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Parametric characteristics for a broadband Gaussian beam in free space

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Abstract Based on the theory of the second-order intensity moment and the method of Fourier transformation, some parameters for a broadband Gaussian beam in free space are studied analytically and numerically. It is shown that the beam width and the divergence angle for a broadband Gaussian beam in free space are both larger than those for a monochromatic Gaussian beam. The broader the bandwidth of a light beam, the larger the beam width and the divergence angle of the light beam. The beam propagation factor for a broadband Gaussian beam in free space is larger than that for a monochromatic Gaussian beam and increases slowly with bandwidth. The kurtosis parameter for a broadband Gaussian beam in free space increases with propagation distance and bandwidth. The kurtosis parameter for a broadband Gaussian beam with much broader bandwidth increases more remarkably with propagation distance.

1 Introduction

With the great development of laser technology and invention of new materials, more and more ultra-short and high-power laser beams have been achieved [1–3]. Most of these ultra-short beams are broadband in spectrum. Dealing with propagation and transformation of a broadband light beam in an optical system is different from that of a monochromatic light beam [4, 5]. Deng et al. [6] applied the concept of a broadband light beam to an intense laser system in order to improve the output power. From then on, researches on characteristics for a broadband light beam in free space [7], modulated by a hard-edged aperture [8], through a dispersive wedge [9] were studied.

The beam propagation factor (M^2 factor) based on the second-order moments is a useful parameter describing laser beam quality [10, 11]. The ABCD law and the M^2 factor are generalized for the propagation of general polychromatic light beams through a first-order optical system [12]. The beam propagation factor for chromatic laser beams was also measured by a simple method [13]. Ponomarenko and Agrawal discussed the lower bound of the phase-space beam quality factor M^2 of an ultra-short pulse [14]. The kurtosis parameter related to the fourth- and second-order intensity moments is used to describe the sharpness or flatness of any laser beam [15–17].

In this paper, a broadband Gaussian beam, whose spatial distribution is Gaussian and spectral distribution is a hard-aperture shape, is taken as an example. Parameters for describing a broadband Gaussian beam in free space are studied analytically and numerically. Firstly, the energy density for a broadband light beam in free space is analyzed in Sect. 2. Secondly, the analytical expressions of parameters such as the beam width, the divergence angle, the beam propagation factor and the kurtosis parameter for a broadband Gaussian beam in free space are obtained in Sect. 3. Thirdly, numerical calculation is given in Sect. 4. Finally, a simple conclusion is outlined in Sect. 5.

2 Energy density for a broadband light beam in free space

For the sake of simplicity, the one-dimensional case is considered. According to the Collins formula [18] and matrix

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optics [19], the output field for a monochromatic light beam $E_1(x_0, z=0, \omega)$ with its angular frequency ω in free space is described by

$$E_2(x, z, \omega) = \left(\frac{\omega}{i2\pi zc}\right)^{1/2} \exp\left(i\frac{\omega}{c}z\right) \times \int_{-\infty}^{+\infty} E_1(x_0, z=0, \omega) \times \exp\left[\frac{i\omega}{2zc}(x-x_0)^2\right] dx_0. \tag{1}$$

Based on the method of Fourier transformation [20, 21], a broadband light beam can be described as a superposition of its monochromatic components. So, the complex amplitude for a broadband light beam in the output plane is given by

$$E_2(x, z, t) = \int_{-\infty}^{+\infty} d\omega \exp(-i\omega t) E_2(x, z, \omega). \tag{2}$$

Here the constant factor of the Fourier transform is omitted. The energy density for a broadband light beam in the output plane can be written as

$$I_2(x, z) = \int_{-\infty}^{+\infty} \left|E_2(x, z, t)\right|^2 dt. \tag{3}$$

On inserting (2) into (3), and using the properties of the delta function [20],

$$\int_{-\infty}^{+\infty} \exp\left[-i(\omega-\omega')t\right] dt = \delta(\omega-\omega'), \tag{4}$$

$$\int_{-\infty}^{+\infty} d\omega' \, g(\omega')\delta(\omega-\omega') = g(\omega), \tag{5}$$

after some calculations, the energy density in the output plane for a broadband light beam in free space is given by

$$I_2(x, z) = \int_{-\infty}^{+\infty} d\omega \left|E_2(x, z, \omega)\right|^2. \tag{6}$$

Equation (6) shows that the energy density for a broadband light beam can also be written as the integration of the intensity for its monochromatic components over the angular frequency ω. Equations (3) and (6) satisfy Parseval's theorem in Fourier transform theory [20].

