J. Nonlinear Sci. Vol. 8: pp. 103–129 (1998)

Journal of Nonlinear Science

Attractors for Robust Heteroclinic Cycles with Continua of Connections

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Original received April 29, 1996; revised manuscript accepted for publication April 16, 1997
Communicated by Martin Golubitsky

Summary. For systems with symmetry (or more generally, systems with invariant subspaces) it is possible to find robust heteroclinic cycles with multi-dimensional connecting manifolds.

Motivated by a problem of rotating convection with low Prandtl number, Swift and Barany [23] considered generic Hopf bifurcation with tetrahedral symmetry. In this situation it is possible to get bifurcation from a steady state directly to a homoclinic cycle with a two-dimensional set of connections. We numerically investigate the dynamics near such cycles.

We conjecture that if a heteroclinic cycle is asymptotically stable then all connections corresponding to the most positive expanding eigenvalues of the linearisation at the fixed points will generically form part of an attractor. This attractor may fail to be asymptotically stable and is, to our knowledge, the first example of this for a homoclinic (as opposed to a heteroclinic) cycle. We prove this conjecture for homoclinic cycles with distinct real expanding and contracting eigenvalues, and present evidence to support it for other cases. An example due to Kirk and Silber [15] (two competing cycles in $\mathbf{R}^4$ with $(\mathbf{Z}_2)^4$ symmetry) is discussed and continua of connections are found in this example.

Key words. heteroclinic cycle, nonlinear dynamics, symmetry

MSC numbers. 34C37, 58F13

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1. Introduction

Heteroclinic behaviour is an important mechanism for creating intermittent behaviour in nonlinear systems. For systems with symmetry, there can be a network of dynamically invariant subspaces which cause the dynamics to possess such heteroclinic cycles in a robust manner, provided that the connections between the fixed points respect the invariant subspaces in a certain way. The structural stability of these cycles in such systems has been cited as possibly causing intermittent complex dynamics in several physical systems, notably rotating convection [4].

Until now, a major emphasis has been to understand the dynamics and stability of robust heteroclinic cycles with one dimensional connections. However, heteroclinic cycles with two or more dimensional manifolds can also be robust, and as we discuss in Section 2 there are new aspects of cycles that need to be considered, in particular their closedness.

To motivate our investigation, we consider an example of Hopf bifurcation with tetrahedral symmetry, originally investigated by Swift and Barany [23] as a model for the onset of convection at low Prandtl number in a rotating fluid layer. The rest of the paper is organised as follows: In Section 2 we define a *heteroclinic chain*, a chain of connections between equilibria, and a *heteroclinic cycle* where the connections imply a cyclic ordering of the equilibria. Although continua of homoclinic connections are of infinite codimension in systems without symmetry, they can be robust (i.e., codimension zero) in systems with symmetry.

In Section 3 we focus our attention on homoclinic cycles. We conjecture that if the cycle is asymptotically stable then the unstable manifold corresponding to the most positive eigenvalues determines a subset that is a Milnor attractor, i.e., a subset that attracts a large measure set of nearby points, but this attractor is not Liapunov stable. In other words, we conjecture that generically a subset of the continuum of connections will be *observed* to have attracting behaviour. This conjecture is proved for some classes of homoclinic cycles where the expanding and contracting eigenvalues are real and distinct. We give an example with complex expanding and contracting eigenvalues where almost all nearby points have ω-limit sets containing the whole two-dimensional homoclinic cycle. Moreover, in this case they visit the connections in a uniformly distributed manner.

In Section 4 we discuss some examples. In Section 4.1 we investigate the dynamics of the local return map. In Section 4.2 we discuss an example of a heteroclinic cycle with $\mathbf{Z}_2^4$ symmetry given by Kirk and Silber [15]. We show that there are continua of connections in addition to the ones they have discussed. Section 5 discusses some generalisations and open problems.

1.1. Generic Hopf Bifurcation with Tetrahedral Symmetry

Swift and Barany [23] noticed that rotating planar convection with low Prandtl number can undergo Hopf bifurcation with tetrahedral symmetry, if one restricts examination to the interaction of three standing waves at 120° angles to each other.

The bifurcation displays a variety of interesting dynamical behaviour. Swift and Barany show that there can be bifurcation from a steady state directly to chaotic dynamics in certain generic (i.e., codimension one) Hopf bifurcations.

