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Singular Hopf Bifurcation in Systems with Fast and Slow Variables

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Summary. We study a general nonlinear ODE system with fast and slow variables, i.e., some of the derivatives are multiplied by a small parameter. The system depends on an additional bifurcation parameter. We derive a normal form for this system, valid close to equilibria where certain conditions on the derivatives hold. The most important condition concerns the presence of eigenvalues with singular imaginary parts, by which we mean that their imaginary part grows without bound as the small parameter tends to zero. We give a simple criterion to test for the possible presence of equilibria satisfying this condition. Using a center manifold reduction, we show the existence of Hopf bifurcation points, originating from the interaction of fast and slow variables, and we determine their nature. We apply the theory, developed here, to two examples: an extended Bonhoeffer–van der Pol system and a predator-prey model. Our theory is in good agreement with the numerical continuation experiments we carried out for the examples.

1. Introduction

In this paper we study the behavior of a class of singularly perturbed systems of ordinary differential equations close to a Hopf bifurcation. This class consists of sufficiently smooth autonomous systems of the form

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y, a), \qquad 0 < \varepsilon \ll 1, \\ \dot{y} &= g(x, y, a), \end{aligned} \tag{1}$$

with fast variables $x \in \mathbb{R}^m$, slow variables $y \in \mathbb{R}^n$, and a bifurcation parameter $a \in \mathbb{R}$. Because ε is a small parameter, the x-variables change significantly faster than the y-variables.

We are in particular interested in the case where oscillations emerge as a result of coupling of the fast and slow variables. A theory for oscillations in singularly perturbed

systems of arbitrary fast and slow dimensions has been established by Fenichel (1983). He distinguishes between 'fast' and 'slow' oscillations. Assuming the existence of oscillations for either the fast or the slow variables in the singular limit $\varepsilon = 0$ (in a suitable reduced system), he shows that such oscillations survive the coupling for small ε with, respectively, the slow and the fast dynamics. However, he does not consider the case where oscillations *originate* from the coupling of fast and slow motions.

Baer & Erneux (1986) develop a theory for what they call *singular Hopf bifurcation* in the case of one fast and one slow variable (that is, $m = n = 1$ in system (1)). Their two-dimensional setting implies in a trivial way that both timescales are involved in the bifurcation mechanism. Here we extend their theory to higher-dimensional systems. In order to avoid some technical difficulties, we first focus on the case of a single fast variable and an arbitrary number of slow variables. Later on we consider the case where there is also an arbitrary number of fast variables.

Hopf bifurcation. The theory of Hopf bifurcation is a tool to investigate the onset of oscillatory behavior in a dynamical system; see for example Marsden & McCracken (1976). It predicts that (under generic conditions) a small periodic oscillation is born when a pair of eigenvalues crosses the imaginary axis (with nonzero speed) at an equilibrium point. This small periodic oscillation may evolve to more complex oscillatory behavior, but that is beyond the scope of Hopf bifurcation theory.

The characteristic feature of a singular Hopf bifurcation, as studied in this paper, is that both fast and slow variables are involved actively in the bifurcation mechanism. This implies that the imaginary part of the crossing eigenvalues grows without bound as $\varepsilon \to 0$; see Section 2 for details. We use this growth behavior as the starting point of our analysis for the case of a single fast variable.

Surprisingly, it turns out to be possible to carry out a fairly complete singular Hopf bifurcation analysis for system (1). We need only one special assumption and a few generic assumptions. The special assumption leads to a simple criterion for locating singular Hopf bifurcation points in a given system of the form (1). We give relatively simple expressions for a normal form, valid close to these bifurcation points, and for determining the stability type of singular Hopf points. The presence of the small parameter ε is the main reason for our (approximate) formulas to be much simpler than standard counterparts for nonsingularly perturbed systems.

We emphasize that, because both timescales are actively involved, it is not possible to study systems of the class (1) close to a singular Hopf bifurcation through a standard dimension reduction method (like the quasi–steady-state method popular among mathematical biologists; see for example the textbooks by Segel (1987) and Murray (1989)).

Organization. The organization of this paper is as follows. In sections 2, 3, and 4 we consider the case of a single fast variable (that is, $m = 1$ in (1)). First, in Section 2, we study a family of matrices, parametrized by ε. For this family we establish the conditions for eigenvalues with singular imaginary parts (that is, eigenvalues whose imaginary parts grow without bound as $\varepsilon \to 0$). Then we apply this result to Jacobian matrices for system (1). We may expect singular Hopf bifurcations to occur close to equilibria where $\partial f/\partial x$ vanishes, if an additional (open) condition holds. In Section 3 we study system (1),

starting at a point where these criteria, and a few additional generic conditions, hold. We derive a normal form to analyze the system in the vicinity of the starting point. This is the first main result of the paper. We do not use a standard method, like for example averaging or a Lie algebra approach, to derive the normal form. Instead we exploit the singular nature of the system, and other characteristics that we meet in the course of the construction, in order to obtain a sequence of explicit scalings and transformations. The normal form has the structure (see (5))

$$\dot{u} = v + \tfrac{1}{2}u^2 + \mathcal{O}(\sqrt{\varepsilon}), \qquad \dot{v} = -u + \mathcal{O}(\sqrt{\varepsilon}), \qquad \dot{w} = \mathcal{O}(\sqrt{\varepsilon}),$$

with u and v scalars and w a vector. In Theorem 1 we give explicit expressions for the $\mathcal{O}(\sqrt{\varepsilon})$ terms, which are derived from certain derivatives at the starting point. The normal form contains four groups of coefficients: a scalar coefficient describing the nonlinear behavior of the fast subsystem, two vectors describing the coupling between the fast (u, v) and the slow w subsystem, and a square matrix describing the linear behavior of the slow system. In the case of two slow variables, all four coefficients are scalars. In the case of only one slow variable, there is no w-system, and we are left with only one (scalar) coefficient. In the normal form, the original parameter a is replaced by a scaled parameter α, which only occurs in the $\mathcal{O}(\sqrt{\varepsilon})$ part of the equation for $\dot{u}$.

In Section 4 we carry out the actual bifurcation analysis. We start from the above normal form, under the additional assumption that the linearized w-equations are hyperbolic. We perform a center manifold reduction of the form $w = \Phi(u, v)$. The dynamics on the two-dimensional center manifold determines the type of Hopf bifurcation. Using standard methods for two-dimensional Hopf bifurcation, we compute an asymptotic approximation for the Hopf bifurcation point (which is close to, but does not necessarily coincide with, the starting point).

The theory, developed for the case of a single fast variable, can be extended without much difficulty to the case of multiple fast variables. In Section 5 we discuss generalizations of the results of sections 2, 3, and 4 in this direction. Some technical difficulties arise from the fact that we have to deal with matrices instead of scalars in the fast subsystem. These complications are mainly of a computational nature, and we omit the details. The additional fast variables lead to a number of fast, $\mathcal{O}(1/\sqrt{\varepsilon})$, equations in the normal form; see (17).

In Section 6 we apply our theory to two examples: an extended Bonhoeffer–van der Pol system and a predator-prey model. We compare our asymptotic predictions to numerical results, obtained with the continuation package LOCBIF. In the appendix we derive one of the transformations used in the construction of the normal form. Although this derivation is interesting in itself, we did not include it in the main text in order to prevent an interruption of the main exposition.

Related results. Our study comprises the case of one fast variable and one slow variable (that is, $m = n = 1$ in system (1)), which previously has been studied by Baer & Erneux (1986). By a local analysis in the vicinity of the bifurcation point, they show how a small, harmonically modulated, periodic solution becomes progressively pulsating as the parameter a deviates from criticality. They also explain the sudden appearance of a large amplitude relaxation oscillation, which is associated with so-called canard limit cycles; see also Callot, Diener, & Diener (1978), Eckhaus (1983), and Dumortier &

Roussarie (1995). Diener (1984) gives a nice informal introduction to canards from the point of view of Nonstandard Analysis. Here we consider only the Hopf bifurcation analysis itself. The analysis farther from the Hopf bifurcation point, for the higher-dimensional case, is the subject of a sequel to this paper. For preliminary results we refer to Braaksma (1993). It turns out that the behavior in systems with more than one slow variable is very different from the two-dimensional case, and a number of essentially different bifurcation scenarios apply. As the topological restrictions imposed by the plane are removed, more complex oscillations and other essentially higher-dimensional phenomena become possible. A striking difference is also that the transition to relaxation oscillations tends to be much smoother in higher dimensions than in the two-dimensional ('canard') case, where it is exponentially fast.

Il'yashenko gives a classification for 'typical' systems with one fast and two slow variables; see Arnol'd, Aframovich, Il'yashenko, & Shil'nikov (1994), chapter 4. He distinguishes three situations. In only one of these situations is there an equilibrium point present, which is nondegenerate when the system is projected on the slow surface. Our singular Hopf bifurcation study is not contained in his results, because the equilibrium is degenerate after projection.

Often the dynamics for systems with fast and slow variables can be studied effectively from a reduction to a (hyperbolically attracting) subsystem consisting of slow variables only. This approach is discussed in detail by Takens (1975) and Fenichel (1979). In our case the reduction would lead to the n-dimensional subsystem $\dot{y} = g(x, y, a)$ on the normally hyperbolic manifold $f(x, y, a) = 0, \partial f/\partial x < 0$. Takens (1975) calls this a *constrained system* when f is the gradient of some function, which is a trivial condition for scalar functions. He gives a topological classification for several generic types of constrained systems. From this topological viewpoint, a singular Hopf bifurcation scenario is described by the situation depicted in Figure 1. The value of a where the equilibrium drops over the edge (case b) $\partial f/\partial x = 0$ of the manifold $f = 0$ corresponds to the singular Hopf bifurcation value. In the terminology of Takens, case b is called a degenerate funnel. It is not generic when considering isolated systems, but it becomes generic for one-parameter *families* of constrained equations. This topological description, however, is too rough to study the singular Hopf bifurcation in more detail. It excludes the direction transverse to the constraining manifold, and this is just the direction in which limit cycles are found.

Applications. Systems with fast and slow variables are used in many fields of applications. For example, such systems arise naturally when a model includes processes that evolve on different timescales; see Grasman (1987). We mention a few examples.

Models for nerve conduction often contain one variable describing a membrane potential and a number of variables describing concentrations of, for example, Na^+, K^+, and Ca^{2+} ions. The potential typically varies much faster than the ionic concentrations. Bertram, Butte, Kiemel, & Sherman (1995) give a mathematical classification of such models from the perspective of bursting.

Other examples of physiological models with fast and slow variables can be found in, for example, Segel (1987), who includes a discussion of models for the *cAMP* signalling system of *D.Discoideum* amoebae; Chernavskii, Palamarchuk, Polezhaev, Solyanik, & Burlakova (1977), who study a model describing lipid peroxidation in the cell division

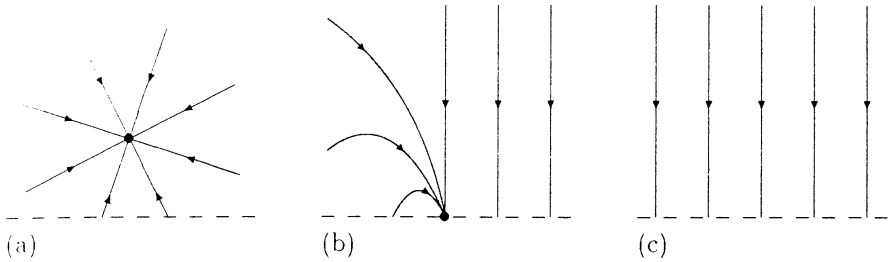

Fig. 1. Topological description of a singular Hopf bifurcation, viewed as a constrained system. Shown is a sketch of the stable part of the slow manifold. Its boundary is represented by dashed lines. The dot in the left two pictures represents an equilibrium point, which has fallen 'over the edge' of the stable manifold in the right picture.

process; and Murray (1989), who gives many examples that are, or can be, scaled to such a form.

Chemical models often involve fast and slow reaction processes. See for example Petrov, Scott, & Showalter (1992), who carry out a numerical study of a model equation displaying singular Hopf characteristics. Koper & Gaspard (1992) analyze a three-dimensional model for an electrochemical oscillator, which is an extension of an earlier two-dimensional model due to Koper & Sluyters (1991).

Ecological models describing the interaction of a number of species may exhibit singular Hopf bifurcations, for example if the life cycles of some species are short compared to others. In Section 6 we discuss an example. A classical field of application, which dates back to van der Pol (1926), is the field of semiconductors. In a completely different field, Cugno & Montrucchio (1983) study a macroeconomic model for business cycles where one of the parameters (the accelerator, or rather, its inverse) can be considered a small parameter. In all these examples, singular Hopf bifurcations occur at parameter values relevant to the study of the underlying real system.

A different kind of application concerns the numerical study of bifurcations. Especially in systems depending on many parameters it is useful to have an idea of where to look for interesting behavior, as this often tends to be confined to narrow regions. Moreover, singularly perturbed systems are notorious for numerical difficulties, due to stiffness problems caused by the small parameter. With our analysis it is both possible to transform a system to a less stiff form (the normal form) and to find good starting points in order to search for oscillatory behavior. Moreover, it might even be useful to introduce an artificial small parameter in a system that in itself is not singularly perturbed, in order to provide a starting point for a Hopf bifurcation analysis, and then eliminate this parameter again by increasing its value to unity via a numerical continuation method. This approach is known as the *homotopy method*.

2. Eigenvalues with Singular Imaginary Parts

In this section we develop necessary and sufficient conditions for a family of matrices to have eigenvalues with singular imaginary parts. Although the derivation of these

conditions is essentially an exercise in linear algebra, we include it in full detail because our further analysis depends on Proposition 1. We apply this result to (1) in order to obtain a criterion for singular Hopf bifurcation.

