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Calculation of average intensity via semi-analytic method

H.T. Eyyuboğlu · E. Sermutlu

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Abstract We present a semi-analytic approach to the solution of the quadruple Huygens-Fresnel integral which is used to calculate the average receiver intensity of a source beam after it has propagated in a turbulent atmosphere. Our approach is based on a self-designed MATLAB function that reduces a quadruple integral to a single one by sequential operations using a form that is readily available from tables. In this manner exact numerical evaluations are obtained, whilst lengthy hand derivations are avoided. Additionally, the computation time of the new approach is not much different from that of the complete analytic solution. Two application examples are cited, also establishing agreement with our previously published results.

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1 Introduction

The Huygens-Fresnel integral that is used to find the receiver field from a given source field is naturally a double integral over the source coordinates either in Cartesian or polar form. Due to averaging over the randomness of the propagating medium, the receiver field is converted into average intensity by multiplying the field by its conjugate, hence the double integral becomes raised to a quadruple integral [1–5]. Solving this quadruple integral analytically entails many lengthy and tedious steps [2–5], where utmost care is needed to obtain accurate results. Of course in some situations, an analytic solution is desirable, since the mathematical appearance of the receiver intensity offers useful information and valuable insight into the behavior of intensity in general and under certain limiting cases. But it is not always possible to solve the quadruple integral analytically; Bessel and modified Bessel, Mathieu and Laguerre Gaussian beams are such examples [6–10]. These last four beam types are generally written on the source side in polar coordinates. Thus when inserted into the Huygens–Fresnel integral, coupling occurs between the radial and angular components, which makes it impossible to proceed beyond two stages of integration [6–10]. Below we show that in our self-designed MATLAB function, such cases can be tackled by reverting to Cartesian coordinates.

For clarity, it should be emphasized that our function truly provides a result identical to the exact analytic solution of the quadruple integration, i.e. for an unapertured source field, it actually performs the fourfold integration from minus infinity to plus infinity. As explained in the Appendix, this is achieved by separating the individual parts of the given integral and putting those parts into a form whose formulation is readily available from tables. In this sense, our method should not be confused with numerically evaluating the quadruple integral, which firstly consumes an excessively long computation time, and secondly may produce unreliable results due to ambiguities in how the infi-

H.T. Eyyuboğlu (✉)
Department of Electronic and Communication Engineering, Çankaya University, Öğretmenler Cad. No. 14 Yüzüncüyıl, 06530 Balgat, Ankara, Turkey
e-mail: h.eyyuboglu@cankaya.edu.tr
Fax: +90-312-2848043

E. Sermutlu
Department of Mathematics and Computer Science, Çankaya University, Öğretmenler Cad. No. 14 Yüzüncüyıl, 06530 Balgat, Ankara, Turkey
e-mail: sermutlu@cankaya.edu.tr
Fax: +90-312-2868962

nite integral limits should be represented during computation [11].

To the best of our knowledge, our self-designed MATLAB function is the first to serve the task outlined above.

2 Formulation and discussion of application examples

We initially take the previously derived case of a general beam whose source field expression is given as [2, 3, 12]

$$u_s(s_x,s_y)=\sum_{\ell=1}^{N}A_\ell H_{x\ell}(a_{x\ell}s_x)H_{y\ell}(a_{y\ell}s_y) \times\exp\left[-\left(0.5k\alpha_{x\ell}s_x^2+jV_{x\ell}s_x\right)\right] \times\exp\left[-\left(0.5k\alpha_{y\ell}s_y^2+jV_{y\ell}s_y\right)\right] \quad (1)$$

where N is the number of beams to be summed, s_x and s_y designate the source coordinates, A_ℓ is the amplitude factor of the ℓth component of the source field, $H_{x\ell}(a_{x\ell}s_x)$ and $H_{y\ell}(a_{y\ell}s_y)$ are the Hermite polynomials shaping the beam distribution for the s_x and s_y directions, where $x\ell$ and $y\ell$ are the respective orders, $a_{x\ell}$ and $a_{y\ell}$ stand for the width, $\alpha_{x\ell}$ and $\alpha_{y\ell}$ are related to source sizes and the focusing parameters along again the s_x and s_y directions, $k=2\pi/\lambda$ is the wave number with λ being the wavelength and $j=(-1)^{0.5}$, $V_{x\ell}$ and $V_{y\ell}$ are the complex displacement parameters. All these source parameters can be combined in various arrangements to create beam types such as sinusoidal and hyperbolic Gaussian beams, annular Gaussian beams and their corresponding higher order counterparts [12]. Partial coherence, measured by the parameter σ_s can be added to (1) in the manner described in [13]. This way, the mutual coherence function for the source beam in (1) will be