Using invariant theory to construct a normal form, they show that generic Hopf bifurcation with the symmetry of a tetrahedron T (i.e., the group of rotations that map a regular solid tetrahedron onto itself) is locally governed by a third-order normal form equation for $z = (z_1, z_2, z_3) \in \mathbf{C}^3$ with $T \times \mathbf{S}^1$ symmetry given by

$$\dot{z}_1 = z_1 \left(\lambda + \gamma|z|^2 + \alpha_n|z_2|^2 + \alpha_s|z_3|^2\right) + \bar{z}_1 \left(\beta_n z_2^2 + \beta_s z_3^2\right), \tag{1}$$

where $|z|^2 = |z_1|^2+|z_2|^2+|z_3|^2$. The other two equations are found by cyclic permutation of the subscripts. This commutes with a group action of $T \times \mathbf{S}^1$ given by

$$\begin{aligned} z &\to Cz = (z_2, z_3, z_1), \\ z &\to Rz = (z_1, -z_2, -z_3), \\ z &\to \theta z = e^{i\theta} z. \end{aligned}$$

The group elements C and R generate T while the phase shifts θ generate $\mathbf{S}^1$. The normal form (1) has six complex coefficients: λ, γ, α_n, α_s, β_n, and β_s; the real part of λ is the bifurcation parameter. We write $\lambda = \lambda_R + i\lambda_I$, etc., to denote the real and imaginary parts of the parameters.

Swift and Barany [23] reduce (1) to an ODE in four dimensions by taking a quotient by $\mathbf{S}^1$ to reduce it to a steady-state bifurcation problem. They note there can be an invariant sphere on which the dynamics occur. We follow their approach, and so relative equilibria correspond to periodic orbits of the original system. Since the $\mathbf{S}^1$ symmetry is a normal form symmetry, one should be aware that the original dynamics will have broken symmetry and this will perturb what we describe here. We shall not discuss this issue in any detail but refer the reader to [10, p. 313] for a discussion of some related issues.

There are three fixed point spaces of two (real) dimensions forced by the action of the symmetry group:

$$\text{Fix } \Sigma_1 = (z_1, 0, 0), \qquad \text{Fix } \Sigma_2 = (0, z_2, 0), \qquad \text{Fix } \Sigma_3 = (0, 0, z_3).$$

For $\lambda_R > 0$ and $\gamma_R < 0$, within each subspace Fix Σ_i, there are fixed points, e.g. $\xi_1 = (z_1, 0, 0)$ for $|z_1|^2 = -\lambda_R/\gamma_R$ with ξ_2, ξ_3 similarly defined. These ξ_i are on a single group orbit. We study homoclinic cycles between this group orbit of $\mathbf{S}^1$-relative equilibria.

There are also three fixed point spaces of four (real) dimensions given by

$$\text{Fix } \Delta_1 = (z_1, z_2, 0), \qquad \text{Fix } \Delta_2 = (0, z_2, z_3), \qquad \text{Fix } \Delta_3 = (z_1, 0, z_3).$$

These have three (real) dimensions in the orbit space. Note that Fix Δ_j contains both ξ_j and ξ_{j+1}. There are other fixed point spaces that do not concern us here; see [23].

The eigenvalues of the relative equilibrium ξ_1 are as follows: One eigenvalue in the direction of[1] Fix $\Delta_1 \cap$ Fix Δ_3 is $-2\lambda_R$; this is negative. The remaining eigenvalues in Fix Δ_1 have sum $(\alpha_n)_R$ and product $|\alpha_n|^2 - |\beta_n|^2$ while those in Fix Δ_3 have sum $(\alpha_s)_R$

[1] By "direction of an eigenvalue" we refer of course to the corresponding eigenspace.

Table 1. Parameter values for the simulations of homoclinic cycles in the tetrahedral Hopf bifurcation. $\lambda = 1$ and $\gamma = -1$ are fixed and the contracting and expanding eigenvalues of the fixed points tangent to the cycle are shown. The "Figure" column refers to the illustrations in Figure 1.

α_n	α_s	β_n	β_s	Contracting	Expanding	Figure
$0.8 + 0.5i$	$-1 - 5i$	0.52	0	$-1 \pm 5i$	0.9428, 0.6572	(i),(ii)
$0.3 + i$	$-0.65 - 0.5i$	0	0.6	-0.9816, -0.3183	$0.3 \pm i$	(iii)
$0.65 + 0.5i$	$-0.9 + 0.5i$	0.51	0.49	$-0.9 \pm 0.099i$	0.7505, 0.5495	(iv)

and product $|\alpha_s|^2 - |\beta_s|^2$. By specifying the coefficients arbitrarily we can place these eigenvalues as we wish.