First, we give a precise definition of an eigenvalue with singular imaginary part.

Definition 1. Let $\mathcal{L}_\varepsilon$ be a smooth family of square matrices, defined for $0 < \varepsilon < \varepsilon_0$ with ε_0 some fixed (small, positive) number. We call the eigenvalue $\lambda = \lambda(\varepsilon)$, $\lambda = \sigma + \mathrm{i}\Omega$ of $\mathcal{L}_\varepsilon$ an *eigenvalue with singular imaginary part* iff $\sigma = \sigma(\varepsilon) = \mathcal{O}(1)$ and

$$\Omega = \Omega(\varepsilon) \to \infty \qquad \text{as } \varepsilon \to 0.$$

On the other hand, we call an eigenvalue $\lambda = \lambda(\varepsilon)$ *regular* if $\lambda = \mathcal{O}(1)$ as $\varepsilon \to 0$.

Our main goal in this section is to prove the following:

Proposition 1. *Consider the family*

$$\mathcal{L}_\varepsilon = \begin{pmatrix} \varepsilon^{-1}A & \varepsilon^{-1}B \\ C & D \end{pmatrix}, \qquad 0 < \varepsilon < \varepsilon_0, \tag{2}$$

of $(n+1)$-dimensional square matrices with $\varepsilon_0 > 0$ some fixed number and $A \in \mathbb{R}$, $B \in \mathbb{R} \times \mathbb{R}^n$, $C \in \mathbb{R}^n \times \mathbb{R}$, $D \in \mathbb{R}^n \times \mathbb{R}^n$. Let $\lambda_i = \lambda_i(\varepsilon)$, $1 \le i \le n+1$, be the eigenvalues of $\mathcal{L}_\varepsilon$.

Then $\mathcal{L}_\varepsilon$ has exactly two eigenvalues with singular imaginary part and $n-1$ regular eigenvalues if, and only if, the following two conditions are satisfied:

[S1] $A = 0$,

[S2] $BC < 0$.

Note that $\mathcal{L}_\varepsilon$ has, in general, at least one eigenvalue with $|\lambda_i| \to \infty$ as $\varepsilon \to 0$, since $\prod_{i=1}^{n+1} \lambda_i = \det(\mathcal{L}_\varepsilon) = \mathcal{O}(1/\varepsilon)$.

Necessity of the conditions. Let us first assume that $\mathcal{L}_\varepsilon$ has two (complex conjugate) eigenvalues λ_n, $\lambda_{n+1} = \sigma \pm \mathrm{i}\Omega$ with singular imaginary part and $n-1$ regular eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Then, using the fact that the sum of the eigenvalues of a matrix equals its trace, we find (note that the singular imaginary parts $\pm\mathrm{i}\Omega$ cancel)

$$2\sigma + \sum_{i=1}^{n-1} \lambda_i = \varepsilon^{-1}A + \operatorname{trace}(D).$$

This equation can only be satisfied if $A = \mathcal{O}(\varepsilon)$, which gives us condition [S1]. Next, using the formula

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \lambda_i \lambda_j = \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} L_{ii}L_{jj} - L_{ij}L_{ji},$$

for the sum of pairwise products of the eigenvalues of $\mathcal{L}_\varepsilon = (L_{ij})$, we find

$$\Omega^2 + 2\sigma \sum_{i=1}^{n-1} \lambda_i + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \lambda_i \lambda_j = \varepsilon^{-1}A \cdot \operatorname{trace}(D) - \varepsilon^{-1}BC + \mathcal{O}(1),$$

or $\varepsilon\Omega^2 + BC = \mathcal{O}(\varepsilon)$, since we already know that $A = \mathcal{O}(\varepsilon)$. Since by assumption $\Omega \to \infty$ as $\varepsilon \to 0$, the above equation can only be satisfied if $BC < 0$, which yields condition [S2]. Thus we obtain, introducing $\omega = \sqrt{-BC}$, that $\Omega = \omega/\sqrt{\varepsilon} + \mathcal{O}(\sqrt{\varepsilon})$. This completes the proof that the above conditions [S1] and [S2] are necessary for eigenvalues with singular imaginary parts.

Sufficiency of the conditions. Next we show that the above conditions are not only necessary, but also sufficient for the existence of eigenvalues with singular imaginary parts. Let $\mathcal{L}_\varepsilon$ be a family of matrices of the form (2). Assume that $\mathcal{L}_\varepsilon$ satisfies both conditions [S1] and [S2], and let $\omega^2 = -BC$. Then the characteristic polynomial $p(\lambda) = \det(\mathcal{L}_\varepsilon - \lambda I)$ is given by

$$p(\lambda) = (-\lambda)^{n+1} + c_n(-\lambda)^n + \varepsilon^{-1}\sum_{i=0}^{n-1} c_i(-\lambda)^i,$$

with

$$\begin{aligned} c_n &= \operatorname{trace}(\mathcal{L}_\varepsilon) = \mathcal{O}(1), \\ c_{n-1} &= \omega^2 + \mathcal{O}(\varepsilon), \\ c_i &= \mathcal{O}(1), \qquad i = 0, \ldots, n-2. \end{aligned}$$

If we multiply $p(\lambda)$ by ε, we obtain $\varepsilon p(\lambda) = \sum_{i=0}^{n-1} c_i(-\lambda)^i + \mathcal{O}(\varepsilon)$, which in the limit $\varepsilon \to 0$ becomes the $(n-1)$-th order polynomial $\sum_{i=0}^{n-1} c_i(-\lambda)^i$. The roots of the latter polynomial approximate (for small values of ε) the roots of $p(\lambda)$ corresponding to the $n-1$ regular eigenvalues of $\mathcal{L}_\varepsilon$.

On the other hand, if we multiply $p(\lambda)$ by $(\sqrt{\varepsilon})^{n+1}$ and substitute $\lambda = \Lambda/\sqrt{\varepsilon}$, we obtain

$$(\sqrt{\varepsilon})^{n+1} p(\Lambda/\sqrt{\varepsilon}) = (-\Lambda)^{n+1} + \omega^2(-\Lambda)^{n-1} + \mathcal{O}(\sqrt{\varepsilon}).$$

In the limit $\varepsilon \to 0$, we find $n-1$ vanishing roots and two nonvanishing roots $\Lambda = \pm i\omega$. For small ε, the nontrivial roots are perturbed to $\Lambda = \mathcal{O}(\sqrt{\varepsilon}) \pm i(\omega + \mathcal{O}(\varepsilon))$. They correspond to singular eigenvalues $\lambda = \mathcal{O}(1) \pm i(\omega/\sqrt{\varepsilon} + \mathcal{O}(\sqrt{\varepsilon}))$ of $\mathcal{L}_\varepsilon$.

Thus we have shown that $\mathcal{L}_\varepsilon$ has $n-1$ regular eigenvalues and two singular eigenvalues if conditions [S1] and [S2] hold.

Remark. The above result can be extended somewhat by allowing A, B, C, and D to depend on ε (in a regular way). Condition [S1] then changes to $A = \mathcal{O}(\varepsilon)$. This extension is useful when dealing with ε-dependent right-hand sides in (1).

Equilibria with eigenvalues with singular imaginary parts. Let system (1) have an equilibrium $(x, y) = (\bar{x}, \bar{y})$ for the parameter value $a = \bar{a}$. Then we say that this equilibrium has eigenvalues with singular imaginary parts iff the Jacobian matrix

$$\mathcal{L}_\varepsilon = \begin{pmatrix} \varepsilon^{-1}\bar{f}_x & \varepsilon^{-1}\bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}, \tag{3}$$

evaluated at the given equilibrium, has eigenvalues with singular imaginary parts in the sense of the above definition. Here we use bars to denote derivatives, evaluated at the given equilibrium. For example, $\bar{f}_x$ denotes $\partial f/\partial x(\bar{x}, \bar{y}, \bar{a})$; the other notations are analogous. From conditions [S1] and [S2] we know that $(\bar{x}, \bar{y})$ has eigenvalues with singular imaginary parts iff $\bar{f}_x = 0$, while $\bar{f}_y \bar{g}_x < 0$.

In Section 3 we take this approach as the starting point for the construction of a normal form. The $\mathcal{O}(1/\sqrt{\varepsilon})$ size of the singular imaginary parts implies that the characteristic timescale for oscillations in the corresponding eigenspace is also $\mathcal{O}(1/\sqrt{\varepsilon})$. This observation agrees with the timescaling in the normal form construction of Section 3.

3. Construction of a Normal Form

3.1. Statement of the Result

In this section we derive a normal form for system (1), with a single fast variable, close to a singular Hopf bifurcation. The precise result is as follows:

Theorem 1. *Let U be an open subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$, and let $f: U \to \mathbb{R}$, $g: U \to \mathbb{R}^n$ be sufficiently smooth functions. Let $(\bar{x}, \bar{y}, \bar{a}) \in U$ be a point where the following conditions on f, g, and their derivatives hold:*

[N1] $\bar{f} = 0,$

[N2] $\bar{g} = 0,$

[N3] $\det \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix} \neq 0,$

[N4] $\bar{f}_x = 0,$

[N5] $\bar{f}_y \bar{g}_x < 0,$

[N6] $\bar{f}_{xx} \neq 0,$

[N7] $-\begin{pmatrix} \bar{f}_{xx} & \bar{f}_{xy} \end{pmatrix} \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1} \begin{pmatrix} \bar{f}_a \\ \bar{g}_a \end{pmatrix} + \bar{f}_{xa} \neq 0.$

(As before, we use bars to denote the values of expressions, evaluated at the point $(\bar{x}, \bar{y}, \bar{a})$.)

Then the system

$$\begin{aligned} \varepsilon \frac{dx}{dt} &= f(x, y, a), \qquad 0 < \varepsilon \ll 1, \\ \frac{dy}{dt} &= g(x, y, a), \end{aligned} \tag{4}$$

can be put in the following normal form:

$$\begin{aligned} \dot{u} &= v + \tfrac{1}{2}u^2 + \delta\, F(u, w, \alpha) + \mathcal{O}(\delta^2), \\ \dot{v} &= -u + \mathcal{O}(\delta^2), \\ \dot{w} &= \delta\, H(u, w) + \mathcal{O}(\delta^2), \end{aligned} \tag{5}$$

with $\delta = \mathcal{O}(\sqrt{\varepsilon})$ *and* $u, v, \alpha \in \mathbb{R}, w \in \mathbb{R}^{n-1}$. *The dots denote differentiation with respect to the slow time variable* $\tau = \delta^{-1}t$. *The functions* F *and* H *have the structure*

$$F = \alpha u + F_{uw}uw + \tfrac{1}{6}F_{uuu}u^3, \qquad H = H_w w + \tfrac{1}{2}H_{uu}u^2.$$

Their coefficients can be expressed in terms of derivatives of f *and* g *at* $(\bar{x}, \bar{y}, \bar{a})$. *The matrix* H_w *is invertible. The normal form is valid for* $(x, y, a) = (\bar{x} + \mathcal{O}(\sqrt{\varepsilon}), \bar{y} + \mathcal{O}(\varepsilon), \bar{a} + \mathcal{O}(\varepsilon))$.

Explicit expressions for the coefficients are given by

$$\begin{aligned} \delta &= \omega^{-1}\sqrt{\varepsilon}, \\ F_{uw} &= (\bar{f}_y\bar{g}_y + \omega^2\, \bar{f}_{xx}^{-1}\, \bar{f}_{xy})\, \mathcal{D}_k, \\ F_{uuu} &= \omega^2\, \bar{f}_{xx}^{-2}\, \bar{f}_{xxx} + \omega^{-2}\, \bar{f}_y\bar{g}_y\bar{g}_x + \bar{f}_{xx}^{-1}(\bar{f}_{xy}\bar{g}_x + \bar{f}_y\bar{g}_{xx}), \\ H_w &= \mathcal{C}_k\, \bar{g}_y\, \mathcal{D}_k, \\ H_{uu} &= \mathcal{C}_k\, (\bar{f}_{xx}^{-1}\bar{g}_{xx} + \omega^{-2}\bar{g}_y\bar{g}_x), \end{aligned}$$

with

$$\begin{aligned} \omega &= \sqrt{-\bar{f}_y\bar{g}_x}, \\ \mathcal{C}_k &= \mathbf{1}_k(\mathbf{1} + \omega^{-2}\bar{g}_x\, \bar{f}_y), \\ \mathcal{D}_k &= (\mathbf{1} - \mathbf{e}_k\, \bar{f}_{y_k}^{-1}\, \bar{f}_y)\mathbf{1}_k^T. \end{aligned}$$

Here, $\mathbf{1}$ denotes the n-dimensional identity matrix and $\mathbf{1}_k$, resp. $\mathbf{1}_k^T$ denote $\mathbf{1}$ with the kth row, resp., the kth column deleted. $\mathbf{e}_k$ denotes the kth n-dimensional unit vector. k is any integer between 1 and n for which $\partial f/\partial y_k \neq 0$ at $(\bar{x}, \bar{y}, \bar{a})$. The relation between α and the original parameter a is given by

$$\alpha = \left\{ -\begin{pmatrix} \bar{f}_{xx} & \bar{f}_{xy} \end{pmatrix} \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1} \begin{pmatrix} \bar{f}_a \\ \bar{g}_a \end{pmatrix} + \bar{f}_{xa} \right\} A - \omega^{-2}\, \bar{f}_y\bar{g}_y\bar{g}_x, \tag{6}$$

where A, the (scaled) deviation from the starting point $\bar{a}$, is defined by $A = (a - \bar{a})/\varepsilon$.

