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Effect of nonlinear gradient terms on pulsating, erupting and creeping solitons

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ABSTRACT The effect of nonlinear gradient terms on the pulsating, erupting and creeping solitons, respectively, of the cubic–quintic complex Ginzburg–Landau equation is investigated. It is found that the nonlinear gradient terms result in dramatic changes in the soliton behavior. They eliminate the periodicity of the pulsating and erupting solitons and transform them into fixed-shape solitons. This is important for potential use, such as to realize experimentally the undistorted transmission of femtosecond pulses in optical fibers. However, the nonlinear gradient terms cause the creeping soliton to breathe periodically at different frequencies on one side and spread rapidly on the other side.

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1 Introduction

The complex Ginzburg–Landau equation (CGLE) is one of the well-established and studied nonlinear equations [1]. The equation and its different modifications describe a variety of phenomena [2–7]. Researchers have paid much attention to these equations. Recently, the CGLE has been used to describe pattern formation near a Hopf bifurcation and has become a paradigmatic model for the study of spatiotemporal chaos [3, 8–11]. In the past several decades, three forms of the CGLE had been analyzed. The simplest one is the cubic CGLE including only cubic terms. This equation has been analyzed mainly in the context of plasma physics [12]. It admits an exact solution which can be found relatively easily. In optics, the solution was introduced by Bélanger et al. [13]. But it has been realized that the pulse-like solutions of this equation are unstable. Later, the quintic CGLE model was introduced, as it admits stable solutions, i.e. both the pulse and the background are stable [14]. But the quintic CGLE is more difficult to analyze than the cubic one. The third form of the CGLE was introduced by Haus et al. [15] to describe passively mode-locked fiber lasers, which is the cubic CGLE with a linear gain term and it admits stable solutions for some range of parameters.

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So far, different types of solutions of the one-dimensional CGLE, such as pulse-like, shock-like, sources, sinks and periodic and quasiperiodic solutions, have been analyzed [16–18]. Recently, three novel solutions of the cubic–quintic CGLE have been presented, which are pulsating, erupting and creeping solitons [19, 20]. In 2002, Akhmediev and co-workers reported the experimental evidence of the erupting solitons in a laser, where the higher-order effects such as the nonlinear gradient terms might have some influence [21]. But almost all those researches have been limited to the CGLE with cubic or quintic terms only except Deissler and Brand, who studied the effect of the nonlinear gradient terms on breathing localized solutions in the CGLE [22]. In fact, the CGLE with some higher-order terms such as nonlinear gradient terms is worth further investigation as mentioned by van Saarloos and Hohenberg [23], since the more general model is useful for understanding a variety of experimental phenomena. A useful example is an optical pulse transmission line. For an intense and short optical pulse in the subpicosecond or femtosecond regime, the nonlinear gradient terms are related to the effect of self-steepening and self-frequency shift, which are the key factors affecting the stable transmission of femtosecond optical pulses.

In this paper, we study the effect of nonlinear gradient terms on the three novel solutions: pulsating, erupting and creeping solitons, respectively, of the cubic–quintic CGLE. It is found that the nonlinear gradient terms can dramatically alter the behavior of these solitons. Small nonlinear gradient terms can eliminate the periodicity of the pulsating and erupting solitons and transform them into fixed-shape solitons. This effect will be of much use in practice such as to realize experimentally the undistorted transmission of femtosecond pulses in optical fibers. However, larger nonlinear gradient terms make the pulsating soliton spread. With the increase of its energy, the pulsating soliton will be transformed into a rectangle-like pulse with a sink on the top. Meanwhile, for the erupting soliton, a little larger nonlinear gradient term only makes it split on the top without increasing its energy. Moreover, when the nonlinear gradient terms are high enough, the erupting soliton will decay to zero. In addition, the nonlinear gradient terms in some range cause the creeping soliton to breathe periodically at different frequencies on one side and spread rapidly on the other side.