Swift and Barany recognised that for open regions in the space of normal-form coefficients, one can get bifurcation directly to a set of three $\mathbf{S}^1$-relative equilibria robustly connected by three $\mathbf{S}^1$-orbits of two-dimensional manifolds. By choosing the parameters, it is easy to make this union of 2-manifolds asymptotically stable.

The question we address is: What are the ω-limit sets in the case that this set is asymptotically stable? In the case of an asymptotically stable homoclinic cycle which is a union of one-dimensional connections there are only a finite number of possible ω-limit sets, but if the connections are on two-dimensional manifolds, the possibilities are uncountable. To get some idea of these possibilities, we report on some numerical experiments.

Numerical simulations. For the parameter values shown in Table 1 we have performed simulations using the program `dstool` [14]. The column headed "Figure" refers to the trajectory shown in Figure 1. To remove the group orbit given by the action of $T \times \mathbf{S}^1$ we use a projection onto the complex plane defined by Swift and Barany (they call this Π_1):

$$\Pi(z) = 3\left((z_1\bar{z}_2)^2 + (z_2\bar{z}_3)^2 + (z_3\bar{z}_1)^2\right)/|z|^4.$$

The projection Π corresponds to a projection of the orbit space $\mathbf{R}^6/(T \times \mathbf{S}^1)$ onto $\mathbf{C}$. Figure 1 shows some typical simulations for this system in cases where there are attracting homoclinic cycles. (i) shows an irregular-looking transient apparently converging to the single connection (in the orbit space), namely (ii). Figure (iii) shows a trajectory converging apparently to the whole of the cycle, while (iv) shows an attractor for another case of complex contracting and real expanding eigenvalues.

In general, we have observed that an *isolated connection* within the continuum attracts most initial conditions if the expanding eigenvalues are *real* whereas the two dimensional *set of all connections* is an attractor if the expanding eigenvalues are *complex*. We conjecture that this is generically the case. For the case of real and distinct expanding and contracting eigenvalues we present a proof, and show that it is possible to have an attractor which is the whole cycle if both expanding and contracting eigenvalues are complex.

