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Light scattering and transmission spectra of the Menger sponge fractal

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ABSTRACT Selection rules for light scattering and transmission by the Menger sponge fractal were derived by the group theory based on the symmetry of its localized electromagnetic eigenmodes. The light scattering and transmission spectra were calculated by the finite-difference time-domain method and compared with the eigenfrequencies of the localized modes obtained from the dipole radiation spectra. Their correspondence is quite good and supports the accuracy of the numerical calculation and the correctness of the selection rules.

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1 Introduction

Localization of electromagnetic (EM) waves is one of the most interesting properties of photonic crystals [1–7]. Defect regions surrounded by regular photonic crystals act as optical cavities. Quality (Q) factors over 10^4 were attained with photonic crystal slabs [8] and used to observe the Rabi splitting of emission lines of quantum dots caused by strong coupling of excitons and the cavity EM mode [9].

On the other hand, another type of EM localization in self-similar structures known as fractals [10, 11] has been attracting much interest in the last two decades [12–16]. Dimensionality of the fractal structure is an important factor for the phenomenon, since the localization takes place more easily in low dimensions due to the small number of scattering channels. In three dimensions, waves may be scattered into all solid angles, which decreases the lifetime of the eigenmodes and prevents the formation of well-defined localized modes. This is especially true for fractals made of transparent materials that do not have resonant polarization in the relevant frequency range.

However, we recently reported a theoretical analysis of the Menger sponge (Fig. 1), which is a three-dimensional cubic fractal [17]. We calculated the eigenfrequency, field distribution, and the Q factor of localized EM modes and showed that Q factor up to 840 is attainable assuming common sample parameters. Although our first experimental study in the microwave region [18] apparently overestimated the Q factor, we presented a new possibility to realize resonance cavities with three-dimensional fractals.

The Menger sponge is the three-dimensional version of the Cantor bar fractal. The Cantor bar fractal is made by removing the center segment from three equivalent segments obtained by dividing an initial bar, and repeating this procedure to the two remaining segments. Similarly, the Menger sponge may be made from a dielectric cube. The initial cube is divided into 27 ($= 3^3$) identical cubic pieces, and seven pieces at the body and face centers are removed. By repeating the same procedure to the 20 remaining pieces, we obtain the Menger sponge. Its fractal dimension is 2.7268. The number of the repetition of the removal procedure is called the stage number. The ideal Menger sponge is obtained by repeating the removal infinite times. But, of course, this infinite structure cannot be realized physically.

In this paper, we will present the light scattering and transmission spectra calculated for the Menger sponge of stage three and show that each peak in the scattering spectra

FIGURE 1 The Menger sponge fabricated by the stereolithography with a photoreactive epoxy resin as a raw material

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corresponds to an localized EM eigenmode. We will also derive selection rules for the light scattering and transmission by the group theory based on the symmetry of the EM eigenmodes.

2 Theory

Light scattering and transmission by the Menger sponge were calculated by the FDTD (finite-difference time-domain) method with the PML (perfectly matched layer) boundary conditions [19]. The Menger sponge of stage three with a dielectric constant of 8.8 (epoxy resin mixed with metal oxides) was assumed as a specimen. The configuration for the calculation is shown in Fig. 2, where S is the light source, and D_1 and D_2 denote the location of two detectors. Absorbing boundary planes were placed perpendicular to the x, y, or z axis and $4a$ away from the origin. We assumed an oscillating dipole as a light source and evaluated the intensity of the scattered and transmitted light at D_1 and D_2, respectively. For the FDTD calculation, the space and time were discretized. The length a was divided into 40 parts and one cycle of oscillation of the dipole was divided into 1024 parts.

Gaussian pulses with several central frequencies were generated by the dipole and their propagation was calculated by the FDTD method. The temporal profile of the electric fields at D_1 and D_2 were Fourier transformed to give the scattering and transmission spectra. The spectra were normalized by the spectral intensity of the light source.

If the dipole is pointed parallel to the y axis (denoted by $\leftrightarrow$), the electric field of the generated spherical wave is symmetric about the xy plane and antisymmetric about the xz plane. In other words, the electric field has an even parity for σ_z and an odd parity for σ_y. Here, σ_y (σ_z) denotes the mirror reflection about the xz (xy) plane, and so, it changes the sign of the y (z) coordinate. On the other hand, if the dipole is pointed perpendicular to the xy plane (denoted by $\odot$), the electric field of the generated wave has an odd parity for σ_z and an even parity for σ_y. These characteristics of the incident waves generated by the light source leads to selection rules for the 90° light scattering and straight transmission due to the spatial symmetry of the localized EM modes.