2 CGLE with the nonlinear gradient terms

The quintic CGLE with the nonlinear gradient terms can be written as

$$\psi_z = \frac{\mathrm{i}D}{2}\psi_{tt} + \mathrm{i}\,|\psi|^2\,\psi + \delta\psi + \varepsilon\,|\psi|^2\,\psi + \beta\psi_{tt} + \mu\,|\psi|^4\,\psi + \mathrm{i}\nu\,|\psi|^4\,\psi - \lambda\,|\psi|^2\,\psi_t - \kappa\psi^2\psi_t^*\,, \quad (1)$$

where D, δ, ε, β, μ and ν are real constants, and ψ is a complex field. The physical meaning of each variable depends on the particular problem. In optics, t is the retarded time, z is the propagation distance, D represents dispersion, which can be fixed to take the values of ± 1 by a proper rescaling in t, ψ is the complex envelope of the electric field, δ is the linear gain ($\delta > 0$) or loss ($\delta < 0$) coefficient, β describes the effect of spectral limitation due to bandwidth-limited amplification, ε accounts for nonlinear gain/absorption processes, μ represents a higher-order correction to the nonlinear amplification/absorption, namely, the saturation to the nonlinear gain or loss, ν is a parabolic correction term to the nonlinear refractive index and λ and κ, which are complex constants related to the effect of self-steepening and self-frequency shift, represent the nonlinear gradient terms. The nonlinear gradient terms cause the fixed-shape solution to be asymmetric and to move at a velocity other than the group velocity [24]. For breathing localized solutions, the nonlinear gradient terms cause them to breathe periodically or chaotically on only one side or to rapidly spread [22]. In the following, we will investigate numerically the effect of nonlinear gradient terms on pulsating, erupting and creeping solitons, respectively. The numerical method used is a time-splitting Fourier method [25]. The time step is 0.1 and the number of Fourier modes and grid points is 1024. The distance step is 1/125. The initial condition is given as a sech-shape pulse.

3 Simulation results

3.1 *Effect of the nonlinear gradient terms on the pulsating soliton*

At first we consider the effect of the nonlinear gradient terms on the pulsating soliton. Figure 1a, b and c show the space–time plots of the modulus of ψ for different values of the nonlinear gradient terms, while the other parameters are fixed as $D = 1$, $\varepsilon = 0.66$, $\delta = -0.1$, $\beta = 0.08$, $\mu = -0.1$ and $\nu = -0.1$. Figure 1a shows the case of $\lambda = \kappa = 0$, which corresponds to the pulsating soliton found by Soto-Crespo et al. [19]. Figure 1b shows the case of $\lambda = \kappa = 0.01 + 0.01\,\mathrm{i}$. It can be seen that, under this condition, the behavior of the soliton is not periodic. It tends to a fixed-shape soliton traveling very slowly towards the right. However, when the nonlinear gradient terms increase further, the fixed-shape soliton will not remain any more because the right-hand side of the pulse will travel faster than the left-hand side. As a result, the soliton changes into a rectangular-like pulse with a sink on the top, as shown in Fig. 1c, for which $\lambda = \kappa = 0.02 + 0.02\,\mathrm{i}$.

To characterize the evolution of these pulses, the areas under the modulus ($s = \int \mathrm{d}x\,|\psi|$) for Fig. 1a, b and c as functions of distance are shown in Fig. 1d, where the dotted line, solid line and dashed line correspond to the cases of $\lambda = \kappa = 0$, $\lambda = \kappa = 0.01 + 0.01\,\mathrm{i}$ and $\lambda = \kappa = 0.02 + 0.02\,\mathrm{i}$, respectively. We note that the area S is independent of the pulse velocity. From Fig. 1d it can be seen that, for $\lambda = \kappa = 0.01 + 0.01\,\mathrm{i}$, the area approaches a constant at large z but, if $\lambda = \kappa = 0.02 + 0.02\,\mathrm{i}$, it increases linearly. Detailed numerical simulations show that although the nonlinear gradient terms change the pulsating soliton into a fixed-shape soliton, the range for admitting a fixed-shape soliton is limited. Even a little larger values will make the pulsating soliton spread on one side because of the increase of energy. It should be pointed out that in Fig. 1a, b and c only the evolution up to distance $z = 50$ is shown and, in Fig. 1d, the evolution characteristics are also limited within the distance of $z = 100$. In order to recognize the behavior of these patterns at much longer distance, we extend the transmission distance of these pulses up to $z = 1000$. It is found that the pulse in Fig. 1b still remains stable; its area is stabilized to a value of 9.6301. The results of Fig. 1b for much longer distance are shown in Fig. 1e and f. This means that, at the long-distance limit, the pattern of Fig. 2b can still remains localized. Similarly, for the case of Fig. 1c, after a long distance, the evolution trend still remains unchanged. The opposite sign of the nonlinear gradient terms will lead to the same result except for the change of the propagation direction.