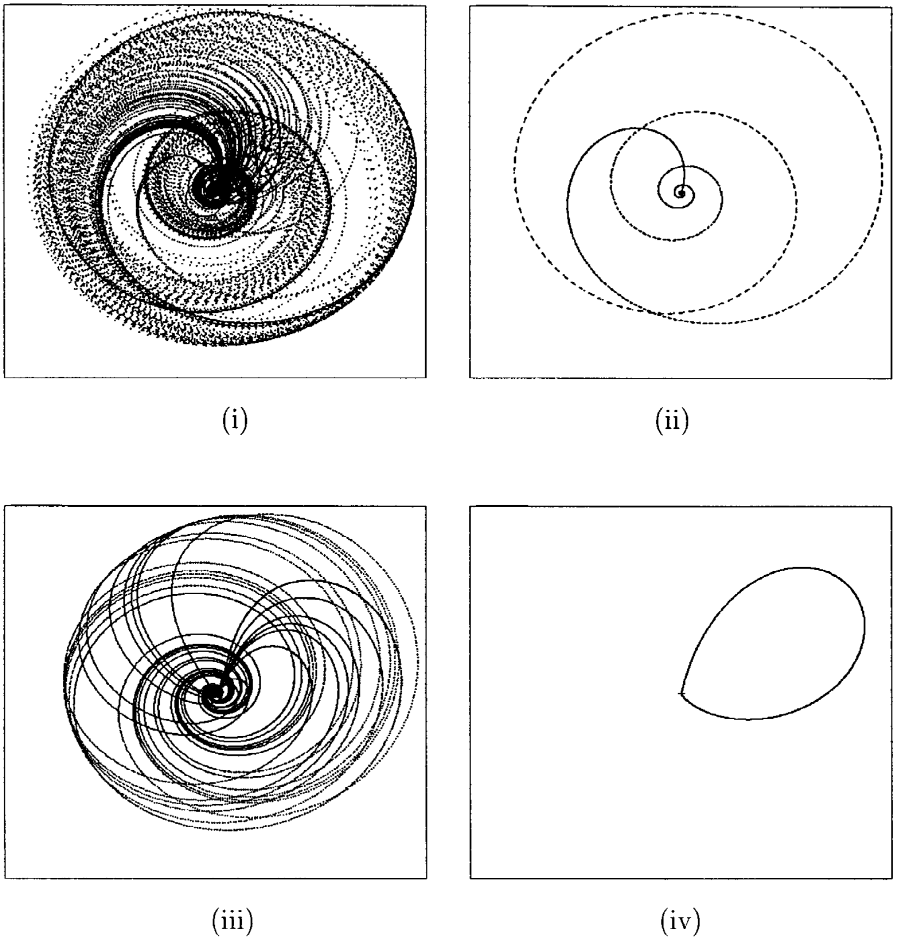


Fig. 1. Examples of trajectories evolving towards asymptotically stable homoclinic cycles with two-dimensional sets of connections for the tetrahedral Hopf bifurcation. The projection of trajectories by Π are shown, and each bounding box corresponds to the square $[-0.25, 0.25] + i[-0.25, 0.25]$. The parameter values are shown in Table 1. (i) Shows a transient converging to the attracting cycle shown in (ii). (iii) and (iv) correspond to behaviour after transients have apparently decayed. In this projection, the group orbit of the fixed points are all projected onto the origin. In all cases there exists a closed one-parameter continuum of homoclinic connections.

2. Definitions

Consider a finite Lie group Γ acting smoothly on a smooth manifold M.[2] For Σ an isotropy subgroup of Γ, we define M_Σ to be the manifold of all points in M with

[2] If $\dim \Gamma \geq 1$ we can examine the orbit space M/Γ_0 and restrict to a connected stratum of this. In this way we can restrict to the action of a finite group on a manifold. The example of Section 1.1 can be handled this way, because the cycle in question lies in the trivial stratum for the action of $\mathbf{S}^1$.

isotropy equal to Σ (a subset of Fix Σ). Flows on M equivariant under the action of Γ leave each M_Σ invariant. Let

$$\dot{x} = f(x) \tag{2}$$

be an ODE with C^∞ equivariant vector field, and let $X_\Gamma^\infty(M)$ be the class of all such f with the topology of C^∞ convergence on compact sets.

Consider a set $E = \{\xi_1, \ldots, \xi_k\}$ of hyperbolic equilibria of (2). The stable and unstable manifolds $W^s(\xi)$ and $W^u(\xi)$ of $\xi \in E$ are defined in the usual way. There is a graph naturally defined on E as follows:

Definition 1. The directed graph $G(E)$ with vertices E is defined as having an edge from $\xi_i \in E$ to $\xi_j \in E$ for each nonempty

$$C_{ij} = W^u(\xi_i) \cap W^s(\xi_j) \setminus \{\xi_i, \xi_j\}.$$

Such a trajectory in C_{ij} is called a *connecting orbit* and C_{ij} is called the set of *connections*.

Note that $\dim(C_{ij}) > 0$ if it is nonempty; we do not assume that $i \neq j$. A directed graph $G(E)$ is *transitive* if there is a directed path between any pair of points in E. It is *cyclic* if it is transitive and each vertex has unique edges entering and leaving it.

Definition 2. If $G(E)$ is transitive then we define the *heteroclinic chain*

$$H(E) = \bigcup_{(\xi_i, \xi_j) \in E} W^u(\xi_i) \cap W^s(\xi_j)$$

of all links between equilibria in E. Often there is a nontrivial Γ-action on $H(E)$; because of this we consider the quotient $\tilde{G}(E) = G(E)/\Gamma$ by identifying vertices and connections on the same group orbits. If $\tilde{G}(E)$ is a cyclic graph then we say that $H(E)$ is a *heteroclinic cycle*. If $\tilde{G}(E)$ has a single vertex then we call $H(E)$ a *homoclinic cycle*.

Such a heteroclinic chain $H(E)$ is said to be *robust* if for all $\epsilon > 0$ there exists a neighbourhood N_ϵ of f in $X_\Gamma^\infty(M)$ such that each $g \in N_\epsilon$ has a set of fixed points, $\tilde{E} = \{\eta_1, \ldots, \eta_k\}$ and a heteroclinic chain $H(\tilde{E})$ with

(a) $\|\xi_i - \eta_i\| < \epsilon$,
(b) $\tilde{C}_{ij}$ is equivariantly diffeomorphic to C_{ij} for any pair (i, j).

There may in fact be an infinite number of connections (this occurs in a system investigated by Guckenheimer and Worfolk [13]), but we will assume that this is not the case.

