Appl. Phys. B 81, 983–987 (2005)
DOI: 10.1007/s00340-005-2018-8

Applied Physics B
Lasers and Optics

M. BÖHM
F. MITSCHKE✉

Soliton propagation in a dispersion map with deviation from periodicity

Physics Department, Universität Rostock, Rostock, Germany

Received: 18 May 2005/Revised: 26 August 2005
Published online: 25 October 2005 •

ABSTRACT Solitons in dispersion-managed fibers are a particular case of a periodically perturbed nonlinear system. Since in practice random deviations from strict periodicity of the dispersion alternation are unavoidable, we consider such deviations and study their impact on the soliton's stability with a view to optical telecommunications. We find a range over which solitons remain stable in a specified sense; this range is sufficient for technical application.

PACS 42.65.Tg; 42.81.Dp

1 Introduction

Nonlinear dynamic systems subject to periodic perturbations constitute a particularly interesting and rich topical area. If the perturbation occurs in the temporal domain, this encompasses the kicked rotator or the driven pendulum and gives rise to synchronization phenomena, and sometimes generation of chaos. Typically, small variations of the period, or small deviations from precise periodicity, have strong consequences. Examples of spatially periodic perturbations run the gamut from the wiggler in free-electron lasers to vehicles moving on corrugated surfaces, like ships at sea. In the realm of optics, periodically poled nonlinear crystals have generated a lot of excitement in recent years because they open an avenue to tailor-made material properties that are not available from any homogeneous materials, and are often far superior for applications.

An important quantity to consider is the scale of the periodic perturbation as compared to the scale of the optical wave. In the case of Bragg gratings, the periodicity is of the order of the wavelength. For periodically poled crystals it may be quite a bit longer than the wavelength, but still is within the coherence length. We will here discuss a spatially periodic system in which the period may even exceed the coherence length. Specifically, we will consider optical solitons traveling down a fiber the dispersion of which is switched between positive and negative in an almost, but not quite, periodic fashion ('dispersion-managed fiber'). We will ask whether pulse propagation in this system is stable, but in doing so we will have to carefully consider what we mean by 'stable'.

In mathematical context, stability refers to bounded behavior when some distance goes to the limit of infinity. In a finite world, however, the limit of infinity provides a needlessly strict criterion: it fully suffices when, say, for a system bounded by the surface of the planet, there is no divergence when the distance goes towards the planet's circumference.

2 Solitons in data transmission

Communication by light pulses sent over optical fibers has become so common today that in everyday life it is hardly noticed; without it, however, most long-distance communication and almost all internet traffic would quickly break down. Light pulses propagating in fibers are subject to the simultaneous effects of group-velocity dispersion, measured by the parameter β_2, and Kerr nonlinearity (i.e. power dependence of the refractive index), measured by the nonlinearity coefficient γ, which, in typical silica fibers, takes values around 1 $\mathrm{W}^{-1}\,\mathrm{km}^{-1}$. The interplay of these effects is captured in the nonlinear Schrödinger equation [1] that describes the propagation of the envelope of light pulses $A(T, z)$:

$$\frac{\partial A}{\partial z} = -\frac{\mathrm{i}}{2}\,\beta_2\,\frac{\partial^2 A}{\partial T^2} + \mathrm{i}\gamma\,|A|^2\,A\,. \quad (1)$$

Here z is the coordinate in the propagation direction and T is time in the comoving frame. Higher-order perturbations like attenuation, Raman scattering, higher-order dispersion, etc., are not included in this equation, but for our present purposes where we consider many-ps pulses, this is justified.

For $\beta_2 < 0$ a particular solution of this equation is the soliton

$$A(T, z) = \sqrt{\hat{P}}\,\mathrm{sech}\left(\frac{T}{T_0}\right)\exp\left(\mathrm{i}\frac{1}{2}\frac{z}{L_\mathrm{D}}\right).$$

Here T_0 is the pulse width and $L_\mathrm{D} = T_0^2/|\beta_2|$ is a characteristic length scale over which the phase of the pulse evolves. The peak power is $\hat{P} = 1/(\gamma L_\mathrm{D})$. Solitons are nonlinear pulses that are able to maintain a stable balance between dispersion and Kerr nonlinearity and therefore do not change their shape. Even when the inevitable power losses in real-world systems are taken into account, solitons remain a useful concept; amplifiers can offset the loss at least on average.