3 Parameters for a broadband Gaussian beam in free space

A broadband light beam in the input plane $z=0$ is supposed to be separable into spatial and spectral distributions. Its spatial distribution is assumed to be of Gaussian shape [19],

$$E_1(x_0, z=0) = \exp\left(-x_0^2/w_0^2\right), \tag{7}$$

where w_0 denotes the waist size of the Gaussian part. Its spectral distribution is assumed to be a hard-aperture shape

$$s(\omega) = \begin{cases} 1, & |\omega-\omega_0| \le \Delta\omega/2, \\ 0, & |\omega-\omega_0| > \Delta\omega/2, \end{cases} \tag{8}$$

where ω_0 denotes the central angular frequency for a broadband light beam and $\Delta\omega$ denotes the bandwidth of its spectrum.

3.1 The beam width, the divergence angle and the beam propagation factor

According to the above analysis, the theory of second-order moments [22, 23] for a monochromatic light beam can be generalized to the case for a broadband light beam. The second-order moments of the spatial variable x, the spatial-frequency variable u and the cross term of the spatial variable x and the spatial-frequency variable u can be respectively defined as

$$\langle x^2 \rangle = \frac{1}{I_0}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} x^2 \left|E_2(x, z, \omega)\right|^2 dx, \tag{9}$$

$$\langle u^2 \rangle = \frac{1}{I_0}\int_{-\infty}^{+\infty} d\omega \frac{c^2}{\omega^2} \int_{-\infty}^{+\infty} \left|\frac{\partial E_2(x, z, \omega)}{\partial x}\right|^2 dx, \tag{10}$$

$$\langle xu \rangle = \frac{1}{I_0}\int_{-\infty}^{+\infty} d\omega \frac{c}{2i\omega} \int_{-\infty}^{+\infty} x\left[\frac{\partial E_2^*(x, z, \omega)}{\partial x} E_2(x, z, \omega) - E_2^*(x, z, \omega)\frac{\partial E_2(x, z, \omega)}{\partial x}\right] dx, \tag{11}$$

where I_0 denotes the total energy of the beam and is given by

$$I_0 = \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} \left|E_2(x, z, \omega)\right|^2 dx. \tag{12}$$

Using the following integration formulae [24]:

$$\int_{-\infty}^{+\infty} dx \exp\left[-(ax^2+2bx+c)\right] = \left(\frac{\pi}{a}\right)^{1/2} \exp\left(\frac{b^2-ac}{a}\right), \tag{13}$$

$$\int_{-\infty}^{+\infty} dx \exp\left[-(ax^2+2bx+c)\right] x^2 = \frac{a+2b^2}{2a^2}\left(\frac{\pi}{a}\right)^{1/2} \exp\left(\frac{b^2-ac}{a}\right), \tag{14}$$

on inserting the related equations into (9)–(11), after tedious calculation, the second-order moments for a broadband Gaussian beam in free space are obtained:

$$\langle x^2 \rangle = \frac{w_0^2}{4} + \frac{z^2c^2}{w_0^2(\omega_0^2 - \Delta\omega^2/4)}, \tag{15}$$

$$\langle u^2\rangle = \frac{c^2}{w_0^2(\omega_0^2-\Delta\omega^2/4)}, \tag{16}$$

$$\langle xu\rangle = \frac{c^2 z}{w_0^2(\omega_0^2-\Delta\omega^2/4)}. \tag{17}$$

The beam width $W(z)$ for a broadband Gaussian beam, which relates to (15), is defined by

$$W(z) = 2\langle x^2\rangle^{1/2}. \tag{18}$$

On inserting (15) into (18), the beam width for a broadband Gaussian beam can be expressed as

$$W(z) = 2\left[\frac{w_0^2}{4} + \frac{z^2c^2}{w_0^2(\omega_0^2-\Delta\omega^2/4)}\right]^{1/2}. \tag{19}$$