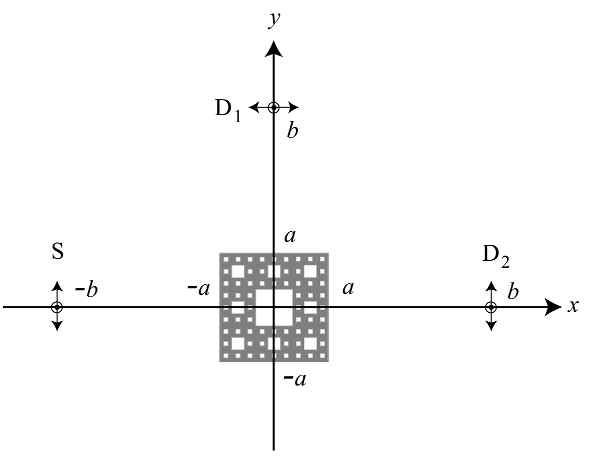

FIGURE 2 Configuration for the calculation of the light scattering and transmission intensity. The center of the Menger sponge whose size was $2a$ was located at the origin of the coordinates. Its surface was perpendicular to the x, y, and z axes. An oscillating dipole was assumed as the light source S and the electric fields at D_1 and D_2 were examined. The distance b was $3.65a$. Both polarizations, perpendicular (*denoted by* $\odot$) and parallel (*denoted by* $\leftrightarrow$) to the xy plane, were analysed

O_h	σ_x	σ_y	σ_z	In	Out (D_1)	Out (D_2)
T_{1g}	1	−1	−1	x	$\odot$	x
	−1	1	−1	$\odot$	x	$\odot$
	−1	−1	1	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$
T_{1u}	−1	1	1	x	$\leftrightarrow$	x
	1	−1	1	$\leftrightarrow$	x	$\leftrightarrow$
	1	1	−1	$\odot$	$\odot$	$\odot$
T_{2g}	1	−1	−1	x	$\odot$	x
	−1	1	−1	$\odot$	x	$\odot$
	−1	−1	1	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$
T_{2u}	−1	1	1	x	$\leftrightarrow$	x
	1	−1	1	$\leftrightarrow$	x	$\leftrightarrow$
	1	1	−1	$\odot$	$\odot$	$\odot$

TABLE 1 Symmetry of triply degenerate eigenmodes and their coupling to the incident, scattered, and transmitted light waves. $\leftrightarrow$ and $\odot$ denote coupling to light waves with the horizontal and vertical polarization, respectively. x shows the absence of coupling

Because the Menger sponge has the cubic symmetry, its EM eigenmodes are irreducible representations of the O_h point group [20]. It has ten irreducible representations, four of which (A_{1g}, A_{1u}, A_{2g}, A_{2u}) are non-degenerate (one-dimensional), two (E_g, E_u) are doubly degenerate (two-dimensional), and the remaining four (T_{1g}, T_{1u}, T_{2g}, T_{2u}) are triply degenerate (three-dimensional). Each eigenmode has its own spatial symmetry, which results in a selection rule for the coupling to incident waves. This fact is known well for photonic crystals with high geometrical symmetry [7, 21], and the same holds for the Menger sponge. For the present geometry, only the triply degenerate eigenmodes have the same symmetry as the incident waves, and thus, contribute to the light scattering and transmission.

Table 1 shows the parity of three independent eigenfunctions of the triply degenerate mode and their coupling to the incident, scattered, and transmitted waves [17]. Let us examine the T_{1g} mode as an example. The first function of the T_{1g} mode does not couple to the incident wave due to the mismatching of symmetry. So, it does not contribute to either the 90° scattering or the straight transmission. The second function couples to the incident and transmitted waves with the perpendicular polarization, but does not couple to the scattered wave. So, it contributes only to the straight transmission. The third function couples to all three waves, and thus, contributes to both the 90° scattering and the straight transmission. Similarly, we can judge the presence or absence of coupling for the remaining three representations. As a result, we can conclude that all four modes contribute to the straight transmission, whereas two modes contribute to the 90° scattering for each polarization, that is, the T_{1u} and T_{2u} modes for the perpendicular polarization, and the T_{1g} and T_{2g} modes for the parallel polarization. Thus we have stronger selection rules for the 90° light scattering.