3.2 *Effect of the nonlinear gradient terms on the erupting soliton*

Next, we consider the effect of the nonlinear gradient terms on the erupting soliton. The space–time evolution plots of the modulus of ψ for different values of the nonlinear gradient terms are shown in Fig. 2a, b and c, where $\lambda = \kappa = 0$, $\lambda = \kappa = 0.3 + 0.3\,\mathrm{i}$ and $\lambda = \kappa = 0.7 + 0.7\,\mathrm{i}$, respectively. The other parameters are fixed as $D = 1$, $\varepsilon = 1.0$, $\delta = -0.1$, $\beta = 0.125$, $\mu = -0.1$ and $\nu = -0.6$. Figure 2d shows corresponding areas S as functions of the distance, where the dotted, solid and dashed lines correspond to the cases for Fig. 2a, b and c, respectively. It is shown in Fig. 2b that the nonlinear gradient terms can eliminate the chaotic structure of the erupting soliton and change it into a fixed-shape soliton. Similar to the fixed-shape soliton described in Sect. 3.1, here the fixed-shape soliton also travels slowly towards the right. However, the range for admitting the fixed-shape soliton is larger than that in Sect. 3.1 The fixed-shape soliton can exist even if the parameter value of the nonlinear gradient term is changed from 0.28 to 0.5. In addition, as shown clearly in Fig. 2b and d, the stable fixed-shape soliton can be formed in a very short distance. There is nearly no evolution process of oscillatory shape as shown in Fig. 1d. When the nonlinear gradient terms are only increased a little, the stable soliton will split on its top but its area is almost unchanged (see, for example, Fig. 2c and the dashed line of Fig. 2d, where $\lambda = \kappa = 0.7 + 0.7\,\mathrm{i}$). By enlarging the figure to show the details of the curve, it is found that S is oscillatory within a very small range. Further calculation indicates that even when the nonlinear gradient terms increase so that $\lambda = \kappa = 1.5 + 1.5\,\mathrm{i}$, the area curve of the pulse is still stabilized in a certain range. It means that the values of the nonlinear gradient terms between $\lambda = \kappa = 0.7 + 0.7\,\mathrm{i}$ and $\lambda = \kappa = 1.5 + 1.5\,\mathrm{i}$ only affect severely the pulse shape but nearly do not affect their energy. However, when the nonlinear gradient terms are much larger (for example,

