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Effective Rayleigh range of partially coherent beams propagating through the turbulent atmosphere

X. Ji · X. Li

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Abstract The concept of the Rayleigh range of partially coherent beams is extended from free space to the turbulent atmosphere. The general analytical expression for the effective Rayleigh range of partially coherent beams propagating through the turbulent atmosphere is derived. It is shown that the longer the free-space Rayleigh range is, the more the effective Rayleigh range is affected by turbulence, which is illustrated by numerical calculation examples.

1 Introduction

It is very important to study the propagation of laser beams through the turbulent atmosphere for many practical applications such as the remote sensing and atmospheric optical communication etc. [1, 2]. However, the turbulence-induced spatial broadening of fully coherent beams is a limiting factor in most applications. Recently, the spreading of partially coherent beams propagating through the turbulent atmosphere has been studied extensively [3–10]. In those studies, the beam width, angular spread, turbulence distance, power in the bucket (*PIB*) and β parameter (i.e., $\beta = \sqrt{S/S_0}$, where S, S_0 is the measured area of 63% of the power in a real beam and a ideal beam respectively) have been taken as the characteristic parameters of beam spreading in turbulence. In 2007, a criterion has been introduced for testing whether a beam retains its beam-like form after it propagates any particular distance through the turbulent atmosphere [11]. On the other hand, the Rayleigh range is used in the theory of lasers to characterize the distance over which a beam may be considered effectively non-spreading [12]. The Rayleigh range of fully and partially coherent beams in free space has been studied in Refs. [12–14]. However, the above-mentioned studies are only restricted to the Rayleigh range of laser beams propagating in free space. The aim of this paper is to extend the concept of the Rayleigh range of partially coherent beams from free space to the turbulent atmosphere, which is called the effective Rayleigh range. The general analytical expression for the effective Rayleigh range of partially coherent beams propagating through the turbulent atmosphere is derived by using the definition of the effective curvature radius of an arbitrary field [15] and analogy method.

X. Ji (✉) · X. Li
Department of Physics, Sichuan Normal University,
Chengdu 610068, China
e-mail: jiXL100@163.com

2 Theoretical model

Based on the extended Huygens–Fresnel principle, the cross-spectral density function of the partially coherent beam propagating through the turbulent atmosphere reads [2]

$$\begin{aligned} W(\boldsymbol{r}, \boldsymbol{r}_d, z) &= \left(\frac{k}{2\pi z}\right)^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(\boldsymbol{r}', \boldsymbol{r}'_d, 0) \\ &\quad \times \exp\left\{\frac{\mathrm{i}k}{z}\left[(\boldsymbol{r}-\boldsymbol{r}')\cdot(\boldsymbol{r}_d-\boldsymbol{r}'_d)\right] - H(\boldsymbol{r}_d, \boldsymbol{r}'_d, z)\right\} \mathrm{d}^2 r' \, \mathrm{d}^2 r'_d \end{aligned} \tag{1}$$

where k is the wave number related to the wave length λ by $k = 2\pi/\lambda$, $\boldsymbol{r}' = (\boldsymbol{r}'_1 + \boldsymbol{r}'_2)/2$, $\boldsymbol{r}'_d = \boldsymbol{r}'_1 - \boldsymbol{r}'_2$, $\boldsymbol{r} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$, $\boldsymbol{r}_d = \boldsymbol{r}_1 - \boldsymbol{r}_2$, $\boldsymbol{r}'_1$ and $\boldsymbol{r}'_2$ are the two-dimensional position vectors in the incident plane $z = 0$, and $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the two-dimensional position vectors in the plane z. The first term in the integrand of (1) is the cross-spectral density function $W(\boldsymbol{r}', \boldsymbol{r}'_d, 0)$ of the incident partially coherent beam. The last term in the same integrand represents the effect of turbulence, and it can be written as [2]

$$H(\boldsymbol{r}_d, \boldsymbol{r}'_d, z) = 4\pi^2 k^2 z \int_0^1 \mathrm{d}\xi \int_0^\infty \kappa \Phi_n(\kappa) \times \left[1 - J_0\left(\kappa\left|\xi \boldsymbol{r}'_d + (1-\xi)\boldsymbol{r}_d\right|\right)\right] \mathrm{d}\kappa, \quad (2)$$

where J_0 being the Bessel function of the first kind and order zero, and $\Phi_n(\kappa)$ being the spatial power spectrum of the refractive-index fluctuations of the turbulent medium.