3 Results and Discussion

First, let us examine the eigenfrequencies of localized EM modes of the Menger sponge. In our previous paper [17], we showed that we can obtain them from the dipole radiation spectra. We assumed an oscillating point dipole located in the central hole of the Menger sponge and calculated the emitted EM field by the FDTD method. By applying boundary conditions that matches the symmetry of the localized eigenmodes, we could extract contribution of eigenmodes with a designated symmetry to the dipole radiation [4].

Figure 3 is such dipole radiation spectra calculated for the triply degenerate eigenmodes. Each peak denoted by an arrow corresponds to a localized EM mode. Some peaks in the low frequency region are very small, but we can recognize them when we magnify the spectra.

Next, let us examine the straight transmission observed at D_2. Note that two different polarizations of the light source at S give the same transmission intensity for this geometry. From Table 1, eigenmodes of all four symmetries couple to both incident and transmitted light waves, and thus, contribute to the straight transmission. Figure 4 is the transmission intensity calculated by the FDTD method, where the eigenfrequencies found in Fig. 3 are denoted by arrows. We can find some agreement between the peaks of the transmission spectrum and the eigenfrequencies of the localized modes. But because of too dense distribution of the eigenfrequencies, the agreement is not very clear.

On the other hand, we have a stronger selection rule for the 90° light scattering when we specify the polarization of the light source. Figure 5 shows the scattering spectrum for the perpendicular polarization to which only the T_{1u} and T_{2u} modes contribute. Apparently the light scattering spectrum is simpler than the transmission spectrum and we can find much better correspondence between the peaks of the spectrum and the eigenfrequencies. The maximum deviation is 1.5%. This fact is an evidence for the accuracy of the two independent numerical calculations and the correctness of the selection rule.

In addition, the light scattering spectrum is background free, i.e., we observe a peak only when there is an eigenmode. This is another reason why the light scattering spectrum looks simpler than the transmission spectrum. As for the latter, we observe complex interference structures caused by multiple internal reflections as well as the peaks due to the localized modes. A further advantage of the light scattering spectrum is that the ratio of the center frequency of the peak to its full width at half maximum gives the Q factor of the relevant localized eigenmode [17].

Figure 6 is the 90° scattering spectrum calculated for an incident wave with the parallel polarization. The T_{1g} and T_{2g} modes are relevant to this case, and their eigenfrequencies found in Fig. 3 are denoted by arrows. The agreement between the peaks of the scattering spectrum and the eigenfrequencies is good again.

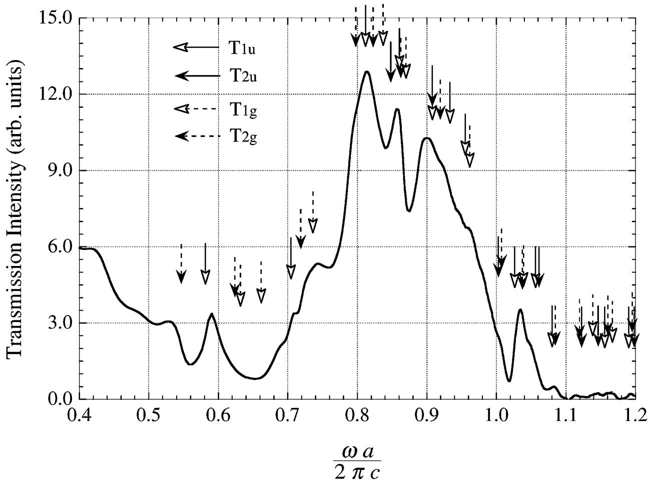

FIGURE 4 Transmission intensity evaluated at D_2. Eigenfrequencies found in the dipole radiation spectra (Fig. 3) are denoted by arrows. The abscissa is the dimensionless frequency normalized by the size of the Menger sponge a and the light velocity in free space c