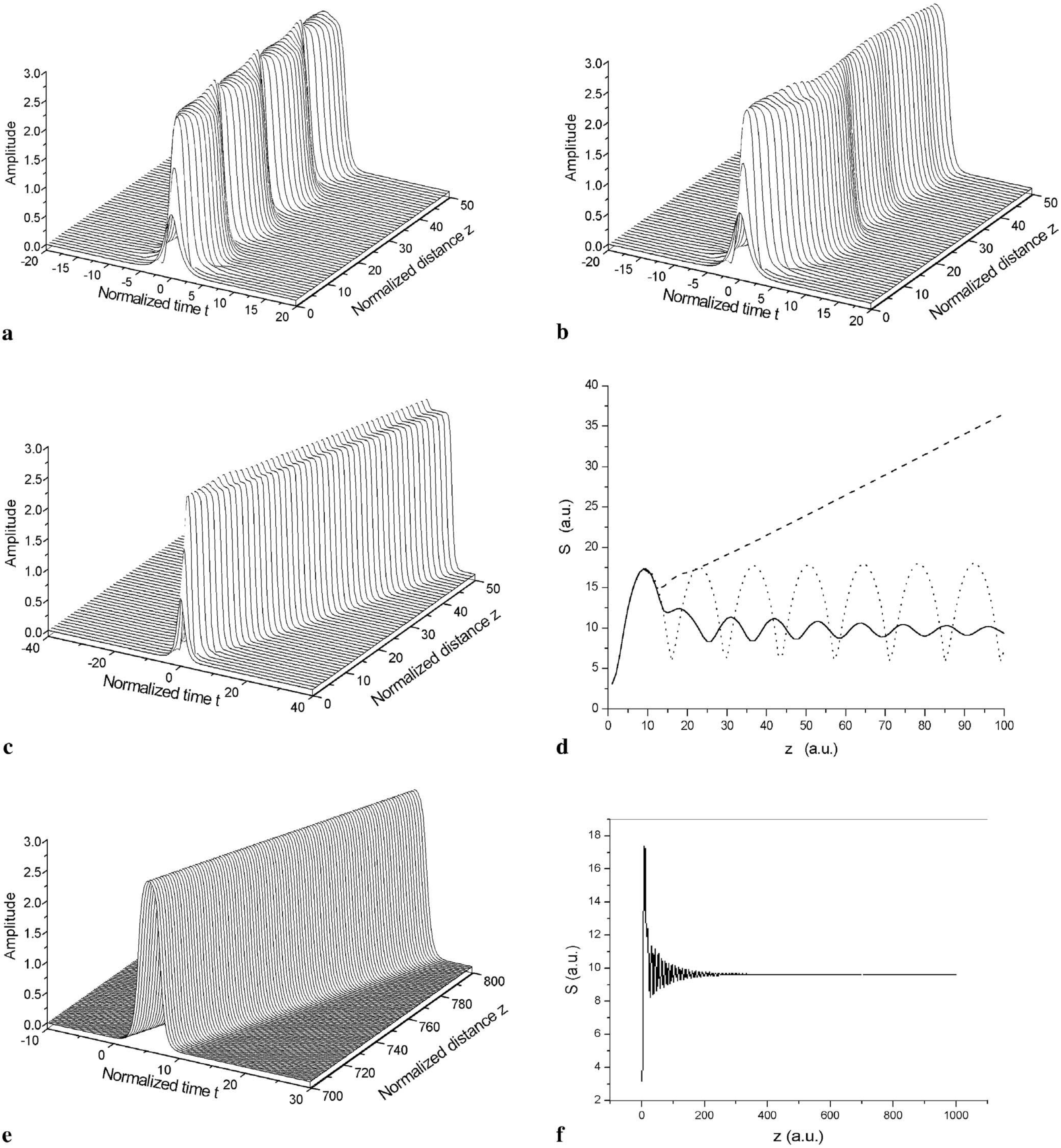

FIGURE 1 **a** Space–time plot of plain pulsating soliton, $\lambda = \kappa = 0$. **b** Space–time plot of fixed-shape soliton, $\lambda = \kappa = 0.01 + 0.01\,\mathrm{i}$. **c** Space–time plot of rectangle-like pulse, $\lambda = \kappa = 0.02 + 0.02\,\mathrm{i}$. **d** Area S under the modulus versus distance z for the cases shown in Fig. 1a, b and c. **e** Space–time plot of fixed-shape soliton between the distances of 700 and 800, $\lambda = \kappa = 0.01 + 0.01\,\mathrm{i}$. **f** Area S under the modulus versus distance z for the cases of Fig. 1b up to the distance of $z = 1000$

$\lambda = \kappa = 1.6 + 1.6\,\mathrm{i}$) the pulse will decay to zero instead of splitting to a chaotic pulse.

It should be mentioned that, although all the results shown in Fig. 2 are only within the distance of $z = 50$ or $z = 100$, the conclusions are the same for up to 10 times as long as the distance. It implies that what are shown in Fig. 2 are also the evolution characteristics of pulses for long distance.

4 Effect of the nonlinear gradient terms on the creeping soliton

The results mentioned above show that the nonlinear gradient terms can change both the pulsating and the erupting solitons into fixed-shape solitons, which are meaningful for practical use such as to realize experimentally the undis-