Equation (19) denotes the beam width in the plane z when a broadband Gaussian beam propagates in free space. It can be seen from (19) that the beam width for a broadband Gaussian beam in free space reaches its minimum in the plane $z=0$. The minimum beam width denotes the beam waist width for a broadband Gaussian beam in free space and the value is $W(z=0)=w_0$. So, the location and size of the beam waist for a broadband Gaussian beam in free space are the same as those for its monochromatic components as shown by (7), respectively. According to the above equation, the relative beam width $W(z)/W(z=0)$, which denotes the ratio of beam width for a broadband Gaussian beam in the output plane z to that in the input plane $z=0$, can be written as

$$W(z)/W(z=0) = \left[1 + \frac{4z^2c^2}{w_0^4(\omega_0^2-\Delta\omega^2/4)}\right]^{1/2}. \tag{20}$$

Equation (20) shows that the relative beam width increases with propagation distance and bandwidth. The relative beam width $W(z)/W_0(z)$, which denotes the ratio of beam width for a broadband Gaussian beam in the output plane z to that for a monochromatic Gaussian beam in the same plane, can be written as

$$W(z)/W_0(z) = \left[\frac{w_0^2}{4} + \frac{z^2c^2}{w_0^2(\omega_0^2-\Delta\omega^2/4)}\right]^{1/2} \Bigg/ \left(\frac{w_0^2}{4} + \frac{z^2c^2}{w_0^2\omega_0^2}\right)^{1/2}. \tag{21}$$

The divergence angle for a broadband Gaussian beam, which relates to (16), is defined by

$$\theta = 2\langle u^2\rangle^{1/2}. \tag{22}$$

On inserting (16) into (22), the divergence angle for a broadband Gaussian beam in free space can be written as

$$\theta = 2\left[\frac{c^2}{w_0^2(\omega_0^2-\Delta\omega^2/4)}\right]^{1/2}. \tag{23}$$

The relative divergence angle θ/θ_0, which denotes the ratio of divergence angle for a broadband Gaussian beam to that for a monochromatic Gaussian beam, can be written as

$$\theta/\theta_0 = \left[\frac{1}{1-\Delta\omega^2/(2\omega_0)^2}\right]^{1/2}. \tag{24}$$

Equations (23) and (24) show that the divergence angle and relative divergence angle increase with bandwidth.

According to the definition for the beam propagation factor M^2 [23],

$$M^2 = 2\frac{\omega_0}{c}\left(\langle x^2\rangle\langle u^2\rangle - \langle xu\rangle^2\right)^{1/2}, \tag{25}$$

on inserting (15)–(17) into (25), we obtain an analytical expression of the beam propagation factor for a broadband Gaussian beam in free space:

$$M^2 = \left[\frac{1}{1-\Delta\omega^2/(2\omega_0)^2}\right]^{1/2}. \tag{26}$$

It can be seen that the beam propagation factor for a broadband Gaussian beam in free space increases with the bandwidth of the spectrum. When $\Delta\omega=0$, $M^2=1$. This result corresponds to the beam propagation factor for a monochromatic Gaussian beam in free space. Comparing (24) with (26), we find that the change of the relative divergence angle with propagation distance is the same as that of the beam propagation factor with propagation distance. The beam propagation factor, also called the beam quality factor, is also defined by the ratio of beam waist radius multiplied by far field divergence angle for a real beam to that for a monochromatic Gaussian beam [10]. So, the change of the beam propagation factor with bandwidth is equal to that of the relative divergence angle with bandwidth.

3.2 The kurtosis parameter

The kurtosis parameter, which describes sharpness or flatness of a light beam, is defined by [25]

$$K = \frac{\langle x^4\rangle}{\langle x^2\rangle^2}. \tag{27}$$

Similarly, the fourth-order moment of the spatial variable x is defined as

$$\langle x^4\rangle = \frac{1}{I_0}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} x^4 \left|E_2(x,z,\omega)\right|^2 dx; \tag{28}$$

