$ synthesizes the paraxial propagation through an optical system in terms of a linear transformation of the ray variables, namely the position $\mathbf{q} = (x, y)$ of the intersection point of the ray with the reference plane and the momentum $\mathbf{p} = (p_x, p_y)$ (basically the angles at $\mathbf{q}$, that the projections of the ray on the (x, z) and (y, z) planes form with the z-axis). The relation between the ray momentum p_s and the spatial frequency k_s, which reads as $k_s = kp_s$, $s = x, y$, imposes the k-scaling of the matrices $\mathbf{B}$ and $\mathbf{C}$ (whose entries have respectively dimensions of length and length^{-1}) to account for the natural variables of the WDF.

In particular, the free propagation from $z_i = 0$ to z can be analyzed separately with respect the x and y coordinates. In such a case, in fact, $\mathbf{A} = \mathbf{D} = \mathbf{I}$, $\mathbf{C} = \mathbf{0}$ and $\mathbf{B} = z\mathbf{I}$. Accordingly, the WDF for the Lorentz–Gauss vortex beam (1) propagating in free space is simply given by

$$\mathcal{W}_{\text{LGV}}(x, y, k_x, k_y, z) = \mathcal{W}_{\text{LGV}}\left(x - \frac{z}{k}k_x, y - \frac{z}{k}k_y, k_x, k_y, 0\right),$$

where, of course, $\mathcal{W}_{\text{LGV}}(x, y, k_x, k_y, 0) = \mathcal{W}_{\varphi_0}(x, y, k_x, k_y)$.

For exemplificative purposes, we firstly show in Fig. 2 the Wigner charts of the WDF $\mathcal{W}_\Psi$ of the Lorentz–Gauss beam (7) at $z = 0$ and $z = kw^2$ for some values of β; precisely, for $\beta = 0$, amounting to a purely Lorentzian beam, $\beta = 0.5$ and $\beta = 1$. We see that with increasing β the contour lines in the Wigner plane (s, k) tend to become ellipses, which are well known to characterize the Wigner chart of the WDF of a Gaussian beam. As said, throughout in the plots, we will deal with normalized coordinates, namely s/w and kw, thus implying that the longitudinal coordinate z is normalized to the inherent diffraction length $z_{\text{d}} = kw^2$, i.e., z/z_{d}.

Correspondingly, Fig. 3 shows the Wigner charts of the WDF $\mathcal{W}_\Phi$ of the Lorentz–Hermite–Gauss function (8) at $z = 0$ and $z = kw^2$ for the same values of β as in Fig. 2. In both cases, as expected, the free propagation amounts to an s-shear of the Wigner chart in the Wigner phase plane (s, k).

Evidently, the dynamics of $\mathcal{W}_{\text{LGV}}$ would be much more complex involving four variables. The panels in Fig. 4 display the contour plots of $\mathcal{W}_{\text{LGV}}$ on some fixed 2D planes at $z = 0$ and $z = kw^2$. In the graphs, we have fixed $\beta_x = \beta_y = \frac{1}{\sqrt{2}}$, and once again normalized the space and Fourier variables by the Lorentzian width parameter $w = w_x = w_y$. The behavior of $\mathcal{W}_{\text{LGV}}$ is somewhat intermediate between that of $\mathcal{W}_\Psi$ and $\mathcal{W}_\Phi$. Interestingly, on the $(x/w, k_x w)$-plane for fixed values of y/w and $k_y w$ (frames (c) and (d) in the figure), we can recognize an x-shear of the Wigner chart, whereas a rotation emerges from frame (a), conveying the Wigner chart on the $(k_x w, k_y w)$-plane for $x/w = y/w = 0$. The correctness of the displayed plots has been checked by the direct evaluation of the 2D Wigner integral (2) with $\psi = \phi = \varphi_0$.