Attractors. There is a fair amount of debate over what definition of an attractor is most useful. The strongest definition usually used is asymptotic stability, but previous work on heteroclinic cycles (notably Melbourne's concept of *essential asymptotic stability* [19]) and other dynamics in invariant subspaces (e.g., [1]) suggests that this is too strong for some uses. We use the definition of Milnor [21]; between these there are a variety of definitions, see for instance [19], [17], [18], [3], [15]. We use Milnor's definition even though what we prove will be stable in a stronger sense; the examples of Brannath [3] and Kirk and Silber [15] have heteroclinic cycles that are Milnor attractors but are not essentially asymptotically stable.

Definition 3. A compact invariant set S is a (Milnor) *attractor* if its basin $\mathcal{B}(S) = \{x\colon \omega(x) \subset S\}$ has positive Lebesgue measure. It is a *minimal attractor* if there is no proper compact invariant subset with basin equal to $\mathcal{B}(S)$ up to a set of Lebesgue measure zero.

2.1. Robust Heteroclinic Chains

As shown by dos Reis [22], Field [7], and Guckenheimer and Holmes [12] among others, the invariant subspace structure caused by a group action can cause heteroclinic chains to be robust (for a review of such behaviour, see [16]). If all connections are robust within invariant subspaces, they cannot be broken by perturbations that preserve the invariant subspaces. Notably, if the connections are of the form saddle-sink or source-saddle when restricted to the invariant subspace, they will be robust. We consider the following hypotheses:

(H)

(a) For any nonempty connection C_{ij} there exists an isotropy subgroup Σ such that $C_{ij} \subset M_\Sigma$, i.e., all trajectories in C_{ij} have the same isotropy.
(b) For any Σ in (a), ξ_j is a sink for the flow restricted to Fix (Σ), i.e., $\dim(W^s(\xi_j) \cap M_\Sigma) = \dim M_\Sigma$.
(c) $H(E)$ contains all unstable manifolds of points in E.
(d) The eigenspaces tangent to C_{ij} and C_{ji} at $\xi_j \subset \text{Fix}\,\Delta_j$ lie within a single isotypic component of Δ_j.

Note that (H) can be broken in a variety of ways. If (Ha) is not satisfied then the connection C_{ij} contains connections of more than one isotropy type (an example where this is the case is discussed in Section 4.2). If (Hb) or (c) are not satisfied, it is still possible to have robustness of connections, e.g., between two fixed points that are both saddles within M_Σ, as long as the connections are Γ-transverse (see Field [6], [7], [8], [9]), necessarily meaning than $\dim(W^u(\xi_i) \cap M_\Sigma) + \dim(W^s(\xi_j) \cap M_\Sigma)$ is larger that $\dim(M_\Sigma)$. Heteroclinic chains subject to assumptions (H) are robust to equivariant perturbations. (Hc) implies among other things that there are no unstable eigenvalues in directions that are transverse to the heteroclinic cycle. (Hd) effectively means that there are no further constraints caused by symmetry to the flow on the connecting space; this is discussed at length in [17], in particular their Section 4.4.

We write $B_\delta(\xi) = \{x \in M\colon \|x - \xi\| < \delta\}$. The following lemma is interesting in that it highlights a new phenomenon present only when the connecting set has dimension greater than one. For one-dimensional connections (which are always closed) a chain need not be a cycle.

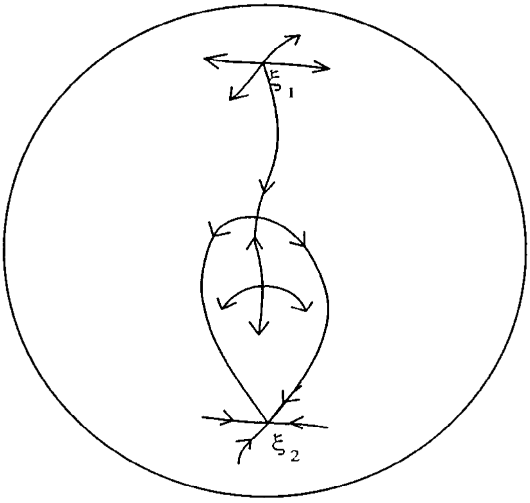


Fig. 2. This two-dimensional set of connections between ξ_1 and ξ_2 on the sphere is not closed; the closure contains the saddle point, one side of its stable manifold, and its unstable manifold.