✉ Fax: +49 381 498 6822, E-mail: fedor.mitschke@uni-rostock.de

We note in passing that for $\beta_2 > 0$ dark solitons, i.e. dark pulses on a bright background, exist. While these are physically entirely viable, they seem to be of less applicability to telecommunications; therefore, we restrict our present discussion to bright solitons.

2.1 *Dispersion management*

Recently, it has become quite common in optical telecommunications that fiber lines are composed of segments of opposite-sign dispersion, concatenated in an alternating fashion. This creates a periodic dispersion structure. The path-average dispersion

$$\beta_2 \to \beta_2(z) \quad \text{with} \quad \beta_2^{\text{ave}} = \frac{L^+\beta_2^+ + L^-\beta_2^-}{L^+ + L^-} \tag{2}$$

can be set as desired by choosing the lengths $L^\pm$ and individual dispersion values $\beta_2^\pm$ appropriately. Such fibers, called dispersion-managed fibers, have advantages over homogeneous fibers: there is suppression of channel crosstalk caused by four-wave mixing due to the high local dispersion [2], and there is reduction of the Gordon–Haus effect [3] because of the low average dispersion.

Even in dispersion-managed fibers, solitons exist, if with properties somewhat different from their counterparts in homogeneous (constant-dispersion) fibers [4]: the shape is different, and the power is different ('power enhancement'). The most prominent difference, however, is that the pulse shape is no longer constant: the soliton 'breathes' over a dispersion period, and thus reshapes after each map period $L_{\text{map}} = L^+ + L^-$. Its shape is therefore stable in a stroboscopic sense. In technical systems deployed today L_{map} is of the order of several kilometers and therefore exceeds the coherence length of the light pulses. However, it may be of a similar order as L_D, if the latter is defined using β_2^{ave}.

2.2 *Random perturbation*

With few exceptions, theoretical descriptions of dispersion-managed systems (such as, for example, Ref. [5]) take for granted that the structure is precisely periodic. However, this assumption is only fulfilled in closed-loop systems, such as fiber lasers, where the same pieces of fiber are revisited on every round trip. In the world of optical telecommunications this is not a realistic assumption: in a fiber span connecting one point to a distant other, exact periodicity of fiber segments cannot be obtained; deviations are expected at least of the order of the precision with which individual segments of fiber can be prepared for concatenation. This applies to submarine systems; in terrestrial systems additional constraints lead to even greater variability in lengths. It turns out that there has been no previous full systematic study of the effects of deviation from periodicity.

We here consider the case of small deviations of the segment length from precise periodicity. Note that this is different from the situation where the individual same-length segments have slightly different β_2 values [6, 7]. Deviations of dispersion are a realistic possibility because in real fibers β_2 fluctuates along the fiber due to fabrication tolerances [8]. Effects of such fluctuations on propagation in a homogeneous fiber were discussed in Refs. [9, 10]; for a dispersion-managed fiber these was studied in Ref. [11]. Nonetheless, we will not consider dispersion fluctuations here but will restrict ourselves to random length deviations for clarity.

Deviations from the mean fiber lengths were already considered theoretically in Ref. [12], where the individual fiber segments were attributed a length which was uniformly distributed in a ±80% interval around the mean value. We reproduce the result that the solitons suffer from energy loss. However, in part by way of the ansatz used, and in part due to the assumed zero dispersion, break-up (splitting) of solitons went undetected. Also, Abdullaev and Baizakov [6] and Ablowitz and Moeser [7] considered random perturbations and noted deterioration of solitons but did not describe pulse splitting. In this work we report that splitting up of solitons is indeed the leading degradation mechanism.

