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Spatial optical solitons in generic nonlocal nonlinear media

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Abstract A generic nonlocal nonlinear optical system with a diffusive type of nonlinearity is investigated analytically, using the homogeneous balance principle and the F-expansion technique. Exact traveling wave and soliton solutions are discovered. Numerical simulation of their propagation and interaction properties is carried out. Our results demonstrate that the nonlocal solitary waves can be manipulated and controlled by changing the nonlocality parameter.

1 Introduction

It is well known that nonlinear (NL) optical media support the formation of spatially localized structures—spatial optical solitons [1]. Such localized waves form when the dispersion and/or the diffraction associated with the finite size of the wave is balanced by the NL change in the properties of the medium, induced by the wave itself. Spatial optical solitons hold ongoing promise for applications in telecommunications and all-optical switching [2]. The soliton concept is also important for diverse physical systems, such as Bose-Einstein condensates [3, 4], fluids, plasmas, solids, matter waves and classical field theory [5].

Nonlocality is an aspect of profound importance in many physical systems. In optics, nonlocal nonlinearities [6] appear when the NL mechanism involves transport [7], long-range forces [8], or many-body interactions [9]. Interestingly, even though nonlocality generally involves some spatial averaging, even highly nonlocal nonlinearities do support solitons, which can attain a wide variety of structures, ranging from one-dimensional (1D) solitons [10], to 3D solitons [11–17]. In all of these "nonlocal solitons" of varying structure, the underlying physical principle is that when the nonlocal nonlinear (NN) effects are long-range, the induced change in the refractive index depends only on the total power carried by the beam, and not on the particular structure of its intensity [6]. However, previous studies on spatial solitons in NN optical systems typically provide solutions of the linear equation in the highly NN limit, or solutions obtained numerically or perturbatively [18, 19].

The construction of exact solutions of NL partial differential equations is one of the essential and most important tasks in NL science. The objective of this paper is to identify traveling wave solutions of a generic NN optical system of equations, with both diffractive and diffusive type of equations, by utilizing the homogeneous balance principle and the F-expansion technique, and to extend the analysis to include the solitary wave solutions.

Owing to the importance of NN model equations, there have naturally been attempts at their exact solution before. An early attempt was presented by Z. Xu et al., who on the basis of a system of equations have discussed in detail the stationary fundamental and many excited solitary mode solutions and their stability in NN media [18]. However, it is worth noting that these stationary-state modes could only be discovered by numerical methods, and not by analytic methods. In order to get a good understanding of the properties of stationary-state modes in such NL media, it is essential to obtain an analytic solution. Until now, no one has been able to present exact analytical solutions of these models under general conditions. Furthermore, only single NL equations have been treated by the homogeneous balance principle and

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the F-expansion technique. For the treatment of systems of NN equations, we face a new challenge.

In this paper we study spatial solitary waves in a generic NN optical system with a diffusive type of nonlinearity. We solve NL equations for the dynamics of the wave using the homogeneous balance principle and the F-expansion technique, which are suitable for finding analytical solutions of different NN optical systems. We discover a variety of traveling periodic and localized solutions.

The paper is organized as follows. Section 2 of the paper introduces the generic nonlocal NL optical system and the solution method for the problem. Section 3 analyzes different forms of spatial solitons. Section 4 gives the conclusions.

2 The model and the soliton solutions

We start our analysis by considering laser beam propagation of a paraxial laser beam along the z axis in a generic NN optical medium. The beam propagation in (1 + 1) D is described by the system of equations for the slowly varying field amplitude u and the NL contribution to the change of the refractive index n [18–22]:

$$\begin{cases} i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+un=0, \\ n-d\frac{\partial^2 n}{\partial x^2}=|u|^2, \end{cases} \tag{1}$$