Lemma 1. *If a heteroclinic chain has connecting sets that satisfy (H), are closed, and have dimension greater than or equal to two, then it is a cycle.*

Proof. Suppose we have a closed heteroclinic chain between equilibria ξ_j. Consider any ξ_j; for small enough δ, (H) implies that if $\dim(W^u(\xi_j)) = d_j$ then $B_\delta(\xi_j) \cap W^u_{\text{loc}}(\xi_j)$ is topologically a $(d_j - 1)$-sphere and therefore connected if $d_j > 1$. For each k, $B_\delta(\xi_j) \cap W^u(\xi_j) \cap W^s(\xi_k)$ is either empty or a $d_j - 1$-manifold without boundary and so equal to $B_\delta(\xi_j) \cap W^u(\xi_j)$. Thus there is a unique connection away from any ξ_j, and the graph is cyclic. □

A homoclinic cycle with one-dimensional connections is always closed; however, for more general homoclinic cycles, this is not the case, as the following example shows.

Example of a homoclinic cycle that is not closed. Consider a homoclinic cycle between three equilibria $\{\xi_1, \xi_2, \xi_3\}$ on a group orbit with connections from ξ_1 to ξ_2 shown in Figure 2 (and cyclic permutations). This shows a 2-sphere containing all connections in a homoclinic cycle including both fixed points ξ_1 and ξ_2. One can robustly get a saddle point on the connection, as shown in the figure, that prevents the homoclinic cycle from being closed.

One can of course include the saddle point shown in Figure 2 and obtain a closed noncyclic heteroclinic chain. However in general there may be limit cycles or even chaotic sets in the closure of a heteroclinic chain. The example from tetrahedral Hopf bifurcation is numerically observed to be closed.

It is possible to give sufficient conditions for the stability of heteroclinic cycles (see Theorem 1) as a direct application of results of Krupa and Melbourne [17]. Assume (by renumbering if necessary) that there are connections from ξ_i to ξ_{i+1}. We define, as in Krupa and Melbourne, four sets of eigenvalues of the linearisation of $df(\xi_i)$. Set P_i to be the flow-invariant subspace containing $C_{i(i+1)}$. We decompose the tangent space at ξ_i into

$$T_{\xi_i}\mathbf{R}^n = Tr_i \oplus Tc_i \oplus Te_i \oplus Tt_i,$$

where

$$\begin{aligned} Tr_i &= P_{i-1} \cap P_i && \textit{radial} \text{ eigenspace,} \\ Tc_i &= P_{i-1} \cap (Tr_i)^{\perp} && \textit{contracting} \text{ eigenspace,} \\ Te_i &= P_i \cap (Tr_i)^{\perp} && \textit{expanding} \text{ eigenspace,} \\ Tt_i &= (P_{i-1} + P_i)^{\perp} && \textit{transverse} \text{ eigenspace.} \end{aligned}$$

These linear subspaces are mapped into themselves by virtue of the invariance of the P_i. We shall only consider the case where Tr_i and Tt_i are trivial; if all eigenvalues in these directions are negative they essentially do not change the picture, although they do complicate calculations.

We call the matrices in Te_i and Tc_i corresponding to this "block diagonalisation" E_i and $-C_i$, respectively. Note that the eigenvalues of E_i need not all have positive real parts; we assume merely that the generalised eigenspace of the eigenvalues with positive real part has dimension ≥ 1. Following the notation of [17] we set

$$\begin{aligned} c_i &= \text{minimum real part of eigenvalue of } C_i, \\ e_i &= \text{maximum real part of eigenvalue of } E_i. \end{aligned}$$

3. A Conjecture about the Dynamics of Heteroclinic Cycles

We recall a restriction of a more general theorem of Krupa and Melbourne giving sufficient conditions for the asymptotic stability of a heteroclinic cycle (their definition implies that the cycle is closed).

Theorem 1. *([17]) Suppose we have a closed heteroclinic cycle satisfying (H), with no radial or transverse eigenvalues and such that*

$$\prod_{j=1}^{m} c_j > \prod_{j=1}^{m} e_j. \tag{3}$$

Then the cycle is asymptotically stable.

Define the principal unstable manifold [17] $W^{pu}(\xi_j)$ to be the invariant manifold of trajectories tangent to the generalised eigenspace of e_j. We define the *principal connections*

$$C_{ij}^p = W^{pu}(\xi_i) \bigcap W^s(\xi_j).$$

Note that C_{ij}^p is nonempty if C_{ij} is nonempty and closed. We define the *principal heteroclinic cycle* $H^p(E)$ to be the union of the equilibria E and the principal connections C_{ij}^p.