In this communication we concentrate on the case of small random length deviations as they are likely encountered when equal segment lengths are intended, as in submarine long-distance cables. We treat the amplitude of the random deviation – the width of the interval over which the actual lengths scatter – as a continuous variable which we can increase from zero. In this way we have the limiting case of strict periodicity as a benchmark reference, and – other than in Refs. [7, 12] – see the continuous transition from a periodic to a randomized map, and can define thresholds beyond which data integrity is jeopardized.

3 Method

To study the propagation of the envelope of light pulses $A(T, z)$ through fibers we solve (1) by the usual split-step Fourier method [1]. We use the common safeguards against numerical inaccuracy, like monitoring preserved quantities (e.g. energy).

We restrict ourselves to a nominally symmetric dispersion map with $L^+ = L^-$. An important measure of a dispersion-managed fiber is the map strength

$$S = \frac{\left|\beta_2^+ L^+ - \beta_2^- L^-\right|}{\tau^2}\,, \tag{3}$$

with τ the full pulse width at half maximum (FWHM). S is zero in the homogeneous (i.e. constant-dispersion) case; appreciable dispersion management begins at $S \approx 1$, and $S \gg 1$ is commonly called strong dispersion management.

To introduce random length variations, we replace

$$L_i \to L_i(1 + j\zeta_i)\,, \tag{4}$$

where each individual fiber segment L_i is modified by a term $j\zeta$. j is the amplitude of the length deviation (the 'jitter' amplitude on top of the periodicity) (see Fig. 1), and ζ_i is a random number taken from a uniform distribution in the interval $[-1, +1]$. For its good suppression of spurious correlations, we used the Marsaglia–Zaman random number generation method [13] as implemented in the Mathematica computer algebra program [14]. Note that in the split-step Fourier algorithm the terms $jL_i\zeta_i$ are rounded to integer multiples of the step size.

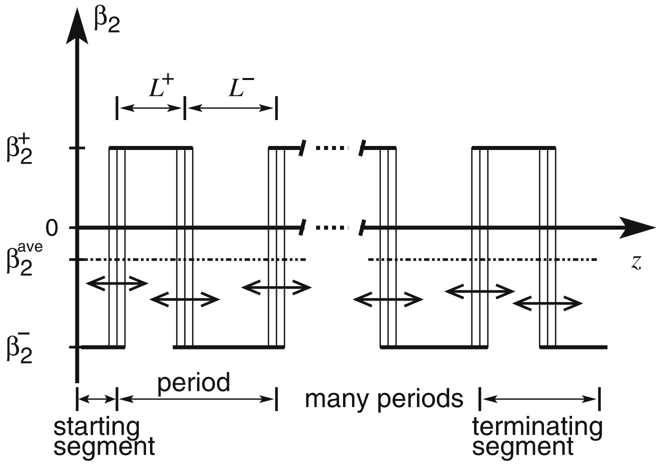

FIGURE 1 Schematic representation of a dispersion-managed fiber. It consists of alternating segments with positive and negative dispersion. Segments are concatenated either in a periodic fashion or – in this work – with random deviations from strict periodicity (*arrows*)

A rectangular distribution is not necessarily the best description of the length variations intended here; a Gaussian might be a better approximation. However, the exponential wings of a Gaussian could render some lengths negative, which does not make sense; one would have to use a truncated Gaussian anyway. In any event, we believe that details of the distribution are unlikely to have a major influence given that the fiber lengths usually span no more than a few hundred map periods. Indeed, to improve statistics and to safeguard against pathological effects from particular choices of random numbers, we routinely repeat calculations with several sets of independently generated random numbers, to better identify the specific effect of the randomness.

We need to introduce a further constraint on the sets of random numbers: we want to distinguish the true effect of the jitter from the accompanying shift of the path-average dispersion because both L^+ and L^- enter in (2). For each fiber segment (β_2^+, L^+) a corresponding segment with the same length but the opposite dispersion β_2^- was generated. The order in which the segments are spliced together was then randomly rearranged. In this way we make sure that the path-average dispersion is kept constant when the degree of randomness is varied.

4 Application to the non-random case

Since we introduced j as a continuously variable quantity, we can use $j = 0$ as a natural point of reference. Increasing j from zero will then allow us to follow the effect of increasing randomness.