where x and z stand for the normalized transverse and longitudinal coordinates, respectively, and the coefficient d stands for the degree of nonlocality of the NL response. Thus, system (1) consists of the paraxial propagation equation for the field envelope and the diffusion equation for the nonlocal nonlinearity. It describes the local NL response when $d \to 0$ and reduces to the simple NL Schrödinger equation in Kerr medium; on the other hand when $d \to \infty$ it describes the strongly nonlocal response. A spatiotemporal model of the form (1) but in (1 + 3)D was considered by Mihalache et al. [23, 24] as a model for the generation of stable 3D spatiotemporal solitons. The analysis in [23, 24] was numerical. A similar model in (1 + 2)D with $d = 1$, but with the n term absent from the second of (1), was considered in Refs. [25, 26]. It is clear that the coefficient d can be eliminated from the equations by rescaling the coordinates, however we prefer to keep it as is, because $\sqrt{d}$ determines the correlation length of the nonlocal nonlinear response.

In general, (1) can be used to depict optical propagation in generic NN media. As a special example, (1) describe the propagation of an electromagnetic wave in a nematic liquid crystal [27]. It is noted that the nonlocality degree d can be modulated externally by changing an applied voltage. Moreover, the same model can also be used to describe the optical beam propagation in partially ionized plasmas [28]. It is easily shown that (1) possess several conserved quantities, for example, the energy flow

$$E=\int_{-\infty}^{+\infty}|u|^2\,dx$$

and the Hamiltonian

$$H=\frac{1}{2}\int_{-\infty}^{+\infty}\left(\left|\frac{\partial u}{\partial x}\right|^2-n|u|^2\right)dx.$$

To determine the amplitude profiles of spatial solitons with $u, n \to 0$ at $x \to \pm\infty$, we assume $u(z,x) = A(\theta)e^{iB}$, $n(z,x) = D(\theta)$, where $A(\theta)$ and $D(\theta)$ are real functions of their argument and $B = lx + \Omega z + B_0$ is the phase of the amplitude; here $\theta = kx + \omega z + \theta_0$ is the traveling wave variable. θ_0 and B_0 represent the initial values at $z = 0$. There are two waves combined into the presumed solutions: the linear plane wave $\exp(iB)$ and the nonlinear traveling waves $A(\theta)$ and $D(\theta)$ which, as it will be seen in a moment, can be connected with the Jacobi elliptic functions (JEFs). The parameters k, ω, l, and Ω describe the wave vector characteristics of these waves. In this manner the system (1) is reduced to the following coupled equations:

$$\begin{cases} \frac{k^2}{2}\frac{\partial^2 A}{\partial\theta^2}+DA-(\frac{l^2}{2}+\Omega)A=0, \\ D-k^2 d\frac{\partial^2 D}{\partial\theta^2}=A^2, \end{cases} \tag{2}$$

and the following dispersion relation is established: $\omega = -kl$. Looking at (2), it is clear that the *linear* waves cannot be a proper solution; the second derivatives of A and D are proportional not only to the functions themselves, but also to the product and the square of the functions. Therefore, we seek the traveling *nonlinear* wave solutions of (2) that satisfy the mentioned dispersion relation. According to the balance principle and the F-expansion technique [29, 30], such solutions are most easily found in the form of the expansions in terms of JEFs:

$$A(\theta)=a_0+a_1F(\theta)+a_2G(\theta)+a_3F^2(\theta)+a_4F(\theta)G(\theta), \tag{3a}$$

$$D(\theta)=e_0+e_1F(\theta)+e_2G(\theta)+e_3F^2(\theta)+e_4F(\theta)G(\theta), \tag{3b}$$

where a_j, e_j $(j = 0, 1, 2, 3, 4)$ are the parameters to be determined, and $F(\theta)$ and $G(\theta)$ are JEFs. These functions satisfy the following NL differential equations:

