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M^2-factors of a non-circular partially coherent flat-topped beam

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Abstract Based on the extended Huygens-Fresnel integral and the second-order moments of the Wigner distribution function, analytical formulae for the propagation factors (known as M^2-factors) of a non-circular (i.e., rectangular or elliptical) partially coherent flat-topped beam in turbulent atmosphere are derived. The properties of the M^2-factors of a non-circular partially coherent flat-topped beam in turbulent atmosphere and in free space are studied numerically and comparatively. It is found that the evolution properties of the M^2-factors are mainly determined by the parameters of the beam and the turbulent atmosphere. The relative M^2-factors of a non-circular partially coherent flat-topped beam can be smaller than a circular partially coherent flat-topped beam and a Gaussian Schell-model beam, particularly at long propagation ranges in turbulent atmosphere. Our results will be useful in long-distance free-space optical communications.

1 Introduction

In many applications, such as material thermal processing, inertial confinement fusion, second-harmonic generation, electron acceleration, and optical communication, a laser beam with a flat-topped spatial profile is required [1–5]. Several theoretical models have been proposed to describe a coherent flat-topped beam of circular symmetry [6–9]. Theoretical models for describing a coherent flat-topped beam of non-circular (elliptical or rectangular) symmetry were also proposed [10–12]. Propagation properties of various coherent flat-topped beams through free space, paraxial optical system and turbulent atmosphere have been studied in detail [13–24]. It has been found that a non-circular flat-topped beam has advantages over a circular Gaussian beam or a circular flat-topped beam for overcoming the destructive effect of atmospheric turbulence from the aspect of scintillation, and it has potential applications in free-space optical communications [24].

In the past decades, partially coherent beams have been investigated widely [25–41]. Partially coherent beams have been applied in optical projection, free-space optical communication, laser scanning, optical imaging, nonlinear optics and optical trapping [33–41]. Most of previous papers on partially coherent beams have been confined to Gaussian Schell-model (GSM) beams, whose spectral density and spectral degree of coherence have Gaussian shapes. Recently, more and more attention is being paid to partially coherent flat-topped beams [42–55]. Mode decomposition of a partially coherent flat-topped beam was studied in [42]. Coutts has generated a partially coherent flat-topped beam with a copper vapor laser [43]. A laser beam generated by an excimer laser is also partially coherent and flat-topped [44]. Wang and Cai have generated a partially coherent flat-topped beam by focusing a GSM beam with a truncated thin lens [45]. Theoretical models for circular and non-circular partially coherent flat-topped beams have been proposed [46–49]. Propagation of a partially coherent flattened Gaussian beam through apertured ABCD optical systems was studied in [50]. Zhao et al. studied the radiation force of partially coherent partially coherent flat-topped beams on a Rayleigh particle [51]. Liu and Zhou studied the propagation of partially coherent flat-topped beams in uniaxial crystals orthogonal to the optical axis [52]. Zhang et al. studied the propagation properties of partially coherent flat-

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topped beams in dispersive or gain media [53, 54]. Propagation properties such as average intensity, spectral changes, polarization characteristics, scintillation, relay propagation, propagation factor, directionality and spatial correlation, of circular partially coherent flat-topped beam in turbulent atmosphere have been studied in [55–63].

Up to now, only few papers have been published on the properties of non-circular partially coherent flat-topped beams [48, 64]. The properties of the M^2-factors of a non-circular partially coherent flat-topped beam have not been studied. The M^2-factor proposed by Siegman [65], which is closely connected with the method of moments proposed by Valsov et al. [66], is a particularly important property of an optical laser beam, and plays an important role in the characterization of beam propagation [62, 67–75]. The purpose of this paper is to investigate the properties of the M^2-factors of a non-circular (i.e., rectangular or elliptical) partially coherent flat-topped beam in turbulent atmosphere and in free space comparatively. Some interesting and useful results are found.

2 Analytical formulae for the M^2-factors of a non-circular partially coherent flat-topped beam in turbulent atmosphere

The electric field of a coherent rectangular flat-topped beam at $z = 0$ can be expressed as the following finite sum of fundamental astigmatic Gaussian beams in the rectangular coordinate [11, 18]

$$E_{MN}(\rho'_x, \rho'_y, 0) = \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{(-1)^{m+n}}{MN}\binom{M}{m}\binom{N}{n} \times \exp\left(-\frac{m\rho'^2_x}{w^2_{0x}} - \frac{n\rho'^2_y}{w^2_{0y}}\right), \tag{1}$$

where w_{0x} and w_{0y} are the waist sizes of the fundamental astigmatic Gaussian beam in x- and y-directions, respectively, M and N are the beam orders of the rectangular flat-topped beam. Under the condition of $M = N = 1$ and $w_{0x} = w_{0y}$, (1) reduces to the expression for the electric field of a stigmatic Gaussian beam. The electric field of a coherent elliptical flat-topped beam at $z = 0$ is expressed as the following finite sum of fundamental astigmatic Gaussian beams [12, 22]

$$E_N(\rho'_x, \rho'_y, 0) = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{N}\binom{N}{n} \times \exp\left(-\frac{n\rho'^2_x}{w^2_{0x}} - \frac{n\rho'^2_y}{w^2_{0y}}\right), \tag{2}$$

where N is the beam order of the elliptical flat-topped beam. Under the condition of $N = 1$ and $w_{0x} = w_{0y}$, (2) reduces to the expression for the electric field of a stigmatic Gaussian beam. The beam profile of a rectangular or elliptical flat-topped beam becomes more flat as the beam orders increase as shown in Figs. 1 and 2.

A partially coherent beam is generally characterized by the cross-spectral density $W(\rho'_{1x}, \rho'_{1y}, \rho'_{2x}, \rho'_{2y}; z) = \langle E^*(\rho'_{1x}, \rho'_{1y}; z)E(\rho'_{2x}, \rho'_{2y}; z)\rangle$ [25], where $\langle\ \rangle$ denotes the ensemble average and "*" is the complex conjugate. The cross-spectral density of a rectangular partially coherent flat-topped beam generated by a Schell-model source (at $z = 0$) can be expressed in the following form

$$\begin{aligned} W_{MN}(\rho'_{1x}, \rho'_{1y}, \rho'_{2x}, \rho'_{2y}; 0) &= \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{h=1}^{M}\sum_{j=1}^{N} \frac{(-1)^{m+n+h+j}}{M^2N^2} \times \binom{M}{m}\binom{N}{n}\binom{M}{h}\binom{N}{j} \\ &\times \exp\left(-\frac{m\rho'^2_{1x}}{w^2_{0x}} - \frac{n\rho'^2_{1y}}{w^2_{0y}} - \frac{h\rho'^2_{2x}}{w^2_{0x}} - \frac{j\rho'^2_{2y}}{w^2_{0y}} - \frac{(\rho'_{1x} - \rho'_{2x})^2 + (\rho'_{1y} - \rho'_{2y})^2}{2\sigma^2_g}\right), \end{aligned} \tag{3}$$

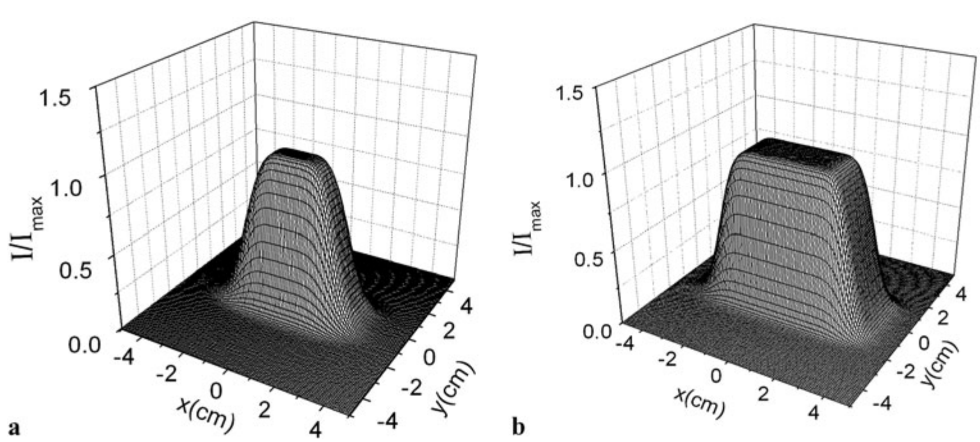

Fig. 1 Normalized intensity distributions of a rectangular flat-topped beam for different values of M and N with $w_{0x} = 2$ cm and $w_{0y} = 1$ cm at $z = 0$. (**a**) $M = N = 3$, (**b**) $M = N = 10$