There is no closed-form solution for the pulse shape in dispersion-managed fibers. However, it is known that it resembles a Gaussian more closely than a sech^2 shape. We therefore choose a chirp-free Gaussian-shaped pulse

$$A(T, 0) = NA(0, 0) \exp\left(-\frac{1}{2}\frac{T^2}{T_0^2}\right) \qquad (5)$$

as a launching condition. $A(0, 0)$ is the square root of the peak power of the fundamental soliton at $S = 0$. N is the soliton order; for the fundamental soliton in a homogeneous fiber ($S = 0$), $N = 1$. N^2 therefore denotes the pulse energy in units of the fundamental soliton energy.

We let the fiber begin with a half-segment of length $L^-/2$. This brings us close to the usual situation that the chirp-free points of the pulses is at the mid-segment positions. Since the initial pulse shape is not the asymptotic one, energy will be drained from the pulse during the early stages of propagation. We minimize this loss through judicious choice of S. It turns out that for $S = 1.424$ the loss is minimal and amounts to only 10^{-5} of the total power. We therefore pick this particular value of S; we consider values around $S = 1$ reasonable anyway because they describe appreciable, but not excessive, map strength. For this S, incidentally, the pulse energy is enhanced by a factor of $N^2 = 1.97$.

Changes in the functional form of the pulse shape render it non-trivial to find a characterization of its width that is unique and universally applicable. For example, the FWHM is easily fooled by wiggles due to interference effects. At the same time, the root-mean-square (rms) width is too sensitive to the far wings and background radiation to be of much usefulness. We settled for the full width at half energy (FWHE), i.e. the width of that temporal interval centered about the pulse center that contains 50% of the total energy. For simplicity we read pulse widths at mid-segment points, which are not necessarily identical to the minimum-chirp points; we may thus have slightly high readings. However, as will become clear below, this is irrelevant for our central point that pulses split up; splitting is detected unambiguously in spite of the simplification.

In principle we could vary many parameters, like the pulse width, the chirp, the pulse shape, etc. However, in a practical situation the only parameter that can be readily controlled is the launch energy. We therefore restrict ourselves to variations of the total energy (or N^2, equivalently) and keep all other parameters constant.

Figure 2 shows the evolution of the FWHE T_{FWHE} as a function of propagation distance z for different values of the soliton order parameter N. There are three different regions: for low energies there is too little energy to form a soliton, and the pulse just broadens due to dispersion. If

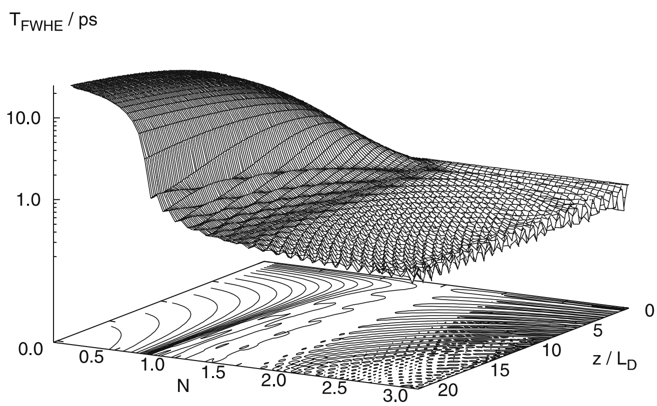

FIGURE 2 Evolution of the pulse duration (FWHE) in a dispersion-managed fiber with $S = 1.424$. The fiber has 256 dispersion periods and is $20.4L_{\mathrm{D}}$ long. The position z is normalized to L_{D}. Gaussian pulses are launched; their energy is measured by N^2. Calculations were performed with 1024 split steps per dispersion segment

$N \approx 1.4$ ($N^2 \approx 1.97$) a dispersion-managed soliton is generated. The contour lines show perhaps more clearly than the three-dimensional graph that the pulse width becomes independent of z. For higher energies one observes rapid variations in the pulse width. These are mainly caused by beating between the soliton and the radiated background; there are also contributions from beats between several solitons when at elevated power more than one is generated.