$$\left(\frac{dF}{d\theta}\right)^2=c_0+c_2F^2+c_4F^4,$$

$$\frac{d^2F}{d\theta^2}=c_2F+2c_4F^3,$$

and

$$\left(\frac{dG}{d\theta}\right)^2 = b_0 + b_2G^2 + b_4G^4,$$

$$\frac{d^2G}{d\theta^2} = b_2G + 2b_4G^3.$$

Here c_0, c_2, c_4 and b_0, b_2, b_4 are real constants related to the elliptic modulus m of JEFs [29, 30]. Substituting (3) into (2), along with the relations mentioned above, yields a system of equations for different powers of the functions $F(\theta)$ and $G(\theta)$. It is found that

$$G^2(\theta) = \varepsilon F^2(\theta) + \lambda, \qquad c_2^2 - 3c_0c_4 = b_2^2 - 3b_0b_4,$$

$$2b_2^2 = c_2^2 + c_2b_2 + 9b_0b_4,$$

where

$$\varepsilon = \frac{c_4}{b_4}, \qquad \lambda = \frac{c_2 - b_2}{3b_4} \quad (\lambda \neq 0).$$

The b_j coefficients can be expressed in terms of the c_j coefficients. Setting the coefficients, which also fixes the values of epsilon and lambda of F^jG^n $(n = 0, 1)$ and $\sqrt{c_0 + c_2F^2 + c_4F^2}\sqrt{b_0 + b_2F^2 + b_4F^2}$ in the system of equations to zero, a set of algebraic polynomials for the parameters is obtained. From the coefficient of $\sqrt{c_0 + c_2F^2 + c_4F^2}\sqrt{b_0 + b_2F^2 + b_4F^2}$ we get $a_4 = e_4 = 0$; hence no solutions with the mixed FG terms can be found with the present method. The other parameters a_j, e_j, k, Ω, and ω are determined from the remaining equations. By solving these equations self-consistently, one can find exact periodic solutions of (1). The following solutions of (1) are found in the form:

$$u(z,x) = \frac{\sqrt{6|d|c_0c_4}}{2d(2c_2^2 - 3c_0c_4)}\left[c_2 + \frac{3c_4}{2d}F^2(\theta)\right] \times e^{i[lx + (\frac{3c_0c_2c_4}{8d^2(2c_2^2-3c_0c_4)^2}\sqrt{6|d|c_0c_4} - \frac{l^2}{2})z + B_0]}, \tag{4a}$$

$$n(z,x) = -\frac{3c_2c_4}{2d(2c_2^2 - 3c_0c_4)}F^2(\theta), \tag{4b}$$

where

$$\theta = \sqrt{\frac{1}{2|d|}\left|\frac{c_2}{2c_2^2 - 3c_0c_4}\right|}(x - lz) + \theta_0.$$

It is important to note that the traveling wave solution in (4) is a stationary solution as long as l is constant. In the general case, l is independent of the nonlocality parameter d.

Table 1 lists some of JEFs that may appear in the solutions. As long as we choose the constants c_0, c_2, c_4 according to the relationships listed in the table, and substitute the appropriate $F(\theta)$ into (4), we obtain exact periodic traveling wave solutions to (1). The parameter m $(0 \le m \le 1)$ in the table is the elliptic modulus of JEFs. When $m \to 0$ JEFs become the trigonometric functions, and the periodic traveling wave solutions become the periodic trigonometric solutions. When $m \to 1$ JEFs become the hyperbolic functions, and the periodic traveling wave solutions become the soliton solutions.