$$\Gamma_s(s_{1x},s_{2x},s_{1y},s_{2y}) =u_s(s_{1x},s_{1y})u_s^*(s_{2x},s_{2y}) \times\exp\left[-\frac{1}{2\sigma_s^2}\left(s_{1x}^2-2s_{1x}s_{2x}+s_{2x}^2+s_{1y}^2 -2s_{1y}s_{2y}+s_{2y}^2\right)\right] \quad (2)$$

where * represents the conjugate operation.

The average receiver intensity falling onto a receiver plane with Cartesian coordinates of r_x and r_y, that is L distance away from the source, is yielded by inserting (2) into the Huygens-Fresnel integral as shown below,

$$\langle I(r_x,r_y,L)\rangle =\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\frac{k^2}{(2\pi L)^2}A_{\ell_1}A_{\ell_2}^* \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}ds_{1x}\,ds_{2x}\,ds_{1y}\,ds_{2y}\,H_{x\ell_1}(a_{x\ell_1}s_{1x}) \times H_{x\ell_2}(a_{x\ell_2}s_{2x})H_{y\ell_1}(a_{y\ell_1}s_{1y})H_{y\ell_2}(a_{y\ell_2}s_{2y}) \times\exp\left[-\left(0.5k\alpha_{x\ell_1}s_{1x}^2+jV_{x\ell_1}s_{1x}\right)\right] \times\exp\left[-\left(0.5k\alpha_{x\ell_2}^*s_{2x}^2-jV_{x\ell_2}^*s_{2x}\right)\right] \times\exp\left[-\left(0.5k\alpha_{y\ell_1}s_{1y}^2+jV_{y\ell_1}s_{1y}\right)\right] \times\exp\left[-\left(0.5k\alpha_{y\ell_2}^*s_{2y}^2-jV_{y\ell_2}^*s_{2y}\right)\right] \times\exp\left[-\frac{1}{2\sigma_s^2}\left(s_{1x}^2-2s_{1x}s_{2x}+s_{2x}^2+s_{1y}^2 -2s_{1y}s_{2y}+s_{2y}^2\right)\right] \times\exp\left[\frac{jk}{2L}\left(s_{1x}^2-2r_xs_{1x}-s_{2x}^2+2r_xs_{2x}+s_{1y}^2 -2r_ys_{1y}-s_{2y}^2+2r_ys_{2y}\right)\right] \times\exp\left[-\frac{1}{\rho_0^2}\left(s_{1x}^2-2s_{1x}s_{2x}+s_{2x}^2+s_{1y}^2 -2s_{1y}s_{2y}+s_{2y}^2\right)\right] \quad (3)$$

Here $\rho_0=(0.545C_n^2k^2L)^{-3/5}$ indicates the coherence length of a spherical wave propagating in turbulence, with C_n^2 referring to the structure constant. It is worth noting that the last two exponentials in (3) arise respectively due to diffraction phenomena and turbulence, whereas the collection of the prior terms is mainly due to the source.

Now our self-designed MATLAB function is capable of solving the quadruple integral that has the following form:

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt_1\,dt_2\,dt_3\,dt_4\exp\left(-q_{11}t_1^2-q_{22}t_2^2 -q_{33}t_3^2-q_{44}t_4^2\right) \times\exp(-2q_{12}t_1t_2-2q_{13}t_1t_3-2q_{14}t_1t_4-2q_{23}t_2t_3 -2q_{24}t_2t_4-2q_{34}t_3t_4) \times\exp(-2r_1t_1-2r_2t_2-2r_3t_3-2r_4t_4)t_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4} \times H_{m_1}(c_1t_1)H_{m_2}(c_2t_2)H_{m_3}(c_3t_3)H_{m_4}(c_4t_4) \quad (4)$$

I in (4) can be computed by invoking the MATLAB function as $I=\text{ExpPolyHerm4}(Q,R,N,M)$, where the function name is ExpPolyHerm4 and Q, R, N, M are the calling arguments related to the parameters encountered in the integrand of (4) via matrices and row vectors as stated below,

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{pmatrix}, \qquad R = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 \end{pmatrix}$$

$$N = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 \end{pmatrix}, \qquad M = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \tag{5}$$