5 Application to the random case

Now we let $j \neq 0$ and thus introduce the random length deviations ('jitter') described above. For small jitter, there is no dramatic change: solitons may still propagate in the fiber, because they can rearrange e.g. their pulse width to fulfill the balance between dispersion and nonlinearity. In the process they shed some energy; this energy is converted into dispersive waves. A continued energy loss will eventually let the solitons decay. The distance over which the solitons survive may, however, be tremendous, and may be sufficient for all terrestrial purposes.

A typical behavior with a jitter amplitude of $j = 0.074$ is shown in Fig. 3. Comparison with Fig. 2 reveals the following: the low-power dispersive broadening is hardly affected. The point of invariance at $N = 1.4$ gives way to minimal variation somewhere between $N = 1$ and $N = 1.5$, but is less clear than before. This will be shown with more clarity below. Above $N = 1.5$ the randomness destroys the regular interference pattern and, notably, gives rise to an enormous growth of the FWHE at specific N values. This is understood as follows: Fig. 4 shows the pulse shape at the distal fiber end for $1.20 \leq N \leq 1.60$. At specific N values (here, $N = 1.45$ and $N = 1.50$) the pulse is split up into a pulse pair. If one measures pulse width as FWHE as we do here, in such a case one finds the pulse separation and not their individual widths. This explains the enormous growth of the FWHE.

A question of great practical importance is: under which conditions and in which locations does the splitting happen? The question for the location is addressed by a graphical representation of the evolution of the FWHE over the distance with j as a parameter in Fig. 5. Again, for $j \lesssim 0.05$ no dramatic events happen to the soliton. However, at $j > 0.05$ a split can occur; here one sees clearly that it occurs in well-

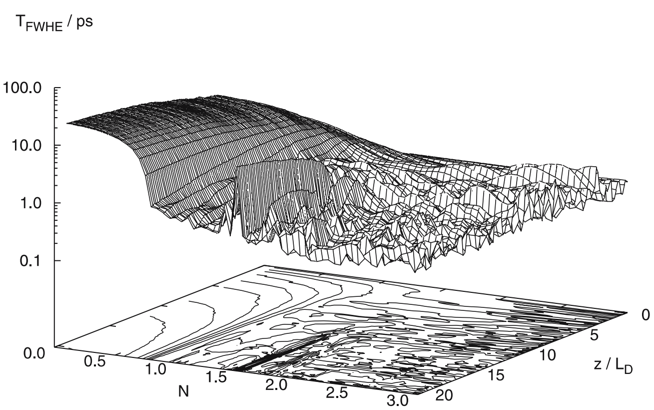

FIGURE 3 Same as in Fig. 2 but with random segment length deviations at $j = 0.074$

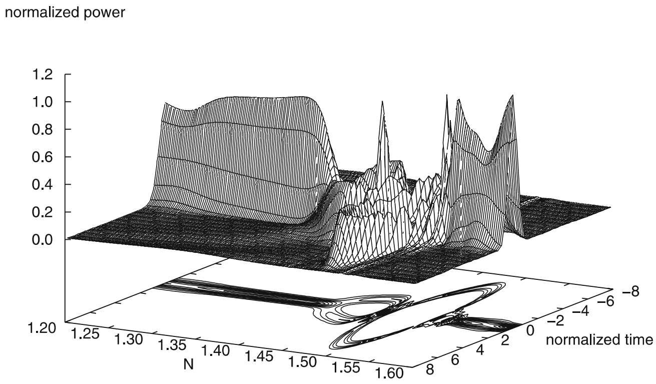

FIGURE 4 Pulse shape at the fiber end at $z = 20.4 L_D$ (compare Fig. 3). Note the 'bubbles' in the contours near $N = 1.45$ and $N = 1.50$, indicative of pulse split-ups