Table 1 Jacobi elliptic functions

Solution	c_0 (b_0)	c_2 (b_2)	c_4 (b_4)	$F(\theta)$
1	1	$-(1+m^2)$	m^2	$\mathrm{sn}(\theta)$
2	$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{cn}(\theta)$
3	m^2-1	$2-m^2$	-1	$\mathrm{dn}(\theta)$
4	m^2	$-(1+m^2)$	1	$\mathrm{ns}(\theta)$
5	$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}(\theta)$
6	-1	$2-m^2$	m^2-1	$\mathrm{nd}(\theta)$
7	1	$2-m^2$	$1-m^2$	$\mathrm{sc}(\theta)$
8	1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{sd}(\theta)$
9	$1-m^2$	$2-m^2$	1	$\mathrm{cs}(\theta)$
10	$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{ds}(\theta)$

As mentioned, the traveling wave and the spatial soliton solutions of the generic nonlocal NL optical system can be categorized in terms of two sets of related functions, F and G. We concentrate here on the solitary solutions. The soliton propagation characteristics crucially depend on the nonlocality coefficient d, which is chosen according to some actual physical modeling requirements. As seen from (4), the shape of the solitons is also controlled by d. To display the effect of the coefficient d on soliton dynamics, as an example we consider a soliton solution containing the tanh function.

Substituting the corresponding JEF from Table 1 into (4), with the condition $m = 1$, and noting the condition

$$\frac{c_2}{(2c_2^2 - 3c_0c_4)} > 0,$$

we obtain the required solitary wave solution of (1). The soliton field u and the refractive index change n of the generic nonlocal NL optical system can be expressed as:

$$u(z,x) = \frac{\sqrt{6}}{5\sqrt{|d|}}\left[-1 + \frac{3}{4d}\tanh^2(\theta)\right]e^{i[lx - (\frac{3\sqrt{6|d|}}{100d^2} + \frac{l^2}{2})z + B_0]}, \tag{5a}$$

$$n(z,x) = \frac{3}{5d}\tanh^2(\theta), \tag{5b}$$

where

$$\theta = \frac{1}{\sqrt{10|d|}}\left(x - lz + \sqrt{10|d|}\theta_0\right)$$

and l is an arbitrary constant. We obtain a simple existence condition for such a soliton solution of (1): $d \neq 0$.

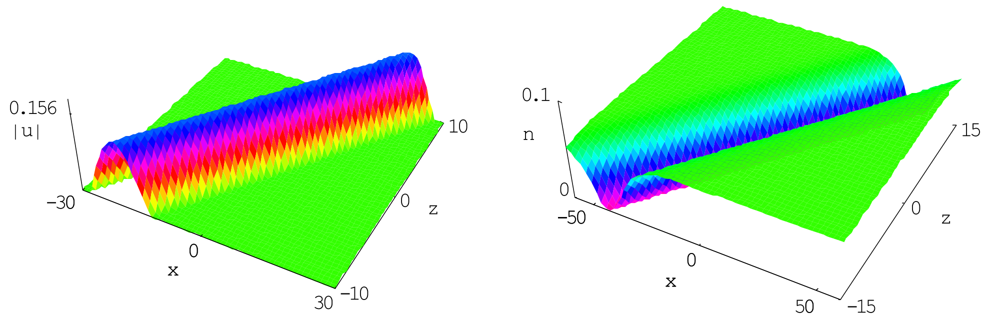


Fig. 1 Exact solitary wave solution in the strongly nonlocal case, with $d = 10$, $l = 1$ and $\theta_0 = 0$

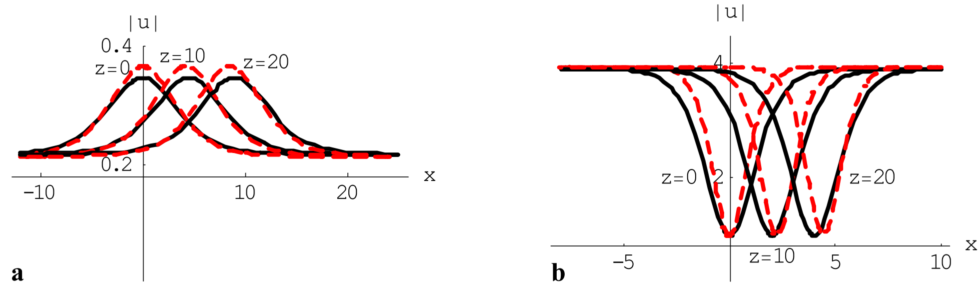


Fig. 2 Comparison of intensity profiles for the exact solution (*solid lines*) and the numerical simulation (*dashed lines*) at different propagation distances $z = 0, 10, 20$ *from left to right*. The parameter are taken to have the values $l = 0.05$ and $\theta_0 = 0$. (**a**) Bright soliton, $d = 2$. (**b**) Dark soliton, $d = -0.25$