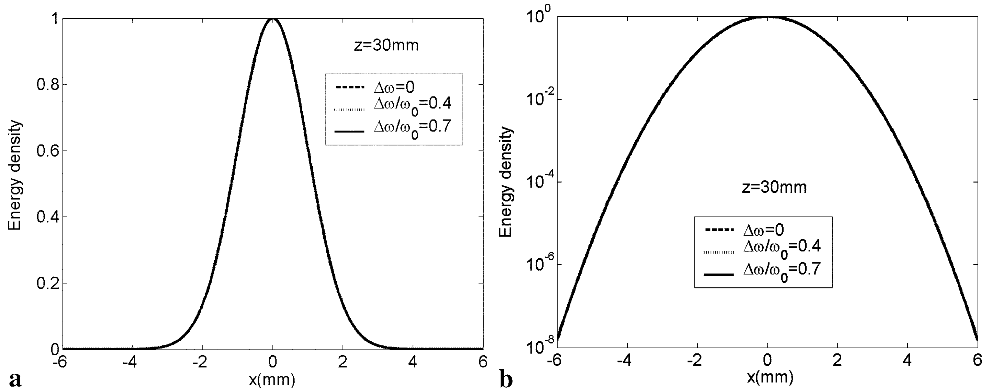

Fig. 1 (**a**) Normalized energy density for different broadband Gaussian beams in the output plane $z = 30$ mm; (**b**) in logarithmic coordinate

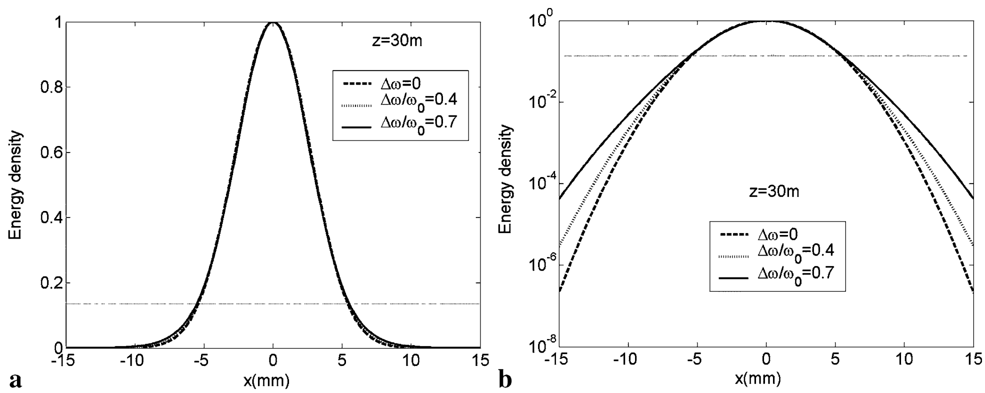

Fig. 2 (**a**) Normalized energy density for different broadband Gaussian beams in the output plane $z = 30$ m; (**b**) in logarithmic coordinate

so, the fourth-order moment of the spatial variable x for a broadband Gaussian beam in free space is obtained:

$$\langle x^4\rangle = 3(zcw_0)^4\left[\frac{3\omega_0^2+\Delta\omega^2/4}{3w_0^8(\omega_0^2-\Delta\omega^2/4)^3}\right.$$

$$\left.+\frac{1}{2c^2z^2w_0^4(\omega_0^2-\Delta\omega^2/4)}+\frac{1}{16c^4z^4}\right]. \tag{29}$$

On inserting (15) and (29) into (27), after tedious calculation, the analytical expression of the kurtosis parameter for a broadband Gaussian beam in free space is obtained:

$$K = 3\left(1+\frac{\Delta\omega^2/\omega_0^2}{3[1-\Delta\omega^2/(4\omega_0^2)]\{1+w_0^4\omega_0^2[1-\Delta\omega^2/(4\omega_0^2)]/(4z^2c^2)\}^2}\right). \tag{30}$$

When $\Delta\omega = 0$, $K = 3$. It corresponds to the kurtosis parameter for a monochromatic Gaussian beam in free space.

4 Numerical calculation

Some parameters are chosen as follows: the center wavelength for a broadband Gaussian beam $\lambda_0 = 1053$ nm, the waist width for the Gaussian part $w_0 = 2$ mm. In Figs. 1–3 and 6, solid lines denote the case $\Delta\omega/\omega_0 = 0.7$, dotted lines denote the case $\Delta\omega/\omega_0 = 0.4$ and dashed lines denote the case of a monochromatic Gaussian beam.