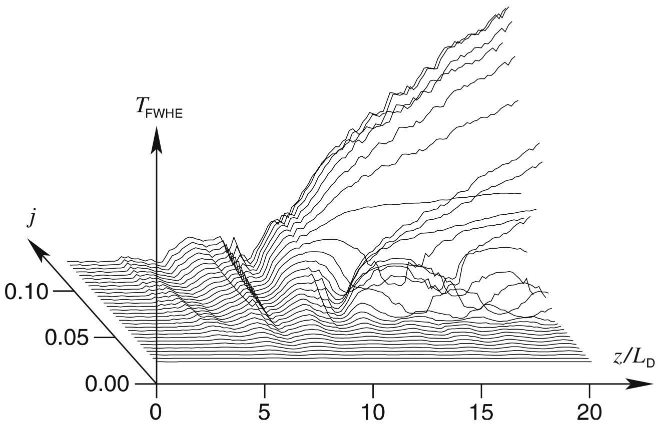

FIGURE 5 Evolution of the pulse duration (FWHE) along the fiber for different jitter amplitudes $0 \leq j \leq 0.113$ with $N = 1.45$. The trace with $j = 0.074$ corresponds to Figs. 3 and 4

localized places (in this case, $z = 7.5 L_D$ at $0.08 \leq j \leq 0.11$ and $z = 11 L_D$ at $0.05 \leq j \leq 0.06$).

The next question is whether the underlying cause is also well localized, or whether it is some cumulative effect. We checked this by operating the same fiber, with the same randomization, in backwards direction (i.e. by launching pulses at the far end and propagating them towards the near end). It turns out that in the reverse direction there may be splits occurring in different positions, or none at all. This shows that the split-ups are dependent on direction: the cause is not strictly localized, and is not fully described by non-directional specifications like the histogram, power spectrum, etc., of the random jitter.

Instead, whatever is responsible apparently accumulates over some distance and prepares the pulses so that they become prone to splitting up. In this context we find it remarkable that the energy intervals for which split-ups occur are so narrow as to remind one of resonance phenomena. For example, in Fig. 4 break-ups occur for $N = 1.45 \pm 0.02$ and for $N = 1.50 \pm 0.02$ (and in other computational runs we encountered some resonances that were an order of magnitude narrower). One might suspect a resonance of the map period L_{map} with L_D; both are indeed of the same order.

However, these 'resonances' are nonlinear phenomena: an alternative view of the pulse splitting, this time as a function of the jitter amplitude j, is presented in Fig. 6. j is increased

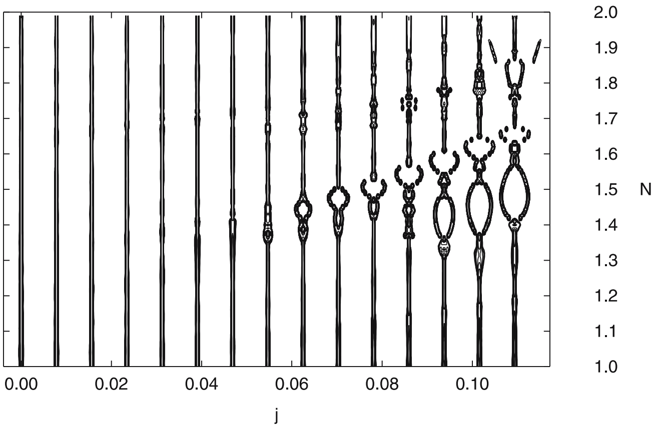

FIGURE 6 Assembly of contour plots as in Fig. 4, for different random amplitudes j. For the individual plots, j was incremented from 0 in steps of $1/128$. Note that the 'bubbles' shift to larger N as j increases

in steps of $1/128$. Again we see that soliton split-ups occur in narrow energy intervals, but now we realize that the 'resonant energies' shift upwards as the jitter amplitude is increased. The ultimate reason for these 'resonances' must therefore be more complex. An observation of considerable practical import is that below the lowest 'resonance' there is no split-up, and the pulses remain intact. In many computational runs we never found any splitting for $j \lesssim 0.05$. On the other hand, for sufficient jitter amplitude we find in every single computational run that pulses split sooner or later along the fiber as long as the energy is varied in sufficiently small increments – it is easy to miss a narrow resonance when using coarse steps.