A formula for the inverse of the matrix of first derivatives, when $\bar{f}_x = 0$, is

$$\begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{R} & \mathcal{S} \end{pmatrix},$$

with

$$\mathcal{P} = -(\bar{f}_y\bar{g}_y^{-1}\bar{g}_x)^{-1}, \qquad \mathcal{Q} = -\bar{f}_y\bar{g}_y^{-1}\mathcal{P}, \qquad \mathcal{R} = -\bar{g}_y^{-1}\bar{g}_x\mathcal{P},$$

$$\mathcal{S} = (\mathbf{1}' + \bar{g}_y^{-1}\bar{g}_x\, \bar{f}_y\mathcal{P})\bar{g}_y^{-1} = \bar{g}_y^{-1}(\mathbf{1}' + \bar{g}_x\, \bar{f}_y\bar{g}_y^{-1}\mathcal{P}),$$

where $\mathbf{1}'$ denotes the $((n-1) \times (n-1))$ identity matrix.

Remarks. In this theorem, 'smooth' means at least C^4. If f and g are of a higher degree of smoothness, then the $\mathcal{O}(\delta^2)$ terms in (5) inherit the same degree of smoothness, since we only apply analytic coordinate changes to arrive at the normal form.

We do not give expressions for the transformation $(x, y) \to (u, v, w)$. If needed, these expressions can be derived from the sequence of transformations below, because all transformations are given by explicit formulas. Equation (8) gives the position of bifurcating equilibria up to $\mathcal{O}(\varepsilon)$ when the proper value for A is substituted.

It is not absolutely necessary to specify the small parameter explicitly in order to apply our theory. If it is known that one variable is significantly faster than the others, the analysis can be carried out when the 'fast' variable is treated like the x-variable. Thus, a rough estimate for the position of singular Hopf points, and even stability information, may be obtained. Only formula (14) in Theorem 2 below, yielding a more accurate approximation for Hopf points, requires explicit knowledge of the magnitude of ε. Even the normal form can be constructed, for an unspecified value of δ.

Normal forms usually are not unique. This is also the case for the above situation. One alternative possibility is to replace the cubic term $\frac{1}{6}F_{uuu}u^3$ by a quadratic term $F_{uv}uv$. We comment on this modification at the appropriate place in the construction below.

In our notations we neglect the difference between scalars, row and column vectors, and matrices. It should be clear from the context which is which. As a general rule, the dimensions can be read off from the dimensions of the function (f or g) and its index (x and/or y). For example, $\bar{f}_{xy}$ is a $1 \times n$-dimensional row vector.

In the case of a two-dimensional system ($n = 1$), the above normal form is also valid, but there are no w-variables and corresponding coefficients. The only coefficient that remains is F_{uuu}. From Theorem 2 below we obtain that its sign determines the type of Hopf bifurcation in the two-dimensional case.

It is not necessary to restrict a priori to a single bifurcation parameter. Often in applications, a number of parameters can be considered as bifurcation (or control) parameters. Then it may not be clear which parameter changes lead to bifurcations, and it is not natural to distinguish one specific parameter from the onset. It is straightforward to adapt the analysis to a vector of parameters $a \in \mathbb{R}^p$, $p > 1$. Note that the parameter α remains a scalar, because Hopf bifurcation is a codimension-one phenomenon. Equation (6) still defines α when making the appropriate changes (for example, A is now a vector), and α becomes a linear combination of the components of the deviation vector A. Hopf bifurcations occur close to points where α vanishes, or equivalently close to a $(p-1)$-dimensional hyperplane specifying a-values. This hyperplane is a linear approximation of the true bifurcation manifold.

The small parameter ε also may be included in the right-hand side of (4). In the latter case, the functions f, g should be defined in a neighborhood of $\varepsilon = 0$, and the starting point must be of the form $(x, y, a, \varepsilon) = (\bar{x}, \bar{y}, \bar{a}, 0)$. This simple modification only affects the first steps in the construction of the normal form. Essentially only formulas (6), (8), and (10) and condition [N7] change. In each of these expressions an ε-dependent term should be included, which is identical to the corresponding (a, A) dependent term with a replaced by ε and A omitted (or replaced by 1).

It is relatively straightforward to extend the result of Theorem 1 to the case of more than one fast variable. In Section 5 we discuss this extension.

The conditions. Before we start the construction of the normal form, we briefly discuss the meaning of conditions [N1]–[N7]. We emphasize that these conditions are very mild. The only special condition is [N4] which, as we will see, enables us to find Hopf bifurcation parameter values.

The meaning of the above conditions is shown by the following:

- Conditions [N1] and [N2] simply express that $(\bar{x}, \bar{y})$ is an equilibrium of system (4) for $a = \bar{a}$. Condition [N3] implies (via the implicit function theorem) that there exists a smooth family of equilibria $(x_0(a), y_0(a))$ in a neighborhood of $\bar{a}$. Furthermore, we use condition [N3] to prove invertibility of H_w.
- Conditions [N4] and [N5] imply that the linearization of system (4) at equilibria close to $(\bar{x}, \bar{y})$ and with a close to $\bar{a}$ admits a pair of eigenvalues with singular imaginary parts for sufficiently small ε, as discussed in Section 2.
- We need condition [N6] to carry out the basic ε-dependent scaling transformations for the normal form. If $\bar{f}_{xx}$ vanishes, we have to use different scalings, yielding a normal form without the quadratic $\mathcal{O}(1)$ term. Depending on the values of other derivatives, different $\mathcal{O}(1)$ terms may appear if $\bar{f}_{xx} = 0$, while the $\mathcal{O}(\varepsilon)$ part also changes. Furthermore we use condition [N6] to obtain some simplifications in the derivation of normal form 5.
- Condition [N7] implies that for the real part $\sigma = \sigma(\bar{x}, \bar{y}, \bar{a})$ of the eigenvalues with large imaginary parts we have $\frac{d\sigma}{da} \neq 0$ at $(\bar{x}, \bar{y}, \bar{a})$. This condition is the counterpart of the condition in the standard Hopf bifurcation theorem that the speed of crossing the imaginary axis at a Hopf parameter point should be nonzero; see Marsden & McCracken (1976) or Guckenheimer & Holmes (1983).
- A geometrical interpretation of conditions [N4] and [N6] is that the equilibrium $(\bar{x}, \bar{y})$ should lie on a fold of the slow curve $f = 0$.

3.2. The Construction

The construction of the normal form (5) consists of a sequence of simple steps. Each of these steps has a single, well-defined goal towards further simplification of the system. This approach is different from the classical algorithmic normalization approach, where one successively removes terms of increasing degree, for example by Lie algebra methods; see Guckenheimer & Holmes (1983).

The first stage in the construction of the normal form consists of a sequence of scalings and linear transformations for a Taylor expansion of (4). Thus, we obtain a system with a constant, parameter free, $\mathcal{O}(1)$ part. The key idea is to introduce a new variable, a linear combination of the slow variables, which describes the $\mathcal{O}(1)$ dependence of the fast variable on the slow variables.

Taylor expansion. We start by expanding f and g in Taylor series around $(x, y, a) = (\bar{x}, \bar{y}, \bar{a})$ as follows:

$$\begin{aligned} f(x, y, a) \;=\; & \bar{f}_y(y - \bar{y}) + \bar{f}_a(a - \bar{a}) \\ & + \tfrac{1}{2}\bar{f}_{xx}(x - \bar{x})^2 + \bar{f}_{xy}(x - \bar{x})(y - \bar{y}) \end{aligned}$$

$$+ \bar{f}_{xa}(x-\bar{x})(a-\bar{a}) + \cdots$$
$$+ \tfrac{1}{6}\bar{f}_{xxx}(x-\bar{x})^3 + \cdots,$$

$$\begin{aligned} g(x,y,a) \;=\; & \bar{g}_x(x-\bar{x}) + \bar{g}_y(y-\bar{y}) + \bar{g}_a(a-\bar{a}) \\ & + \tfrac{1}{2}\bar{g}_{xx}(x-\bar{x})^2 + \cdots. \end{aligned} \tag{7}$$

Note that the terms $\bar{f}$, $\bar{g}$, and $\bar{f}_x(x-\bar{x})$ are absent from this Taylor expansion because of conditions [N1], [N2], and [N4]. The other omitted terms are not important in the sequel, because they are at least of order $\mathcal{O}(\varepsilon^2)$ in the following scaling transformations.

Scaling the parameters and the variables. Since we want to study eigenvalues with singular imaginary parts, we scale the parameter a, motivated by the results in Section 2, with

$$a = \bar{a} + \varepsilon A.$$

Now for all fixed $A \in \mathbb{R}$, a unique point $(x_0, y_0) = (x_0(A,\varepsilon), y_0(A,\varepsilon))$ exists for sufficiently small $\varepsilon > 0$ such that

$$\begin{aligned} f(x_0, y_0, \bar{a} + \varepsilon A) \;&=\; 0, \\ g(x_0, y_0, \bar{a} + \varepsilon A) \;&=\; 0. \end{aligned}$$

This follows from condition [N3] by an application of the implicit function theorem. The degree of smoothness of the functions x_0 and y_0 is the same as the degree of smoothness of f and g. An asymptotic formula for the position of the point (x_0, y_0) is

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} - \varepsilon \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1} \begin{pmatrix} \bar{f}_a \\ \bar{g}_a \end{pmatrix} A + \mathcal{O}(\varepsilon^2). \tag{8}$$

We want to 'regularize' the dynamics close to the A-dependent equilibrium (x_0, y_0) by scaling the x, y, and t variables appropriately. Here, 'appropriately' means that for the resulting system we require that

- the system is regular for $\varepsilon \to 0$,
- the equations for the scaled variables contain as many $\mathcal{O}(1)$ terms as possible.

This is the method of 'significant degenerations' or 'distinguished limits,' cf. Eckhaus (1979). To apply the method we substitute $x = x_0 + \varepsilon^{c_x} X$, $y = y_0 + \varepsilon^{c_y} Y$, $t = \varepsilon^{c_t} T$ with $c_i \geq 0$ in the Taylor series (7). We also write $x_0 = \bar{x} + \varepsilon x_1 + \mathcal{O}(\varepsilon^2)$, where x_1 represents the first-order term in the above formula (8) for x_0. Thus, we obtain a system of the structure

$$\begin{aligned} \frac{dX}{dT} \;&=\; \varepsilon^{-c_x+c_y+c_t-1}\bar{f}_y Y + \varepsilon^{c_t}\bar{f}_{xx}x_1 X + \varepsilon^{c_x+c_t-1}\tfrac{1}{2}\bar{f}_{xx}X^2 + \cdots, \\ \frac{dY}{dT} \;&=\; \varepsilon^{c_x-c_y+c_t}\bar{g}_x X + \varepsilon^{c_t}\bar{g}_y Y + \cdots, \end{aligned}$$

where dots represent omitted terms that are, for all allowed choices of c_x, c_y and c_t, of higher order than the terms shown.

To retain the maximum number of $\mathcal{O}(1)$ terms we have to solve the system of inequalities

$$-c_x + c_y + c_t - 1 \geq 0, \qquad c_t \geq 0, \qquad c_x + c_t - 1 \geq 0, \qquad c_x - c_y + c_t \geq 0,$$

while as many equalities (instead of inequalities) are satisfied as possible. On combining the first and the last equation, we find $c_t \geq 1/2$, so the second inequality can not be made an equality. The remaining three equations, however, can be solved simultaneously by $c_x = 1/2$, $c_y = 1$, $c_t = 1/2$. Therefore, we scale x, y, and t by

$$\begin{aligned} x &= x_0 + \sqrt{\varepsilon} X, \\ y &= y_0 + \varepsilon Y, \\ t &= \sqrt{\varepsilon} T. \end{aligned}$$

The equilibrium point (x_0, y_0) is thus fixed to the new origin $(X, Y) = (0, 0)$. The new $\mathcal{O}(\sqrt{\varepsilon})$ timescale is consistent with the characteristic timescale for singular oscillations as found in Section 2.

Note that condition [N6] is essential for the above scaling. For example, if $\bar{f}_{xx} = 0$ while $\bar{f}_{xxx} \neq 0$, we find $c_x = 1/3$, $c_y = 1$, $c_t = 1/2$. The corresponding normal form has an $\mathcal{O}(1)$ part with a cubic term in X instead of a quadratic term and also different 'small' terms.

The scaled system. If we introduce the above scalings, together with the Taylor expansions for f and g, in the basic system (4), we obtain

$$\begin{aligned} \frac{dX}{dT} &= \bar{f}_y Y + \tfrac{1}{2}\bar{f}_{xx} X^2 + \sqrt{\varepsilon}\,(\alpha_0 X + \bar{f}_{xy} XY + \tfrac{1}{6}\bar{f}_{xxx} X^3) + \mathcal{O}(\varepsilon), \\ \frac{dY}{dT} &= \bar{g}_x X + \sqrt{\varepsilon}\,(\bar{g}_y Y + \tfrac{1}{2}\bar{g}_{xx} X^2) + \mathcal{O}(\varepsilon). \end{aligned} \tag{9}$$

Here we introduce the scaled parameter

$$\alpha_0 = \left\{ -\begin{pmatrix} \bar{f}_{xx} & \bar{f}_{xy} \end{pmatrix} \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1} \begin{pmatrix} \bar{f}_a \\ \bar{g}_a \end{pmatrix} + \bar{f}_{xa} \right\} A, \tag{10}$$

which combines, in terms of the deviation A from the starting point $\bar{a}$, all parameter occurrences on the $\mathcal{O}(\sqrt{\varepsilon})$ level. We essentially keep this parameter for the rest of the paper, including the bifurcation analysis in Section 4. The only modification to α_0 consists in adding a constant in one of the following transformations.

The above formula again shows the importance of nondegeneracy condition [N7]: If this condition is not satisfied, then α_0 vanishes identically, and our approach will (in this form) not be successful.