3 Discussion

The solitary wave solution represented by (5) contains four parameters. They are $\frac{1}{\sqrt{10|d|}}$, which represents the amplitude as well as the inverse of the pulse width of the solitary wave, $-2l$ represents its speed as well as the frequency shift, and two parameters which represent the phase constants, $x_0 = \sqrt{10|d|}\theta_0$ and B_0. From (5a), we can see that when $d = \frac{3}{4}$,

$$|u| = \frac{2\sqrt{2}}{5}\,\mathrm{sech}^2\left[\frac{2}{\sqrt{30}}\left(x - lz + \frac{\sqrt{30}}{2}\theta_0\right)\right],$$

which is analogous to the bright soliton solution describing soliton propagation in Kerr medium. It should be noted that our solution is *not* equivalent to the standard bright solitons solution of Kerr media; these solitons are obtained for $d = 0$, and the amplitude of such solitons is proportional to the sech function, not to the sech^2 function. It should also be noted that when $d > 0$, (5a) represents the bright soliton solutions, whereas when $d < 0$, (5a) represents the dark (or rather the gray) soliton solutions. This is also at variance with the existence condition for the standard dark solitons, which is connected with the *sign* of the nonlinearity coefficient. Dark solitons are *not* possible in (1) when $d = 0$. Obviously, we deal here with the *nonlocal* solitons, which require $d \neq 0$. Our results have been confirmed experimentally, in that the dipole-mode bright solitons were observed by Hutsebaut et al. [31], and the attractive dark solitons were observed by Dreischuh et al. [32].

First, we consider the strongly nonlocal case, $d \gg 1$. For $d = 10$, $l = 1$ and $\theta_0 = 0$, the evolution of the corresponding soliton and the distribution of the refractive index are depicted in Fig. 1. It is seen that the spatial soliton profile and its width remain unchanged with the increasing propagation distance. To demonstrate the stability of such exact solutions, we compare the exact solution (5a) with the numerical simulation. Actually, checking the soliton stability in 1D is neither crucial nor necessary; it is expected. Still, we want to display the stability of the obtained exact solutions. Adding noise changes these solutions, however the stable solutions are again obtained. We perform numerical solution of (1), with the initial fields coming from (5), and with the two values of d: $d = 2$ (bright soliton) and $d = -0.25$ (dark soliton). Other parameters are chosen as $l = 0.05$ and $\theta_0 = 0$. We utilize the split-step beam propagation method adopted from [30]. Figure 2 presents the evolutions of the exact solution and the numerical simulation. As expected, no collapse is seen, and the numerical solution is in good agreement with the analytical solution.

Next, we consider the Kerr-like medium case, d close but not equal to $\frac{3}{4}$. For $d = 1$, $l = 1$ and $\theta_0 = 0$, the shape of the soliton oscillates, as is evident in Fig. 3. The periodic change in the soliton shape during its motion is clearly seen. We find that the soliton executes periodic oscillations of a breather. Such *breathing* behavior is characteristic of nonlocal solitons; it is often seen in media in which the influence of nonlocality is crucial on the formation of localized waves, e.g. in nematic liquid crystals [33–35]. As mentioned, these solitons are different from the standard Kerr solitons.