Conjecture 1. *Suppose a closed heteroclinic cycle $H(E)$ satisfies (H) and (3). Then generically the principal heteroclinic cycle $H^p(E)$ is a (Milnor) attractor which decomposes into a finite number of minimal attractors.*

In this conjecture we do not assume absence radial or transverse eigenvalues to the cycle; just that they are all negative by (Hc). From here on, to simplify the presentation, we will assume that they are not present, although this is not a central assumption for our results.

This conjecture does *not* necessarily hold if we drop the assumption (Ha) and look at heteroclinic chains rather than cycles, as we discuss in Section 4.2. We prove a weaker version of this for homoclinic cycles with distinct real expanding and contracting eigenvalues and no radial or transverse directions in Theorem 2. We also give an example where the expanding and contracting eigenvalues are complex and the whole chain is the ω-limit set of a full measure set of nearby initial conditions, and thus a minimal attractor. We present numerical evidence in Section 4 covering some of the other cases.

3.1. Homoclinic Cycles with d-Dimensional Sets of Connections

We now discuss *homoclinic cycles* in more detail. In this case, we can essentially reduce the problem to looking at some neighbourhood of a single equilibrium, for example ξ_1. Assuming there are no transverse or radial eigenvalues, we decompose the tangent space into

$$T_{\xi_1}\mathbf{R}^n = Tc_1 \oplus Te_1.$$

The expanding and contracting tangent spaces Tc_1 and Te_1 are assumed to be d-dimensional.

In a neighbourhood of ξ_1 we transform the vector field into its linearised form,

$$\begin{aligned} \dot{v} &= M_e v, \\ \dot{w} &= M_c w, \end{aligned} \tag{4}$$

where the $d \times d$ matrices M_e, M_c have all eigenvalues positive, resp., negative.

Approximation of the return map. By scaling a neighbourhood of the equilibrium to put M_e and M_c into Jordan normal form we can define a section $H^{(\mathrm{in})}$:

$$H^{(\mathrm{in})} = \{(v, w)\colon \|w\| = 1\} = \{(v, \theta)\colon v \in \mathbf{R}^d,\ \theta \in \mathbf{S}^{d-1}\}. \tag{5}$$

This will transversely intersect all trajectories approaching some neighbourhood of ξ_1. Similarly, the section $H^{(\mathrm{out})}$ defined by

$$H^{(\mathrm{out})} = \{(v, w)\colon \|v\| = 1\} = \{(\phi, w)\colon \phi \in \mathbf{S}^{d-1},\ w \in \mathbf{R}^d\}$$

intersects all trajectories leaving a neighbourhood of ξ_1. Both of these sets are topologically a product of a d-disk with a $(d-1)$-sphere. The centres of the disks correspond to $W^s_{\mathrm{loc}}(\xi_1) \cap H^{(\mathrm{in})}$ $(v = 0)$, resp., $W^u_{\mathrm{loc}}(\xi_1) \cup H^{(\mathrm{out})}$ $(w = 0)$. Note that if we refer to $\mathbf{S}^{d-1} \subset \mathbf{R}^d \times \mathbf{S}^{d-1}$ then we mean the subset $\{(0, w)\colon\ w \in \mathbf{S}^{d-1}\}$.

We examine the dynamics in a neighbourhood of this cycle by approximating the maps Ψ_i and Φ_i,

$$\cdots \xrightarrow{\Phi_n} H_1^{(\mathrm{in})} \xrightarrow{\Psi_1} H_1^{(\mathrm{out})} \xrightarrow{\Phi_1} H_2^{(\mathrm{in})} \xrightarrow{\Psi_2} H_2^{(\mathrm{out})} \xrightarrow{\Phi_2} H_3^{(\mathrm{in})} \cdots.$$

If χ is a group element such that $\chi\xi_1 = \xi_2$, note that $\chi H_j^{(\mathrm{in})} = H_{j+1}^{(\mathrm{in})}$, etc. We can find the nth root of the return map from $H_1^{(\mathrm{in})}$ to itself by considering the map

$$f_1 = \chi^{-1} \circ \Phi_1 \circ \Psi_1,$$

where $\chi\xi_1 = \xi_2$. We shall therefore drop the subscript, refer to $\chi^{-1} \circ \Phi_1$ as Φ and note that by computing the dynamics of this map and considering the n^{th} iterate, we recover the return map on $H^{(\mathrm{in})}$.

Rather than dwell here on the derivation of the return map, we list below the possible return maps and refer to Appendices A and B for their derivation for arbitrary d with real distinct eigenvalues and for $d = 2$.