The WDF of field is very useful for handling partially coherent beams, which can be defined in terms of the cross-spectral density as [16]

$$h(\boldsymbol{r}, \boldsymbol{\theta}, z) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{+\infty} W(\boldsymbol{r}, \boldsymbol{r}_d, z) \exp(-\mathrm{i}k\boldsymbol{\theta} \cdot \boldsymbol{r}_d)\, \mathrm{d}^2 r_d. \quad (3)$$

In (3), $\boldsymbol{\theta} = (\theta_x, \theta_y)$, $k\theta_x$ and $k\theta_y$ are the wave vector components along the x-axis and y-axis respectively. Integration of function h over the angular variables θ_x and θ_y gives the beam intensity, and its integral over the spatial variables x and y is proportional to the radiant intensity of the field.

The moments of the order $n_1 + n_2 + m_1 + m_2$ of WDF are given by the expression [16]

$$\left\langle x^{n_1} y^{n_2} \theta_x^{m_1} \theta_y^{m_2}\right\rangle = \frac{1}{P} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^{n_1} y^{n_2} \theta_x^{m_1} \theta_y^{m_2} h(\boldsymbol{r}, \boldsymbol{\theta}, z)\, \mathrm{d}^2 r\, \mathrm{d}^2\theta, \quad (4)$$

where $P = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\boldsymbol{r}, \boldsymbol{\theta}, z)\, \mathrm{d}^2 r\, \mathrm{d}^2\theta$ is the total irradiance of the beam.

We assume, for simplicity, that the beam waist is located in the plane $z = 0$. On substituting from (3) into (4), after very tedious integral calculations, we obtain the second moments $\langle r^2\rangle$ and $\langle \boldsymbol{r} \cdot \boldsymbol{\theta}\rangle$ of a partially coherent beam propagating through the turbulent atmosphere, which are given by

$$\left\langle r^2\right\rangle = \left\langle r_0^2\right\rangle + \left\langle \theta_0^2\right\rangle z^2 + 4Tz^3/3, \quad (5)$$

$$\langle \boldsymbol{r} \cdot \boldsymbol{\theta}\rangle = \left\langle \theta_0^2\right\rangle z + 2Tz^2, \quad (6)$$

where $\langle r_0^2\rangle$ and $\langle \theta_0^2\rangle$ denote the second moments in the plane $z = 0$, and

$$T = \pi^2 \int_0^\infty \kappa^3 \Phi_n(\kappa)\, \mathrm{d}\kappa. \quad (7)$$

It is noted that, to derive (5) and (6) we used the formulae

$$\delta^{(j)}(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-\mathrm{i}x)^j \exp(-\mathrm{i}sx)\, \mathrm{d}x, \quad (8)$$

$$\int_{-\infty}^{+\infty} f(x)\delta^{(j)}(x)\, \mathrm{d}x = (-1)^{(j)} f^{(j)}(0) \quad (j = 1, 2) \quad (9)$$

where δ denotes the Dirac delta function and $\delta^{(j)}$ is its jth derivative, f is an arbitrary function and $f^{(j)}$ is its jth derivative.

The physical meaning of second moments is clear, e.g. $\langle r^2\rangle$ represents the beam width, $\langle \theta^2\rangle$ denotes the divergence angle, and $\langle \boldsymbol{r} \cdot \boldsymbol{\theta}\rangle$ is relative to the radius of curvature [15]. According to Ref. [15], the effective radius of curvature of an arbitrary field is defined as

$$R = \frac{\langle r^2\rangle}{\langle \boldsymbol{r} \cdot \boldsymbol{\theta}\rangle}. \quad (10)$$

Upon substituting from (5) and (6) into (10), this yields

$$R = \frac{\langle r_0^2\rangle + \langle \theta_0^2\rangle z^2 + 4Tz^3/3}{\langle \theta_0^2\rangle z + 2Tz^2}. \quad (11)$$

Equation (11) is the effective radius of curvature of a partially coherent beam propagating through the turbulent atmosphere.

For $T = 0$ (i.e., free space), (11) reduces to

$$R|_{\text{free}} = z_R|_{\text{free}} \left(\frac{z_R|_{\text{free}}}{z} + \frac{z}{z_R|_{\text{free}}}\right), \quad (12)$$

where

$$z_R|_{\text{free}} = \sqrt{\langle r_0^2\rangle / \langle \theta_0^2\rangle} \quad (13)$$

is the Rayleigh range of a partially coherent beam in free space [15]. From (12) it can be seen that, in free space the propagation equation of the effective radius of curvature of a partially coherent beam will has the same form as a Gaussian beam.