Identification of a two-dimensional subsystem. The next two steps are crucial for the further analysis, because they partition the system into two weakly coupled subsystems. One subsystem is two-dimensional and has an $\mathcal{O}(1)$ timescale; the other system is $(n-1)$-dimensional (in the first steps n-dimensional) and has a slow $\mathcal{O}(1/\sqrt{\varepsilon})$ timescale.

Consider system (9) and observe that, on the $\mathcal{O}(1)$ lengthscale, dY/dT depends on X only, while dX/dT depends linearly on the Y-variables. Hence, an $\mathcal{O}(\sqrt{\varepsilon})$-decoupled two-dimensional subsystem can be isolated if we introduce a new scalar variable V by

$$V = \bar{f}_y Y.$$

This yields

$$\begin{aligned}
\frac{dX}{dT} &= V + \tfrac{1}{2}\bar{f}_{xx}X^2 + \sqrt{\varepsilon}\,(\alpha_0 X + \bar{f}_{xy}XY + \tfrac{1}{6}\bar{f}_{xxx}X^3) + \mathcal{O}(\varepsilon),\\
\frac{dV}{dT} &= \bar{f}_y\bar{g}_x X + \sqrt{\varepsilon}\;\bar{f}_y(\bar{g}_y Y + \tfrac{1}{2}\bar{g}_{xx}X^2) + \mathcal{O}(\varepsilon),\\
\frac{dY}{dT} &= \bar{g}_x X + \sqrt{\varepsilon}\,(\bar{g}_y Y + \tfrac{1}{2}\bar{g}_{xx}X^2) + \mathcal{O}(\varepsilon).
\end{aligned}$$

Notice that this system contains one superfluous variable, because we introduced the new variable V. It is straightforward to remove one of the Y-variables, using the fact that $\bar{f}_y \neq 0$. For the time being, however, we keep the extra variable, because this simplifies the intermediate expressions. We postpone removal of the superfluous variable to the final transformation.

Slowing down the remaining variables. Condition [N5] states that $\bar{f}_y\bar{g}_x < 0$. Hence, we may remove the zeroth–order term in the third equation through the transformation

$$W = Y + \omega^{-2}\bar{g}_x V,$$

where (cf. Section 2)

$$\omega = \sqrt{-\bar{f}_y\bar{g}_x}.$$

We obtain the system:

$$\begin{aligned}
\frac{dX}{dT} &= V + \tfrac{1}{2}\bar{f}_{xx}X^2 + \sqrt{\varepsilon}\,(\,\alpha_0 X - \omega^{-2}\bar{f}_{xy}\bar{g}_x XV\\
&\qquad + \bar{f}_{xy}XW + \tfrac{1}{6}\bar{f}_{xxx}X^3) + \mathcal{O}(\varepsilon),\\
\frac{dV}{dT} &= -\omega^2 X + \sqrt{\varepsilon}\;\bar{f}_y(-\omega^{-2}\bar{g}_y\bar{g}_x V + \bar{g}_y W + \tfrac{1}{2}\bar{g}_{xx}X^2) + \mathcal{O}(\varepsilon),\\
\frac{dW}{dT} &= \sqrt{\varepsilon}\,(\mathbf{1} + \omega^{-2}\bar{g}_x\bar{f}_y)(-\omega^{-2}\bar{g}_y\bar{g}_x V + \bar{g}_y W + \tfrac{1}{2}\bar{g}_{xx}X^2) + \mathcal{O}(\varepsilon).
\end{aligned}$$

Here $\mathbf{1}$ denotes the n-dimensional identity matrix.

Thus, we have obtained a decomposition in two subsystems, where the two-dimensional (X, V)-subsystem and the n-dimensional W-subsystem are only weakly coupled. For later use (in the final transformation), we observe that $\bar{f}_y W = 0$.

Simplification of the $\mathcal{O}(1)$ part. Our next aim is to simplify the $\mathcal{O}(1)$ part of the above weakly coupled system. It is possible to eliminate all parameter-dependence on the $\mathcal{O}(1)$ level by using the scalings

$$X = \omega\,\bar{f}_{xx}^{-1}u, \qquad V = \omega^2\,\bar{f}_{xx}^{-1}v, \qquad W = \omega^2\,\bar{f}_{xx}^{-1}w, \qquad T = \omega^{-1}\tau.$$

For these scalings we need condition [N6]. We also introduce a new small parameter δ by

$$\delta = \omega^{-1}\sqrt{\varepsilon}.$$

Thus, we obtain

$$\begin{aligned}
\frac{du}{d\tau} &= v + \tfrac{1}{2}u^2 + \delta\,(\alpha_0 u - \bar{f}_{xx}^{-1}\bar{f}_{xy}\bar{g}_x uv \\
&\quad + \omega^2\bar{f}_{xx}^{-1}\bar{f}_{xy}uw + \tfrac{1}{6}\omega^2\bar{f}_{xx}^{-2}\bar{f}_{xxx}u^3) + \mathcal{O}(\delta^2), \\
\frac{dv}{d\tau} &= -u + \delta\ \bar{f}_y(-\omega^{-2}\bar{g}_y\bar{g}_x v + \bar{g}_y w + \tfrac{1}{2}\bar{f}_{xx}^{-1}\bar{g}_{xx}u^2) + \mathcal{O}(\delta^2), \\
\frac{dw}{d\tau} &= \delta\ (\mathbf{1} + \omega^{-2}\bar{g}_x\bar{f}_y)(-\omega^{-2}\bar{g}_y\bar{g}_x v + \bar{g}_y w + \tfrac{1}{2}\bar{f}_{xx}^{-1}\bar{g}_{xx}u^2) + \mathcal{O}(\delta^2).
\end{aligned}$$

Simplification of the $\mathcal{O}(\delta)$ part. Now that we have simplified the $\mathcal{O}(1)$ part to a parameterless form, we focus on further simplification of the $\mathcal{O}(\delta)$ part. In order not to disturb the $\mathcal{O}(1)$ part, we have to restrict ourselves to near-identity transformations from now on, where 'near-identity' means $\mathcal{O}(\delta)$ close to the identity. Our program consists of three steps. First, we remove the linear term in v in the third equation, and we remove the entire $\mathcal{O}(\delta)$ part in the second equation. The main effect of these steps is that they linearly decouple the w-variables from the (u, v)-variables (at least up to the $\mathcal{O}(\delta^2)$ level). This is necessary for the center manifold computations in Section 4. We then proceed to remove a term in the first equation and, finally, we remove the superfluous variable in the third equation.

Linear decoupling of the subsystems. In order to remove the v-term in the third equation, we transform w to

$$\tilde{w} = w + \delta\ (\mathbf{1} + \omega^{-2}\bar{g}_x\bar{f}_y)\omega^{-2}\bar{g}_y\bar{g}_x u.$$

It is clear that this (near-identity) transformation removes the v-dependent term from the $\mathcal{O}(\delta)$ part of the third equation, while it only has an $\mathcal{O}(\delta^2)$ effect on the other equations. A side effect of the transformation is that it creates a linear term in u in the $\mathcal{O}(\delta^2)$ part of the third equation (and after the next transformation, also an $\mathcal{O}(\delta^3)$ linear term in $\tilde{w}$), which did not contain linear terms before. This undesirable effect can be prevented by an $\mathcal{O}(\delta^2)$ change in $\tilde{w}$, but since we neglect $\mathcal{O}(\delta^2)$ terms in the sequel we do not go into the details.

To remove the $\mathcal{O}(\delta)$ part in the second equation, we introduce

$$\tilde{u} = u - \delta\ \bar{f}_y(-\omega^{-2}\bar{g}_y\bar{g}_x v + \bar{g}_y\tilde{w} + \tfrac{1}{2}\bar{f}_{xx}^{-1}\bar{g}_{xx}u^2),$$

and we modify the bifurcation parameter α to

$$\alpha = \alpha_0 - \omega^{-2}\bar{f}_y\bar{g}_y\bar{g}_x.$$

This yields the system

$$\begin{aligned}
\frac{d\tilde u}{d\tau} &= v + \tfrac12\tilde u^2 + \delta\,(\alpha\tilde u - (\omega^{-2}\,\bar f_y\bar g_y\bar g_x + \bar f_{xx}^{-1}(\bar f_{xy}\bar g_x + \bar f_y\bar g_{xx}))\tilde u v \\
&\quad + (\bar f_y\bar g_y + \omega^2\,\bar f_{xx}^{-1}\,\bar f_{xy})\tilde u\tilde w + \tfrac16\omega^2\,\bar f_{xx}^{-2}\,\bar f_{xxx}\tilde u^3) + \mathcal{O}(\delta^2), \\
\frac{dv}{d\tau} &= -\tilde u + \mathcal{O}(\delta^2), \\
\frac{d\tilde w}{d\tau} &= \delta\,(\mathbf{1} + \omega^{-2}\bar g_x\,\bar f_y)(\bar g_y\tilde w + \tfrac12(\bar f_{xx}^{-1}\bar g_{xx} + \omega^{-2}\bar g_y\bar g_x)\tilde u^2) + \mathcal{O}(\delta^2). \qquad (11)
\end{aligned}$$

Removing the uv-term. So far we have used fairly straightforward transformations to transform (4) into a system with fixed $\mathcal{O}(1)$-part and only six terms in the $\mathcal{O}(\delta)$-part. We can remove one more $\mathcal{O}(\delta)$-term, but this requires a less straightforward (near-identity) transformation. We refer to Appendix A for the derivation of the transformation formulas. We introduce

$$\tilde{\tilde u} = \tilde u + \tfrac13\delta C_{uv}(\tfrac12\tilde u^2 v - \tilde u^2 + v^2), \qquad \tilde v = v - \tfrac13\delta C_{uv}\tilde u v,$$

where $C_{uv} = -(\omega^{-2}\,\bar f_y\bar g_y\bar g_x + \bar f_{xx}^{-1}(\bar f_{xy}\bar g_x + \bar f_y\bar g_{xx}))$ represents the coefficient of the $\tilde u v$-term in the first of the previous system equations. Upon application of this transformation, we obtain

$$\begin{aligned}
\frac{d\tilde{\tilde u}}{d\tau} &= \tilde v + \tfrac12\tilde{\tilde u}^2 + \delta\,(\alpha\tilde{\tilde u} + (\bar f_y\bar g_y + \omega^2\,\bar f_{xx}^{-1}\,\bar f_{xy})\tilde{\tilde u}\tilde w \\
&\quad + \tfrac16(\omega^2\,\bar f_{xx}^{-2}\,\bar f_{xxx} + \omega^{-2}\,\bar f_y\bar g_y\bar g_x + \bar f_{xx}^{-1}(\bar f_{xy}\bar g_x + \bar f_y\bar g_{xx}))\tilde{\tilde u}^3) \\
&\quad + \mathcal{O}(\delta^2), \\
\frac{d\tilde v}{d\tau} &= -\tilde{\tilde u} + \mathcal{O}(\delta^2), \\
\frac{d\tilde w}{d\tau} &= \delta\,(\mathbf{1} + \omega^{-2}\bar g_x\,\bar f_y)(\bar g_y\tilde w + \tfrac12(\bar f_{xx}^{-1}\bar g_{xx} + \omega^{-2}\bar g_y\bar g_x)\tilde{\tilde u}^2) \\
&\quad + \mathcal{O}(\delta^2). \qquad (12)
\end{aligned}$$

Remark. It is a matter of taste to remove either the $\tilde u v$-term or the $\tilde u^3$-term. We have chosen to remove the $\tilde u v$-term in order to obtain a system similar to the Bonhoeffer–van der Pol equation (in Liénard form), which is the prototype for two-dimensional slow/fast systems. For some systems, however, it may be more natural to drop the cubic term. For example, for chemical systems, the product $\tilde u \cdot v$ often corresponds to a reaction term. If we remove the cubic term instead of the product term, the $\tilde{\tilde u}\tilde v$-term will have the same coefficient as the cubic term now has (without the factor 1/6), while all other coefficients remain unchanged.

Final transformation. Finally, we remove the superfluous $\tilde w$–variable (recall that we kept all Y–variables when we introduced the additional variable V). We use condition [N5], which implies that $\bar f_{y_k} \neq 0$ for some $k \in \{1, \ldots, n\}$, to eliminate the corresponding variable $\tilde w_k$.

From the observation above that $\bar{f}_y W = 0$, it follows that $\bar{f}_y \tilde{w} = \mathcal{O}(\delta)$ since we have only changed W through multiplication with a scalar coefficient and a near-identity transformation. This relation permits us to express $\tilde{w}_k$ in terms of the other $\tilde{w}$-variables as

$$\tilde{w}_k = -\bar{f}_{y_k}^{-1} \sum_{i=1(i\neq k)}^{n-1} \bar{f}_{y_i} \tilde{w}_i + \mathcal{O}(\delta).$$

Hence, the n-vector of the $\tilde{w}$-variables can, after elimination of $\tilde{w}_k$, be expressed in the $(n-1)$-vector $\hat{w} = (\tilde{w}_1, \ldots, \tilde{w}_{k-1}, \tilde{w}_{k+1}, \ldots, \tilde{w}_n)$ of the remaining variables as

$$\tilde{w} = (\mathbf{1} - \mathbf{e}_k \bar{f}_{y_k}^{-1} \bar{f}_y) \mathbf{1}_k^T \hat{w} + \mathcal{O}(\delta).$$

Using this relation we eliminate $\tilde{w}_k$ from the right-hand sides in the above system (12). Omitting the equation for $\tilde{w}_k$ (which is equivalent to multiplication of the right-hand side of the last equation in (12) on the left by $\mathbf{1}_k$) and dropping the hats and tildes, the system is in the normal form announced in Theorem 1 (dots denote differentiation with respect to the time variable τ):

$$\begin{aligned} \dot{u} &= v + \tfrac{1}{2}u^2 + \delta\,(\alpha u + F_{uw} u w + \tfrac{1}{6} F_{uuu} u^3) + \mathcal{O}(\delta^2), \\ \dot{v} &= -u + \mathcal{O}(\delta^2), \\ \dot{w} &= \delta\,(H_w w + \tfrac{1}{2} H_{uu} u^2) + \mathcal{O}(\delta^2). \end{aligned}$$

It is clear that no further terms can be removed on the $\mathcal{O}(\delta)$ level. The parameter α and the coefficients H_w determine the linear behavior of the (u, v) and w subsystems, while the nonlinear stability of the (u, v) subsystem is governed by F_{uuu}. The other two coefficients, F_{uw} and H_w, represent the coupling strengths.