Figures 1a and b show normalized energy density and normalized energy density in logarithmic coordinate for different broadband Gaussian beams in the output plane $z = 30$ mm, respectively. It can be seen from the figure that the normalized energy density for a broadband Gaussian beam is almost the same as that for a monochromatic Gaussian beam in the near field. The effect of broad bandwidth on the normalized energy density is quite weak. Figures 2a and b show normalized energy density and normalized energy density in logarithmic coordinate for different broadband Gaussian beams in the output plane $z = 30$ m, respectively. It can be seen from the figure that the energy density for a broadband light beam slightly deviates from that for a mono-

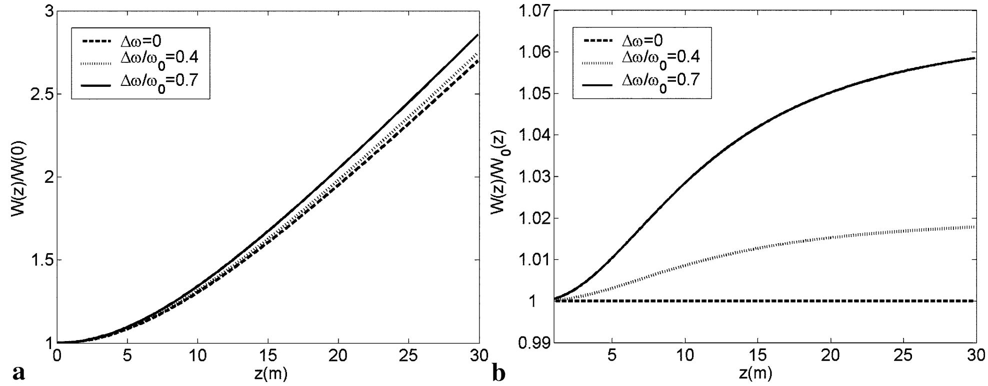

Fig. 3 Change of the relative beam width for different broadband Gaussian beams with the propagation distance

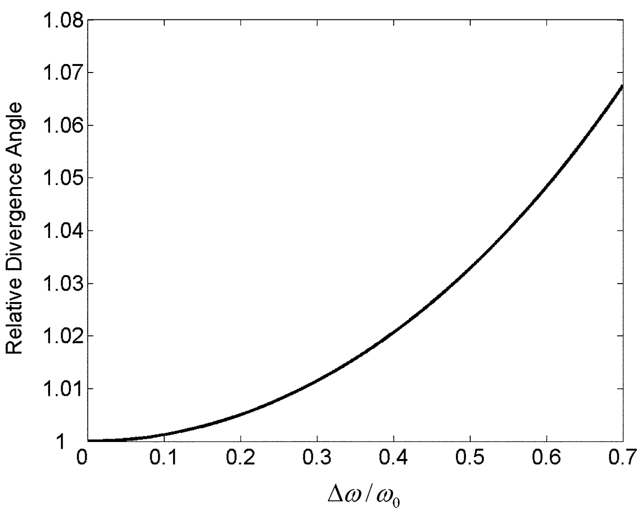

Fig. 4 Relative divergence angle for different broadband Gaussian beams

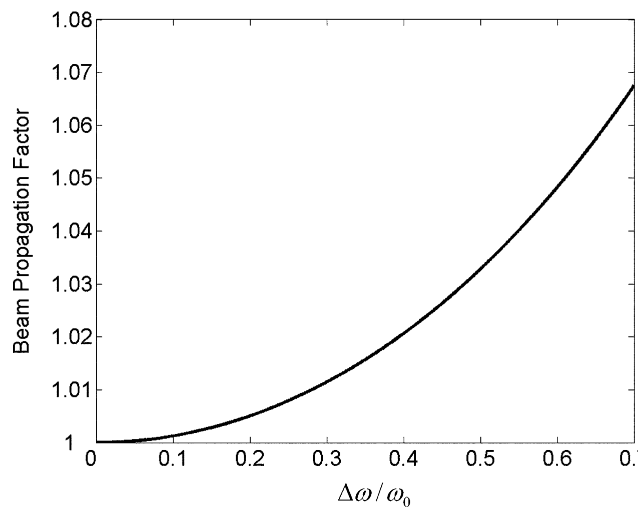

Fig. 5 Beam propagation factor for different broadband Gaussian beams

chromatic light beam in the far field. It is well known that the beam width for a monochromatic Gaussian beam in an optical system is defined by the full width at $1/e^2$ times maximum intensity [19]. The beam width for a broadband Gaussian beam can also be defined by this method. The horizontal line denoted by a dash-dotted line in Fig. 2 is given where the corresponding energy density is $1/e^2$ times the maximum energy density. It can be seen from Fig. 2b that the beam width for a broadband light beam is slightly larger than that for a monochromatic light beam.