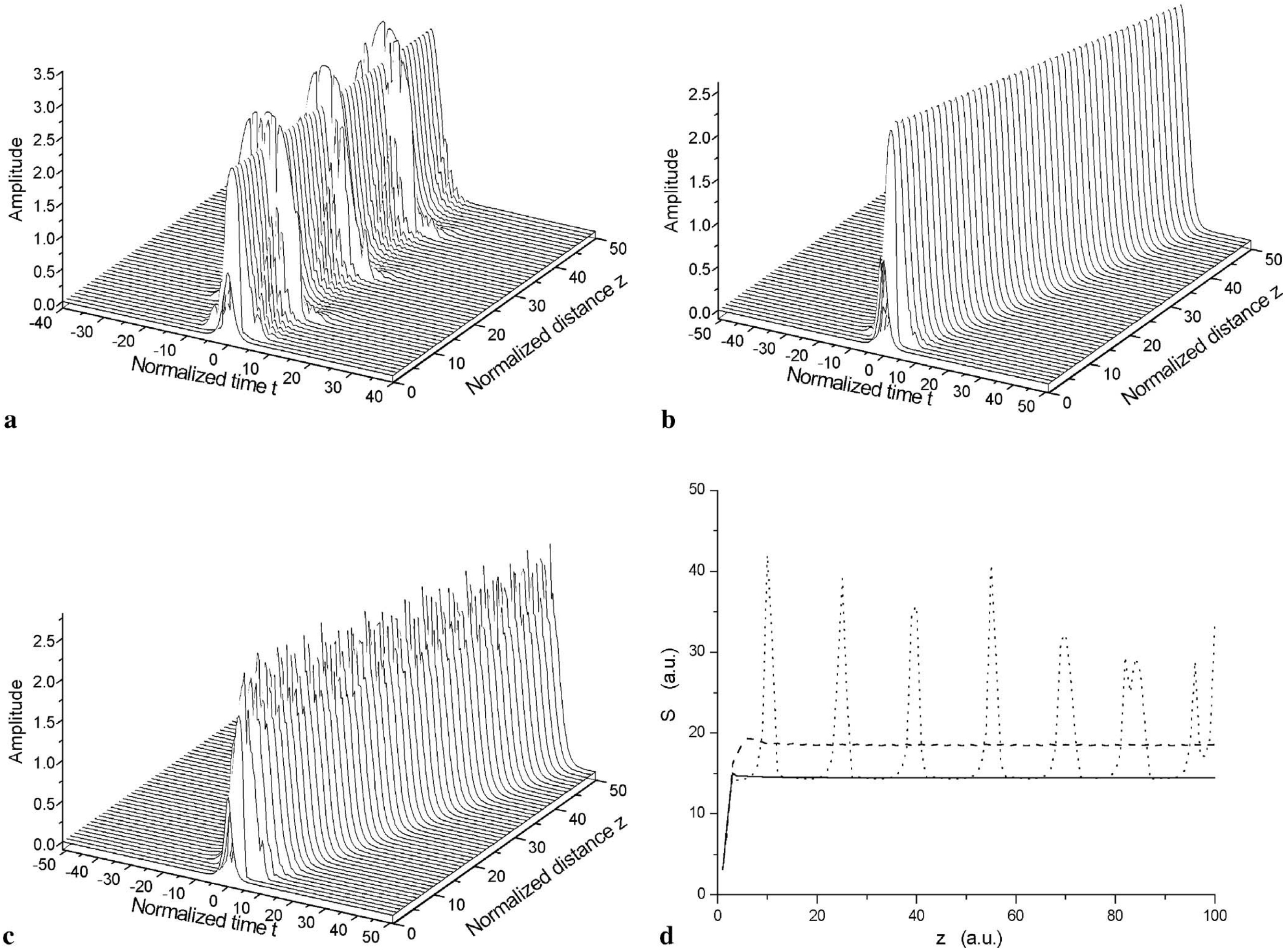

FIGURE 2 **a** Space–time plot of erupting soliton when $\lambda = \kappa = 0$. **b** Space–time plot of fixed-shape soliton when $\lambda = \kappa = 0.3 + 0.3\,\mathrm{i}$. **c** Space–time plot when $\lambda = \kappa = 0.7 + 0.7\,\mathrm{i}$. **d** Area S under the modulus versus distance z for the cases shown in Fig. 2a, b and c