It still is possible to simplify one or more of the coefficients F_{uw}, H_w, and H_{uu} by linear transformations of w. For example, H_w can be brought in Jordan normal form, or $du/d\tau$ can be made to depend on only one w-variable directly by introducing the scalar coordinate $w_* = F_{uw} w$. We do not pursue this matter further, because determining the most convenient simplification depends on the particular system under study.

Invertibility of H_w. The only part of Theorem 1 that remains to be proven is the invertibility of H_w. This can be proven as follows.

The determinant of the Jacobian of (4) at the starting point $(\bar{x}, \bar{y}, \bar{a})$ equals

$$\varepsilon^{-1} \det \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}.$$

The determinant of the Jacobian of the normal form (5) at the origin equals $\delta^n \det H_w$. Since we have transformed (4) into (5) using only invertible transformations, both determinants agree if we take the change of time $t \to \tau$ into account, or

$$\varepsilon^{-1} \det \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix} = \left(\frac{d\tau}{dt} \right)^{n+1} \delta^{n-1} \det H_w.$$

If we evaluate the above expression using that $d\tau/dt = \delta^{-1}$ and $\delta = \sqrt{\varepsilon}/\omega$, we obtain

$$\det H_w = \omega^{-2} \det \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix} \neq 0.$$

Hence, H_w is invertible.

4. Hopf Bifurcation Analysis

In this section we analyze the Hopf bifurcation in system (4). We make one further assumption: The matrix H_w in the normal form (5) has no elliptic part. Because this is a generic assumption, it does not impose a severe restriction on our theory. Below we briefly comment on the case where H_w also has an elliptic part.

For the analysis it is convenient to perform a further reduction of the normal form using center manifold theory.

4.1. Reduction to a Center Manifold

If the matrix H_w in the normal form (5) has no elliptic part, system (5) is in the proper format for a center manifold reduction, since the nonhyperbolic (u, v)-part and the hyperbolic w-part are linearly decoupled.

The full computation of a center manifold is generally a considerable task. We are satisfied, however, with a quadratic $\mathcal{O}(1)$ approximation, since this yields enough information for the Hopf bifurcation analysis below. The center manifold can be expressed as a graph (cf. Guckenheimer & Holmes (1983) or Kuznetsov (1996)),

$$\begin{aligned} w = \Phi(u, v, \delta) \;=\; & \tfrac{1}{2}\Phi^0_{uu}u^2 + \Phi^0_{uv}uv + \tfrac{1}{2}\Phi^0_{vv}v^2 + \mathcal{O}(3) \\ & + \delta\,(\tfrac{1}{2}\Phi^1_{uu}u^2 + \Phi^1_{uv}uv + \tfrac{1}{2}\Phi^1_{vv}v^2 + \mathcal{O}(3)) + \mathcal{O}(\delta^2), \end{aligned}$$

where $\mathcal{O}(3)$ represents cubic and higher-order terms in u and v. Actually, we do not need the $\mathcal{O}(\delta)$ terms in the bifurcation analysis. We do need, however, one of the quadratic $\mathcal{O}(\delta)$ terms to determine the quadratic $\mathcal{O}(1)$ part completely.

The determining condition for Φ is that it satisfies the equation $\dot{\Phi} = \delta(H_w\Phi + \frac{1}{2}H_{uu}u^2) + \mathcal{O}(\delta^2)$, or

$$\begin{aligned} & \Phi^0_{uu}uv + \Phi^0_{uv}v^2 - \Phi^0_{uv}u^2 - \Phi^0_{vv}uv + \delta\,(\Phi^1_{uu}uv + \Phi^1_{uv}v^2 - \Phi^1_{uv}u^2 - \Phi^1_{vv}uv) \\ & \quad = \delta\,(H_w(\tfrac{1}{2}\Phi^0_{uu}u^2 + \Phi^0_{uv}uv + \tfrac{1}{2}\Phi^0_{vv}v^2) + \tfrac{1}{2}H^2_{uu}) + \mathcal{O}(3) + \mathcal{O}(\delta^2). \end{aligned}$$

From inspection of the $\mathcal{O}(1)$ terms, it is clear that $\Phi^0_{uv} = 0$ and that $\Phi^0_{uu} = \Phi^0_{vv}$, but their common value is not yet determined. In order to determine their value, we have to go one step further in the expansion and consider the coefficients of δu^2 and δv^2. For these coefficients we find, respectively,

$$-\Phi^1_{uv} = \tfrac{1}{2}H_w\Phi^0_{uu} + \tfrac{1}{2}H_{uu} \qquad \text{and} \qquad \Phi^1_{uv} = \tfrac{1}{2}H_w\Phi^0_{vv},$$

and on adding these equations we obtain

$$\Phi^0_{uu} = \Phi^0_{vv} = -\tfrac{1}{2}H_w^{-1}H_{uu},$$

such that the center manifold approximation reads

$$w = \Phi(u, v, \delta) = -\tfrac{1}{4} H_w^{-1} H_{uu}(u^2 + v^2) + \mathcal{O}(3) + \mathcal{O}(\delta). \tag{13}$$

The corresponding equations in the center manifold are (up to higher-order terms):

$$\begin{aligned} \dot{u} &= v + \tfrac{1}{2}u^2 \\ &\quad + \delta(\alpha u + (-\tfrac{1}{4}F_{uw}H_w^{-1}H_{uu} + \tfrac{1}{6}F_{uuu})u^3 - \tfrac{1}{4}F_{uw}H_w^{-1}H_{uu}uv^2), \\ \dot{v} &= -u. \end{aligned}$$

Remark. The above approach for computing a center manifold can also be applied when the matrix H_w has an elliptic part. The equations for the center manifold become more complex, because we must incorporate a number of w-variables, equal to the dimension of the elliptic part. This dimension will always be even, because H_w has no vanishing eigenvalues.

4.2. The Hopf Bifurcation

The center manifold equations are in a form that enables us to apply a standard formula for determining the type of Hopf bifurcation in two-dimensional systems. This formula can be found in many textbooks dealing with nonlinear ODEs; see for example Guckenheimer & Holmes (1983) or Kuznetsov (1996). If we combine this formula with previous results, we have the following theorem.

Theorem 2. *Consider system (4), satisfying the assumptions of Theorem 1 and the additional assumption that the matrix H_w in the corresponding normal form (5) has no elliptic part. Then system (4) undergoes, for sufficiently small ε, a Hopf bifurcation at $a = \bar{a} + \varepsilon A + \mathcal{O}(\varepsilon\sqrt{\varepsilon})$, where A is the solution of the equation*

$$\left\{ -\begin{pmatrix} \bar{f}_{xx} & \bar{f}_{xy} \end{pmatrix} \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1} \begin{pmatrix} \bar{f}_a \\ \bar{g}_a \end{pmatrix} + \bar{f}_{xa} \right\} A = \omega^{-2} \bar{f}_y \bar{g}_y \bar{g}_x. \tag{14}$$

This Hopf bifurcation is super(sub)critical if

$$\tfrac{1}{2}F_{uuu} - F_{uw}H_w^{-1}H_{uu} > (<)0. \tag{15}$$

Remarks. In Theorem 2, 'supercritical' means that the bifurcation creates a stable limit cycle for values of A corresponding to $\alpha > 0$ and 'subcritical' means that the bifurcation creates an unstable limit cycle for values of A corresponding to $\alpha < 0$.

If the expression determining the stability type of the Hopf bifurcation vanishes, the Hopf bifurcation is degenerate and cannot be studied using the current normal form, because neglected higher-order terms come into play. Takens (1973) studies degenerate Hopf bifurcations in a general setting.

Our stability formula in the above Theorem 2 agrees in the two-dimensional case with formulas (2.27)–(2.28), obtained by Baer & Erneux (1986). Note that only the $F_{uuu}/2$ term remains.

If H_w has an elliptic part, the situation is degenerate. The center manifold equations, obtained from the normal form (5), do not yield enough information to study this degenerate situation. For example, in the simplest case we need to study a so-called 'Hopf-Hopf' bifurcation on a four-dimensional center manifold; see Guckenheimer & Holmes (1983) or Kuznetsov (1996). If we normalize the center manifold equations, however, the cubic part appears to be degenerate and it is not possible to obtain bifurcation information. One way to resolve this problem might be to compute the normal form (5) up to higher order, but maybe a completely different approach is needed. In the higher-dimensional case, we must also take resonances into account: This is not a problem in the four-dimensional case, because one pair of eigenvalues is $\mathcal{O}(1)$, while the other pair is $\mathcal{O}(\delta)$, such that no (strong) resonances occur.

Alternative methods. There are other ways to obtain Hopf bifurcation information from system (5) apart from the center manifold approach. We have checked that the following approaches are feasible (and give the same results!):

- Express (u, v) in polar coordinates and average over the angular coordinate. We have to carry out averaging up to third order and do some work to prove validity, since standard textbook averaging theorems are not adapted to slow/fast systems.
- Kuznetsov (1996) gives a general formula (originally due to Howard & Kopell and Diekman & van Gils; see the reference in Kuznetsov (1996)) for Hopf bifurcation in finite dimensional systems. This formula requires as its main data eigenvectors and conjugate eigenvectors at the bifurcating equilibrium.
- For $\delta = 0$, system (5) is integrable and admits a family of closed orbits around the stationary point at the origin. For small δ we can construct an approximate Poincaré map. Its fixed points correspond to limit cycles.

The last approach, which uses a Poincaré map, turns out to be particularly fruitful. It gives lots of additional information on the orbit structure close to the bifurcating equilibrium, for example on the shape of limit cycles, secondary bifurcations and transitional behavior. We will devote a separate paper to this approach; see Braaksma (1993) for preliminary results.

If only an approximation for the position of the Hopf bifurcation point is desired (without stability information), it is not necessary to go through the full normalization procedure. Formula (10) gives, with α equated to zero, a value for A and hence an $\mathcal{O}(\varepsilon)$ correction for the starting value $\bar{a}$. Formula (8) gives the corresponding $\mathcal{O}(\varepsilon)$ correction for the position of the bifurcating equilibrium $(\bar{x}, \bar{y})$.

It is also possible to compute asymptotic expansions for singular Hopf points directly from the characteristic polynomial of the linearization matrix of (4), although this requires a considerable computational effort.

5. More than One Fast Variable

In this section we consider the modifications to the results of sections 2, 3, and 4, for the case where system (1) has more than one fast variable. Most remarks made in those sections remain valid.

5.1. Eigenvalues with Singular Imaginary Parts

Suppose that, in the notation of Section 2, A is an m-dimensional square matrix instead of a scalar, with the dimensions of B and C changed accordingly. Now we trivially obtain eigenvalues with singular imaginary parts when A has imaginary eigenvalues. Such eigenvalues, however, do not arise from the interaction of large and small entries in $\mathcal{L}_\varepsilon$, unlike those we obtained in Proposition 1. Eigenvalues of the latter type (which we will call *nontrivial* eigenvalues with singular imaginary parts) arise if, and only if, the following generalized conditions hold:

[S1a] $\det(A) = 0$,

[S2a] $\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m} b_{ij} c_{jk} \det(A_{ik})(-1)^{i+k} / \sum_{i=1}^{m} \det(A_{ii}) < 0$.

Here, b_{ij} and c_{jk} represent the entries of B and C, and $\det(A_{ij})$ denotes the minor of $\det(A)$, obtained when the i-th row and j-th column are deleted. The square root of the negative of the left-hand side of formula [S2a] gives the coefficient of the $\mathcal{O}(1/\sqrt{\varepsilon})$ leading term of the nontrivial eigenvalues with singular imaginary parts. We omit the proof of this generalization of Proposition 1. Apart from some cumbersome bookkeeping, it is analogous to the original proof. Observe that conditions [S1a] and [S2a] reduce to [S1] and [S2] when $m = 1$.

A particular simplification arises when the matrix A is in Jordan normal form. Condition [S1a] now implies that one of the diagonal entries of A, say the lowermost (m-th) one, vanishes and so all minors $\det(A_{ij})$ vanish, except for $\det(A_{mm})$. Hence, condition [S2a] simplifies to

$$\sum_{j=1}^{n} b_{mj} c_{jm} < 0,$$

which is essentially the same as condition [S2]. This simplified condition also holds when a corresponding row and column (same index) in A vanish.

5.2. The Normal Form

It is possible to extend the result of Theorem 1 to the case of more than one fast variable. The main difference is that the inclusion of additional fast variables introduces a third, fast timescale of $\mathcal{O}(1/\sqrt{\varepsilon})$ in the normal form. The generalized theorem reads as follows.