Establishing the equivalence between (3) and (4), we see that in the particular case of (3), i.e. the general beam, the individual elements of Q, R, N, M will become

$$q_{11} = 0.5k\alpha_{x\ell_1} + \frac{1}{2\sigma_s^2} - \frac{jk}{2L} + \frac{1}{\rho_0^2}$$

$$q_{12} = -\frac{1}{2\sigma_s^2} - \frac{1}{\rho_0^2} = q_{34}, \qquad q_{13} = q_{14} = 0$$

$$q_{22} = 0.5k\alpha^*_{x\ell_2} + \frac{1}{2\sigma_s^2} + \frac{jk}{2L} + \frac{1}{\rho_0^2} \tag{6a}$$

$$q_{23} = q_{24} = 0$$

$$q_{33} = 0.5k\alpha_{y\ell_1} + \frac{1}{2\sigma_s^2} - \frac{jk}{2L} + \frac{1}{\rho_0^2}$$

$$q_{44} = 0.5k\alpha^*_{y\ell_2} + \frac{1}{2\sigma_s^2} + \frac{jk}{2L} + \frac{1}{\rho_0^2}$$

$$r_1 = j0.5V_{x\ell_1} + \frac{jkr_x}{2L}, \qquad r_2 = -j0.5V^*_{x\ell_2} - \frac{jkr_x}{2L}$$

$$r_3 = j0.5V_{y\ell_1} + \frac{jkr_y}{2L}, \qquad r_2 = -j0.5V^*_{y\ell_2} - \frac{jkr_y}{2L} \tag{6b}$$

$$n_1 = n_2 = n_3 = n_4 = 0 \tag{6c}$$

$$m_1 = x\ell_1, \quad m_2 = x\ell_2, \quad m_3 = y\ell_1, \quad m_4 = y\ell_2$$

$$c_1 = a_{x\ell_1}, \quad c_2 = a_{x\ell_2}, \quad c_3 = a_{y\ell_1}, \quad c_4 = a_{y\ell_2} \tag{6d}$$

Thus the computation in (3) will turn into

$$\langle I(r_x, r_y, L)\rangle = \sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N} \frac{k^2}{(2\pi L)^2} A_{\ell_1} A^*_{\ell_2} I_{ES} \tag{7}$$

where I_{ES} represents the integral in (4) containing the substitutions given in (6). It should be pointed out that it is a futile attempt to construct plots arising from (7) and compare them with previously published results that are exclusively based on full analytic derivations [2, 3, 11, 12]. This is because both yield identical and exact results whose verification was actually implemented via several numeric evaluations, since such a test was important to ensure the reliability of our new semi-analytic method. A more reasonable assessment for the performance of our new method would be the comparison of computation times for average receiver intensity at selected beam types. To this end, Table 1 displays the comparison between the method outlined here and those previously hand-derived results [2, 3, 11, 12]. For each beam type of this table, in the column labeled "Computation time" the figure of the first row refers to the computation time of the new method, while the second row is the computation time of previously derived analytic expressions. In the calculations of average receiver intensity, the common settings for all beams are $\alpha_{x\ell} = \alpha_{y\ell} = 0.0025\ \text{m}^{-1}$, $\sigma_s \to \infty$, $\lambda = 1.55\ \mu\text{m}$, $C_n^2 = 10^{-15}\ \text{m}^{-2/3}$, $L = 1$ km. The additional

Table 1 Comparison of computation times for general beam

Beam type	Parameter settings	Computation time (seconds)
cosh Gaussian	$A_\ell = [0.5\ 0.5]$, $V_x = V_y = [100j\ -100j]$	1.17 0.84
cos Gaussian	$A_\ell = [0.5\ 0.5]$, $V_x = V_y = [100\ -100]$	1.23 0.84
sinh Gaussian	$A_\ell = [0.5\ -0.5]$, $V_x = V_y = [100j\ -100j]$	1.57 1.3
sine Gaussian	$A_\ell = [0.5j\ -0.5j]$, $V_x = V_y = [100\ -100]$	1.19 0.82
annular Gaussian	$A_\ell = [0.5\ -0.5]$, $\alpha_{x1} = \alpha_{y1} = 0.0025\ \text{m}^{-1}$ $\alpha_{x2} = \alpha_{y2} = 0.0039\ \text{m}^{-1}$	1.55 1.13
higher order Hermite Gaussian	$x\ell = [1\ 1]$, $y\ell = [1\ 1]$	7.47 2.43