6 Conclusion

As soon as the dispersion map deviates from strict periodicity, we have to expect several modifications: (a) if resonances of the soliton dynamics with the dispersion period were a problem, randomization would tend to wash them out. (b) On the other hand, the soliton's average phase will undergo a random walk, which, if it accumulates too much, may eventually lead to the soliton's destruction. This could be corrected without affecting the previous item by local compensation of accumulated dispersion at one or both of the fiber end points, or even at some convenient points along the span. (c) A continued energy loss of the soliton as it travels through the randomized map is unavoidable. This effect is comparable to fiber loss in a non-dispersion-managed system. For excessive fiber loss solitons may split into two. An obvious fix would be to launch pulses such that they have the optimum energy not at the launch point, but at mid-span. In other words, one would deliberately make the launched power high by about one-half of the scattering loss. We caution, however, that this approach may actually provoke pulse split-ups.

How does all this affect the usefulness of dispersion-managed solitons? For fiber spans on this planet, its half-circumference of about 20 Mm should serve as a reasonable maximum distance. As long as degradation over this distance contributes little to the overall bit error rate, the scheme can be considered stable for all practical intents and purposes. Our data show that for all realizations of random numbers that we employed, jitter amplitudes up to a few percent never destroyed a soliton over a distance of $20L_{\mathrm{D}}$; in our calculations and assuming $T_0 = 30$ ps, $20L_{\mathrm{D}}$ corresponds to 21 Mm.

This suggests the following conclusion: when the fiber segment lengths are measured with a 1% accuracy as is typically obtained with standard OTDR (optical time domain reflectometry) equipment, and for typical amounts of random fluctuation of local dispersion along the fiber due to manufacturing tolerances, the individual segments should have their accumulated dispersion matched to within less than about 2%. Even in 40 Gbit-per-second systems with pulse durations 10 ps and shorter, solitons will survive intact for US interstate (or European capital-to capital) distances without problem. It is also quite possible that with some care transoceanic distances can be spanned without significant error increase.

We finally note that we did not yet take into account the effect of optical amplifiers. At this point it remains an open question of how amplified systems are affected. Also, we have not yet considered the effect of channel interaction in wavelength-division multiplex systems. All dispersion-managed systems suffer from background radiation scattered off the solitons; so far this seems to be not too detrimental. However, with a randomized periodicity, the amount of scatter increases, and the problem is aggravated. Further research must show how the results presented here hold in the dense wavelength division multiplex case.

ACKNOWLEDGEMENTS We gratefully acknowledge the German–Israeli Project Cooperation (DIP) for financial support.

REFERENCES

1 G.P. Agrawal, *Nonlinear Fiber Optics* (Academic, London 2001)
2 C. Kurtzke, IEEE Photon. Tech. Lett. **5**, 1250 (1993)
3 M. Suzuki, I. Morita, N. Edagawa, S. Yamamoto, H. Taga, S. Akiba, Electron. Lett. **31**, 2027 (1995)
4 N.J. Smith, F.M. Knox, N.J. Doran, K.J. Blow, I. Bennion, Electron. Lett. **32**, 54 (1996)
5 S.K. Turitsyn, E.G. Shapiro, S.B. Medvedevc, M.P. Fedoruk, V.K. Mezentsev, C.R. Physique **4**, 145 (2000)
6 F.K. Abdullaev, B.B. Baizakov, Opt. Lett. **25**, 93 (2000)
7 M.J. Ablowitz, J.T. Moeser, Opt. Lett. **29**, 821 (2004)
8 J. Gripp, L.F. Mollenauer, Opt. Lett. **23**, 1603 (1998)
9 R. Ohhira, A. Hasegawa, Y. Kodama, Opt. Lett. **20**, 701 (1995)
10 M. Chertkov, Y. Chung, A. Dyachenko, I. Gabitov, I. Kolokolov, V. Lebedev, Phys. Rev. E **67**, 036 615 (2003)
11 E. Poutrina, G.P. Agrawal, J. Lightwave Technol. **21**, 990 (2003)
12 B.A. Malomed, A. Berntson, J. Opt. Soc. Am. B **18**, 1243 (2001)
13 G. Marsaglia, A. Zaman, Ann. Appl. Probab. **1**, 462 (1991)
14 http://www.wolfram.com