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Filamentation length of powerful laser pulses

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ABSTRACT From a theoretical model for the propagation of high-power laser pulses in air over long distances, we derive analytical estimates for the length of an infrared light filament as a function of the pulse duration and beam energy. For a fixed energy per pulse, a maximum filamentation length is shown to be obtained for a specific pulse duration. These estimates are in agreement with the results of numerical simulations and measurements available in the literature.

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When a femtosecond infrared laser pulse propagates in the atmosphere, it is well known to form a light filament, i.e. a narrow structure of about 100-μm diameter covering extended distances and leaving in its wake an ionized channel [1, 2]. This phenomenon also occurs for laser pulses in the ultraviolet [3–5] and in the visible domains [6], as well as in various gases [7], transparent solids [8, 9] and liquids [10] over shorter distances. From remote sensing of the atmosphere [11] to lightning protection [12–14], many potential applications of femtosecond filamentation have been proposed, while several experimental and theoretical studies have been devoted to understand the physics underlying this long-range propagation [15–24]. Measurements in the core of a light filament are difficult because intensities in the range 10^{13}–$10^{14}\,\mathrm{W\,cm^{-2}}$ are reached locally and may damage any optical instrument. Since various nonlinear phenomena are involved, any theoretical model of the phenomenon is not tractable analytically unless very restrictive assumptions are made. So far, very few experimental data have been collected concerning the length of a filament which, however, constitutes one of the most interesting quantities for several applications. This lack of data is due to the difficulty of performing this type of measurement. First, the longest filaments are formed in a collimated beam and require a sufficiently long experimental room. Second, the filamentation length is sensitive to shot to shot fluctuations that induce variations in the starting point of the filament, i.e. in the location of the nonlinear focus of the beam. This location is extremely sensitive to any air turbulence or inhomogeneity of index [25, 26]. Third, multifilamentation generally occurs for high-power pulses; it is induced by the modulational instability of the beam inhomogeneity and makes it difficult to define the length of a single filament without following its track along the whole propagation distance.

Despite these difficulties, the filamentation length has been measured in a restricted area of the parameter space spanned by the laser wavelength, the pulse energy and the pulse duration, together with the parameters characterizing the medium. For 100-fs infrared pulses with energies of a few mJ, filaments in the meter range are formed while several tenths of meters are covered with subpicosecond pulses [1, 2, 15, 18]. For ultra-short laser pulses, the filamentation length is likely to increase with the pulse energy. On the other hand, long high-energy pulses promote avalanche ionization, which leads to breakdown. An optimal pulse duration should therefore allow a maximum filamentation length.

The aim of this paper is to present a very simple model providing analytical estimates for the length of light filaments as a function of the pulse duration and energy, and other parameters depending on the laser wavelength such as ionization rates and nonlinear susceptibilities of the medium. To derive these estimates we consider a connected filament assumed to be obtained from a single-shot experiment. While numerical results obtained under this assumption are available [16, 21], a comparison with experimental results can only be made if the variation in the filamentation length resulting from shot to shot fluctuations is averaged. Similarly, when several interacting filaments are obtained from a powerful beam, the length of each individual filament should be added before comparison with the results of the present model.

We start from a widely used paraxial model which describes the propagation of the laser pulse. The pulse propagates along the z axis and is linearly polarized. It is decomposed into a slowly varying amplitude and a carrier wave with frequency ω_0 and wavenumber $k \equiv n_0\omega_0/c$, where n_0 denotes the linear index of the medium, as $\boldsymbol{E} = \boldsymbol{e}_x \mathrm{Re}[\mathcal{E}\exp(\mathrm{i}kz - \mathrm{i}\omega_0 t_{\mathrm{lab}})]$. The scalar envelope of the electric field $\mathcal{E}(x, y, z, t)$ evolves according to the propagation equations expressed in the reference frame moving at the group velocity