specific settings are shown in the "Parameter settings" column of the table. From the last column of Table 1, we understand that the computation times of our new method compete favorably with the computation times of previous analytic expressions, apart from the case of the higher order Hermite Gaussian beam, where relatively a large increase of computation time is detected as seen from the last row of Table 1. It was observed that as the source settings were simplified toward the fundamental Gaussian beam, the computation time of the previous analytic results was slightly reduced, whilst this had almost no effect on the new method. It should be mentioned that the computation times of previous analytic results are derived from the computation of receiver intensity over the whole transverse plane, i.e. a three dimensional view of the intensity profile [2, 3, 11, 12]. Such a construction is implemented in MATLAB via the readily available meshgrid function in a parallel fashion while consuming almost no extra time than that of an evaluation at a single coordinate location. For the computation in (7) however, the situation is quite different, because no such parallel operation is possible, therefore for each r_x, r_y location of the transverse receiver plane, a separate evaluation of (7), i.e. a new call to the function ExpPolyHerm4, is required. Under these circumstances, the computation time would inevitably and enormously increase with rising size of the square matrix containing r_x, r_y coordinates. Considering that for most beam types, a two dimensional profile is an adequate representative of the whole three dimensional picture, the evaluation of (7) can then be confined to a two dimensional case. Thus in Table 1, the given computation times for the semi-analytic method refer to those of the two dimensional views cut along the diagonal axis of the transverse receiver plane matrix.

As the second example, we take a Laguerre Gaussian beam whose source plane formulations in polar and Cartesian coordinates are [1, 14]

$$u_s(s,\phi_s) = (-j)^m \left(\frac{\sqrt{2}s}{\alpha_s}\right)^m \exp(jm\phi_s) \times \exp\left(-\frac{s^2}{\alpha_s^2}\right) L_n^m\left(\frac{2s^2}{\alpha_s^2}\right) \tag{8a}$$

$$u_s(s_x,s_y) = (-j)^m \left(\frac{s_x + js_y}{\alpha_{cs}}\right)^m \times \exp\left[-0.5\left(\frac{s_x^2}{\alpha_{sx}^2} + \frac{s_y^2}{\alpha_{sy}^2}\right)\right] \times L_n^m\left(\frac{s_x^2}{\alpha_{sx}^2} + \frac{s_y^2}{\alpha_{sy}^2}\right) \tag{8b}$$

where s and ϕ_s designate the polar coordinates. In (8), we have reverted to source size notation instead of the α parameter, thus $\alpha_s = \sqrt{2}\alpha_{sx}$ or $\alpha_s = \sqrt{2}\alpha_{sy}$ and α_{cs} stands for α_{sx} or α_{sy}, and $L_n^m()$ is the associated Laguerre polynomial with n and m respectively defining radial and angular mode numbers. Note that the polar form, i.e. (8a), is inappropriate for our purposes here, because, as explained in the Introduction, the coupling that emerges in the Huygens–Fresnel integral between the radial and the angular coordinates cannot be sorted by the transformations executed within the function ExpPolyHerm4. So we are forced to continue with (8b). To make (8b) compatible with (4), we expand $(s_x + js_y)^m$ in a binomial series, and the Laguerre polynomial in a finite series. In this way the average receiver intensity expression for the Laguerre Gaussian beam can be written as