Theorem 3. *Let U be an open subset of $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ and let $f: U \to \mathbb{R}^m$, $g: U \to \mathbb{R}^n$ be sufficiently smooth functions. Let $(\bar{x}, \bar{y}, \bar{a}) \in U$ be a point where the following conditions on f, g, and their derivatives hold:*

[N0a] *The Jacobian matrix $\bar{f}_x$ is in real Jordan form,*

[N1a] $\bar{f} = 0$,

[N2a] $\bar{g} = 0$,

[N3a] $\det \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix} \neq 0,$

[N4a] $(\bar{f}_m)_{x_m} = 0,$

[N5a] $(\bar{f}_m)_y \, \bar{g}_{x_m} < 0,$

[N6a] $(\bar{f}_m)_{x_m x_m} \neq 0,$

[N7a] $$-\left((\bar{f}_m)_{x_m x} \quad (\bar{f}_m)_{x_m y}\right) \begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1} \begin{pmatrix} \bar{f}_a \\ \bar{g}_a \end{pmatrix} + (\bar{f}_m)_{x_m a} \neq 0.$$

Then the system

$$\begin{aligned} \varepsilon \frac{dx}{dt} &= f(x, y, a), \qquad 0 < \varepsilon \ll 1, \\ \frac{dy}{dt} &= g(x, y, a), \end{aligned} \tag{16}$$

can be put in the following normal form:

$$\begin{aligned} \delta \, \dot{\xi} &= E(u, \xi) + \mathcal{O}(\delta), \\ \dot{u} &= v + \tfrac{1}{2}u^2 + \delta \, F(u, w, \alpha) + \mathcal{O}(\delta^2), \\ \dot{v} &= -u + \mathcal{O}(\delta^2), \\ \dot{w} &= \delta \, H(u, w) + \mathcal{O}(\delta^2), \end{aligned} \tag{17}$$

with $\delta = \mathcal{O}(\sqrt{\varepsilon})$ and $\xi \in \mathbb{R}^{m-1}$, $u, v, \alpha \in \mathbb{R}$, $w \in \mathbb{R}^{n-1}$. The dots denote differentiation with respect to the slow time variable $\tau = \delta^{-1}t$. The functions E, F, and H have the structure

$$E = E_\xi \xi + \tfrac{1}{2}E_{uu}u^2, \qquad F = \alpha u + F_{u\xi}u\xi + F_{uw}uw + \tfrac{1}{6}F_{uuu}u^3, \qquad H = H_w w + \tfrac{1}{2}H_{uu}u^2.$$

Their coefficients can be expressed in terms of derivatives of f and g at $(\bar{x}, \bar{y}, \bar{a})$. The matrices E_ξ and H_w are invertible. The normal form is valid in an $\mathcal{O}(\sqrt{\varepsilon})$ neighborhood of $\bar{x}_m$ and in an $\mathcal{O}(\varepsilon)$ neighborhood of $\bar{x}_1, \ldots, \bar{x}_{m-1}, \bar{y}, \bar{a}$.

Remarks. For the sake of simplicity, the assumptions in this theorem have not been formulated to be as general as possible. In particular, the Jordan assumption [N0a] is superfluous. Without this assumption, however, the formulation of conditions [N4a]–[N7a] would become much more complicated. Moreover, a transformation of $\bar{f}_x$, in order to decouple the additional fast variables, would anyhow be the first step in the construction of the normal form.

For a 'general' system (that is, $\bar{f}_x$ is not in Jordan form), condition [N4a] changes to $\det(\bar{f}_x) = 0$, a condition that is easily tested. Hence, it is not difficult to check beforehand, for a given system, whether it is worthwhile to put $\bar{f}_x$ in Jordan form in order to compute the above normal form.

The construction. Since the construction of the normal form for Theorem 3 is analogous to that for Theorem 1, we only briefly sketch the relevant modifications.

We can essentially copy the construction of Section 3.2, while keeping track of the additional fast variables. We need to make some preparations, however.

First we have to consider the scaling of the variables. The m-th fast variable, which corresponds to the vanishing eigenvalue, has to be scaled with $\sqrt{\varepsilon}$, like the single fast variable in Section 3.2. The other fast variables, however, have to be scaled with ε instead. This is a consequence of the fact that the corresponding rows in $\bar{f}_x$ do not vanish. The slow variables have to be scaled with ε, again like Section 3.2. The scaled system has the structure (compare also (9)):

$$\begin{aligned}
\sqrt{\varepsilon}\frac{d\widehat{X}}{dT} &= \widehat{\bar{f}}_{\hat{x}}\widehat{X} + \widehat{\bar{f}}_y Y + \tfrac{1}{2}\widehat{\bar{f}}_{x_m x_m} X_m^2 + \mathcal{O}(\sqrt{\varepsilon}),\\
\frac{dX_m}{dT} &= (\bar{f}_m)_y Y + \tfrac{1}{2}(\bar{f}_m)_{x_m x_m} X_m^2\\
&\quad + \sqrt{\varepsilon}\,(\alpha_0 X_m + (\bar{f}_m)_{x_m \hat{x}} X_m \widehat{X} + (\bar{f}_m)_{x_m y} X_m Y\\
&\quad + \tfrac{1}{6}(\bar{f}_m)_{x_m x_m x_m} X_m^3) + \mathcal{O}(\varepsilon),\\
\frac{dY}{dT} &= \bar{g}_x X_m + \sqrt{\varepsilon}\,(\bar{g}_{\hat{x}}\widehat{X} + \bar{g}_y Y + \tfrac{1}{2}\bar{g}_{xx} X_m^2) + \mathcal{O}(\varepsilon),
\end{aligned} \tag{18}$$

where X_m denotes the scaled fast variable, corresponding to the vanishing eigenvalue in $\bar{f}_x$; $\widehat{X} = (X_1, \ldots, X_{m-1})$ denotes the remaining scaled fast variables; and $\bar{f}_m$, $\widehat{\bar{f}}$ denote the corresponding right-hand sides.

After the system has been scaled, we have to (linearly) decouple the additional fast variables $\widehat{X}$ from the slow variables, while preserving the Jordan structure of $\bar{f}_x$. This is accomplished by the following linear near-identity transformation:

$$\begin{aligned}
\widehat{X} &\to \widehat{X} - \sqrt{\varepsilon}\,(\widehat{\bar{f}}_{\hat{x}})^{-2}\widehat{\bar{f}}_y \bar{g}_{x_m} X_m - \varepsilon\,(\widehat{\bar{f}}_{\hat{x}})^{-1}\widehat{\bar{f}}_y Y,\\
X_m &\to X_m,\\
Y &\to Y + \varepsilon\,\bar{g}_{\hat{x}}(\widehat{\bar{f}}_{\hat{x}})^{-1}\widehat{X}.
\end{aligned}$$

This transformation removes the linear terms $\widehat{\bar{f}}_y Y$ and $\sqrt{\varepsilon}\bar{g}_{\hat{x}}\widehat{X}$ from the scaled equations, while modifying the term

$$\sqrt{\varepsilon}\,\bar{g}_y Y \qquad \text{to} \qquad \sqrt{\varepsilon}\,(\bar{g}_y - \bar{g}_{\hat{x}}\widehat{\bar{f}}_{\hat{x}}\widehat{\bar{f}}_y)Y,$$

and leaving the remaining three linear terms unchanged. Note that this transformation also modifies some of the nonlinear coefficients.

After these preparations have been made, the construction of Section 3.2 can be carried out without any difficulties. The additional fast variables are not involved in the further transformations.

Explicit expressions for the coefficients of the functions E, F, and H and the parameter α can be obtained, as those in the single fast variable case were, by keeping track of the effects of the transformations.

The Hopf bifurcation. Extension of the results of Section 4 is straightforward. The graph of a center manifold is given by

$$\xi = -\tfrac{1}{2}E_\xi^{-1}E_{uu}u^2 + \mathcal{O}(3) + \mathcal{O}(\delta), \qquad w = \Phi(u, v, \delta),$$

where Φ is given by (13). This yields the following Hopf bifurcation theorem.

Theorem 4. *Consider system (16), satisfying the assumptions of Theorem 3 and the additional assumption that the matrices E_ξ and H_w in the corresponding normal form (17) have no elliptic parts. Then system (16) undergoes, for sufficiently small ε, a Hopf bifurcation at $a = \bar{a} + \varepsilon A + \mathcal{O}(\varepsilon\sqrt{\varepsilon})$, where A is the solution of the equation*

$$\left\{ -\left(\left(\bar{f}_m\right)_{x_m x} \quad \left(\bar{f}_m\right)_{x_m y}\right)\begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1}\begin{pmatrix} \bar{f}_a \\ \bar{g}_a \end{pmatrix} + \left(\bar{f}_m\right)_{x_m a} \right\} A$$
$$= -\left(\bar{f}_m\right)_y g_y g_{x_m} / \left(\bar{f}_m\right)_y g_{x_m}. \tag{19}$$

This Hopf bifurcation is super(sub)critical if

$$\tfrac{1}{2}F_{uuu} - F_{uw}H_w^{-1}H_{uu} - \tfrac{3}{2}F_{u\xi}E_\xi^{-1}E_{uu} > (<)0. \tag{20}$$

6. Examples

In this section we discuss two examples of systems (with one fast variable) exhibiting a singular Hopf bifurcation. We also compare the asymptotic results, obtained from application of Theorems 1 and 2, to numerical results.

Based on the results of the previous sections, we have the following useful criterion for locating singular Hopf bifurcation points: One should search for Hopf points in the neighborhood of equilibria where f_x vanishes, while $f_y g_x < 0$, or equivalently, solve the set of $n+2$ equations $f = 0, g = 0, f_x = 0$ and test if the obtained solutions satisfy $f_y g_x < 0$. The solution of this set of equations gives a rough approximation for the position of the bifurcation point. After having computed the normal form coefficients using Theorem 1, an application of Theorem 2 gives information on the type of Hopf bifurcation and a better estimate for the position of the bifurcation point.

An extended Bonhoeffer–van der Pol system. Consider the system

$$\begin{aligned} \varepsilon\dot{x} &= y_1 + x - \tfrac{1}{3}x^3, \\ \dot{y}_1 &= y_2 - x, \\ \dot{y}_2 &= a + by_1 - y_2, \end{aligned} \tag{21}$$

with $0 < \varepsilon \ll 1$ and $a, b \in \mathbb{R}$. We consider the parameter a as the bifurcation parameter, while the secondary parameter b may take arbitrary values. This system is an extension of the Bonhoeffer–van der Pol system. The latter system has been studied by several authors as a prototype model for two-dimensional singular Hopf bifurcation and canard behavior; see the references given in the introduction.

We will look for equilibria that satisfy the conditions of Theorem 1. Put $f(x, y, a) = y_1 + x - \frac{1}{3}x^3$ and $g(x, y, a) = (y_2 - x, a + by_1 - y_2)$. First we look for equilibria $f = 0$, $g = 0$ where additionally $f_x = 1 - x^2$ vanishes. A straightforward computation shows that $\pm(1, -2/3, 1)$ are the only such equilibria. Because the system is point-symmetric, we can restrict to $(1, -2/3, 1)$. The corresponding parameter value is $\bar{a} = 2/3\ b + 1$, where $b \in \mathbb{R}$ is arbitrary. This means that there is a one-parameter family of candidates for singular Hopf bifurcation. The relevant partial derivatives at these points are

$$\bar{f}_x = 0, \qquad \bar{f}_{xx} = -2, \qquad \bar{f}_{xxx} = -2,$$

$$\bar{f}_y = (1 \quad 0), \qquad \bar{f}_{xy} = (0 \quad 0), \qquad \bar{f}_a = (0 \quad 0), \qquad \bar{f}_{xa} = (0 \quad 0),$$

$$\bar{g}_x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \bar{g}_y = \begin{pmatrix} 0 & 1 \\ b & -1 \end{pmatrix}, \qquad \bar{g}_{xx} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \bar{g}_a = \begin{pmatrix} 0 & 0 \\ 1 & -2/3 \end{pmatrix},$$

and

$$\begin{pmatrix} \bar{f}_x & \bar{f}_y \\ \bar{g}_x & \bar{g}_y \end{pmatrix}^{-1} = \begin{pmatrix} b & -1 & -1 \\ 1 & 0 & 0 \\ b & 0 & -1 \end{pmatrix}.$$

It is easy to check that the conditions of Theorem 1 are satisfied, such that we can use that theorem (we must take $k = 1$) to obtain the normal form

$$\begin{aligned} \dot{u} &= v + \tfrac{1}{2}u^2 + \delta\ (\alpha u + uw - \tfrac{1}{12}u^3) + \mathcal{O}(\delta^2), \\ \dot{v} &= -u + \mathcal{O}(\delta^2), \\ \dot{w} &= \delta\ (-w - \tfrac{1}{2}bu^2) + \mathcal{O}(\delta^2). \end{aligned}$$

The bifurcation parameter α is defined by

$$\alpha = -2A,$$

with A given by $a = \bar{a} + \varepsilon A$. Since $H_w = (-1)$ clearly has no elliptic part, Theorem 2 gives the existence (for sufficiently small ε) of a Hopf bifurcation point at $\alpha = 0$. This point corresponds to $a = 2/3\ b + 1 + \mathcal{O}(\varepsilon\sqrt{\varepsilon})$, where b is arbitrary. The corresponding equilibrium is given by $(x, y_1, y_2) = (1, -2/3, 1) + \mathcal{O}(\varepsilon^2)$, cf. (8). Stability formula (15) yields that the bifurcation is supercritical for $b > b_0$, subcritical for $b < b_0$, and degenerate for $b = b_0$, where $b_0 = -1/4 + \mathcal{O}(\sqrt{\varepsilon})$.

Thus, we have found a line of Hopf bifurcations in the (b, a) plane. For $b > b_0$ we find stable limit cycles below this line, and for $b < b_0$ we find unstable limit cycles above the line. Due to the symmetry, the reflection of this line in the b-axis is also a line of Hopf bifurcations.

For $b = 0$ we have a special situation, since the equation for w decouples. On the (hyperbolically attracting) invariant plane $w = 0$ the system behaves exactly like the two-dimensional Bonhoeffer–van der Pol system studied by several authors; see the references in the introduction. This behavior includes the 'canard' explosion of the size of the limit cycle in an exponentially small parameter region. For $b \neq 0$ there is no such explosion; the limit cycle grows more gradually, and more complex oscillations may develop as well. We come back to this matter in a sequel to this paper.