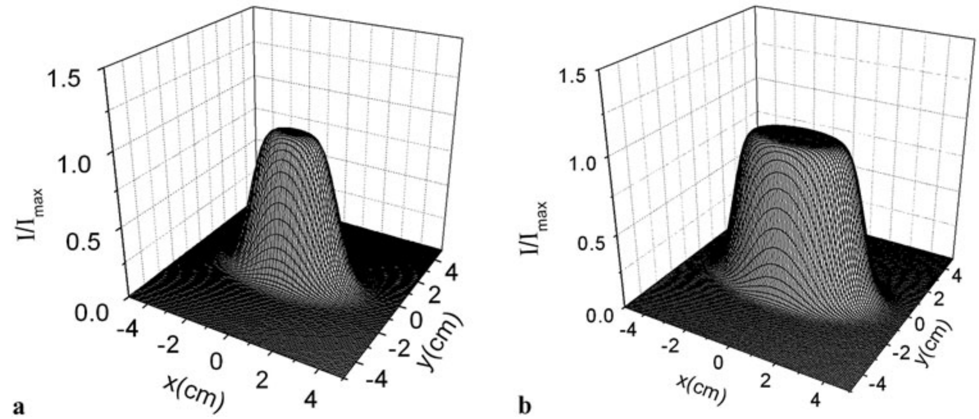

Fig. 2 Normalized intensity distributions of an elliptical flat-topped beam for different values of N with $w_{0x} = 2$ cm and $w_{0y} = 1$ cm at $z = 0$. (**a**) $N = 3$, (**b**) $N = 10$

where σ_g is the transverse coherence width. The intensity of a rectangular partially coherent flat-topped beam can be determined from the relation $I(\rho'_x, \rho'_y, 0) = W(\rho'_x, \rho'_y, \rho'_x, \rho'_y, 0)$. Similarly, the cross-spectral density of an elliptical partially coherent flat-topped beam can be expressed as follows

$$\begin{aligned} &W_N(\rho'_{1x}, \rho'_{1y}, \rho'_{2x}, \rho'_{2y}; 0) \\ &\quad = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m} \\ &\qquad \times \exp\left(-\frac{n\rho'^2_{1x}}{w_{0x}^2} - \frac{n\rho'^2_{1y}}{w_{0y}^2}\right) \\ &\qquad \times \exp\left(-\frac{m\rho'^2_{2x}}{w_{0x}^2} - \frac{m\rho'^2_{2y}}{w_{0y}^2}\right. \\ &\qquad \left. - \frac{(\rho'_{1x} - \rho'_{2x})^2 + (\rho'_{1y} - \rho'_{2y})^2}{2\sigma_g^2}\right). \end{aligned} \tag{4}$$

Under the condition of $\sigma_g \to \infty$, a non-circular partially coherent flat-topped beam becomes a coherent beam. Under the condition of $M = 1$ and $N = 1$, a non-circular partially coherent flat-topped beam becomes a GSM beam [25–32]. Although the intensity of the non-circular partially coherent flat-topped beam at $z = 0$ is independent of the coherence width, the M^2-factors are closely determined by the coherence width as shown later.

Within the validity of the paraxial approximation, the propagation of the cross-spectral density of a partially coherent beam in the turbulent atmosphere can be studied with the help of the following extended Huygens-Fresnel integral [62, 74, 75]:

$$\begin{aligned} &W(\rho_x, \rho_{dx}, \rho_y, \rho_{dy}; z) \\ &\quad = \left(\frac{k}{2\pi z}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\rho'_x, \rho'_{dx}, \rho'_y, \rho'_{dy}; 0) \\ &\qquad \times \exp\left[\frac{ik}{z}(\rho_x - \rho'_x)\cdot(\rho_{dx} - \rho'_{dx}) \right. \\ &\qquad \left. - H(\rho_{dx}, \rho'_{dx}, z)\right] d\rho'_x\, d\rho'_{dx} \\ &\qquad \times \exp\left[\frac{ik}{z}(\rho_y - \rho'_y)\cdot(\rho_{dy} - \rho'_{dy}) \right. \\ &\qquad \left. - H(\rho_{dy}, \rho'_{dy}, z)\right] d\rho'_y\, d\rho'_{dy}, \end{aligned} \tag{5}$$

where $k = 2\pi/\lambda$ is the wave number with λ being the wavelength. In (5) we have used the following sum and difference vector notation

$$\begin{aligned} \rho'_x &= \frac{\rho'_{1x} + \rho'_{2x}}{2}, & \rho'_{dx} &= \rho'_{1x} - \rho'_{2x}, \\ \rho'_y &= \frac{\rho'_{1y} + \rho'_{2y}}{2}, & \rho'_{dy} &= \rho'_{1y} - \rho'_{2y}, \\ \rho_x &= \frac{\rho_{1x} + \rho_{2x}}{2}, & \rho_{dx} &= \rho_{1x} - \rho_{2x}, \\ \rho_y &= \frac{\rho_{1y} + \rho_{2y}}{2}, & \rho_{dy} &= \rho_{1y} - \rho_{2y}, \end{aligned} \tag{6}$$

where ρ_{1x}, ρ_{1y} and ρ_{2x}, ρ_{2y} are the coordinates of two arbitrary points in the receiver plane, perpendicular to the direction of propagation of the beam. We can express the cross-spectral density in the source plane as

$$\begin{aligned} &W(\rho'_{1x}, \rho'_{1y}, \rho'_{2x}, \rho'_{2y}; 0) \\ &\quad = W(\rho'_x, \rho'_y, \rho'_{dx}, \rho'_{dy}; 0) \\ &\quad = W\left(\rho'_x + \frac{\rho'_{dx}}{2}, \rho'_y + \frac{\rho'_{dy}}{2}, \rho'_x - \frac{\rho'_{dx}}{2}, \rho'_y - \frac{\rho'_{dy}}{2}; 0\right). \end{aligned} \tag{7}$$

The term $H(\rho_{dx}, \rho'_{dx}, z)$ and $H(\rho_{dx}, \rho'_{dx}, z)$ in (5) are the contribution from the atmospheric turbulence expressed as

$$\begin{aligned} &H\left(\rho_{dx}, \rho'_{dx}, z\right) \\ &\quad = 4\pi^2 k^2 z \int_0^1 d\xi \int_0^\infty \left[1 - J_0\left(\kappa \left|\rho'_{dx}\xi + (1-\xi)\rho_{dx}\right|\right)\right] \\ &\qquad \times \Phi_n(\kappa)\kappa\, d\kappa, \\ &H\left(\rho_{dy}, \rho'_{dy}, z\right) \\ &\quad = 4\pi^2 k^2 z \int_0^1 d\xi \int_0^\infty \left[1 - J_0\left(\kappa \left|\rho'_{dy}\xi + (1-\xi)\rho_{dy}\right|\right)\right] \\ &\qquad \times \Phi_n(\kappa)\kappa\, d\kappa, \end{aligned} \tag{8}$$

where J_0 is the Bessel function of zero order, Φ_n represents the one-dimensional power spectrum of the index-of-refraction fluctuations [74, 75].

Equation (5) can be expressed in the following alternative form:

$$\begin{aligned} &W(\rho_x, \rho_{dx}, \rho_y, \rho_{dy}; z) \\ &\quad = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W\left(\rho''_x, \rho_{dx} + \frac{z}{k}\kappa_{dx}, \rho''_y,\right. \\ &\qquad \left.\rho_{dy} + \frac{z}{k}\kappa_{dy}; 0\right) \exp\left[-i\rho_x\kappa_{dx} + i\rho''_x\kappa_{dx} - i\rho_y\kappa_{dy}\right. \\ &\qquad + i\rho''_y\kappa_{dy} - H\left(\rho_{dx}, \rho'_{dx}, z\right) \\ &\qquad \left.- H\left(\rho_{dy}, \rho'_{dy}, z\right)\right] d\rho'_x\, d\rho'_{dx}\, d\rho'_y\, d\rho'_{dy}, \end{aligned} \tag{9}$$

where $\boldsymbol{\kappa}_d \equiv (\kappa_{dx}, \kappa_{dy})$ is the position vector in spatial-frequency domain. For a rectangular partially coherent flat-topped beam, using (3), we can express its cross-spectral density $W(\rho''_x, \rho_{dx} + \frac{z}{k}\kappa_{dx}, \rho''_y, \rho_{dy} + \frac{z}{k}\kappa_{dy}; 0)$ as

$$\begin{aligned} &W_{MN}\left(\rho''_x, \rho_{dx} + \frac{z}{k}\kappa_{dx}, \rho''_y, \rho_{dy} + \frac{z}{k}\kappa_{dy}; 0\right) \\ &\quad = \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{h=1}^{M}\sum_{j=1}^{N} \frac{(-1)^{m+n+h+j}}{M^2N^2} \\ &\qquad \times \binom{M}{m}\binom{N}{n}\binom{M}{h}\binom{N}{j} \\ &\qquad \times \exp\left[-A_1\rho''^2_x + B_1\rho''_x\left(\rho_{dx} + \frac{z}{k}\kappa_{dx}\right)\right. \\ &\qquad \left.- C_1\left(\rho_{dx} + \frac{z}{k}\kappa_{dx}\right)^2\right] \\ &\qquad \times \exp\left[-A_2\rho''^2_y + B_2\rho''_y\left(\rho_{dy} + \frac{z}{k}\kappa_{dy}\right)\right. \\ &\qquad - C_2\left(\rho_{dy} + \frac{z}{k}\kappa_{dy}\right)^2 \\ &\qquad \left.- \frac{(\rho_{dx} + \frac{z}{k}\kappa_{dx})^2 + (\rho_{dy} + \frac{z}{k}\kappa_{dy})^2}{2\sigma_g^2}\right], \end{aligned} \tag{10}$$

where $A_1 = \frac{h+m}{w_{0x}^2}$, $A_2 = \frac{j+n}{w_{0y}^2}$, $B_1 = \frac{h-m}{w_{0x}^2}$, $B_2 = \frac{j-n}{w_{0y}^2}$, $C_1 = \frac{h+m}{4w_{0x}^2}$, $C_2 = \frac{j+n}{4w_{0y}^2}$. For an elliptical partially coherent flat-topped beam, using (4), we can express its cross-spectral density $W(\rho''_x, \rho_{dx} + \frac{z}{k}\kappa_{dx}, \rho''_y, \rho_{dy} + \frac{z}{k}\kappa_{dy}; 0)$ as

$$\begin{aligned} &W_N\left(\rho''_x, \rho_{dx} + \frac{z}{k}\kappa_{dx}, \rho''_y, \rho_{dy} + \frac{z}{k}\kappa_{dy}; 0\right) \\ &\quad = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{(-1)^{m+n}}{N^2}\binom{N}{n}\binom{N}{m} \\ &\qquad \times \left[-A_3\rho''^2_x + B_3\rho''_x\left(\rho_{dx} + \frac{z}{k}\kappa_{dx}\right)\right. \\ &\qquad \left.- C_3\left(\rho_{dx} + \frac{z}{k}\kappa_{dx}\right)^2\right] \\ &\qquad \times \exp\left[-A_4\rho''^2_y + B_4\rho''_y\left(\rho_{dy} + \frac{z}{k}\kappa_{dy}\right)\right. \\ &\qquad - C_4\left(\rho_{dy} + \frac{z}{k}\kappa_{dy}\right)^2 \\ &\qquad \left.- \frac{(\rho_{dx} + \frac{z}{k}\kappa_{dx})^2 + (\rho_{dy} + \frac{z}{k}\kappa_{dy})^2}{2\sigma_g^2}\right], \end{aligned} \tag{11}$$

where $A_3 = \frac{m+n}{w_{0x}^2}$, $A_4 = \frac{m+n}{w_{0y}^2}$, $B_3 = \frac{m-n}{w_{0x}^2}$, $B_4 = \frac{m-n}{w_{0y}^2}$, $C_3 = \frac{m+n}{4w_{0x}^2}$, $C_4 = \frac{m+n}{4w_{0y}^2}$.