Figure 3 shows the change of the relative beam width for different broadband Gaussian beams with propagation distance. Figure 3a is plotted according to (20). Figure 3a shows that the beam width for a broadband Gaussian beam increases with propagation distance and bandwidth. The beam width for a broadband Gaussian beam with broader bandwidth increases more quickly with propagation distance. As for a broadband Gaussian beam with its bandwidth $\Delta\omega = 0.7\omega_0$, the beam width in the plane $z = 30$ m is about 2.8 times larger than that in the input plane. Figure 3b is plotted according to (21). Figure 3b shows that beam width is slightly affected by bandwidth. As for a broadband Gaussian beam with its bandwidth $\Delta\omega = 0.7\omega_0$, the beam width in the output plane $z = 30$ m is about 1.06 times larger than that for a monochromatic Gaussian beam in the same plane. So, it can be seen from Fig. 3a and b that the beam width for a broadband light beam is greatly affected by propagation distance rather than bandwidth. Both the propagation distance and bandwidth make a broadband light beam in free space divergent.

Figure 4 is plotted according to (24). Figure 4 shows the change of the relative divergence angle with bandwidth. It can be seen that the relative divergence angle increases with bandwidth. The broader the bandwidth, the larger the divergence angle. Figure 5 shows the change of the beam propagation factor with bandwidth. It can be seen that the beam propagation factor for a broadband Gaussian beam in free space increases slowly with bandwidth. The influence of bandwidth on the beam propagation factor is weak. Comparing Fig. 4 with Fig. 5, we find that the changes of relative divergence angle and beam propagation factor with the bandwidth are the same. According to (20), the beam waist

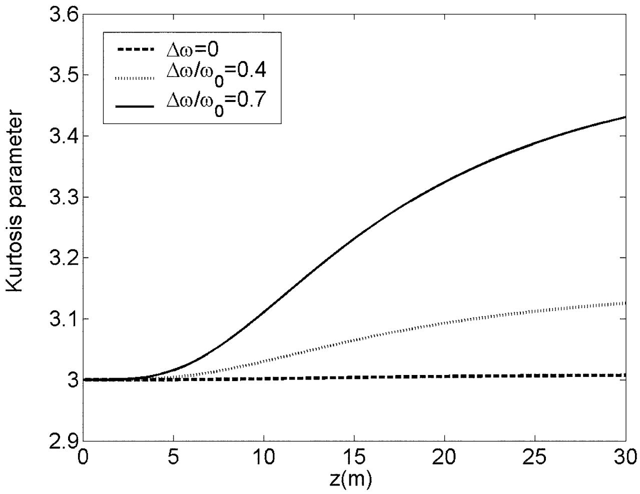

Fig. 6 Change of the kurtosis parameter for different broadband Gaussian beams with the propagation distance

for a broadband Gaussian beam in free space is the same size as that for a monochromatic Gaussian beam in free space and both of the beam waists are located in the plane $z = 0$. The beam propagation factor can be defined as the ratio of the beam waist width multiplied by the divergence angle for a real beam to that for a monochromatic fundamental mode Gaussian beam. The beam propagation factor for a monochromatic fundamental mode Gaussian beam in free space is $M^2 = 1$. So, the change of the beam propagation factor for a broadband Gaussian beam with bandwidth is the same as the change of the relative divergence angle with bandwidth.

Figure 6 shows the change of the kurtosis parameter with propagation distance. It is shown that the kurtosis parameter for a broadband Gaussian beam in free space is larger than that for a monochromatic Gaussian beam. According to the definition of the kurtosis parameter, the larger the kurtosis parameter, the sharper the beam profile. So, the transverse energy density distribution for a broadband Gaussian beam in free space becomes sharper than that for a monochromatic Gaussian beam. The kurtosis parameter for a broadband Gaussian beam with much broader bandwidth increases more obviously with propagation distance.

5 Conclusion

Parameters for characterizing a broadband Gaussian beam in free space were studied analytically and numerically. The beam width, divergence angle, beam propagation factor and kurtosis parameter for a broadband Gaussian beam in free space are larger than those for a monochromatic light beam. A broadband Gaussian beam in free space becomes more divergent and the transverse energy density distribution becomes sharper compared with a monochromatic Gaussian beam. The beam quality for a broadband Gaussian beam becomes worse than that for a monochromatic light beam. The method introduced here can also be used to deal with other types of broadband light beams in free space. It provides a theoretical basis for practical applications.

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