$$\begin{aligned}
&\langle I(r_x,r_y,L)\rangle \\
&= \frac{k^2}{(2\pi L)^2}\frac{[(n+m)!]^2}{\alpha_{cs}^{2m}} \\
&\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds_{1x}\,ds_{1y}\,ds_{2x}\,ds_{2y} \\
&\times \exp\left[-0.5\left(\frac{s_{1x}^2}{\alpha_{sx}^2} + \frac{s_{1y}^2}{\alpha_{sy}^2}\right)\right] \\
&\times \exp\left[-0.5\left(\frac{s_{2x}^2}{\alpha_{sx}^2} + \frac{s_{2y}^2}{\alpha_{sy}^2}\right)\right] \\
&\times \sum_{m_{11}=0}^{m}\sum_{m_{12}=0}^{m}\sum_{n_{11}=0}^{n}\sum_{n_{12}=0}^{n}\sum_{n_{21}=0}^{n_{11}}\sum_{n_{22}=0}^{n_{12}} \binom{m}{m_{11}}\binom{m}{m_{12}} \\
&\times \binom{n_{11}}{n_{21}}\binom{n_{12}}{n_{22}}\left(\frac{1}{\alpha_{sx}^2}\right)^{n_{11}-n_{21}+n_{12}-n_{22}}\left(\frac{1}{\alpha_{sy}^2}\right)^{n_{21}+n_{22}} \\
&\times \frac{j^{m_{11}-m_{12}}(-1)^{n_{11}+n_{12}} s_{1x}^{m-m_{11}+2n_{11}-2n_{21}} s_{2x}^{m-m_{12}+2n_{12}-2n_{22}} s_{1y}^{m_{11}+2n_{21}} s_{2y}^{m_{12}+2n_{22}}}{(n-n_{11})!(n-n_{12})!(n_{11}+m)(n_{12}+m)!n_{11}!n_{12}!} \\
&\times \exp\left[\frac{jk}{2L}\left(s_{1x}^2 - 2r_x s_{1x} - s_{2x}^2 + 2r_x s_{2x} + s_{1y}^2\right.\right. \\
&\left.\left. - 2r_y s_{1y} - s_{2y}^2 + 2r_y s_{2y}\right)\right] \\
&\times \exp\left[-\frac{1}{\rho_0^2}\left(s_{1x}^2 - 2s_{1x}s_{2x} + s_{2x}^2 + s_{1y}^2\right.\right. \\
&\left.\left. - 2s_{1y}s_{2y} + s_{2y}^2\right)\right]
\end{aligned} \tag{9}$$

where the format $\binom{n_1}{n_2}$ represents the binomial coefficient, and ! means the factorial. Again by associating (9) with (4), the elements of the Q, R, N, M matrices can be determined as illustrated in (6) for the general beam case. Supplying those terms to the MATLAB function ExpPolyHerm4 will then accomplish the calculation of the average receiver intensity for the Laguerre Gaussian beam.

It is clear from above that there exist more summations in (9) than in (3). Therefore it is reasonable to expect an increase in computation times of Laguerre Gaussian beams,

Table 2 Computation times for Laguerre Gaussian beams at different n and m values

n and m values	Computation time (seconds)
$n=0, m=0$ Fundamental Gaussian	0.53
$n=1, m=0$	3.77
$n=2, m=0$	39.75
$n=1, m=1$	29.2
$n=2, m=1$	338.83
$n=2, m=1$	2093 (approximately 35 minutes)

particularly with rising n and m values. By adopting the source and propagation settings of the first example, in Table 2, we provide the variation of computation time, for some selected n and m values. The trend in this table confirms the above expectation; that is, dramatic computation time increases are experienced with rising n and m values.

It should be noted that (9) is actually a replacement for the hand-driven analytic expression of the average receiver intensity for a Laguerre Gaussian beam propagating in a turbulent atmosphere, a topic that has so far been investigated only in a limited scope [9]. But here our focus is on the efficiency of the self-designed MATLAB routine. Thus we leave the analysis of propagation aspects of a Laguerre Gaussian beam in turbulence via the use of the new MATLAB function to a future study.

3 Conclusion

A self-designed MATLAB function that is capable of solving analytically the quadruple integral that is used to find the average receiver intensity of a beam propagating in a turbulent atmosphere has been introduced. The attractive feature of this new routine is that it completely avoids lengthy and laborious hand derivations, also minimizing the risk of making mistakes. Presently it can handle many types of beams, including sinusoidal and hyperbolic Gaussian beams, annular Gaussian beams and their higher order counterparts involving Hermite polynomials. Furthermore, some types of beams, such as Laguerre Gaussian beams, can be covered using series expansions. For the future, our aim is to extend this MATLAB function to accommodate more source beam formulations.

We have tested the reliability of our self-designed MATLAB function by comparing receiver beam profiles against the previously hand-derived and published cases. The comparisons on the basis of computation times indicate that the new method does not appreciably differ from the computation time spent by previous analytic formulations provided that the new MATLAB function calculates the average receiver intensity in two dimensions.