The Wigner distribution of a partially coherent beam on propagation in turbulent atmosphere can be expressed in terms of the cross-spectral density function by the formula [50]

$$\begin{aligned} &h(\rho_x, \theta_x, \rho_y, \theta_y; z) \\ &\quad = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\rho_x, \rho_{dx}, \rho_y, \rho_{dy}, z) \\ &\qquad \times \exp(-ik\theta_x \cdot \rho_{dx} - ik\theta_y \cdot \rho_{dy})\, d\rho_{dx}\, d\rho_{dy}, \end{aligned} \tag{12}$$

where $\theta \equiv (\theta_x, \theta_y)$ denotes an angle which the vector of interest makes with the z-direction, $k\theta_x$ and $k\theta_y$ are the wave vector components along the x-axis and y-axis, respectively.

Substituting from (9) and (10) into (12), we obtain (after tedious integration) the following expression for the Wigner distribution of a rectangular partially coherent flat-topped beam in the receiver plane:

$$h(\rho_x, \theta_x, \rho_y, \theta_y; z) = h(\rho_x, \theta_x; z)h(\rho_y, \theta_y; z), \tag{13}$$

where

$$h(\rho_x,\theta_x;z) = \frac{k}{4\pi\sqrt{\pi A_1}}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h} \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Big[a_1\rho_{dx}^2+b_1\rho_{dx}\kappa_{dx}+c_1\kappa_{dx}^2 - i\rho_x\kappa_{dx}-ik\theta_x\cdot\rho_{dx} - H\Big(\rho_{dx},\rho_{dx}+\frac{z}{k}\kappa_{dx},z\Big)\Big]d\rho_{dx}\,d\kappa_{dx},$$

$$h(\rho_y,\theta_y;z) = \frac{k}{4\pi\sqrt{\pi A_2}}\sum_{n=1}^{N}\sum_{j=1}^{N}\frac{(-1)^{n+j}}{N^2}\binom{N}{n}\binom{N}{j} \times\exp\Big[a_2\rho_{dy}^2+b_2\rho_{dy}\kappa_{dy}+c_2\kappa_{dy}^2 - i\rho_y\kappa_{dy}-ik\theta_y\cdot\rho_{dy} - H\Big(\rho_{dy},\rho_{dy}+\frac{z}{k}\kappa_{dy},z\Big)\Big]d\rho_{dx}\,d\rho_{dy}\,d\kappa_{dx}\,d\kappa_{dy}, \tag{14}$$

$$a_1=-\frac{hm}{(h+m)w_{0x}^2}-\frac{1}{2\sigma_g^2},$$

$$b_1=\frac{i(h-m)}{2(h+m)}-\frac{2zhm}{k(h+m)w_{0x}^2}-\frac{z}{k\sigma_g^2},$$

$$c_1=\frac{i(h-m)z}{2(h+m)k}-\frac{hmz^2}{k^2(h+m)w_{0x}^2}-\frac{z^2}{2k^2\sigma_g^2}-\frac{w_{0x}^2}{4(h+m)},$$

$$a_2=-\frac{jn}{(j+n)w_{0y}^2}-\frac{1}{2\sigma_g^2},$$

$$b_2=\frac{i(j-n)}{2(j+n)}-\frac{2zjn}{k(j+n)w_{0y}^2}-\frac{z}{k\sigma_g^2},$$

$$c_2=\frac{i(j-n)z}{2(j+n)k}-\frac{jnz^2}{k^2(j+n)w_{0y}^2}-\frac{z^2}{2k^2\sigma_g^2}-\frac{w_{0y}^2}{4(j+n)}.$$

In a similar way, we obtain the Wigner distribution of an elliptical partially coherent flat-topped beam in the receiver plane as

$$h(\rho_x,\theta_x,\rho_y,\theta_y z) = \frac{k^2}{16\pi^3\sqrt{A_3A_4}}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m} \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Big[a_3\rho_{dx}^2+b_3\rho_{dx}\kappa_{dx}+c_3\kappa_{dx}^2 - i\rho_x\kappa_{dx}-ik\theta_x\cdot\rho_{dx}-H\Big(\rho_{dx},\rho_{dx}+\frac{z}{k}\kappa_{dx},z\Big)\Big] \times\exp\Big[a_4\rho_{dy}^2+b_4\rho_{dy}\kappa_{dy}+c_4\kappa_{dy}^2 - i\rho_y\kappa_{dy}-ik\theta_y\cdot\rho_{dy} - H\Big(\rho_{dy},\rho_{dy}+\frac{z}{k}\kappa_{dy},z\Big)\Big]d\rho_{dy}\,d\kappa_{dy}\,d\rho_{dx}\,d\kappa_{dx}, \tag{15}$$

where

$$\begin{aligned} a_3&=-\frac{mn}{(m+n)w_{0x}^2}-\frac{1}{2\sigma_g^2},\\ b_3&=\frac{i(m-n)}{2(m+n)}-\frac{2zmn}{k(m+n)w_{0x}^2}-\frac{z}{k\sigma_g^2},\\ c_3&=\frac{i(m-n)z}{2(m+n)k}-\frac{mnz^2}{k^2(m+n)w_{0x}^2}-\frac{z^2}{2k^2\sigma_g^2} -\frac{w_{0x}^2}{4(m+n)},\\ a_4&=-\frac{mn}{(m+n)w_{0y}^2}-\frac{1}{2\sigma_g^2},\\ b_4&=\frac{i(m-n)}{2(m+n)}-\frac{2zmn}{k(m+n)w_{0y}^2}-\frac{z}{k\sigma_g^2},\\ c_4&=\frac{i(m-n)z}{2(m+n)k}-\frac{mnz^2}{k^2(m+n)w_{0y}^2}-\frac{z^2}{2k^2\sigma_g^2} -\frac{w_{0y}^2}{4(m+n)}. \end{aligned} \tag{16}$$

Based on the second-order moments of the Wigner distribution function, the M^2-factor of a partially coherent beam in x or y direction is defined as [68, 74]

$$M_s^2(z)=2k\big(\langle\rho_s^2\rangle\langle\theta_s^2\rangle-\langle\rho_s\theta_s\rangle^2\big)^{1/2}\quad (s=x,y), \tag{17}$$

where

$$\langle\rho_x^{n1}\rho_y^{n2}\theta_x^{m1}\theta_y^{m2}\rangle = \frac{1}{P}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_x^{n1}\rho_y^{n2}\theta_x^{m1}\theta_y^{m2} \times h(\rho_x,\theta_x,\rho_y,\theta_y;z)\,d\rho_x\,d\theta_x\,d\rho_y\,d\theta_y, \tag{18}$$

$$P=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(\rho_x,\theta_x,\rho_y,\theta_y z)\,d\rho_x\,d\theta_x\,d\rho_y\,d\theta_y. \tag{19}$$

Substituting (13) into (18) and (19), we obtain

$$\langle\rho_x^2\rangle=\frac{1}{P}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_x^2 \times h(\rho_x,\theta_x,\rho_y,\theta_y;z)\,d\rho_x\,d\theta_x\,d\rho_y\,d\theta_y = \frac{1}{P_xP_y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_x^2h(\rho_x,\theta_x;z)\,d\rho_x\,d\theta_x$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\rho_y,\theta_y;z)\,d\rho_y\,d\theta_y$$
$$= \frac{1}{P_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_x^2 h(\rho_x,\theta_x;z)\,d\rho_x\,d\theta_x, \tag{20}$$

where

$$P_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\rho_x,\theta_x,z)\,d\rho_x\,d\theta_x,$$
$$P_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\rho_y,\theta_y;z)\,d\rho_y\,d\theta_y. \tag{21}$$