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$v_g \equiv \partial\omega/\partial k|_{\omega_0}$:

$$\frac{\partial\mathcal{E}}{\partial z} = \frac{\mathrm{i}}{2k}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathcal{E} + \mathrm{i}k_0 n_2 \left[(1-f)I(t) + f\int_{-\infty}^{t} R(t-t')I(t')\mathrm{d}t'\right]\mathcal{E}(t) - \frac{\sigma}{2}(1+\mathrm{i}\omega_0\tau_c)\varrho\mathcal{E} - \frac{\beta_K}{2}I^{K-1}\mathcal{E}, \tag{1}$$

where $I \equiv |\mathcal{E}(x,y,z,t)|^2$, $\varrho(x,y,z,t)$ denotes the electron density of the plasma generated by ionization and t refers to the retarded time variable $t_{\mathrm{lab}} - z/v_g$. The first term on the right-hand side (rhs) of (1) accounts for diffraction within the transverse plane. The second and third terms on the rhs of (1) account for the Kerr effect with an instantaneous component due to the electronic response in the polarization and a delayed component, of fraction $f = 0.5$, due to stimulated molecular Raman scattering [27], merely described by an exponential response function $R(t)$ with a characteristic time of 70 fs. In air, at the laser wavelength $\lambda_0 = 800$ nm, the nonlinear index of refraction is $n_2 = 3.2\times10^{-19}\,\mathrm{cm^2\,W^{-1}}$ [28] and the critical power for self-focusing is expressed as $P_{\mathrm{cr}} \equiv \lambda_0^2/2\pi n_0 n_2 = 3.2$ GW. The fourth term on the rhs of (1) accounts for plasma absorption and defocusing. The cross section for inverse Bremsstrahlung follows the Drude model [29] and reads $\sigma = k\omega_0\tau_c/n_0^2\varrho_c \times (1+\omega_0^2\tau_c^2) = 5.1\times10^{-18}\,\mathrm{cm^2}$, where τ_c denotes the characteristic time for electron-neutral inverse bremsstrahlung: $\tau_c = 3.5\times10^{-13}$ s. In the limit $\tau_c \gg \omega_0^{-1}$, the defocusing term is expressed as $k\sigma\omega_0\tau_c\varrho \simeq k_0^2\varrho/2\varrho_c$, where ϱ_c denotes the critical plasma density $\varrho_c = 2\times10^{21}\,\mathrm{cm^{-3}}$ above which the plasma becomes opaque. The last term in (1) accounts for energy absorption due to multiphoton ionization; the coefficient β_K is related to the multiphoton ionization rate (see below).

To keep the theory analytically tractable, some physical effects commonly taken into account in this model [21, 30] have been neglected here: these effects, namely, group-velocity dispersion, space–time focusing and self-steepening are negligible in air unless sub-10-fs pulses are considered [21].

The losses due to multiphoton absorption and plasma absorption are described by the coefficients σ and β_K in the equation derived from (1) for the evolution of the pulse power $P(z,t) = \int |\mathcal{E}(\boldsymbol{r},z,t)|^2\mathrm{d}\boldsymbol{r}$:

$$\partial P/\partial z = -\beta_K\int|\mathcal{E}(\boldsymbol{r},z,t)|^{2K}\mathrm{d}\boldsymbol{r} - \sigma\int\varrho|\mathcal{E}(\boldsymbol{r},z,t)|^2\mathrm{d}\boldsymbol{r}, \tag{2}$$

where $\boldsymbol{r} \equiv (x,y)$ denotes the coordinates in the transverse diffraction plane. The generation of the plasma by multiphoton and avalanche ionization is described by the evolution equation for the electron density ϱ:

$$\partial\varrho/\partial t = \sigma_K I^K(\varrho_{\mathrm{at}} - \varrho) + \eta\varrho I - \alpha\varrho^2, \tag{3}$$

where $I \equiv |\mathcal{E}(\boldsymbol{r},z,t)|^2$. For multiphoton ionization of oxygen molecules with the potential $U_i = 12.1$ eV, $K = 8$ photons are necessary to liberate an electron. The coefficient $\sigma_K = 3.7\times10^{-97}\,\mathrm{s^{-1}\,cm^{16}\,W^{-8}}$ has been computed from Keldysh's theory [31] and is linked to β_K as $\beta_K = \sigma_K\hbar\omega_0\varrho_{\mathrm{at}}$, where $\varrho_{\mathrm{at}} = 0.2\varrho_{\mathrm{air}} = 5\times10^{18}\,\mathrm{cm^{-3}}$ denotes the density of oxygen molecules. For avalanche ionization $\eta = \sigma/U_i$. Radiative electron recombination may be added to the model with $\alpha = 5\times10^{-7}\,\mathrm{cm^3\,s^{-1}}$ [16].