Letting the first derivative $\mathrm{d}R/\mathrm{d}z = 0$ in (11), yields

$$\frac{8}{3}T^2 z^4 + \frac{8}{3}\left\langle \theta_0^2\right\rangle T z^3 + \left\langle \theta_0^2\right\rangle^2 z^2 - 4\left\langle r_0^2\right\rangle T z - \left\langle r_0^2\right\rangle\left\langle \theta_0^2\right\rangle = 0. \quad (14)$$

Equation (14) is a quartic equation, and there are four solutions. However, among the four solutions only one is real and positive, which denotes the position $z_{\min}$ where the effective radius of curvature R reaches its minimum $R_{\min}$ along the propagation direction. After very tedious calculations, we obtain

$$z_{\min} = -\frac{\langle \theta_0^2\rangle}{4T} + \frac{1}{2}\sqrt{C} + \frac{1}{4}\left[\frac{1}{\sqrt{C}}\left(\frac{\langle \theta_0^2\rangle^3}{2T^3} + \frac{12\langle r_0^2\rangle}{T}\right) - 4C\right]^{1/2}, \quad (15)$$

where

$$C=\frac{1}{4\sqrt[3]{2}T^2}\left(\frac{3\langle\theta_0^2\rangle^4}{\sqrt[3]{2}B}+\frac{1}{6}B\right), \tag{16}$$

$$B=\left(A-\sqrt{A^2-54^2\langle\theta_0^2\rangle^{12}}\right)^{1/3}, \tag{17}$$

$$A=54\left(\langle\theta_0^2\rangle^3+24\langle r_0^2\rangle T^2\right)^2. \tag{18}$$

From (12) it can be concluded that in free space the effective radius of curvature of a partially coherent beam reaches its minimum at the position $z_R|_{\text{free}}$. By using the analogy method, we can define the effective Rayleigh range of a partially coherent beam propagating through the turbulent atmosphere as the propagation distance $z_{\min}$ where the effective radius of curvature reaches its minimum, i.e.,

$$z_R=z_{\min}. \tag{19}$$

The effective Rayleigh range defined by (19) can be used to characterize the distance over which a partially coherent beam in turbulence may be considered without spreading appreciably. Equations (15) and (19) are the main analytical results obtained in this paper.

3 Examples

In this paper, the Gaussian Schell-model (GSM) beam is taken as a typical example of partially coherent beams, and the influence of turbulence on the effective Rayleigh range is studied by numerical calculation examples.

For GSM beams we have $\langle r_0^2\rangle = w_0^2/2$, $\langle\theta_0^2\rangle = 2(1+1/\alpha^2)/(k^2w_0^2)$ [5, 17], where w_0 is the waist width, α is the beam coherence parameter related to the spatial correlation length σ_0 by $\alpha=\sigma_0/w_0$. Thus, from (15)–(19) the effective Rayleigh range of GSM beams in turbulence can be expressed as

$$z_R=-\frac{1+1/\alpha^2}{2k^2w_0^2T}+\frac{\sqrt{F}}{2}+\frac{1}{2}\left\{\frac{1}{\sqrt{F}}\left[\left(\frac{1+1/\alpha^2}{k^2w_0^2T}\right)^3+\frac{3w_0^2}{2T}\right]-F\right\}^{1/2}, \tag{20}$$

where

$$F=\frac{1}{4\sqrt[3]{2}T^2}\left[\frac{48}{\sqrt[3]{2}E}\left(\frac{1+1/\alpha^2}{k^2w_0^2}\right)^4+\frac{1}{6}E\right], \tag{21}$$

$$E=\left[D-\sqrt{D^2-2^{14}3^6\left(\frac{1+1/\alpha^2}{k^2w_0^2}\right)^{12}}\right]^{1/3}, \tag{22}$$