Substituting (14) into (20), we obtain

$$\begin{aligned}\langle\rho_x^2\rangle &= \frac{k\sqrt{\pi}}{4\pi^2 P_x\sqrt{A_1}}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_x^2\exp\bigl[a_1\rho_{dx}^2+b_1\rho_{dx}\kappa_{dx}\\
&\quad+c_1\kappa_{dx}^2-i\rho_x\kappa_{dx}-ik\theta_x\cdot\rho_{dx}\bigr]\\
&\quad\times\exp\Bigl[-H\Bigl(\rho_{dx},\rho_{dx}+\frac{z}{k}\kappa_{dx},z\Bigr)\Bigr]d\rho_{dx}\,d\kappa_{dx}\,d\rho_x\,d\theta_x\\
&= \frac{k\sqrt{\pi}}{4\pi^2 P_x\sqrt{A_1}}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h}\\
&\quad\times\int_{-\infty}^{\infty}\rho_x^2\cdot\exp[-i\rho_x\kappa_{dx}]\,d\rho_{dx},\\
&\int_{-\infty}^{\infty}\exp[-ik\theta_x\cdot\rho_{dx}]\,d\theta_x\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Bigl[a_1\rho_{dx}^2+b_1\rho_{dx}\kappa_{dx}\\
&\quad+c_1\kappa_{dx}^2-H\Bigl(\rho_{dx},\rho_{dx}+\frac{z}{k}\kappa_{dx},z\Bigr)\Bigr]d\rho_{dx}\,d\kappa_{dx}\\
&= -\frac{\sqrt{\pi}}{P_x\sqrt{A_1}}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h}\\
&\quad\times\int_{-\infty}^{\infty}\exp\Bigl[\frac{a_1\rho_{dx}^2}{k^2}+\frac{b_1\rho_{dx}\kappa_{dx}}{k}\Bigr],\\
&\exp\Bigl[-4\pi^2k^2z\int_0^1 d\xi\int_0^{\infty}\Bigl[1-J_0\Bigl(\kappa\Bigl|\frac{z}{k}\kappa_{dx}\xi+\frac{\rho_{dx}}{k}\Bigr|\Bigr)\Bigr]\\
&\quad\times\Phi_n(\kappa)\kappa\,d\kappa\Bigr]\delta(\rho_{dx})\,d\rho_{dx}\\
&\quad\times\int_{-\infty}^{\infty}\exp[c_1\kappa_{dx}^2]\delta''(\kappa_{dx})\,d\kappa_{dx}\\
&= -\frac{\sqrt{\pi}}{P_x\sqrt{A_1}}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h}\\
&\quad\times\int_{-\infty}^{\infty}\exp\Bigl[c_1\kappa_{dx}^2-4\pi^2k^2z\\
&\quad\times\int_0^1 d\xi\int_0^{\infty}\Bigl[1-J_0\Bigl(\kappa\Bigl|\frac{z}{k}\kappa_{dx}\xi\Bigr|\Bigr)\Bigr]\\
&\quad\times\Phi_n(\kappa)\kappa\,d\kappa\Bigr]\delta''(\kappa_{dx})\,d\kappa_{dx}\\
&= -\frac{\sqrt{\pi}}{P_x\sqrt{A_1}}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h}\\
&\quad\times\int_{-\infty}^{\infty}\exp\Bigl[c_1\kappa_{dx}^2-\pi^2z^3\kappa_{dx}^2\\
&\quad\times\int_0^1\xi^2d\xi\int_0^{\infty}\Phi_n(\kappa)\kappa^3\,d\kappa\Bigr]\delta''(\kappa_{dx})\,d\kappa_{dx}\\
&= -\frac{\sqrt{\pi}}{P_x\sqrt{A_1}}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h}\\
&\quad\times\Bigl[\exp\Bigl(c_1\kappa_{dx}^2-\frac{1}{3}\pi^2Tz^3\kappa_{dx}^2\Bigr)\Bigr]''\\
&= \frac{\sqrt{\pi}}{P_x}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h}\\
&\quad\times\Bigl(\frac{-2c_1+\frac{2}{3}\pi^2Tz^3}{\sqrt{A_1}}\Bigr),\end{aligned} \tag{22}$$

where

$$T = \int_0^{\infty}\Phi_n(\kappa)\kappa^3\,d\kappa. \tag{23}$$

In the above derivations, we have used the following expansion formula [76]:

$$J_0(x)\approx 1-\frac{1}{4}x^2, \tag{24}$$

and integral formulae [77]

$$\begin{aligned}&\int_{-\infty}^{\infty}\exp(-s^2x^2\pm qx)\,dx = \frac{\sqrt{\pi}}{s}\exp\Bigl(\frac{q^2}{4s^2}\Bigr)\quad (s>0),\\
&\delta(s) = \frac{1}{2\pi}\int\exp(-isx)\,dx,\\
&\delta^{(n)}(s) = \frac{1}{2\pi}\int(-ix)^n\exp(-isx)\,dx\quad (n=1,2),\\
&\int f(x)\delta^{(n)}(x)\,dx = (-1)^n f^{(n)}(0)\quad (n=1,2).\end{aligned} \tag{25}$$

The use of (24) in (22) means that we in fact have used the quadratic approximation for the wave structure function. The quadratic approximation has been approved reliable in [75, 78] and has been used widely [18–24, 55–63]. The difference between the result obtained with this quadratic approximation and the one calculated numerically by the extended Huygens-Fresnel integral is quite small and they give the same trends. Such a difference becomes much greater toward the complete incoherent limit and at relatively higher structure constant values [75, 78].

In a similar way, we obtain the following expressions for P_x, P_y, $\langle\rho_y^2\rangle$, $\langle\theta_x^2\rangle$, $\langle\theta_y^2\rangle$, $\langle\rho_x\theta_x\rangle$ and $\langle\rho_y\theta_y\rangle$,

$$P_x = \sqrt{\pi}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{1}{\sqrt{A_1}}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h},$$
$$P_y = \pi\sum_{n=1}^{N}\sum_{j=1}^{N}\frac{1}{\sqrt{A_2}}\frac{(-1)^{n+j}}{N^2}\binom{N}{n}\binom{N}{j}, \tag{26}$$

$$\langle\rho_y^2\rangle = \frac{1}{P_y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_y^2 h(\rho_y,\theta_y;z)\,d\rho_y\,d\theta_y = \frac{\sqrt{\pi}}{P_y}\sum_{n=1}^{N}\sum_{j=1}^{N}\frac{(-1)^{n+j}}{N^2}\binom{N}{n}\binom{N}{j} \times\left(\frac{-2c_2+\frac{2}{3}\pi^2Tz^3}{\sqrt{A_2}}\right), \tag{27}$$

$$\langle\theta_x^2\rangle = \frac{1}{P_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_x^2 h(\rho_x,\theta_x;z)\,d\rho_x\,d\theta_x = \frac{\sqrt{\pi}}{P_x}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h} \times\left(\frac{2\pi^2zT-\frac{2a_1}{k^2}}{\sqrt{A_1}}\right), \tag{28}$$

$$\langle\theta_y^2\rangle = \frac{1}{P_y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_y^2 h(\rho_y,\theta_y;z)\,d\rho_y\,d\theta_y = \frac{\sqrt{\pi}}{P_y}\sum_{n=1}^{N}\sum_{j=1}^{N}\frac{(-1)^{n+j}}{N^2}\binom{N}{n}\binom{N}{j}\left(\frac{2\pi^2zT-\frac{2a_2}{k^2}}{\sqrt{A_2}}\right), \tag{29}$$

$$\langle\rho_x\cdot\theta_x\rangle = \frac{1}{P_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_x\cdot\theta_x h(\rho_x,\theta_x;z)\,d\rho_x\,d\theta_x = \frac{\sqrt{\pi}}{P_x}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h} \times\left(\frac{\pi^2z^2T-\frac{b_1}{k}}{\sqrt{A_1}}\right), \tag{30}$$

$$\langle\rho_y\theta_y\rangle = \frac{1}{P_y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_y\cdot\theta_y h(\rho_y,\theta_y;z)\,d\rho_y\,d\theta_y = \frac{\sqrt{\pi}}{P_y}\sum_{n=1}^{N}\sum_{j=1}^{N}\frac{(-1)^{n+j}}{N^2}\binom{N}{n}\binom{N}{j} \times\left(\frac{\pi^2z^2T-\frac{b_2}{k}}{\sqrt{A_2}}\right). \tag{31}$$

Substituting (22), (26)–(31) into (17), we obtain the following expression for the M^2-factors of a rectangular partially coherent flat-topped beam on propagation in turbulent atmosphere:

$$M_x^2(z) = 2k\left\{\left[\frac{\sqrt{\pi}}{P_x}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h} \times\frac{\frac{2}{3}\pi^2Tz^3-2c_1}{\sqrt{A_1}}\right] \times\left[\frac{\sqrt{\pi}}{P_x}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h} \times\frac{2\pi^2zT-\frac{2a_1}{k^2}}{\sqrt{A_1}}\right] - \left[\frac{\sqrt{\pi}}{P_x}\sum_{m=1}^{M}\sum_{h=1}^{M}\frac{(-1)^{m+h}}{M^2}\binom{M}{m}\binom{M}{h} \times\frac{\pi^2z^2T-\frac{b_1}{k}}{\sqrt{A_1}}\right]^2\right\}^{1/2}, \tag{32}$$

$$M_y^2(z) = 2k\left\{\left[\frac{\sqrt{\pi}}{P_y}\sum_{n=1}^{N}\sum_{j=1}^{N}\frac{(-1)^{n+j}}{N^2}\binom{N}{n}\binom{N}{j} \times\frac{\frac{2}{3}\pi^2Tz^3-2c_2}{\sqrt{A_2}}\right] \times\left[\frac{\sqrt{\pi}}{P_y}\sum_{n=1}^{N}\sum_{j=1}^{N}\frac{(-1)^{n+j}}{N^2}\binom{N}{n}\binom{N}{j} \times\frac{2\pi^2zT-\frac{2a_2}{k^2}}{\sqrt{A_2}}\right] - \left[\frac{\sqrt{\pi}}{P_y}\sum_{n=1}^{N}\sum_{j=1}^{N}\frac{(-1)^{n+j}}{N^2}\binom{N}{n}\binom{N}{j} \times\frac{\pi^2z^2T-\frac{b_2}{k}}{\sqrt{A_2}}\right]^2\right\}^{1/2}. \tag{33}$$