More in general, the evolution of the WDF under paraxial propagation through an optical system is conveyed by the relation [11, 25, 26]

$$\mathcal{W}(x, y, k_x, k_y, z_o) = \iiiint \mathsf{G}(x, y, k_x, k_y, x', y', k'_x, k'_y) \\ \mathcal{W}(x', y', k'_x, k'_y, z_i) \mathrm{d}x' \mathrm{d}y' \mathrm{d}k'_x \mathrm{d}k'_y, \tag{46}$$

whose kernel $\mathsf{G}(x, y, k_x, k_y, x', y', k'_x, k'_y)$ is given by

$$\mathsf{G}(x, y, k_x, k_y, x', y', k'_x, k'_y) = \\ \frac{1}{4\pi^2} \iiiint \mathsf{g}^*\left(x - \frac{x''}{2}, y - \frac{y''}{2}, x' - \frac{x'''}{2}, y' - \frac{y'''}{2}\right) \mathsf{g}\left(x + \frac{x''}{2}, y + \frac{y''}{2}, x' + \frac{x'''}{2}, y' + \frac{y'''}{2}\right) \\ \cdot e^{-i(k_x x'' - k'_x x''')} e^{-i(k_y y'' - k'_y y''')} \mathrm{d}x'' \mathrm{d}y'' \mathrm{d}x''' \mathrm{d}y'''. \tag{47}$$

Fig. 2 Wigner charts of the WDF $\mathcal{W}_\psi$ of the Lorentz–Gauss function (7) at $z = 0$ (on the *left*) and $z = kw^2$ (on the *right*) for **(a)** , **(b)** $\beta = 0$, **(c)**, **(d)** $\beta = 0.5$, **(e)**, **(f)** $\beta = 1$
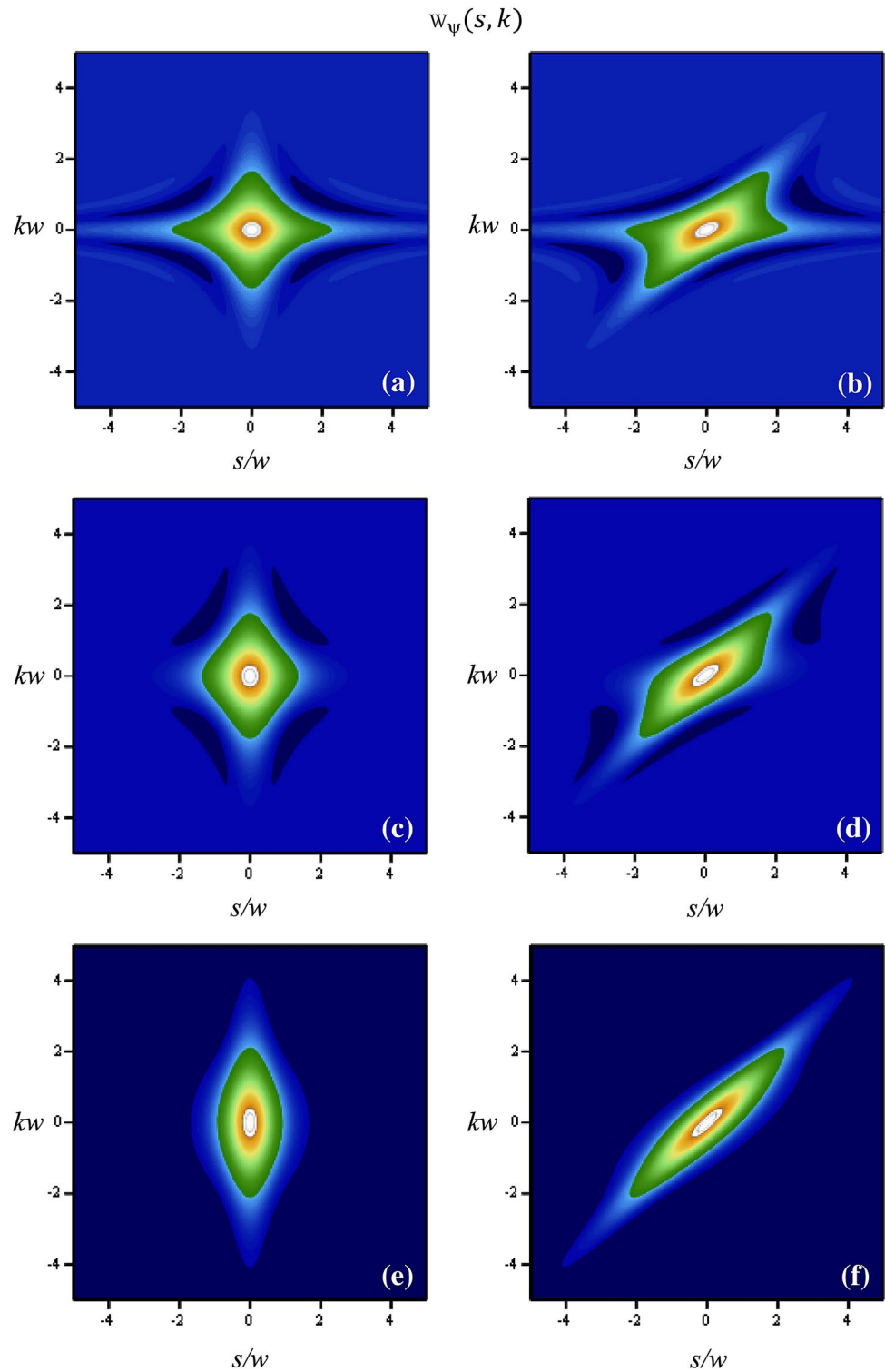
It is completely determined by the point-spread function $g(x, y, x', y')$ of the system of concern, which in turn rules the evolution of the wave function propagating through the system, being the kernel of the Collins (or, Huygens–Fresnel diffraction) integral [3, 4].