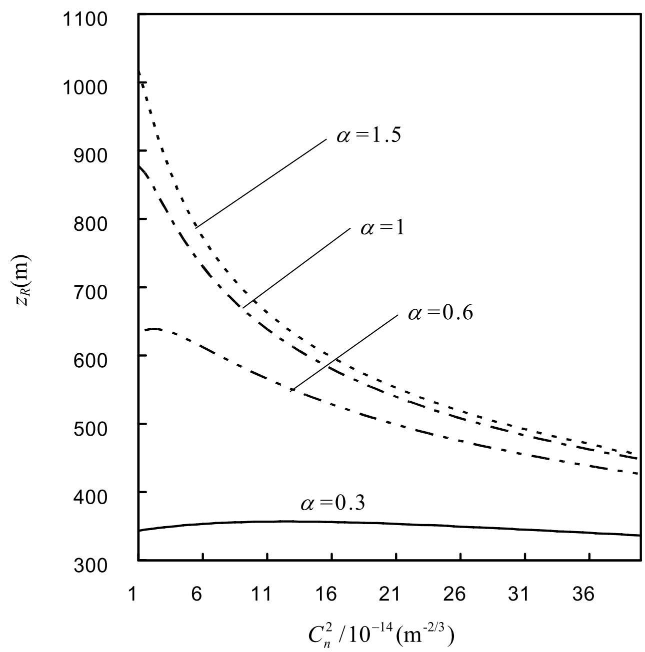

Fig. 1 Effective Rayleigh range z_R versus the refraction index structure constant C_n^2. $\lambda = 1.06$ µm, $w_0 = 2$ cm

$$D=2^5 3^3\left[2\left(\frac{1+1/\alpha^2}{k^2w_0^2}\right)^3+3w_0^2T^2\right]^2. \tag{23}$$

From (20) together with (21)–(23) it follows that the effective Rayleigh range of GSM beams in turbulence depends on the beam parameters (i.e., α, w_0 and λ) and the strength of turbulence.

In the numerical calculations, a von Kármán spectrum is adopted, i.e. [2],

$$\Phi_n(\kappa)=0.033C_n^2\left(\kappa^2+1/L_0^2\right)^{-11/6}\exp\left(-\kappa^2/\kappa_m^2\right), \tag{24}$$

where C_n^2 is the refraction index structure constant, $\kappa_m=5.92/l_0$, l_0 and L_0 are the turbulence inner scale and outer scale, respectively. If typical values of $l_0=0.01$ m and $L_0=10$ m are taken, from (7) and (24) yields $T=7.067C_n^2$.

Figures 1–4 give the curves of the effective Rayleigh range z_R of GSM beams propagating through the turbulent atmosphere versus parameters C_n^2, α, w_0 and λ. From Figs. 1–4 it can be seen that z_R decreases due to turbulence. The stronger the strength of turbulence is, the shorter the effective Rayleigh range is. In free space, z_R increases with increasing α and w_0, and decreasing λ. However, z_R of GSM beams with larger α and w_0, and smaller λ is more affected by turbulence than that of GSM beams with smaller α and w_0 (see Figs. 2 and 3 respectively), and larger λ (see Fig. 4). In particular, z_R in turbulence is almost the same as that in free space when α is small enough (e.g., $\alpha=0.3$ in Fig. 1, $\alpha<0.5$ in Fig. 2), w_0 is small enough (e.g., $w_0<0.03$ m in Fig. 3), and λ is large enough (e.g., $\lambda>0.72$ µm in Fig. 4).

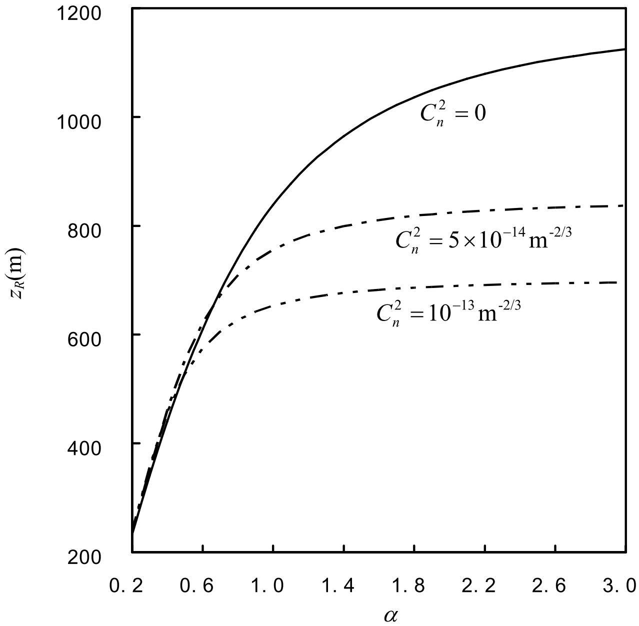

Fig. 2 Effective Rayleigh range z_R versus the beam coherence parameter α. $\lambda = 1.06$ μm, $w_0 = 2$ cm