In a similar way, substituting from (15) into (17)–(19), we obtain (after tedious integration and operation) following expression for the M^2-factors of an elliptical partially coherent flat-topped beam on propagation in turbulent atmosphere

$$M_x^2(z) = 2k\left\{\left[\frac{\pi}{P}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m} \times\frac{\frac{2}{3}\pi^2Tz^3-2c_3}{\sqrt{A_3A_4}}\right]\right.$$

$$\times \left[\frac{\pi}{P} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n} \binom{N}{m} \times \frac{2\pi^2 z T - \frac{2a_3}{k^2}}{\sqrt{A_3 A_4}} \right] - \left[\frac{\pi}{P} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n} \binom{N}{m} \times \frac{\pi^2 z^2 T - \frac{b_3}{k}}{\sqrt{A_3 A_4}} \right]^2 \Bigg\}^{1/2}, \quad (34)$$

$$M_y^2(z) = 2k \Bigg\{ \left[\frac{\pi}{P} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n} \binom{N}{m} \times \frac{\frac{2}{3}\pi^2 T z^3 - 2c_4}{\sqrt{A_3 A_4}} \right] \times \left[\frac{\pi}{P} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n} \binom{N}{m} \times \frac{2\pi^2 z T - \frac{2a_4}{k^2}}{\sqrt{A_3 A_4}} \right] - \left[\frac{\pi}{P} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n} \binom{N}{m} \times \frac{\pi^2 z^2 T - \frac{b_4}{k}}{\sqrt{A_3 A_4}} \right]^2 \Bigg\}^{1/2}, \quad (35)$$

with

$$P = \pi \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n} \binom{N}{m} \frac{1}{\sqrt{A_3 A_4}}. \quad (36)$$

Equations (32)–(36) are the main analytical results of present paper. Under the condition of $\Phi_n(\kappa) = 0$ (without turbulence), (32)–(36) reduce to the expression for the M^2-factors of a non-circular partially coherent flat-topped beam in free space.

3 Numerical results

In this section, we study the properties of the M^2-factors of a non-circular partially coherent flat-topped beam in free space and in turbulent atmosphere numerically and comparatively using the formulae derived in above section

Figures 3, 4, 5 show the dependence of the M^2-factors of rectangular and elliptical partially coherent flat-topped beams on the beam orders (M and N), transverse coherence width σ_g, waist sizes w_{0x} and w_{0y} in free space (i.e., $T = 0$ or $\Phi_n(\kappa) = 0$). From Figs. 3–5, one finds that the M^2-factors of a non-circular partially coherent flat-topped beam in free space are mainly determined by the beam parameters in the source plane. Their values increase as the beam orders and waist sizes increase or as the transverse coherence width decreases. Furthermore, the M^2-factors of a non-circular partially coherent flat-topped beam in free space are independent of its wavelength and the propagation distance, as shown in (32)–(36).

Now we study the properties of the normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere. In the following

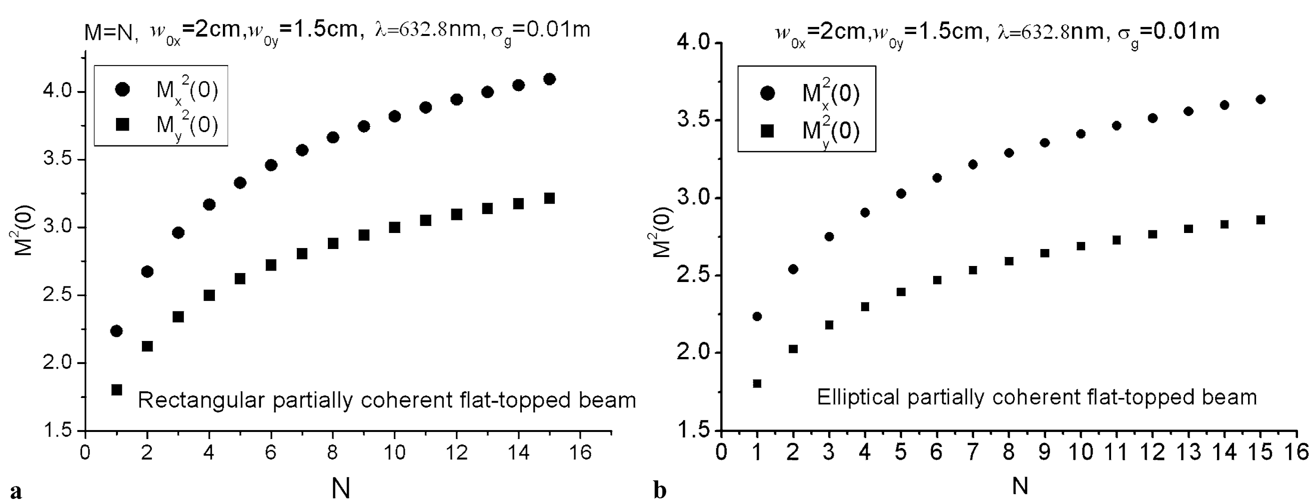

Fig. 3 Dependence of the M^2-factors of rectangular and elliptical partially coherent flat-topped beams on the beam orders (M and N) in free space

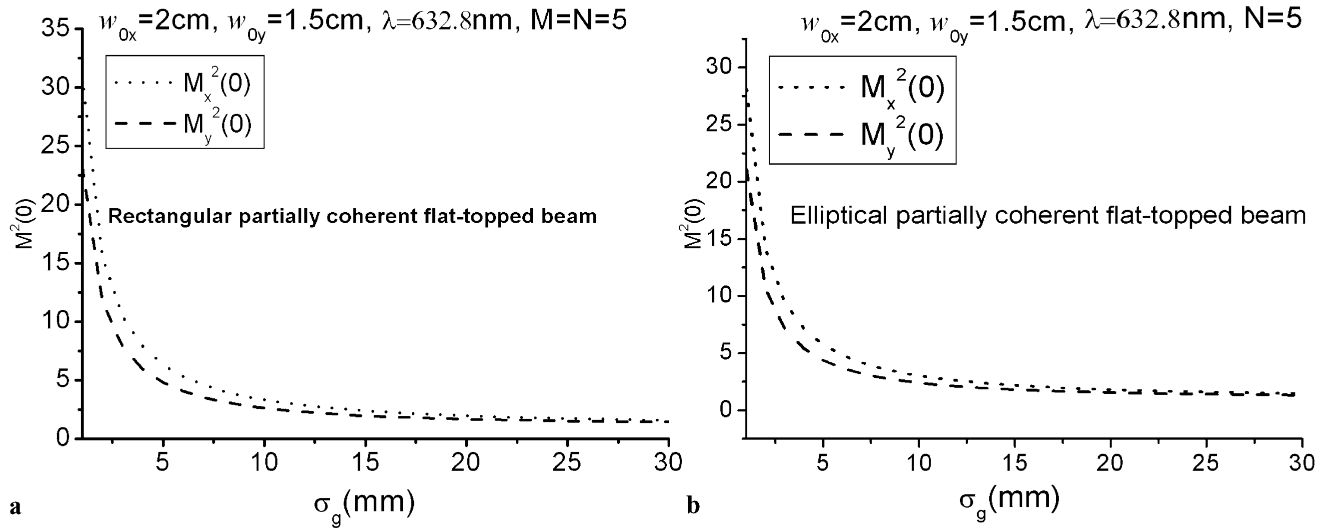

Fig. 4 Dependence of the M^2-factors of rectangular and elliptical partially coherent flat-topped beams on the transverse coherence width σ_g in free space

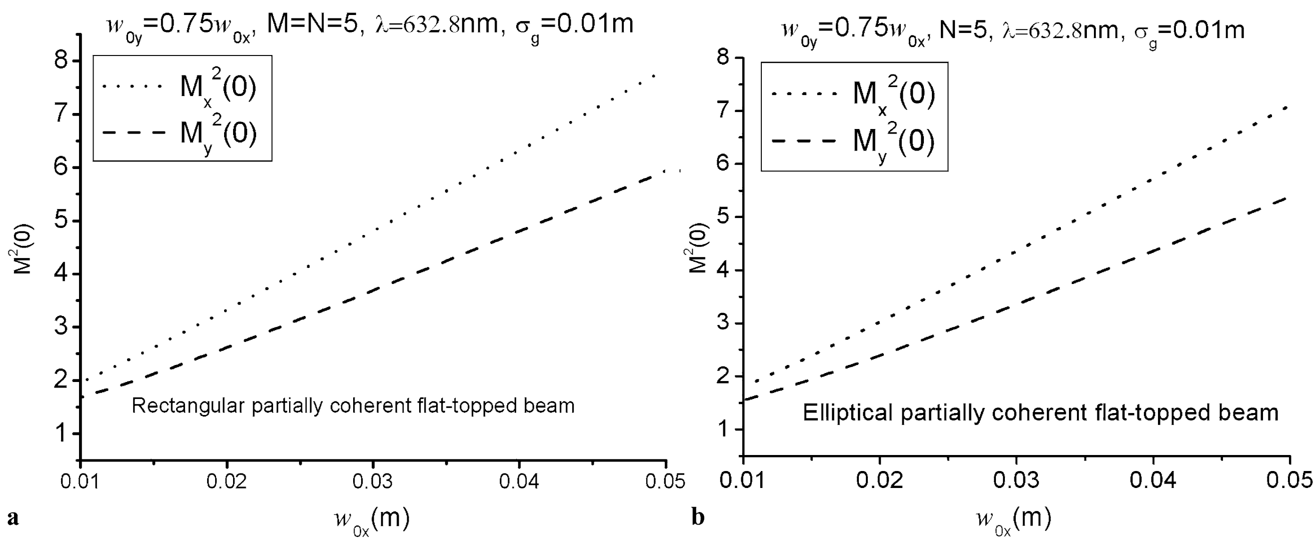

Fig. 5 Dependence of the M^2-factors of rectangular and elliptical partially coherent flat-topped beams on the waist sizes w_{0x} and w_{0y} in free space

numerical examples, we choose the Tatarskii spectrum for the spectral density of the index-of-refraction fluctuations, which is expressed as [62, 74, 75]

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3} \exp\left(-\frac{\kappa^2}{\kappa_m^2}\right), \tag{37}$$

where C_n^2 is the structure constant of the turbulent atmosphere, $\kappa_m = 5.92/l_0$ with l_0 being the inner scale of the turbulence. Substituting from (37) into (23), we obtain

$$T = \int_0^\infty \Phi_n(\kappa)\kappa^3\, d\kappa = 0.1661 C_n^2 l_0^{-1/3}. \tag{38}$$

Substituting from (38) into (32)–(36) we can calculate the M^2-factors of a non-circular partially coherent flat-topped beam numerically.