torted transmission of femtosecond pulses in optical fibers. But in the following we can see that the result is rather different for the creeping soliton. Figure 3a–d show the space–time plots for different values of the nonlinear gradient terms; the other parameters are fixed as $D = 1$, $\varepsilon = 1.3$, $\delta = -0.1$, $\beta = 0.101$, $\mu = -0.3$ and $\nu = -0.101$. Figure 3a shows the creeping soliton for $\lambda = \kappa = 0$, the corresponding area as a function of the distance is shown in Fig. 4a. As reported in [19], it is a rectangular pulse with two fronts and one sink on the top. The two fronts pulsate back and forth relative to the sink asymmetrically at the two sides of the soliton. However, when the nonlinear gradient terms are involved, the evolution result will be different. Figure 3b shows the case for $\lambda = \kappa = 0.005 + 0.005\,\mathrm{i}$. It can be seen that even such small values of the nonlinear gradient terms can cause an obvious change in the behavior of the creeping soliton. Instead of the two fronts pulsating back and forth relative to the sink asymmetrically at the two sides, there is only the one on the left-hand side that pulsates back and forth relative to the sink, while the front on the right-hand side pulsates far away from the sink with a fixed velocity, which leads to the increase of its energy. In fact, the left of the solution also travels slowly towards the right but it travels less slowly than the right. The corresponding area as a function of the distance is shown in Fig. 4b. It clearly shows that the area increases with z, accompanied with a periodic oscillation caused by the breathing movement of the left front.

In addition, it is very clear that the breathing behavior of the left-hand side of the pulse in Fig. 3b is period-1 (see Fig. 4b). One important phenomenon is that for different values of the nonlinear gradient terms, the evolution period of the pulse is different. For instance, if $\lambda = \kappa = 0.012 + 0.012\,\mathrm{i}$, the breathing behavior is period-2, which can be clearly seen in Fig. 4c. The corresponding space–time evolution result is shown in Fig. 3c, which shows that the behavior is similar to that shown in Fig. 3b except that the breathing behavior is period-2. Surprisingly, apart from the breathing behavior of period-1 and period-2, we found that when $\lambda = \kappa = 0.01 + 0.01\,\mathrm{i}$ the breathing behavior of the left-hand side of the pulse will be period-3, namely, between the parameters for the behavior of period-1 and period-2 there exists a parameter for the behavior of period-3. The area as a function of the distance for period-3 is shown in Fig. 4d and its evolution plot is shown in Fig. 3d. Possibly period-3 might be a new result, since it does not belong to the route of the well-known period-doubling bifurcation.

In order to investigate how the behavior of period-3 appears, we have observed the space–time plots and correspond-