For pulse durations up to several tenths of a picosecond and plasma densities below a few percent of the gas density, which is the case in our study, recombination can be neglected and (3) admits the solution:

$$\varrho = \varrho_{\mathrm{at}}\int_{-\infty}^{t}\sigma_K I^K(t') \times \exp\left(\int_{t'}^{t}[\eta I(t'') - \sigma_K I^K(t'')]\mathrm{d}t''\right)\mathrm{d}t'. \tag{4}$$

In order to make quantitative estimates of the filamentation length, we will use pulses with a Gaussian profile in time and space with duration t_p, peak intensity I_p and transverse radius $w(z,t)$. During the propagation, the pulse does not remain Gaussian but, as can be shown by the application of standard variational methods to the problem of femtosecond filamentation [19, 20, 22, 32], this approximation amounts to neglecting corrective factors of the order of unity that would be introduced in the equations below with different spatial and temporal shapes (test functions). This justifies the use of a Gaussian pulse intensity as

$$I = I_p\exp\left(-2t^2/t_p^2 - 2r^2/w^2\right). \tag{5}$$

Here the radius $w(z,t)$ is allowed to vary with the propagation distance and time in general. An evolution equation for $w(z,t)$ may be derived as reported in [24]. Below, for the sake of simplicity, we will however consider the case where $w(z)$ varies with the propagation distance only as assumed in most variational methods [19, 20, 22]. This amounts to neglecting the detailed dynamical aspect of the propagation along the temporal axis, i.e. we assume a connected quasi-steady filament with mean radius $w(z)$ and mean intensity $I_p(z)$, which leads to quantitatively correct estimates in comparison with numerically obtained or experimentally measured filamentation lengths. From (4), the electron density can be estimated as $\varrho(t,r,z) = \varrho_p(z)\varphi(t)\mathrm{e}^{-2Kr^2/w(z)^2}$, where

$$\varrho_p = \varrho_{\mathrm{at}} t_p \sigma_K I_p^K/\sqrt{2K}, \tag{6}$$

$$\varphi(t) = \int_{-\infty}^{t\sqrt{2K}/t_p}\mathrm{e}^{-u^2 + h(u,t_p) - h\left(\sqrt{2K}t/t_p,\,t_p\right)}\mathrm{d}u, \tag{7}$$

$$h(u,t_p) = \tau_{\mathrm{av}}^{-1}\mathrm{erfc}\left(u/\sqrt{K}\right) - \tau_K^{-1}\mathrm{erfc}(u), \tag{8}$$

$$\tau_{\mathrm{av}}^{-1} = t_p\eta I_p\sqrt{\pi/2}/2\,, \quad \tau_K^{-1} = t_p\sigma_K I_p^K\sqrt{\pi/2K}/2\,. \tag{9}$$

Here, the intermediate variable $u = t'\sqrt{2K}/t_p$ has been introduced and a top-hat spatial profile for the intensity distribution has been assumed in the expression of $h(u,t_p)$. The variation of the electron density proportional to $I^K \simeq \exp(-2Kr^2/w^2)$, however, has been kept in the density profile since it is correct in the limit of short pulse durations where mainly multiphoton ionization creates the plasma. The quantities τ_{av}

and τ_K denote the dimensionless ionization rates for avalanche and multiphoton ionizations, respectively.

In a filament the maximum intensity is reached when self-focusing is saturated by plasma defocusing [16, 21, 23]. This enables us to link roughly the peak intensity to the maximum electron density on the propagation axis $r = 0$ as:

$$n_2 I_\mathrm{p} \sim \varrho_\mathrm{max}/2\varrho_\mathrm{c}, \tag{10}$$

where $\varrho_\mathrm{max} = \varrho_\mathrm{p} \int_{-\infty}^{+\infty} \exp\left[-u^2 + h(u, t_\mathrm{p})\right] \mathrm{d}u$. When radiative recombination is not negligible ϱ_max is computed by direct integration of (3). From (2), we make an integration in the transverse diffraction plane and we obtain an evolution equation for the power of the pulse $P(z,t) = \int_0^{+\infty} I_\mathrm{p} \mathrm{e}^{-2t^2/t_\mathrm{p}^2 - 2r^2/w^2} \times 2\pi r \mathrm{d}r = I_\mathrm{p} \frac{\pi}{2} w^2 \mathrm{e}^{-2t^2/t_\mathrm{p}^2}$:

$$\frac{\partial P}{\partial z} = -\frac{\beta_K}{K} I_\mathrm{p}(z)^{K-1} \mathrm{e}^{-2(K-1)\tau^2} P - \frac{\sigma}{K+1} \varrho_\mathrm{p}(z) \varphi(\tau) P, \tag{11}$$

where $\tau = t/t_\mathrm{p}$. At this stage, the pulse radius $w(z)$ and intensity $I_\mathrm{p}(z)$ can exhibit small oscillations along the propagation axis. We now consider the averaged intensity in the filament over one oscillation cycle $I_\mathrm{p} = \langle I_\mathrm{p}(z) \rangle$. Its level is determined by the saturation condition (10) and may be considered as constant. This amounts to assuming that the variations of the quantity P/w^2 are slower than the averaged power dissipation, which continuously decreases along z. A direct integration of (11) from the nonlinear focus where the filamentation starts, at $z = 0$, yields $P(z,t) = P_0 \exp[-z/L_\mathrm{d}(\tau)]$, where $P_0 \equiv P(z = 0, t) = P_\mathrm{in} \exp(-2\tau^2)$ and

$$L_\mathrm{d}(\tau) = \left[\frac{\beta_K}{K} I_\mathrm{p}^{K-1} \mathrm{e}^{-2(K-1)\tau^2} + \frac{\sigma}{K+1} \varrho_\mathrm{p} \varphi(\tau)\right]^{-1}. \tag{12}$$

When the power contained in a specific temporal slice becomes smaller than the critical power for self-focusing, this slice cannot self-focus again; it can only diffract. This occurs beyond the distance

$$L(\tau) = L_\mathrm{d}(\tau) \left(\log \frac{P_\mathrm{in}}{P_\mathrm{cr}} - 2\tau^2\right). \tag{13}$$

The length, L_fil, of a homogeneous light filament may be defined as the minimum distance for which energy losses have decreased the peak power of the pulse from its initial value to the critical power for self-focusing. Beyond this distance, the whole pulse should be unable to refocus by self-focusing. We therefore obtain an expression for L_fil by introducing in (13) the temporal slices for which the losses due to multiphoton absorption and plasma absorption are maximum; for multiphoton absorption, this slice corresponds exactly to $t = 0$. For plasma absorption, it corresponds to a neighboring slice very close to $t = 0$ where the maximum density is reached, due to the steep growth of the electron density generated by multiphoton ionization. This approximation lowers the filamentation length. We introduce the pulse energy and duration through $P_\mathrm{in} = E_\mathrm{in}\sqrt{2}/t_\mathrm{p}\sqrt{\pi}$:

$$L_\mathrm{fil} = \left(\frac{\beta_K}{K} I_\mathrm{p}^{K-1} + \frac{\sigma}{K+1} \varrho_\mathrm{max}\right)^{-1} \times \log \frac{E_\mathrm{in}\sqrt{2}}{t_\mathrm{p}\sqrt{\pi} P_\mathrm{cr}}, \tag{14}$$

where ϱ_max and I_p are linked by (10). Our definition for the filament length L_fil, which relies on the assumption of a saturated intensity in the filament, should be relaxed when a comparison is made with the length of a disconnected filament such as, for example, those currently obtained in numerical simulations [21]. In the latter case, only the filament parts where the density exceeds a given threshold should be added and compared to the length given by (14). Figure 1a shows the filamentation length for pulses with input energies $E_\mathrm{in} = 10$, 100 and 300 mJ as a function of the pulse duration. The three maxima on these curves indicate for each energy the largest filamentation lengths 3.16, 13.8 and 29.6 m obtained for the specific pulse durations of 500 fs, 5.6 ps and 15.8 ps, respectively.