For the convenience of comparison, we study the evolution properties of the normalized M^2-factors of a non-circular partially coherent flat-topped beam defined as $M_s^2(z)/M_s^2(0)$ $(s = x, y)$ on propagation in turbulent atmosphere. Figure 6 shows the normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different values of the structure constant C_n^2 of the turbulent atmosphere with $w_{0x} = 2$ cm, $w_{0y} = 1.5$ cm, $l_0 = 0.01$ m, $\lambda = 632.8$ nm and $\sigma_g = 8$ cm. From Fig. 6, we find that the normalized

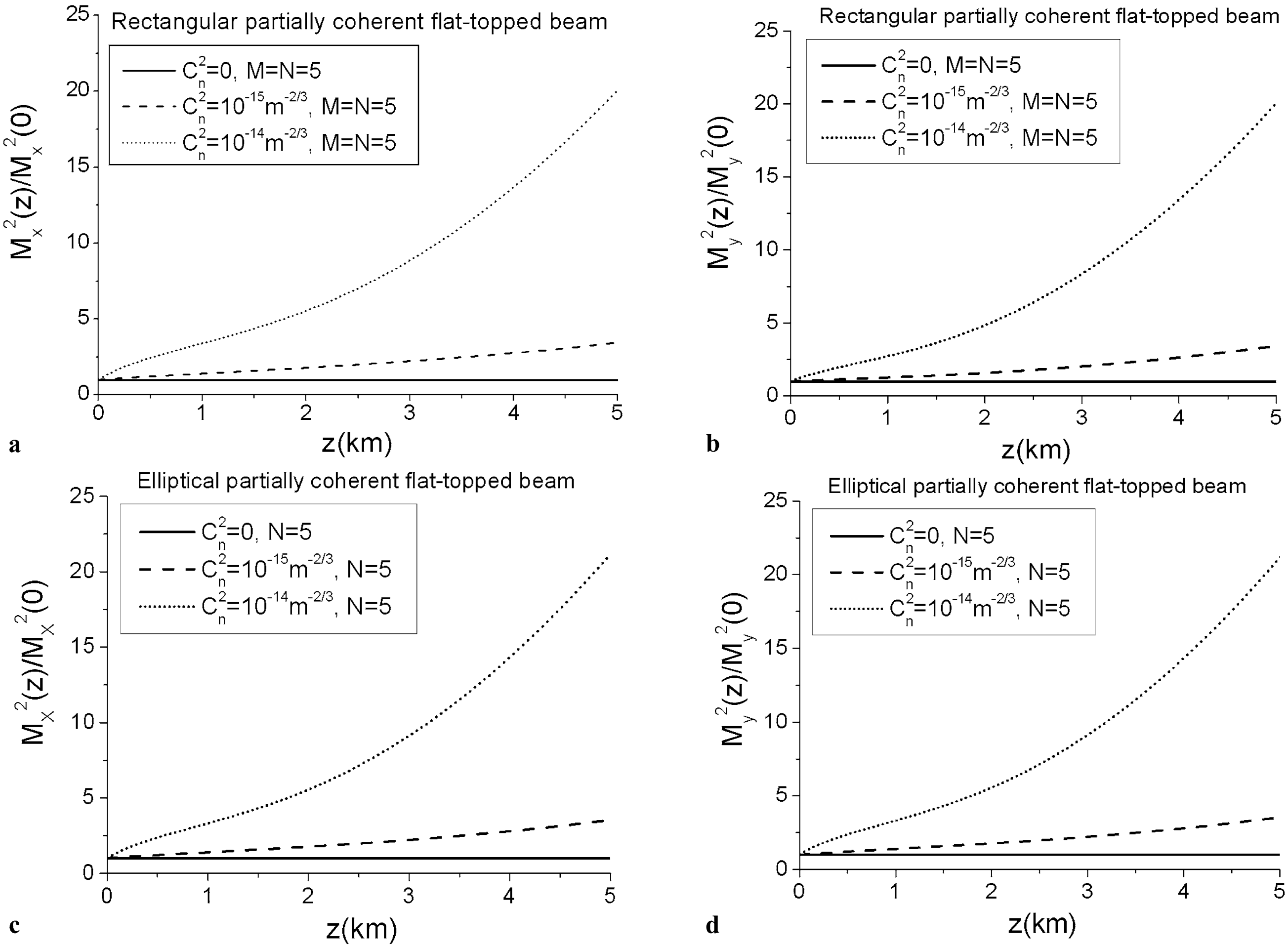

Fig. 6 Normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different values of the structure constant C_n^2 of the turbulent atmosphere

M^2-factors of a non-circular partially coherent flat-topped beam increase on propagation in turbulent atmosphere, which is much different from their propagation-invariant properties in free space. This can be explained by the fact that the turbulence degrades the beam quality of a partially coherent flat-topped beam [55], and this degradation increases as the turbulence becomes strong (i.e., C_n^2 increases). Figure 7 shows the normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different values of the inner scale of the turbulence l_0 with $w_{0x} = 2$ cm, $w_{0y} = 1.5$ cm, $C_n^2 = 10^{-15}$ m$^{-2/3}$, $\lambda = 632.8$ nm and $\sigma_g = 8$ cm. One finds from Fig. 7 that the evolution properties of the normalized M^2-factors of a non-circular partially coherent flat-topped beam are also determined by the inner scale of the turbulence, and the normalized M^2-factors increase more rapidly as the inner scale of the turbulence decreases.

Now we discuss the influence of the initial beam parameters on the evolution properties of the normalized M^2-factors of a non-circular partially coherent flat-topped beam in turbulent atmosphere. We calculate in Fig. 8 the normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different beam orders with $l_0 = 0.01$ m, $C_n^2 = 10^{-15}$ m$^{-2/3}$, $\lambda = 632.8$ nm and $\sigma_g = 8$ cm. As seen in Fig. 8, the normalized M^2-factors of a non-circular partially coherent flat-topped beam ($M > 1$, $N > 1$) are larger than those of a GSM beam ($M = N = 1$) at short propagation distances, but are smaller than those of a GSM beam at long propagation distances, and the normalized M^2-factors decreases as the beam orders increase at long propagation distance. One can come to the conclusion that a non-circular partially coherent flat-topped beam with larger beam orders has advantages over a non-circular partially coherent flat-topped beam with smaller beam orders and GSM beam for long-distance free-space optical communications. Figure 9 shows the normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different values of the waist sizes with $l_0 = 0.01$ m, $C_n^2 = 10^{-15}$ m$^{-2/3}$, $\lambda = 632.8$ nm and $\sigma_g = 8$ cm. It is