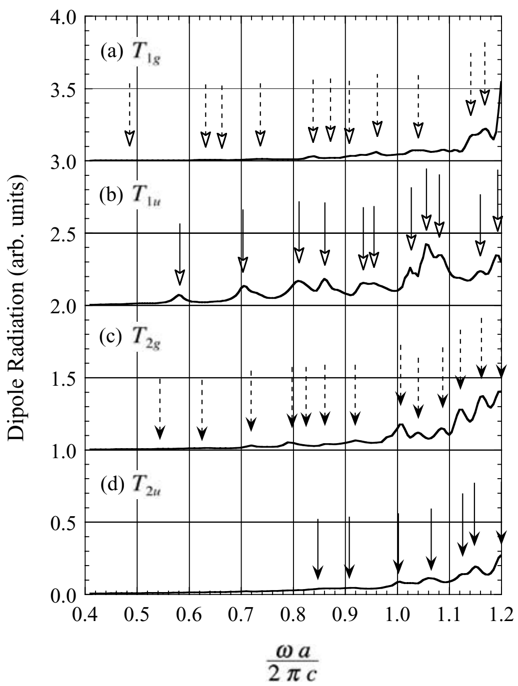

FIGURE 3 Dipole radiation intensity calculated for triply degenerate eigenmodes with (a) T_{1g}, (b) T_{1u}, (c) T_{2g}, and (d) T_{2u} symmetry. Accumulated EM energy after 50 cycles of oscillation is shown. The abscissa is the dimensionless frequency of the dipole oscillation normalized with the size of the Menger sponge, a (see Fig. 2), and the light velocity in free space, c. Peaks in the spectra are denoted by arrows

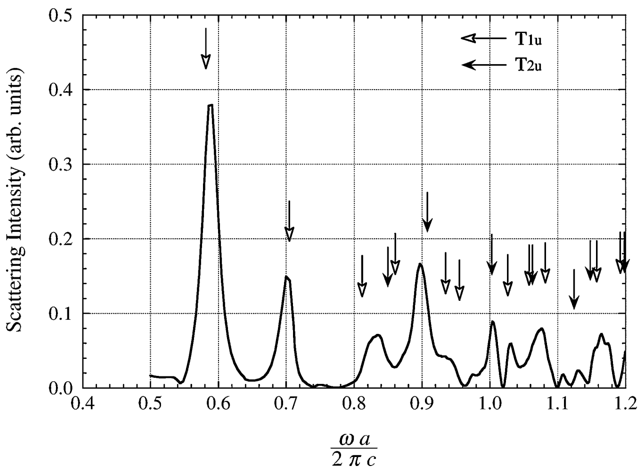

FIGURE 5 90° light scattering intensity evaluated at D_1 for the perpendicular polarization (⊙). Eigenfrequencies of the T_{1u} and T_{2u} modes found in Fig. 3 are denoted by arrows

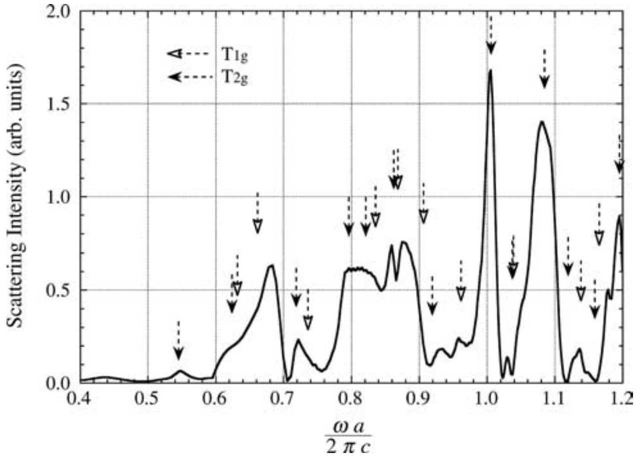

FIGURE 6 90° light scattering intensity evaluated at D_1 for the parallel polarization (↔). Eigenfrequencies of the T_{1g} and T_{2g} modes found in Fig. 3 are denoted by *arrows*

4 Conclusion

Selection rules for the 90° light scattering and straight transmission by the Menger sponge fractal were derived by the group theory based on the symmetry of its localized EM eigenmodes. We found that the selection rules for the 90° scattering is stronger than those for the straight transmission. The light scattering and transmission spectra were calculated by the FDTD method and compared with the eigenfrequencies of the localized modes reported previously. Their correspondence is quite good and supports the accuracy of the two independent numerical calculations and the correctness of the selection rules. Thus the 90° light scattering is a powerful method to investigate the localized EM modes of the Menger sponge.

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