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Appendix

First we consider the quadruple integral given below,

$$\begin{aligned} I = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1\,dt_2\,dt_3\,dt_4 \exp\bigl(-q_{11}t_1^2 - q_{22}t_2^2 \\ &- q_{33}t_3^2 - q_{44}t_4^2\bigr) \\ &\times \exp(-2q_{12}t_1t_2 - 2q_{13}t_1t_3 - 2q_{14}t_1t_4 - 2q_{23}t_2t_3 \\ &- 2q_{24}t_2t_4 - 2q_{34}t_3t_4) \\ &\times \exp(-2r_1t_1 - 2r_2t_2 - 2r_3t_3 - 2r_4t_4)t_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4} \end{aligned} \tag{A.1}$$

where n_p with $p = 1..4$ is zero or a positive integer. To reduce the quadruple integral of (A.1) to the case of a single integral, initially we isolate one integral, namely the one with respect to t_1 as follows:

$$\begin{aligned} I = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_2\,dt_3\,dt_4 \exp\bigl(-q_{22}t_2^2 - q_{33}t_3^2 - q_{44}t_4^2 \\ &- 2q_{23}t_2t_3 - 2q_{24}t_2t_4 - 2q_{34}t_3t_4\bigr) \\ &\times \exp(-2r_2t_2 - 2r_3t_3 - 2r_4t_4)t_2^{n_2}t_3^{n_3}t_4^{n_4} \\ &\times \int_{-\infty}^{\infty} dt_1 \exp\bigl(-q_{11}t_1^2 - 2r_gt_1\bigr)t_1^{n_1} \end{aligned} \tag{A.2}$$

where $r_g = r_1 + q_{12}t_2 + q_{13}t_3 + q_{14}t_4$. Now using a modified version 3.462.2 of [15] which is

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \exp\bigl(-qt^2 - 2rt\bigr)t^n \\ &\quad = n!\exp\left(\frac{r^2}{q}\right)\sum_{i=0}^{\lfloor n/2\rfloor}\frac{\Gamma(i+1/2)}{q^{i+1/2}(n-2i)!(2i)!}\left(-\frac{r}{q}\right)^{n-2i} \end{aligned} \tag{A.3}$$

the isolated integral in (A.1) can be solved, hence (A.1) will reduce to the following triple integral:

$$\begin{aligned} I = &(-1)^{n_1}n_1!\exp\left(\frac{r_1^2}{q_{11}}\right)\sum_{i=0}^{\lfloor n_1/2\rfloor}\frac{\Gamma(i+1/2)}{q_{11}^{n_1-i+1/2}(n_1-2i)!(2i)!} \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_2\,dt_3\,dt_4 \end{aligned}$$

$$\times \exp\left[-\left(q_{22}-\frac{q_{12}^2}{q_{11}}\right)t_2^2-\left(q_{33}-\frac{q_{13}^2}{q_{11}}\right)t_3^2 - \left(q_{44}-\frac{q_{14}^2}{q_{11}}\right)t_4^2\right]$$
$$\times \exp\left[-2\left(q_{23}-\frac{q_{12}q_{13}}{q_{11}}\right)t_2t_3-2\left(q_{24}-\frac{q_{12}q_{14}}{q_{11}}\right)t_2t_4 - 2\left(q_{34}-\frac{q_{13}q_{14}}{q_{11}}\right)t_3t_4\right]$$
$$\times \exp\left[-2\left(r_2-\frac{r_1q_{12}}{q_{11}}\right)t_2-2\left(r_3-\frac{r_1q_{13}}{q_{11}}\right)t_3 - 2\left(r_4-\frac{r_1q_{14}}{q_{11}}\right)t_4\right]$$
$$\times t_2^{n_2}t_3^{n_3}t_4^{n_4}(r_1+q_{12}t_2+q_{13}t_3+q_{14}t_4)^{n_1-2i} \tag{A.4}$$

The last term of (A.4) can be expanded via the binomial formula, and the result can be rearranged so that the integral with respect to t_2 can be managed individually again by the use of (A.3). The development continues in this manner until all the integrations have been performed. To facilitate an easy track of equation development, the MATLAB function ExpPolyHerm4 is organized as the main function plus the others named ExpPoly4, ExpPoly3, ExpPoly2, ExpPoly1, which call each other in numeric sequence to transform the quadruple integral into triple, quadruple and single integrals, while the main function ExpPolyHerm4 initiates the first call and makes the final evaluation. The Hermite polynomials appearing in (4) are handled by writing their series expansions and embedding the arising powers of t_1, t_2, t_3 and t_4 into n_1, n_2, n_3 and n_4 in (A.1).

The MATLAB code of the function ExpPolyHerm4 can be obtained at http://academic.cankaya.edu.tr/~sermutlu/matlab.html.

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