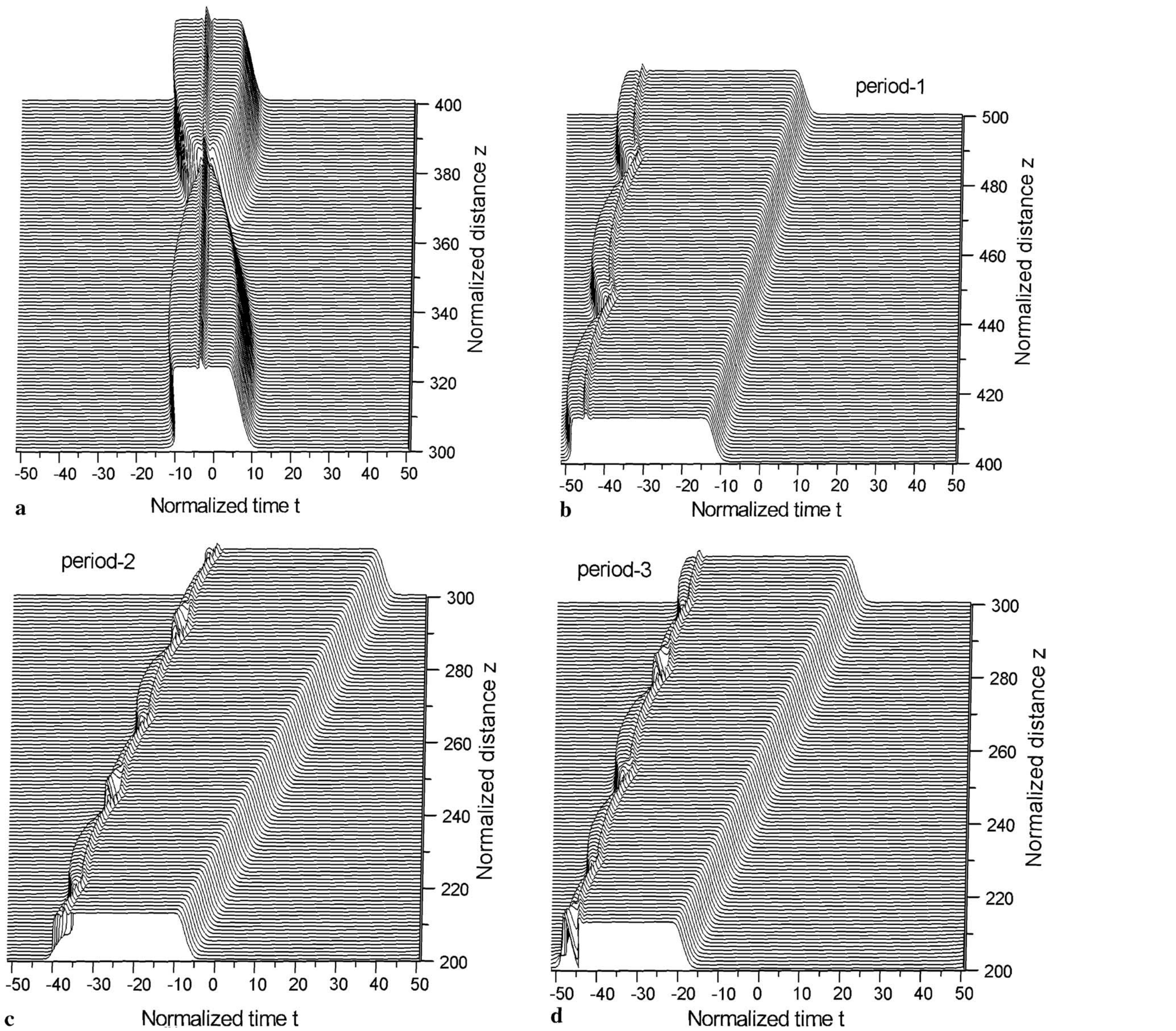

FIGURE 3 **a** Space–time plot of creeping soliton, $\lambda = \kappa = 0$. **b** Space–time plot, $\lambda = \kappa = 0.005 + 0.005\,\mathrm{i}$. **c** Space–time plot, $\lambda = \kappa = 0.012 + 0.012\,\mathrm{i}$. **d** Space–time plot, $\lambda = \kappa = 0.010 + 0.010\,\mathrm{i}$

ing area curves for all values of nonlinear gradient terms between $\lambda = \kappa = 0.005 + 0.005\,\mathrm{i}$ and $\lambda = \kappa = 0.012 + 0.012\,\mathrm{i}$ in detail. It is found that for $\lambda = \kappa = 0.006 + 0.006\,\mathrm{i}$ the behavior is also period-1, which is the same as that for $\lambda = \kappa = 0.005 + 0.005\,\mathrm{i}$. For $\lambda = \kappa = 0.007 + 0.007\,\mathrm{i}$, the behavior looks like period-5, but it is not very certain. However, for even larger values, such as $0.008 + 0.008\,\mathrm{i}$, $0.009 + 0.009\,\mathrm{i}$ and $0.011 + 0.011\,\mathrm{i}$, the evolution of pulses are not periodic any more. Their area functions increase linearly. Furthermore, if $\lambda = \kappa = 0.014 + 0.014\,\mathrm{i}$, the breathing behavior is still period-2. Unfortunately, although we have calculated so many results, we still cannot find whether there is any orderliness for these phenomena. Further results are under investigation.

It has been shown that the nonlinear gradient terms dramatically alter the behavior of the pulsating, erupting and creeping solitons. The results obtained are different from those for the breathing solutions investigated by Deissler [22]. The nonlinear gradient terms transform pulsating and erupting solitons into fixed-shape solitons. Only for the creeping soliton do the nonlinear gradient terms cause it to breathe periodically at different frequencies on one side, while the other side is caused to spread rapidly in a fixed direction.

5 Conclusion

In conclusion, we have studied the effects of the nonlinear gradient terms on three new solitons: pulsating, erupting and creeping solitons, respectively. It is found that the nonlinear gradient terms dramatically change the behavior of these solitons. For the pulsating and erupting solitons, small nonlinear gradient terms eliminate their periodicity and transform them into fixed-shape solitons. This is meaningful for practical use such as to realize experimentally the undistorted transmission of femtosecond pulses in optical fibers. If the nonlinear gradient terms increase, the pulsating soliton will be caused to spread on one side.