For pulses with a few hundreds of femtoseconds duration and a few mJ energy, the filamentation length is in the meter range, in agreement with various experimental data [1, 2, 15, 18] and numerical results [16, 21, 23]. Figure 1b shows the intensity I_p and the electron density ϱ_max entering (14) and satisfying (10), as a function of the pulse duration. These quantities are found to be independent of the pulse energy and vary from less than $10^{14}\,\mathrm{W\,cm^{-2}}$ and a few $10^{16}\,\mathrm{e^-\,cm^{-3}}$ to $10^{12}\,\mathrm{W\,cm^{-2}}$ and a few $10^{14}\,\mathrm{e^-\,cm^{-3}}$ when the pulse duration varies from 10 fs to 100 ps. Longer pulses with more energy therefore promote less intense filaments. This phenomenon mainly results from the nonlinear saturation of the self-focusing by optical field ionization accounted for by (10). When multiphoton ionization prevails over avalanche ionization, which is the case for sufficiently short pulses, the maximum electron density may be estimated as $\varrho_\mathrm{max} \sim \sigma_K I_\mathrm{p}^K t_\mathrm{p} \varrho_\mathrm{at}$ and (10) yields $n_2 \sim \sigma_K I_\mathrm{p}^{K-1} t_\mathrm{p} \varrho_\mathrm{at}/2\varrho_\mathrm{c}$. Therefore, the saturation in the filament occurs at lower intensities for longer pulses with more energy. For the same energy, longer pulse filaments propagate longer because they result in less intense filaments and

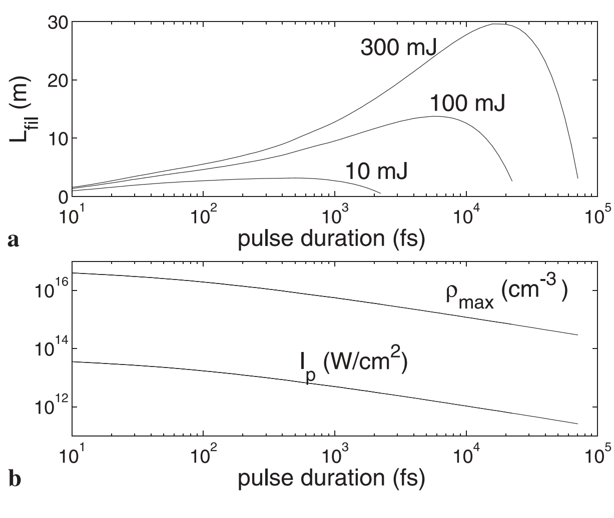

FIGURE 1 **a** Filamentation length as a function of the pulse duration for $E_\mathrm{in} = 10$, 100 and 300 mJ. **b** Maximum intensity and electron density in the filament as a function of the pulse duration

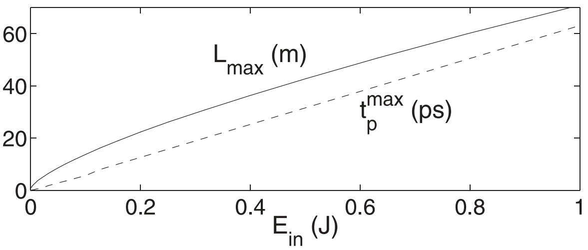

FIGURE 2 Maximum filamentation length (*continuous line*) and optimal pulse duration corresponding to the maximum length as a function of the pulse energy

therefore the power-dissipation processes, i.e. multiphoton and plasma absorption, are weaker. Finally, beyond a certain pulse width, there is a steep drop in the filament length because of an increased avalanche ionization which induces a faster power dissipation.

Figure 2 shows the maximum filamentation length (continuous line) obtained by solving (14) and $\mathrm{d}L/\mathrm{d}t_\mathrm{p} = 0$ as a function of the pulse energy. The corresponding optimal pulse duration is shown as a dashed line. An increase of the filamentation length may be obtained for longer pulses with more energy if the pulse duration and energy can be increased correspondingly. The present analytical model is in agreement with numerical results by Mlejnek et al., according to which a powerful laser beam constitutes a background energy reservoir that can form and feed filaments with individual powers above P_cr, until multiphoton absorption consumes the initial pulse energy [17].

Equation (14) for the filamentation length relies on several simplifications of the physics of light filaments. These approximations amount to replacing a dynamic process by a mean steady propagation. This allows the derivation of a predictive model that is in good agreement with experimental measurements and might trigger further experiments on long-range propagation of short laser pulses.

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