Further, we analyze the impact of different values of the nonlocality parameter d on the soliton intensity profile $|u|$, in the strongly nonlocal bright soliton case ($d \gg 1$) and in the nonlocal dark soliton case ($d < 0$). Representative examples are depicted in Fig. 4. Evidently, a decrease in the de-

Fig. 3 Exact soliton solution for the weakly nonlocal case, with $d = 1$, $l = 1$ and $\theta_0 = 0$

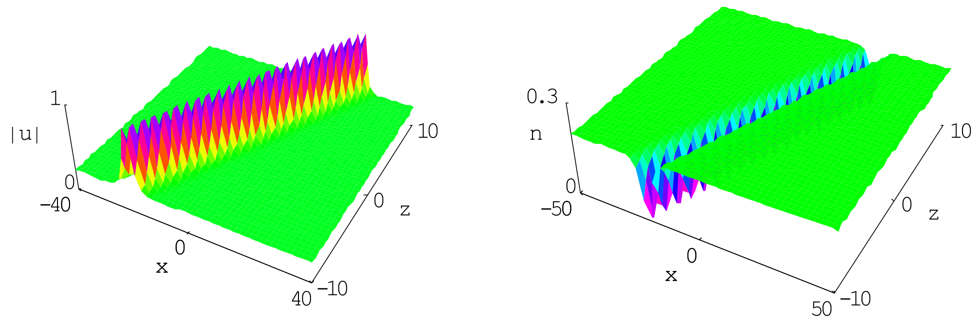
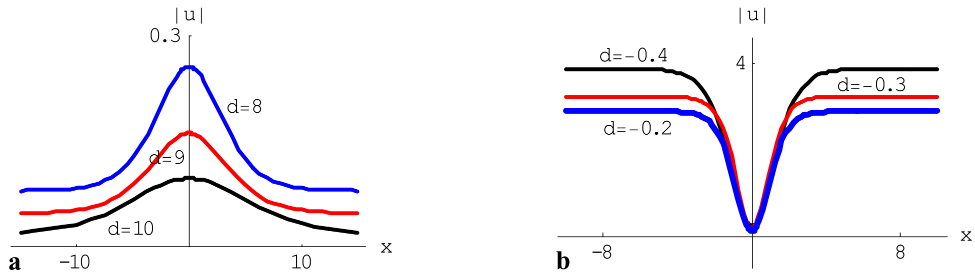


Fig. 4 Soliton intensity profiles for different degrees of nonlocality. (**a**) The strongly nonlocal bright soliton case, $d = 10, 9, 8$ *from bottom to top*; (**b**) The nonlocal dark soliton case, $d = -0.2, -0.3, -0.4$ *from bottom to top*



gree of nonlocality results in an increase of the soliton width. The nonlocality suppresses the change in the refractive index profile, thereby leading to a broadening of the beam. This effect is more clearly seen in (4a), where the soliton amplitude is $\eta = 1/\sqrt{10|d|}$ and its reciprocal $\sqrt{10|d|}$ represents the pulse width of the soliton wave. We can see that the soliton width changes monotonically with the degree of nonlocality d.

Finally, the interaction of two localized beams interacting in a NN medium is considered for the bright solitons ($d > 0$) and the dark solitons ($d < 0$). We again numerically integrate (1) using the split-step Fourier method [30]. We use the exact soliton solution (5) as an initial condition. We set $u(0, x) = u_1(0, x + x_0) + u_2(0, x - x_0)$ as a superposition of two bound soliton states with opposite velocities. It is obvious that the initial separation between the two beams is $2x_0$. First, we study the interaction of two bright solitary wave under the special conditions $d = 3/4$, $l = 1$, $\theta_0 = 0$, $B_0 = 0$, by taking the superposition of the exact solutions (5) as an initial input; that is:

$$u(x,0) = -\frac{2\sqrt{2}}{5}\operatorname{sech}^2\left[\frac{2}{\sqrt{30}}(x + x_0)\right]e^{ix} - \frac{2\sqrt{2}}{5}\operatorname{sech}^2\left[\frac{2}{\sqrt{30}}(x - x_0)\right]e^{ix}.$$