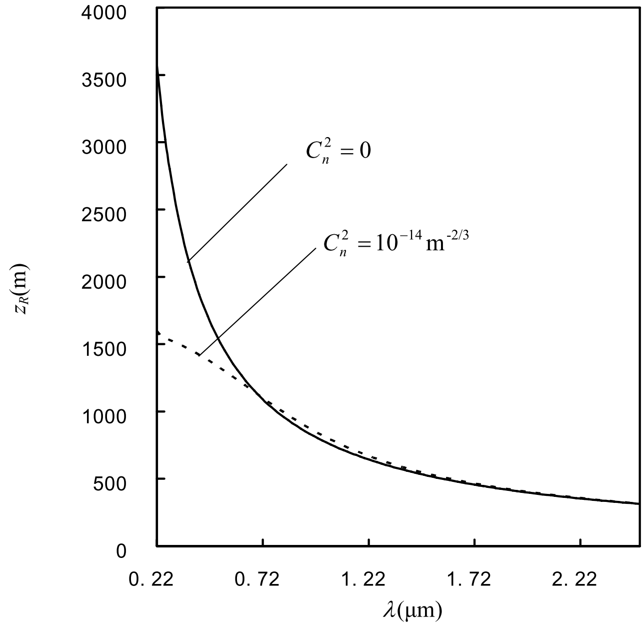

Fig. 4 Effective Rayleigh range z_R versus the wave length λ. $w_0 = 2$ cm, $\alpha = 0.8$

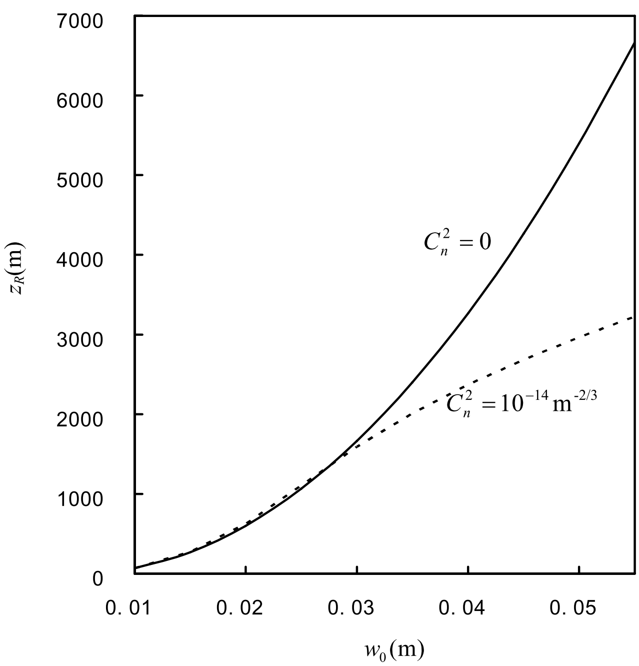

Fig. 3 Effective Rayleigh range z_R versus the waist width w_0. $\lambda = 1.06$ μm, $\alpha = 0.8$

4 Conclusions

In summary, the concept of the Rayleigh range has been extended from free space to the turbulent atmosphere, and the general analytical expression for the effective Rayleigh range of partially coherent beams propagating through the turbulent atmosphere has been derived. It has been shown that the effective Rayleigh range is shortened due to turbulence. The longer the free-space Rayleigh range is, the more the effective Rayleigh range is affected by turbulence.

Very recently, the Rayleigh range of partially coherent beams in atmospheric turbulence is also proposed in Ref. [18]. It is noted that the Rayleigh range in turbulence defined in this paper is different from that defined in Ref. [18]. In Ref. [18], the Rayleigh range defined by the mean-squared beam width, i.e., the Rayleigh range is defined as the distance at which the mean-squared beam width is twice as large as its waist value. However, in this paper the Rayleigh range defined by the effective radius of curvature, i.e., the Rayleigh range is defined as the distance where the effective radius of curvature reaches its minimum. The two types of the Rayleigh range definition are in agreement in free space (see (11) in Ref. [18] and (13) in this paper), but disagreement in turbulence (see (17) in Ref. [18] and (15) in this paper). Two approximation results of the Rayleigh range of partially coherent beams in turbulence are obtained in Ref. [18], and the general analytical expression for the effective Rayleigh range of partially coherent beams in turbulence is derived without any approximation in this paper.

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