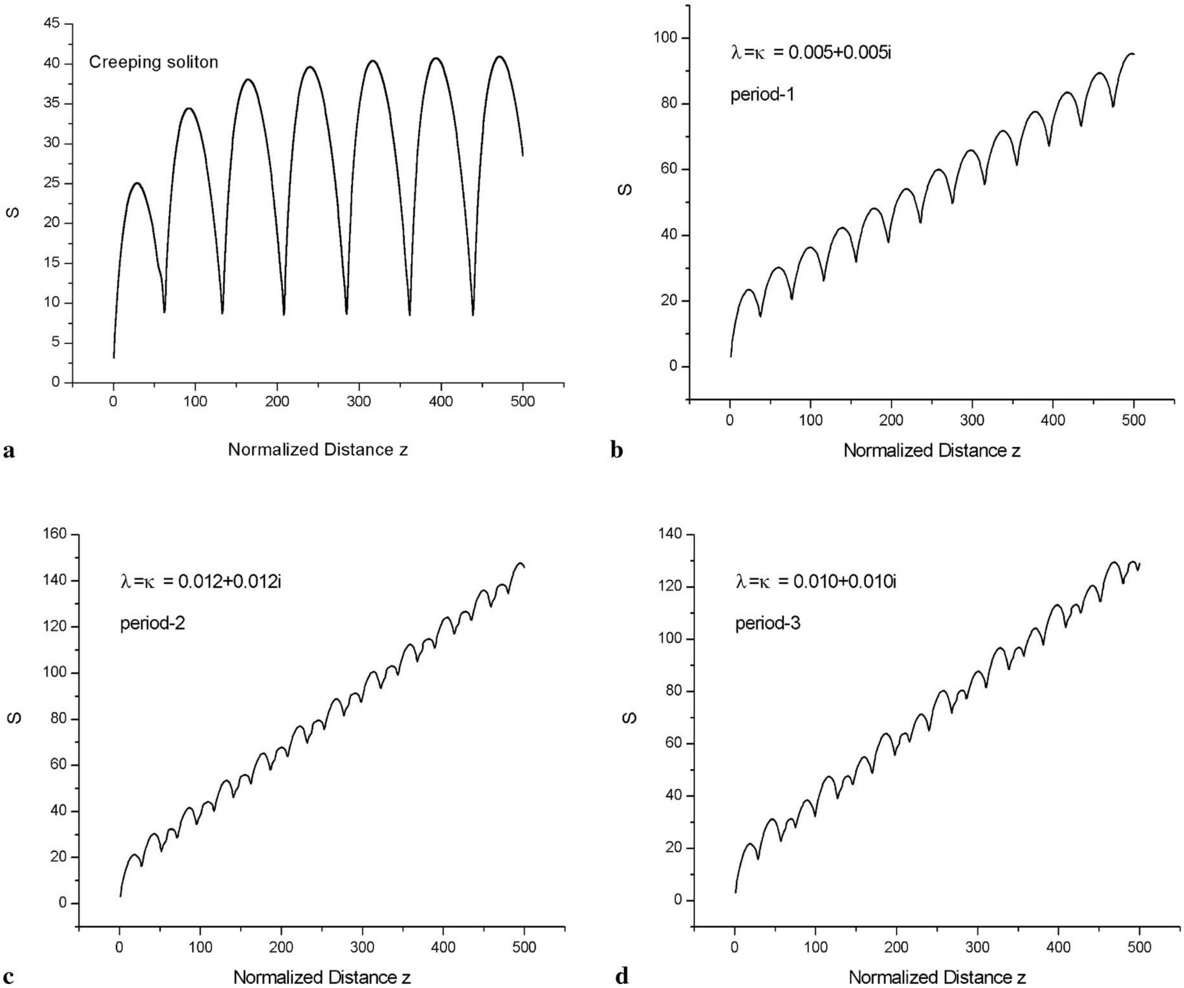

FIGURE 4 **a** Area S versus distance z for the case shown in Fig. 3a. **b** Area S versus distance z for the case shown in Fig. 3b. **c** Area S versus distance z for the case shown in Fig. 3c. **d** Area S versus distance z for the case shown in Fig. 3d

The pulsating soliton will be transformed into a pulse with a front on the right and a sink on the top. Larger nonlinear gradient terms will make the erupting soliton split on the top. For the creeping soliton, the nonlinear gradient terms will make the soliton breathe periodically at different frequencies on one side and rapidly spread on the other side.

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