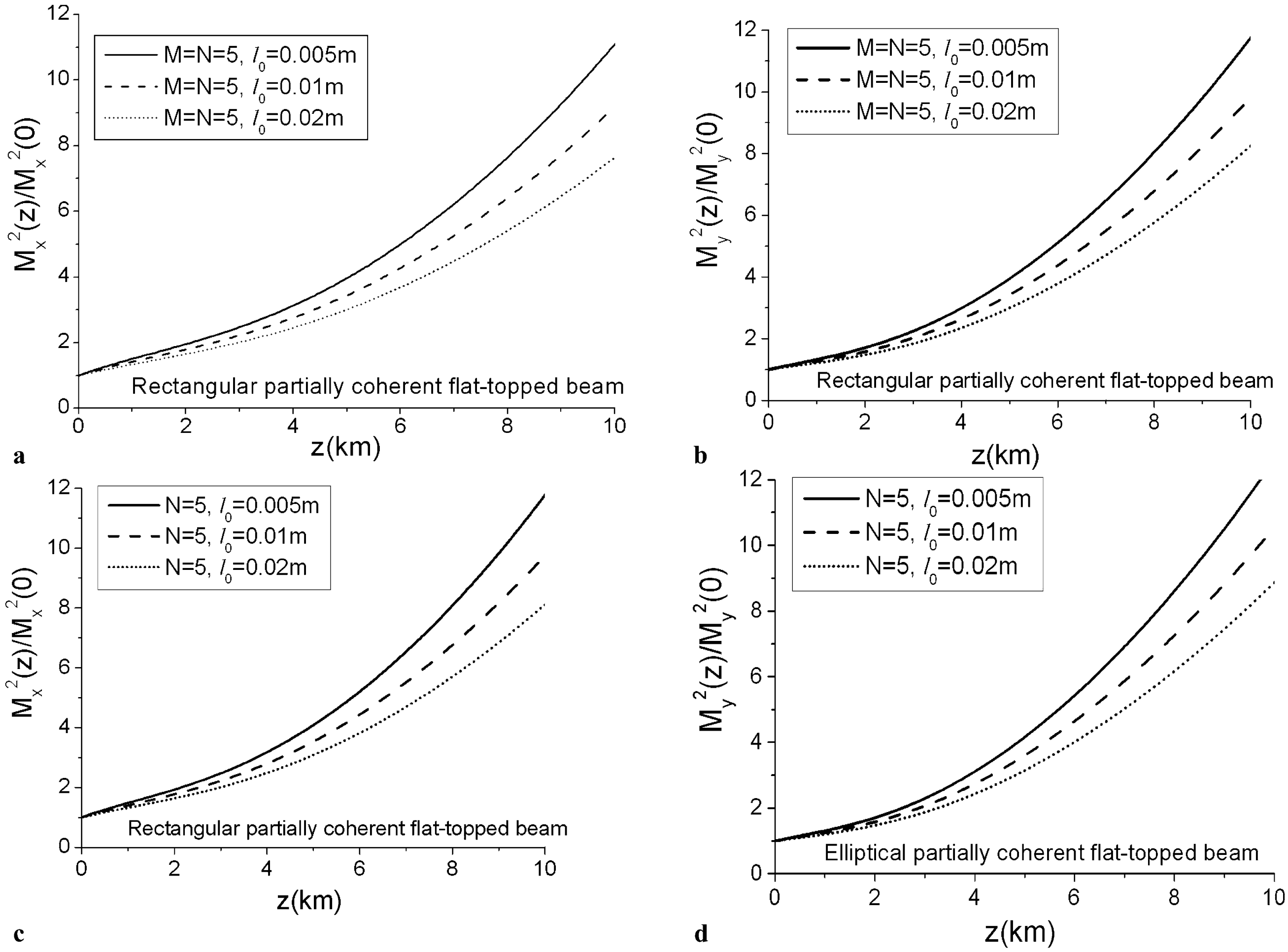

Fig. 7 Normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different values of the inner scale of the turbulence l_0

clear from Fig. 9 that the normalized M^2-factors of a rectangular partially coherent flat-topped beam ($w_{0x} > 2$ cm, $w_{0y} = 2$ cm) are larger than those of a square partially coherent flat-topped beam ($w_{0x} = 2$ cm, $w_{0y} = 2$ cm) at short propagation distances, but are smaller than those of a square partially coherent flat-topped beam at long propagation distances. Similarly, an elliptical partially coherent flat-topped beam ($w_{0x} > 2$ cm, $w_{0y} = 2$ cm) has advantages over a circular partially coherent flat-topped beam ($w_{0x} = 2$ cm, $w_{0y} = 2$ cm) and a GSM beam at long propagation distances. Figure 10 shows the normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different values of the transverse coherence width σ_g with $l_0 = 0.01$ m, $C_n^2 = 10^{-15}$ m$^{-2/3}$, $\lambda = 632.8$ nm, $w_{0x} = 2$ cm, $w_{0y} = 1.5$ cm. As shown by Fig. 10, the normalized M^2-factors of a non-circular partially coherent flat-topped beam increase more rapidly on propagation as its initial coherence increases, which means a non-circular flat-topped beam with lower coherence is less affected by the atmospheric turbulence. A similar phenomenon is also observed for a circular partially coherent flat-topped beam [62]. Figure 11 shows the normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different values of the wavelength with $l_0 = 0.01$ m, $C_n^2 = 10^{-15}$ m$^{-2/3}$, $w_{0x} = 2$ cm, $w_{0y} = 1.5$ cm and $\sigma_g = 8$ cm. Although the M^2-factors of a non-circular partially coherent flat-topped beam in free space is independent of its wavelength, its value is closely related to the wavelength in turbulent atmosphere. The normalized M^2-factors of a non-circular partially coherent flat-topped beam increases more rapidly as its wavelength decreases, which means a non-circular partially coherent flat-topped beam with longer wavelength is less affected by the turbulence.

4 Summary

In conclusion, we have derived the analytical formulae for the M^2-factors of a non-circular partially coherent flat-

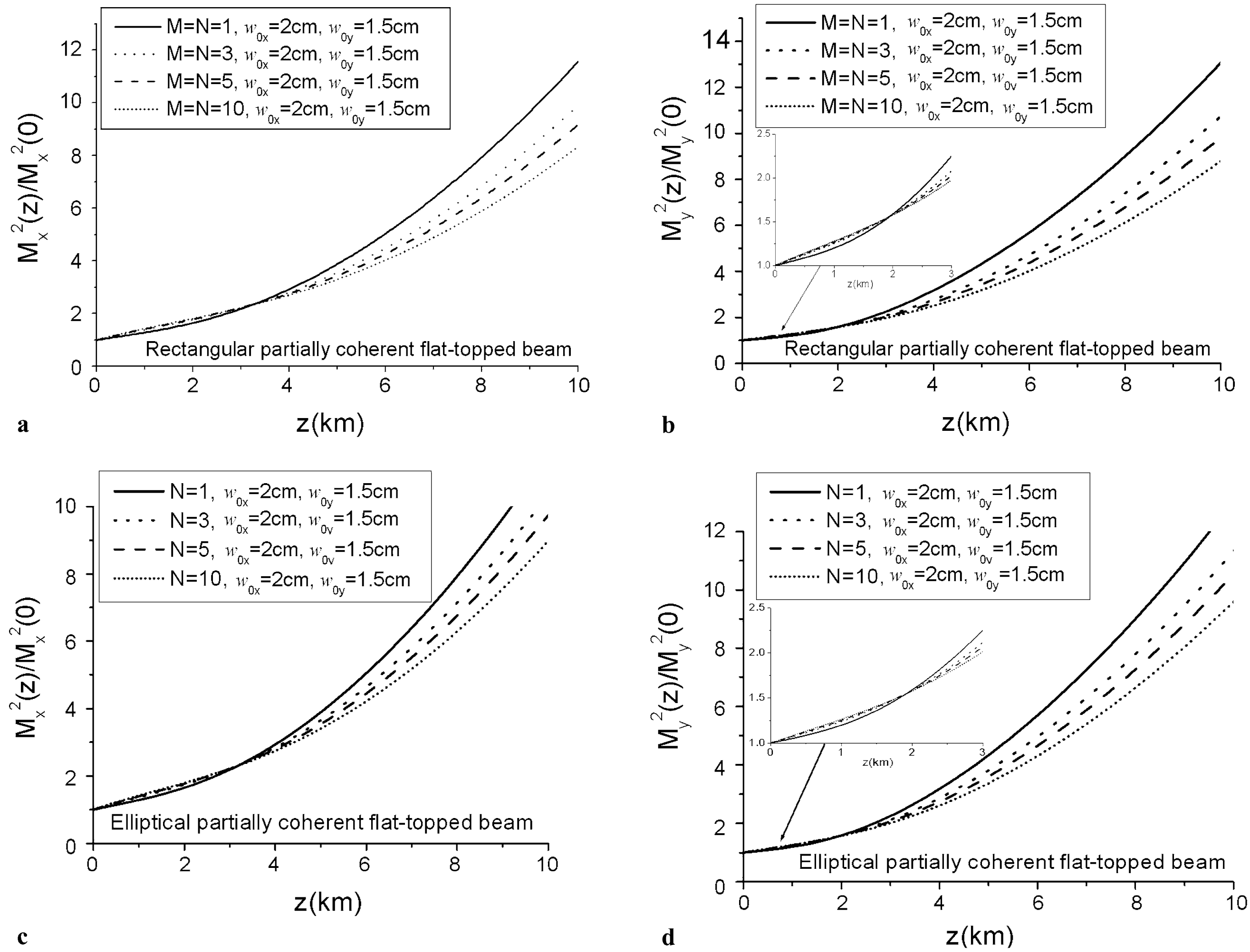

Fig. 8 Normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different beam orders

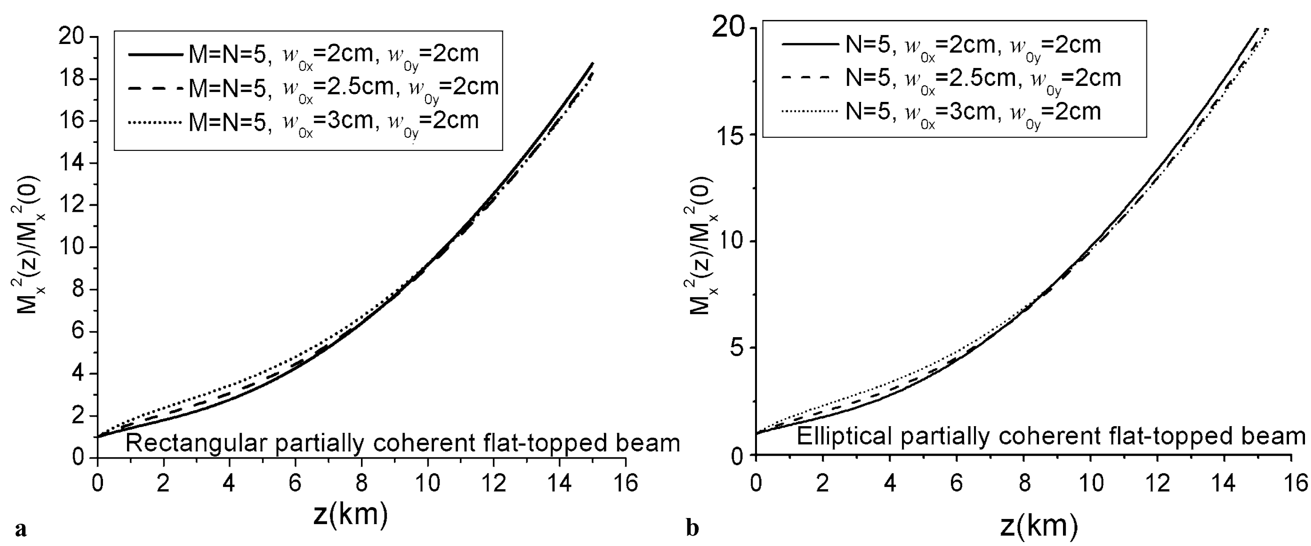

Fig. 9 Normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different waist sizes