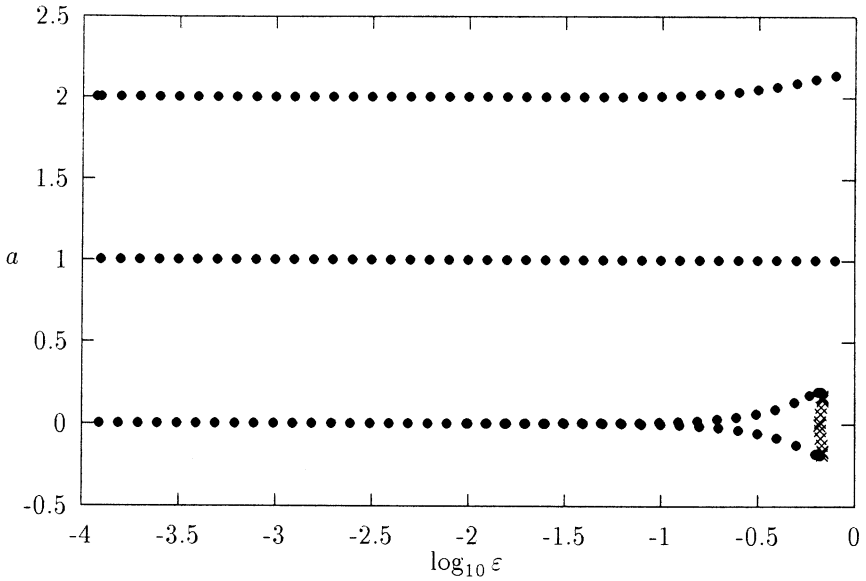

Fig. 2. Hopf bifurcation values for (21) for $b = 3/2$ (upper points), 0 (middle), and $-3/2$ (lower). Plotted is $\log_{10} \varepsilon$ vs. a for the equilibrium close to $(1, -2/3, 1)$, for the range $10^{-4} < \varepsilon < 1$. The corresponding asymptotic values for a are (up to $\mathcal{O}(\varepsilon^{3/2})$) 2, 1, and 0, respectively. Crosses on the lowest curve mark equilibria with a positive third eigenvalue.

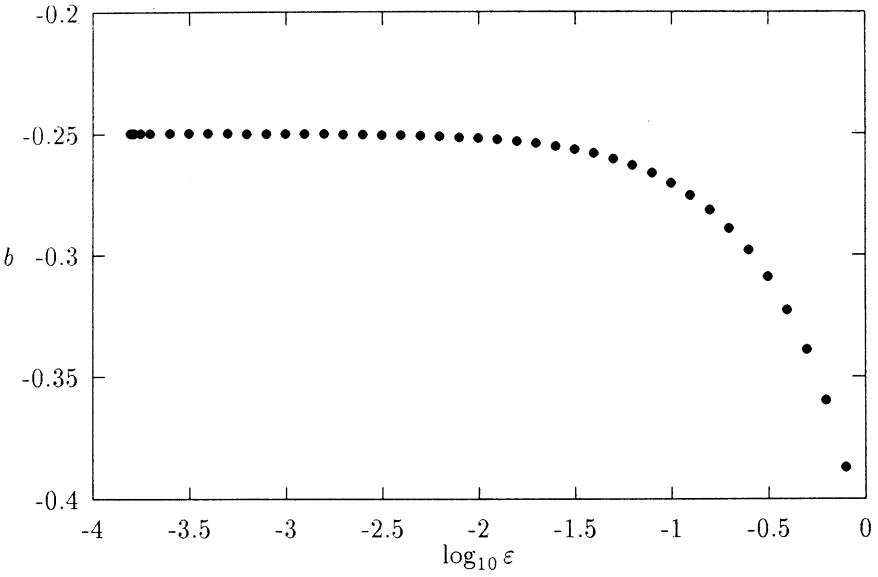

Fig. 3. Curve of degenerate Hopf bifurcation for (21). Plotted is $\log_{10} \varepsilon$ vs. b for the equilibrium close to $(1, -2/3, 1)$. The asymptotic value for b is $-1/4$.

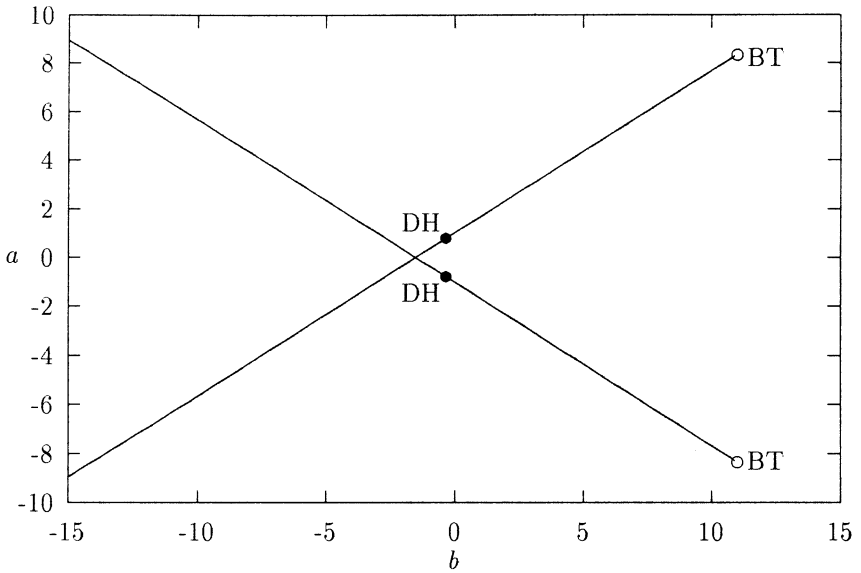

Fig. 4. Hopf bifurcation curves for (21), $\varepsilon = 0.1$. Plotted is b vs. a for both equilibria $\pm(1, -2/3, 1)$. On the right, the bifurcation curves end in Bogdanov-Takens points (BT) at $(b, a) \approx (10.92, \pm 8.30)$. Degenerate Hopf bifurcation points (DH) are located at $(b, a) \approx (-0.271, \pm 0.818)$. Both curves continue indefinitely to the left, and remain almost straight lines.

Koper (1995) carries out a detailed numerical bifurcation study for a system similar to (21) and finds a very complicated bifurcation structure close to the line of singular Hopf bifurcation.

In Figures 2–4 we present numerical results for (21). These results were obtained with the continuation package LOCBIF. The agreement between the numerical experiments and our asymptotic predictions is excellent. Note that for any fixed value of $\varepsilon > 0$, the Hopf bifurcation curve in the (a, b) plane ends in Bogdanov-Takens points. At these points, the Jacobian has a nilpotent part with two zero eigenvalues. A short calculation shows that these points occur at equilibria corresponding to parameter values $b = (1 + \sqrt{1 + 4\varepsilon})/2\varepsilon$, $a = \pm 2/3\ \varepsilon b^2\sqrt{\varepsilon b}$. This result is not in contradiction with our theory, since both a and b are of order $\mathcal{O}(1/\varepsilon)$ here, such that the y_2-equation is as fast as the x-equation. Hence, the system is not of the form (1) anymore, and our theory does not apply.

The lower curve in Figure 2, corresponding to $b = -3/2$, exhibits turning points at $\varepsilon = 2/3$, $a = \pm 1/3\sqrt{3}$, $x = \pm 1/\sqrt{3}$. This turning point behavior is typical for negative values of b and is associated with a vanishing third eigenvalue. At these turning points the Hopf curve folds back on itself. Along the segment marked with crosses, the Hopf equilibrium moves, via the origin and with a positive third eigenvalue, from the neighborhood of $(1, -2/3, 1)$ to the neighborhood of $(-1, 2/3, -1)$ (for other values of $b < 0$, this would yield two asymptotically parallel lines and not two asymptotically coinciding

lines). The combination of a zero eigenvalue with a pair of eigenvalues on the imaginary axis gives rise of course to an additional degeneracy; see for example Guckenheimer & Holmes (1983) or Kuznetsov (1996), and their references, for a discussion of the so-called 'Fold-Hopf' codimension-two bifurcation. It is easy to check that this combination of eigenvalues implies that $\varepsilon = -1/b$. Hence, in the asymptotic limit, this codimension-two bifurcation cannot occur close to singular Hopf points, just as with the Bogdanov-Takens bifurcation.

A predator-prey model. Consider the system

$$\begin{aligned} \varepsilon \dot{x} &= x P(x, y, a), \qquad x \in \mathbb{R}, \qquad 0 < \varepsilon \ll 1, \\ \dot{y} &= y Q(x, y, a), \qquad y \in \mathbb{R}^n. \end{aligned}$$

This is the basic structure for a predator-prey model, where a number of predators hunt for the same prey. For more information we refer to the literature; see for example chapter 3 in Murray (1989). The variable x represents the size of a prey population, and the variables y represent sizes of different predator populations, all feeding on the same prey. The small parameter ε expresses the fact that the life cycle of the prey species is much shorter than that of the predators, which is a reasonable assumption, for example, in the case of insect-eating birds.

To be more specific, we suppose that the growth rate functions P and Q can be written as $P = P_1(x, a) - P_2(y)$ and $Q = Q_1(x) - Q_2(y)$. Furthermore, we suppose that the functions P_2, Q_1, and Q_2 are all positive (which is a standard assumption for predator-prey models), and that the function P_1 is positive on an open set (but not necessarily at $x = 0$). In the absence of predators ($y = 0$), the prey species will evolve until it reaches an equilibrium point where $P_1(x, a) = 0$, or will become extinct. In the absence of prey, all predators will become extinct. Hence, oscillations are only possible if there is a nonzero population of prey and at least one predator species. We will investigate the possibilities of singular Hopf bifurcation. According to condition [N4] in Theorem 1, we should look for equilibria where $\partial/\partial x(xP(x, y, a))$ vanishes. If we exclude the trivial equilibrium, this implies that $P_1'(\bar{x}, \bar{a}) = 0$. Condition [N5] is always satisfied because of the opposite signs of P_2 and Q_1, and condition [N6] translates to $\mathcal{P}_1''(\bar{x}, \bar{a}) \neq 0$. The other conditions may fail for some P and Q of the given type, but since these conditions are generic they will be satisfied for 'most' choices. From this knowledge we conclude that singular Hopf bifurcation is possible for generic P and Q (of the given type) only if the function P_1 is not monotonous. Nonmonotonicity of P_1 is related to the so-called *Allee effect*. Such models typically exhibit a threshold phenomenon; see Murray (1989) for a discussion.

We restrict the class of models even further and suppose that P_1 is a quadratic function, P_2, Q_1, and Q_2 are linear and Q_2 is a diagonal matrix (which means that there are no interactions between the predators other than their competition for the same food resources). After a rescaling of the variables and parameters, we may assume that $P_1(x, a) = -x^2 + 2x + a$, $P_2 = (p_1 y_1 + \cdots + p_n y_n)$, $Q_1 = (q_1 x, \ldots, q_n x)^T$, and $Q_2 = \mathrm{diag}(r_i + y_i)$, where $a > -1$ is the bifurcation parameter and $p_i, q_i, r_i, 1 \leq i \leq n$, are positive constants with $q_i > r_i$.

For singular Hopf bifurcation purposes we seek an equilibrium where P_1' vanishes. The only equilibrium where all $\bar{y}$-values are positive (a zero $\bar{y}_i$-value amounts to a dimension

reduction) is $\bar{x} = 1, \bar{y}_i = q_i - r_i, \bar{a} = -1 + \sum_{i=1}^{n} p_i \bar{y}_i$. A tedious calculation, for which a computer algebra package is useful (we used MAPLE) gives the normal form coefficients of Theorem 1 as (we took $k = n$, but any value is allowed)

$$F_{uw} = \hat{p}, \qquad F_{uuu} = -\omega^2 + \Gamma, \qquad H_w = -R + \omega^{-2}\hat{q}\,\hat{p}, \qquad H_{uu} = \omega^{-2}(-R + \Gamma)\hat{q},$$

and the bifurcation parameter α relates to the scaled parameter $A = (a - \bar{a})/\varepsilon$ via

$$\alpha = -2S^{-1}A - \Gamma,$$

where $\omega^2 = \sum p_i q_i \bar{y}_i, \Gamma = \sum p_i q_i \bar{y}_i^2 / \sum p_i q_i \bar{y}_i, S = \sum p_i q_i, R = \mathrm{diag}(\bar{y}_1, \ldots, \bar{y}_{n-1})$, $\hat{p} = (p_1(\bar{y}_1 - \bar{y}_n), \ldots, p_{n-1}(\bar{y}_{n-1} - \bar{y}_n))$, and $\hat{q} = (q_1\bar{y}_1, \ldots, q_{n-1}\bar{y}_{n-1})^T$. Sums are taken over the range $1 \le i \le n$. The special structure of the functions P and Q enables us to show that, apart from one pair of complex conjugate eigenvalues close to the imaginary axis with $\mathcal{O}(1/\varepsilon)$ imaginary parts, all other eigenvalues of the system, linearized at $(\bar{x}, \bar{y}, \bar{a})$, are real and negative. This follows easily from inspection of the characteristic equation, written in the form $(\varepsilon\lambda + \sum p_i q_i \bar{y}_i/(\bar{y}_i + \lambda)) \prod(\bar{y}_i + \lambda) = 0$, since all constants in this equation are positive.

Hence, an application of Theorem 2 gives a singular Hopf bifurcation at $a = \bar{a} - \frac{1}{2}\varepsilon S\Gamma + \mathcal{O}(\varepsilon\sqrt{\varepsilon})$. The corresponding center manifold is attracting. The stability formula (15) reads

$$-\tfrac{1}{2}\omega^2 + \tfrac{1}{2}\Gamma - \frac{1}{S\omega^2} \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} p_i q_i \; p_j q_j (\bar{y}_i - \bar{y}_j)^2.$$

The Hopf bifurcation is supercritical (subcritical) if the above expression is positive (negative). Due to the large number of parameters, it is not easy to determine the stability type of the Hopf bifurcation, but for any given numerical values this computation is a straightforward exercise.