Numerical results show that their interaction features particle-like properties, as shown in Fig. 5(a). The other example is the collision between two dark solitary waves. We again consider a case with the opposite velocities between the two beams, with the conditions $d = -3/4$, $l = 1$, $\theta_0 = 0$, $B_0 = 0$. In this case, the initial conditions at $z = 0$ are given by

$$u(x,0) = -\frac{2\sqrt{2}}{5}\left(1 + \tanh^2\left[\frac{2}{\sqrt{30}}(x + x_0)\right]\right)e^{ix} - \frac{2\sqrt{2}}{5}\left(1 + \tanh^2\left[\frac{2}{\sqrt{30}}(x - x_0)\right]\right)e^{ix}.$$

Figure 5(b) shows the interaction between the two dark solitary waves.

From the numerical simulations in Fig. 5, we demonstrate that the shape of two colliding solitons does not change after collisions, which indicates no energy exchange between the two solitons; their interaction features are seen to be as of interacting particles. It is evident that the solitons propagate in a stable manner. The collision, on the other hand, involves considerable oscillatory deformation of the soliton amplitude and width.

4 Conclusions

We have solved analytically the generic nonlocal NL optical system, using the homogeneous balance principle and the F-expansion technique. Different exact periodic traveling wave solutions are obtained in terms of JEFs, and new exact soliton solutions are found. Numerical simulations are performed, to ascertain the stability of such soliton solutions. A procedure is presented for controlling NN solitons, in which one may select the nonlocality coefficient d, to control the propagation behavior of solitons.

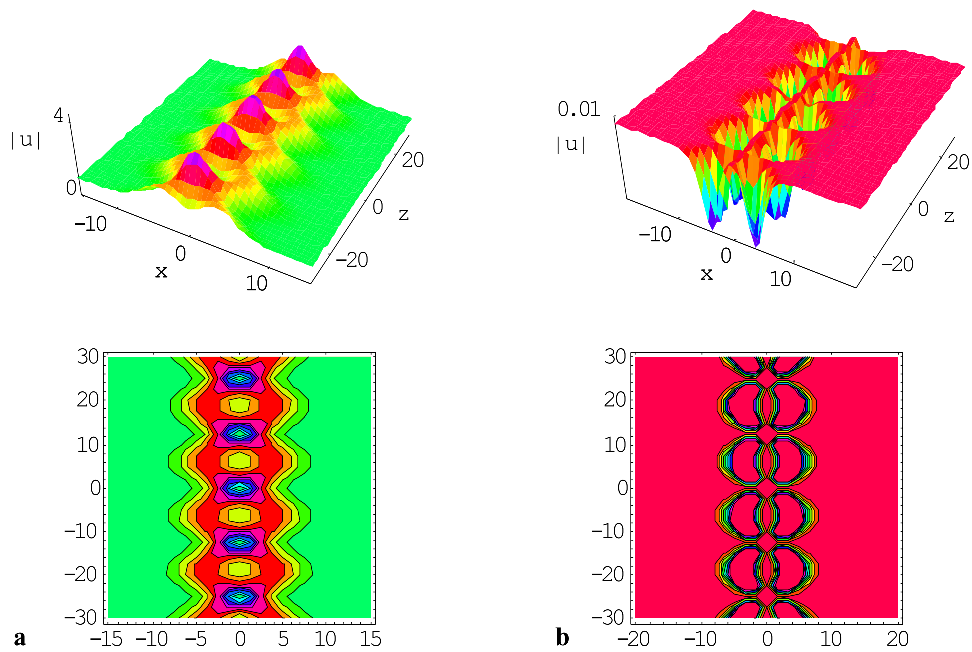


Fig. 5 Collision of two bound solitons for different degrees of nonlocality. (**a**) The weakly nonlocal bright soliton case, with $d = 3/4$; (**b**) The nonlocal dark soliton case, with $d = -3/4$. Other parameters: $l = 1$, $x_0 = 5$ and $\theta_0 = 0$

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