Indeed, the phase-space transfer relation (46) for the WDF is paralleled by the space-domain transfer relation for the wave function,

$$\varphi(x, y, z_o) = \iint g(x, y, x', y') \varphi(x', y', z_i) \mathrm{d}x' \mathrm{d}y'.$$

Fig. 3 Wigner charts of the WDF $\mathcal{W}_\Phi$ of the Lorentz–Hermite–Gauss function (8) at $z = 0$ (on the *left*) and $z = kw^2$ (on the *right*) for **(a)**, **(b)** $\beta = 0$, **(c)**, **(d)** $\beta = 0.5$, **(e)**, **(f)** $\beta = 1$
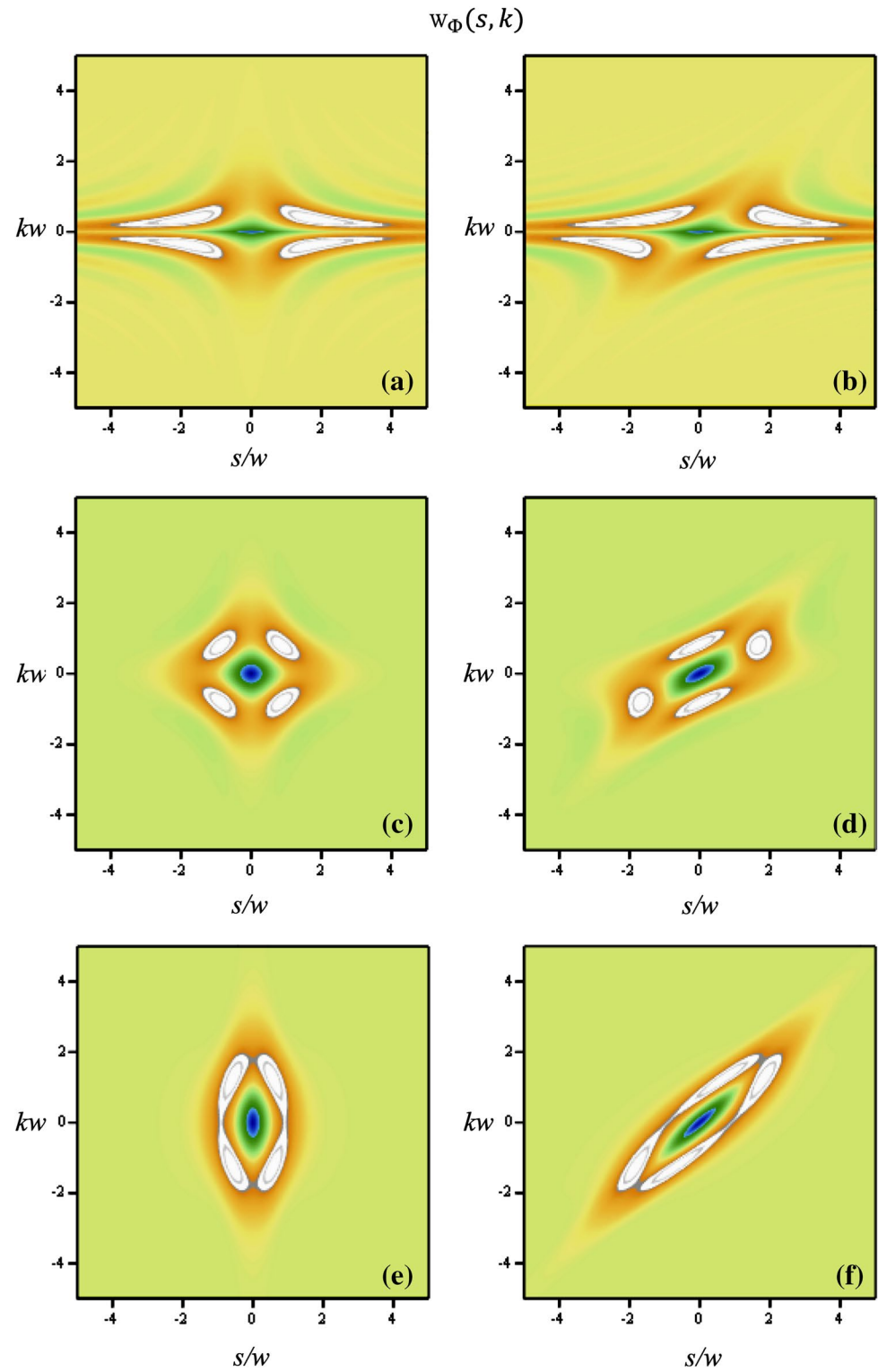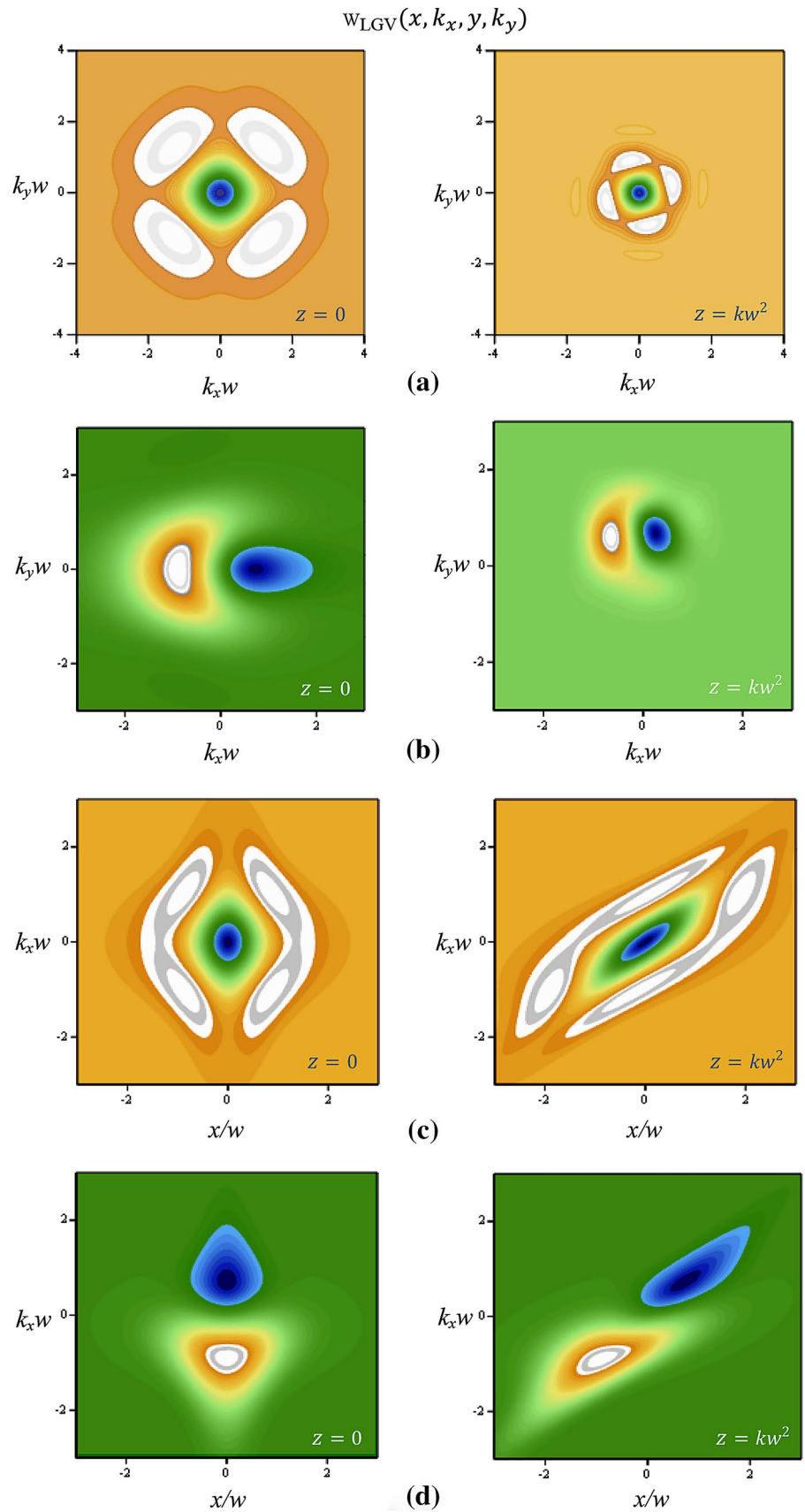
$$\mathrm{w}_\Phi(s, k)$$