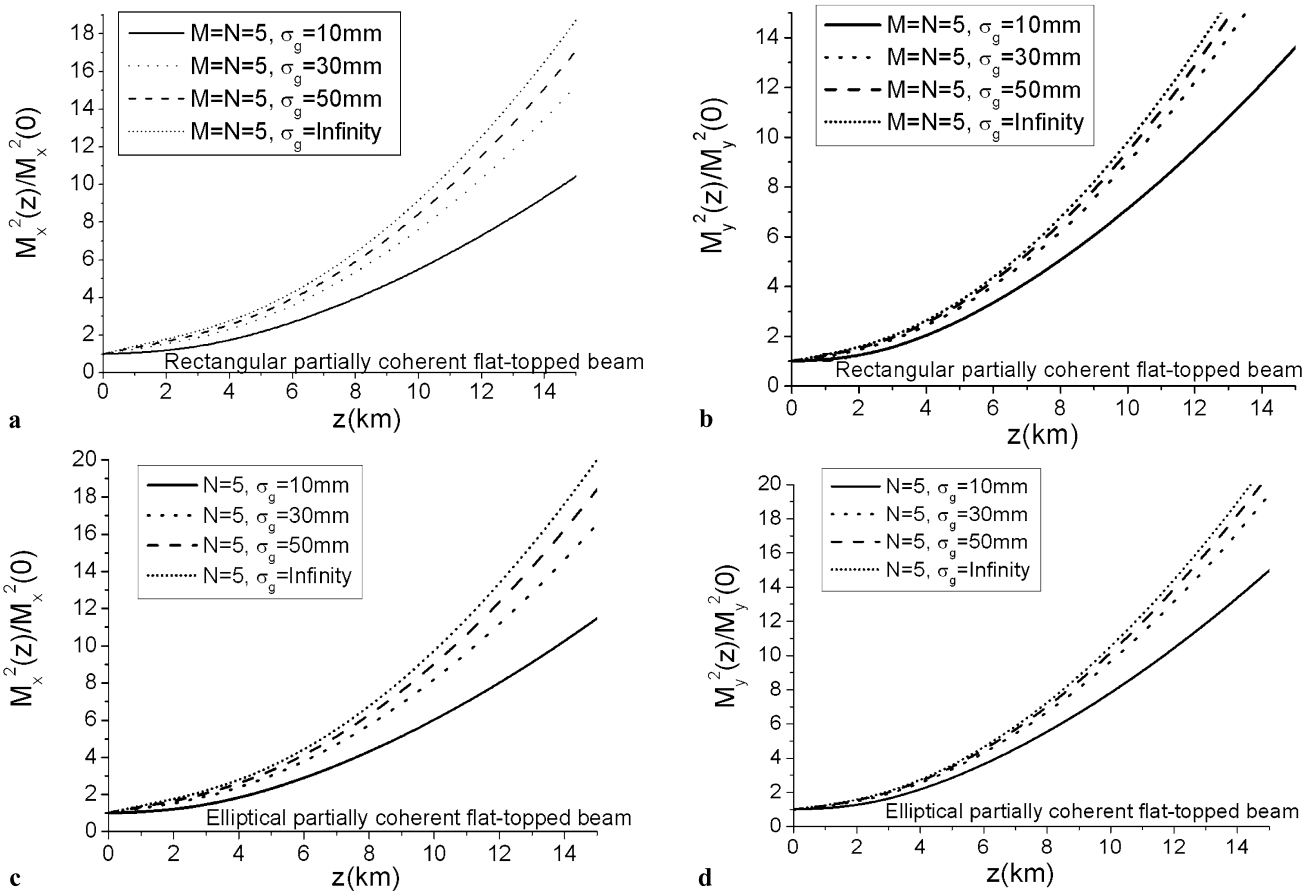

Fig. 10 Normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different transverse coherence width

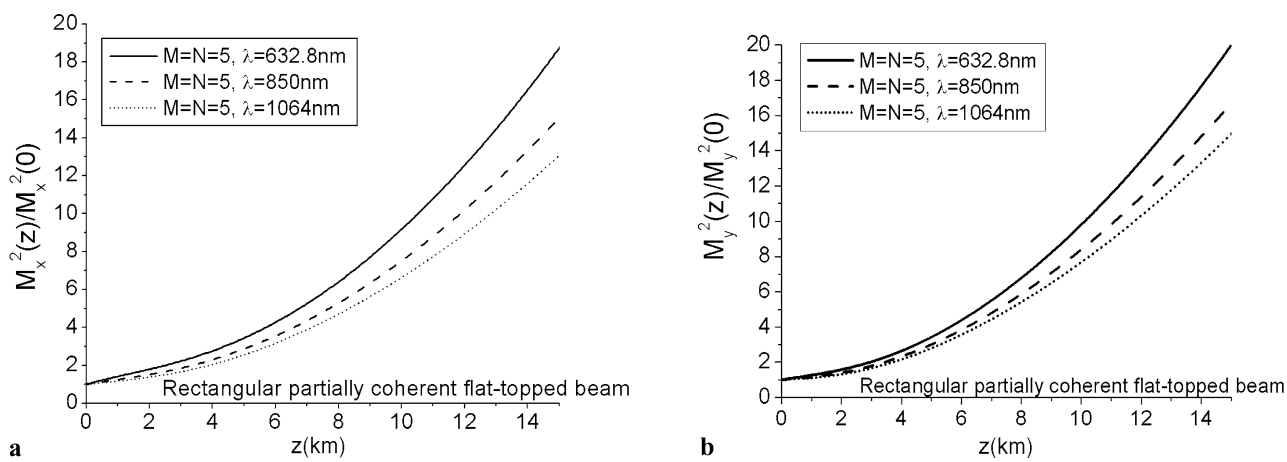

Fig. 11 Normalized M^2-factors of a non-circular partially coherent flat-topped beam on propagation in turbulent atmosphere for different wavelength

topped beam in turbulent atmosphere. Properties of the M^2-factors of a non-circular partially coherent flat-topped beam in free space and in turbulent atmosphere have been studied numerically and comparatively. We have found that the M^2-factors of a non-circular partially coherent flat-topped beam in free space are only determined by the beam orders, beam waist sizes and transverse coherence width, and remain invariant on propagation. The M^2-factors of a non-circular

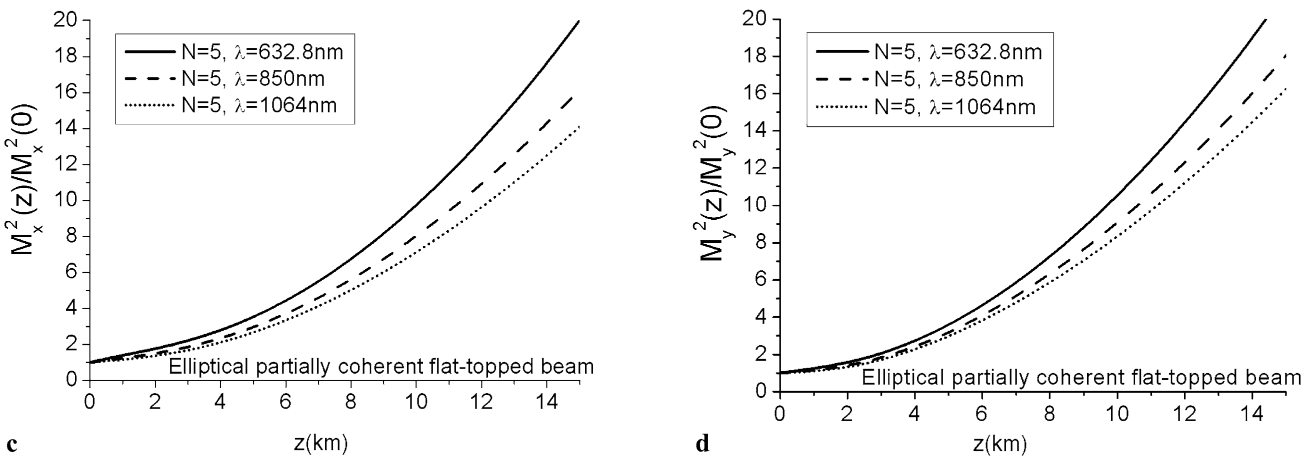

Fig. 11 (*Continued*)

partially coherent flat-topped beam increases on propagation in turbulent atmosphere, and they are determined by the parameters of the beam (i.e., beam orders, beam waist sizes and the transverse coherence width, wavelength) and the turbulent atmosphere (i.e., structure constant and inner scale of the turbulence). A non-circular partially coherent flat-topped beam is less affected by the turbulence than a circular partially coherent flat-topped beam and a GSM beam particular at long propagation distances under suitable condition. Our results may be employed in application in long-distance free-space optical communications.

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