A particular simplification arises when all predator equilibrium populations have the same size, $\bar{y}_i = \bar{y}_0$. The stability formula then simplifies to $\frac{1}{2}\bar{y}_0(1 - S)$, such that stable (unstable) oscillations occur for $S = \sum p_i q_i < (>) 1$. Note that both F_{uw} and H_{uu} vanish in this case, such that the (u, v)-and w-subsystems in the normal form decouple and the situation is essentially two-dimensional. A perturbation argument shows that for $\bar{y}_i = \bar{y}_0 + \eta_i$, with $|\eta_i|$ small, the stability type of the $\eta_i = 0$ situation persists. This implies that both super- and subcritical Hopf bifurcations are possible when the $\bar{y}$-values are not identical.

For the numerical experiments (again performed with LOCBIF), we used a different simplification in order to reduce the number of parameters. We took a three-predator ($n = 3$) system and fixed $r_1 = 1.2, r_2 = 1.0, r_3 = 0.8$. Furthermore, we took all p- and q- values equal to p_0 and q_0, respectively. This leaves us four parameters—ε, a, p_0, and q_0—to play with.

In Figure 5 we present a comparison between asymptotic and numerical results. On the log-log scale it is clearly seen that the agreement is excellent. Even for the not-so-small value of $\varepsilon = 1$, the error is within ten percent. Based on the slope of the curve, it appears that the error decays like ε^2 instead of $\varepsilon\sqrt{\varepsilon}$.

In Figure 6 we present an asymptotic curve and numerical values for the position of degenerate Hopf points. Keeping the other parameters fixed, we let p_0 and q_0 vary and

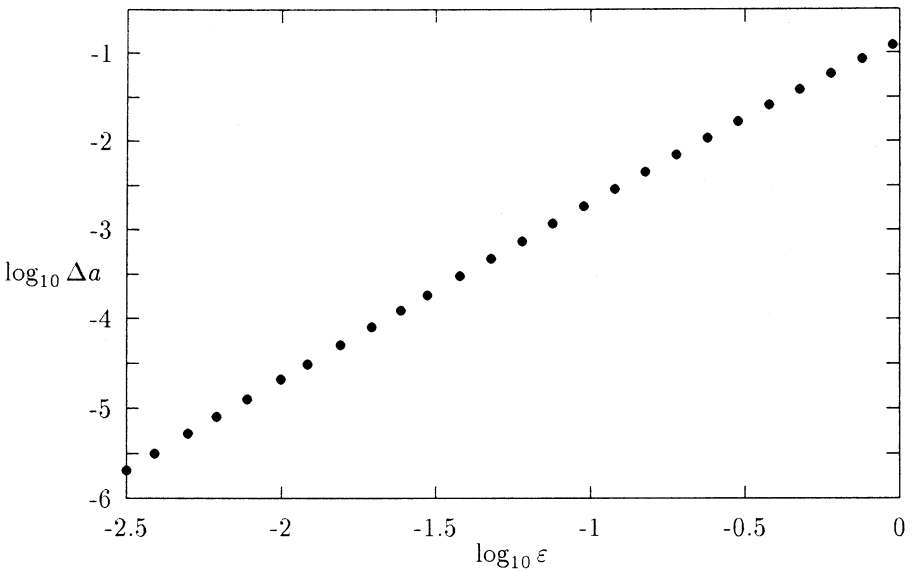

Fig. 5. Plot of the accuracy of asymptotic predictions for Hopf bifurcation for the predator-prey system. Plotted (on logarithmic scales) are the differences Δa between asymptotic and numerical values for $10^{-2.5} < \varepsilon < 1$. Parameter values are $p_0 = 0.25, q_0 = 1.5, r_1 = 1.2, r_2 = 1.0, r_3 = 0.8$. The asymptotic value for a is $a = -0.625 - 0.31125\,\varepsilon + \mathcal{O}(\varepsilon^{3/2})$.

computed the corresponding Hopf bifurcation value for a. The agreement is again good. The asymptotic formula for the degenerate Hopf bifurcation curve, which is obtained by equating the stability formula above to zero, reads $p_0 = ((q_0-1)^2-2/75)/(3q_0(q_0-1)^2)$. Note that both curves end at $(q_0,\ p_0) = (1.2,\ 0.0926)$, because the $\bar{y}_1$ equilibrium value here crosses the zero axis and assumes "impossible" (negative population) values.

Appendix A A Transformation to Simplify (11)

In this Appendix we describe a method to find a transformation which removes the uv-term in (11). We perform the calculations for a reduced (u, v) subsystem only, omitting the α- and w-dependent terms. It is easy to check that these terms are not affected by transformations of the class (24). Consider the system

$$\begin{aligned} \dot{u} &= v + \tfrac{1}{2}u^2 + \delta(C_{uv}uv + \tfrac{1}{6}C_{uuu}u^3) + \mathcal{O}(\delta^2), \\ \dot{v} &= -u + \mathcal{O}(\delta^2). \end{aligned} \tag{22}$$

We seek a transformation that removes the uv-term without affecting the $\mathcal{O}(1)$ part or introducing new $\mathcal{O}(\delta)$ terms. As before, we do not bother about creating additional higher-order terms. The resulting system should have the structure

$$\dot{\tilde{u}} = \tilde{v} + \tfrac{1}{2}\tilde{u}^2 + \delta(\tfrac{1}{6}\tilde{C}_{uuu}\tilde{u}^3) + \mathcal{O}(\delta^2),$$

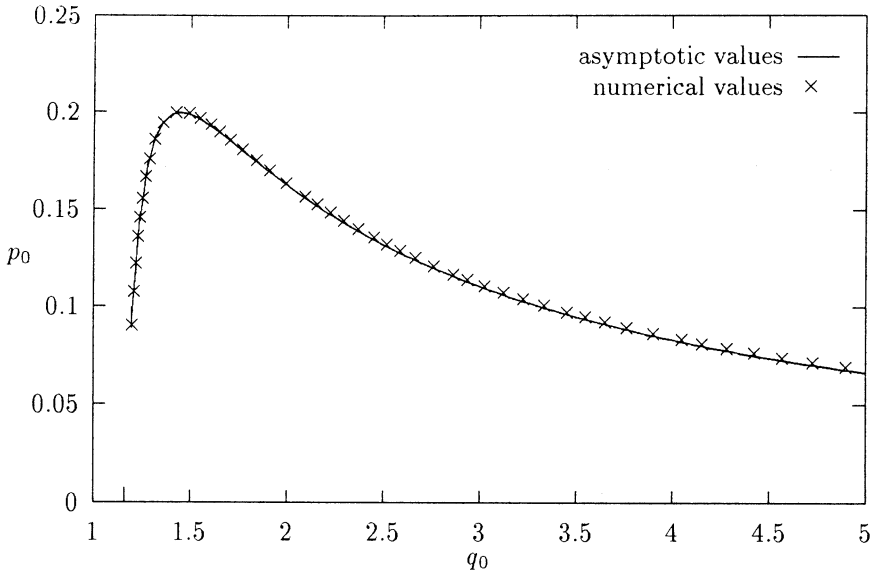

Fig. 6. Curve of degenerate Hopf bifurcation for the predator-prey system. Plotted are both asymptotic and numerical values for q_0 vs. p_0. The corresponding values for a are not shown. Parameter values are $\varepsilon = 0.01, r_1 = 1.2, r_2 = 1.0, r_3 = 0.8$. The Hopf bifurcation is supercritical below and subcritical above this curve. The curves end at $q_0 = 1.2$, where $\bar{y}_1$ vanishes.

$$\dot{\tilde{v}} = -\tilde{u} + \mathcal{O}(\delta^2). \tag{23}$$

Furthermore, we want an explicit (finite) expression for the transformation, and not just a formal series expansion. Therefore we restrict to a class of polynomial transformations. We consider transformations of the form

$$\begin{aligned} \tilde{u} &= u + \delta \sum_{k=0}^{K} \phi_k(v) u^k, \\ \tilde{v} &= v + \delta \sum_{\ell=0}^{L} \psi_\ell(v) u^\ell, \end{aligned} \tag{24}$$

where ϕ_k, ψ_ℓ are polynomials in v and K, L are positive (finite) integers.

The result of introducing this near-identity transformation in system (23) is (always summing over the obvious index variable and range)

$$\begin{aligned} &\dot{u}(1 + \delta \sum k\phi_k u^{k-1}) + \dot{v}(\delta \sum \phi_k' u^k) = v + \delta \sum \psi_\ell u^\ell \\ &\qquad + \tfrac{1}{2}u^2 + \delta u \sum \phi_k u^k + \delta(\tfrac{1}{6}\tilde{C}_{uuu} u^3) + \mathcal{O}(\delta^2), \\ &\dot{u}(\delta \sum \ell\psi_\ell u^{\ell-1}) + \dot{v}(1 + \delta \sum \psi_\ell' u^\ell) = -u - \delta \sum \phi_k u^k + \mathcal{O}(\delta^2). \end{aligned}$$

If we use (22) to substitute for $\dot{u}$ and $\dot{v}$, we find that the $\mathcal{O}(1)$ terms cancel, while the

$\mathcal{O}(\delta)$ terms read

$$(v+\tfrac{1}{2}u^2)\sum k\phi_k u^{k-1} - u\sum \phi_k' u^k + C_{uv}uv + \tfrac{1}{6}C_{uuu}u^3 = \sum \psi_\ell u^\ell + u\sum \phi_k u^k + \tfrac{1}{6}\tilde{C}_{uuu}u^3, \tag{25a}$$

$$(v+\tfrac{1}{2}u^2)\sum \ell\psi_\ell u^{\ell-1} - u\sum \psi_\ell' u^\ell = -\sum \phi_k u^k. \tag{25b}$$

From these equations we try to determine a suitable transformation of the form (24).

First we determine the value of L. The highest-order term in u on the left-hand side of (25b) is $(\frac{1}{2}L\psi_L - \psi_L')u^{L+1}$. Since ψ_L is a nonzero polynomial (by assumption), this term cannot vanish and hence must equal the highest-order term on the right-hand side, $-\phi_K u^K$, which implies that $L = K-1$.

Next we determine K. It can be shown easily that $K \neq 2$ leads to a contradiction. For example, if $K > 2$, the equation for the highest-order coefficients in u in (25a) reads $\frac{1}{2}K\phi_K - \phi_K' = \phi_K$. The only polynomial solution of this equation is $\phi_K = 0$, which is impossible since (by definition) the highest-order coefficient does not vanish. We omit the simple arguments showing that the cases $K = 0$ and $K = 1$ also lead to contradictions. For $K = 2$ the situation is different. The ϕ_K-terms in the equation for the highest-order coefficients cancel, and we must take additional cubic terms into account. The resulting equation for the u^3 terms reads $-\phi_2' + \frac{1}{6}C_{uuu} = \frac{1}{6}\tilde{C}_{uuu}$. It admits solutions of the form $\phi_2 = \frac{1}{6}(C_{uuu} - \tilde{C}_{uuu})v + c_1$. The coefficient $\tilde{C}_{uuu}$ and the integration constant c_1 are as yet undetermined.

Thus, we have $K = 2$, $L = 1$, and it remains to determine the coefficients of ϕ_2 and the polynomials $\phi_0, \phi_1, \psi_0, \psi_1$. These polynomials satisfy the following set of equations, which is obtained when we collect terms with equal powers of u in (25a) and (25b),

$$\tfrac{1}{2}\phi_1 - \phi_1' = \phi_1, \tag{26a}$$
$$2v\phi_2 - \phi_0' + C_{uv}v = \psi_1 + \phi_0, \tag{26b}$$
$$v\phi_1 = \psi_0, \tag{26c}$$
$$\tfrac{1}{2}\psi_1 - \psi_1' = -\phi_2, \tag{26d}$$
$$-\psi_0' = -\phi_1, \tag{26e}$$
$$v\psi_1 = -\phi_0. \tag{26f}$$

These equations actually constitute two independent subsystems. Equations (26a), (26c), and (26e) can be solved simultaneously for ϕ_1 and ψ_0. They only admit the trivial solution $\phi_1 = \psi_0 = 0$. Equations (26b), (26d), and (26f) can be solved for ϕ_2, ϕ_0, and ψ_1. The unique solution is

$$\phi_2 = \tfrac{1}{6}C_{uv}v - \tfrac{1}{3}C_{uv} - \tfrac{1}{2}c_2,$$
$$\phi_0 = \tfrac{1}{3}C_{uv}v^2 - c_2 v,$$
$$\psi_1 = -\tfrac{1}{3}C_{uv}v + c_2,$$

where c_2 is a new integration constant. Because c_2 affects linear terms and we do not want to change the linear part, it is necessary to put $c_2 = 0$. A comparison with our previous result for ϕ_2 yields $\tilde{C}_{uuu} = C_{uuu} - C_{uv}$ and $c_1 = \frac{1}{3}C_{uv}$.

To summarize, the desired transformation (24) reads

$$\tilde{u} = u + \tfrac{1}{3}\delta C_{uv}(\tfrac{1}{2}u^2 v - u^2 + v^2), \qquad \tilde{v} = v - \tfrac{1}{3}\delta C_{uv} uv,$$

and the new coefficient for the cubic term is $\tilde{C}_{uuu} = C_{uuu} - C_{uv}$.

Remark. Using a similar approach, it is possible to remove the u^3-term in (22) instead of the uv-term. In that case, the coefficient for the uv-term in the resulting system will be $\tilde{C}_{uv} = C_{uv} - \tilde{C}_{uuu}$.

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