Fig. 4 Contour plots of $\mathcal{W}_{LGV}$ on some fixed 2D planes at $z = 0$ (to the *left*) and $z = kw^2$ (to the *right*). Specifically, **(a)**, **(b)** on the $(k_x w, k_y w)$-plane for $x/w = 0$, $y/w = 0$ and $x/w = 0$, $y/w = 1$, respectively, and **(c)**, **(d)** on the $(x/w, k_x w)$-plane for $y/w = 0$, $k_y w = 0$ and $y/w = 0$, $k_y w = 1$, respectively. In the graphs, $\beta_x = \beta_y = \frac{1}{\sqrt{2}}$

The point-spread function $g(x, y, x', y')$ is determined by the ray matrix $\mathbf{M}$ according to

$$g(x, y, x', y') = \frac{k}{2\pi i [\det(\mathbf{B})]^{1/2}} e^{i\frac{k}{2}\Phi(\mathbf{r}, \mathbf{r}')},$$

with

$$\Phi(\mathbf{r}, \mathbf{r}') = \mathbf{r'}^\top \mathbf{B}^{-1} \mathbf{A} \mathbf{r}' - 2\mathbf{r'}^\top \mathbf{B}^{-1} \mathbf{r} + \mathbf{r}^\top \mathbf{D} \mathbf{B}^{-1} \mathbf{r},$$

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{r}' = \begin{pmatrix} x' \\ y' \end{pmatrix},$$

the symbol $\top$ denoting transposition and the principal root being conventionally taken for $[\det(\mathbf{B})]^{1/2}$.

Notably, the *ray-spread function* $G(x, y, k_x, k_y, x', y', k'_x, k'_y)$ has the structure of a "double" WDF, and hence, it has all the properties of a WDF.

When real entries are involved in the ray matrix $\mathbf{M}$, the relations (46)–(47) yield the transfer law (45).

Evidently, this is not the case when complex entries are involved. However, complex matrices that are meaningful in optics are those yielding Gaussian aperturing and Gaussian convolution. A circular aperture is, for instance, accounted for by a (spherical) thin lens-like ray matrix with purely imaginary "focal length" $f = ia$ and hence $\mathbf{A} = \mathbf{D} = \mathbf{I}$, $\mathbf{B} = \mathbf{0}$ and $\mathbf{C} = \frac{i}{ka}\mathbf{I}$, the parameter a relating to the characteristic width of the aperture. Accordingly, one would obtain the Gaussian apodization of the signal by $e^{-k\frac{x^2+y^2}{2a}}$ so that $\varphi_0(x, y)$ would be transformed into $\varphi_0(x, y)e^{-k\frac{x^2+y^2}{2a}}$. A well-known and aforementioned property of the WDF establishes that the WDF of the product of signals is the convolution in the Fourier domain of the WDFs of the product components. Therefore, the WDF of the Gaussian apodized Lorentz–Gauss vortex beam (1) would be conveyed by the convolution integral

$$\mathcal{W}_{\text{LGV}}^{(\text{ga})}(x, y, k_x, k_y) = \frac{a}{\pi k} e^{-k\frac{x^2+y^2}{a}}$$
$$\iint \mathcal{W}_{\varphi_0}(x, y, k'_x, k'_y) e^{-\frac{a}{k}[(k_x - k'_x)^2 + (k_y - k'_y)^2]} dk'_x dk'_y. \quad (48)$$

Note that the same property could have been exploited on evaluating $\mathcal{W}_{\varphi_0}$ since $\varphi_0(x, y)$ can be understood as resulting from the Gaussian apodization of the Lorentz vortex waveform.

Similarly, Gaussian convolution, amounting to the Poisson transform of the input signal, is accountable by a free-space section-like matrix with purely imaginary "length" $d = -i\tau$, so that $\mathbf{A} = \mathbf{D} = \mathbf{I}$, $\mathbf{C} = \mathbf{0}$ and $\mathbf{B} = -ik\tau\mathbf{I}$. Gaussian convolution is optically implementable by the propagation through a Gaussian aperture (with a corresponding to τ^{-1}), enclosed between a direct and inverse Fourier transforms. Resorting again to a definite property of the WDF, we could obtain the WDF of the Lorentz–Gauss

vortex beam (1) after undergoing a convolution, by convolving with respect to the space variables $\mathcal{W}_{\varphi_0}$ and the WDF of the Gaussian $e^{-k\frac{x^2+y^2}{2\tau}}$, and hence by the "dual" of $\mathcal{W}_{\text{LGV}}^{(\text{ga})}$ with a replaced by τ.

Evidently, in both case one would deal with a Gaussian smoothed WDF.

Needless to say, the same results would be obtained from the relation (46) with the expressions for the ray-spread function G relevant to the Gaussian aperturing and convolution that we will not report for sake of space.

4 Concluding notes

We have deduced a closed-form expression for the WDF of a Lorentz–Gauss vortex beam at the initial plane. As recalled, the WDF of Lorentz–Gauss beams has been the object of a detailed analysis in [1], where it has been evaluated resorting to an approximation of the Lorentzian function by a finite sum of (even-order) Hermite–Gaussian functions, as suggested in [2]. The expression (41), deduced here, parallels that worked out in the quoted paper. It is our opinion that it is more easily manageable than that proposed in [1], since it resorts to known functions, like the erfc function, which is supported by almost any software.

According to the evolution law obeyed in general by the WDF, expression (41) provides a complete description of the Wigner-space dynamics of the Lorentz–Gauss vortex beams paraxially propagating through first-order optical systems. In particular, as far as real optical systems are concerned, one deals with the simple transfer law (45), which directly relates to that of the light-ray variables in geometrical optics. The more involved propagation law (46) is indeed needed when dealing with complex ray matrices. However, for the cases of interest in optics, addressing Gaussian aperturing and Gaussian convolution, one should deal with the Gaussian smoothed WDF (48), with the parameter a being properly specified to signalize aperturing or convolution.

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