Case of arbitrary *d* with distinct real eigenvalues. Assume all eigenvalues are real and distinct and order them $\lambda_1 > \cdots > \lambda_d > 0$ and $\gamma_1 > \cdots > \gamma_d > 0$ so that a local change of coordinates gives

$$M_e = \mathrm{diag}(\lambda_1, \ldots, \lambda_d), \qquad M_c = \mathrm{diag}(-\gamma_1, \ldots, -\gamma_d).$$

In this case Appendix B gives a return map

$$f(v, w) = (v', w'),$$

where to leading order we have

$$\begin{aligned} v' &= A(\tau(v))(e^{-\gamma_1 T(v)} w_1, \ldots, e^{-\gamma_d T(v)} w_d), \\ w' &= g(\tau(v)). \end{aligned} \tag{6}$$

We define $T\colon\ \mathbf{R}^d \setminus \{0\} \to \mathbf{R}^+$ implicitly by

$$\sum_j v_j^2 e^{2\lambda_j T(v)} = 1$$

and $\tau\colon\ \mathbf{R}^d \setminus \{0\} \to \mathbf{S}^{d-1}$ by

$$\tau_i(v) = e^{\lambda_i T(v)} v_i.$$

$A(\phi)$ is an arbitrary smooth map from $\mathbf{S}^{d-1}$ to invertible linear maps on $\mathbf{R}^d$, and g is a smooth map of $\mathbf{S}^{d-1}$ to itself (this is arbitrary due to hypothesis (Hd); see [17] Proposition 4.9).

Case of $d = 2$. There are four cases for homoclinic cycles, corresponding to the expanding and contracting eigenvalues being real or complex. We write

$$M_e = \mathrm{diag}(\lambda_1, \lambda_2) \qquad \text{or} \qquad \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix},$$

with λ, λ_i, and ω positive for the expanding linearisation, and

$$M_c = \mathrm{diag}(-\gamma_1, -\gamma_2) \qquad \text{or} \qquad \begin{pmatrix} -\gamma & \rho \\ -\rho & -\gamma \end{pmatrix},$$

with γ, γ_i, and ρ positive. To lowest order, the map Φ_1 contributes a linear map $A(\phi)$ transverse to the continuum of connections and a circle map $g(\phi)$ on the continuum of connections.

If the cycle is closed then $A(\phi)$ is an arbitrary continuous map from $\mathbf{S}^1$ to invertible 2×2 matrices, whereas g is a smooth diffeomorphism of the circle. If the cycle is not closed, A and g are only defined for an open domain and range.

As detailed in Appendix A, we obtain maps $f\colon H^{(\mathrm{in})} \to H^{(\mathrm{in})}$ of the form

$$f(v_1, v_2, \theta) = (v_1', v_2', \theta'). \tag{7}$$

Depending on the expanding eigenvalues we compute τ and T thus:

	Real	Complex
$\tau(v)$	$\arg\begin{pmatrix} v_1 \\ v_2 e^{(\lambda_2-\lambda_1)T} \end{pmatrix}$	$\arg\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \frac{\omega}{\lambda} \ln \lVert v \rVert$
$T(v)$	soln. of $v_1^2 e^{2\lambda_1 T} + v_2^2 e^{2\lambda_2 T} = 1$	$-\frac{1}{\lambda} \ln \lVert v \rVert$

and then we define f for real contracting eigenvalues by

$$\begin{aligned} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} &= A(\tau(v)) \begin{pmatrix} \cos\theta e^{-\gamma_1 T(v)} \\ \sin\theta e^{-\gamma_2 T(v)} \end{pmatrix}, \\ \theta' &= g(\tau(v)), \end{aligned} \tag{8}$$

and for complex contracting eigenvalues by

$$\begin{aligned} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} &= e^{-\gamma T(v)} A(\tau(v)) \begin{pmatrix} \cos(\theta - \omega T(v)) \\ \sin(\theta - \omega T(v)) \end{pmatrix}, \\ \theta' &= g(\tau(v)). \end{aligned} \tag{9}$$

3.2. ω-Limit Sets for Arbitrary d with Distinct Real Eigenvalues

We use the local return map to demonstrate that for asymptotically stable homoclinic cycles under certain assumptions, the ω-limit set for most points near the homoclinic cycle generically will be just the principal heteroclinic cycle. In particular, there will be a discrete set of connections (at most two leaving any point) that are essentially asymptotically stable. Define

$$e^{\pm} = (\pm 1, 0, 0, \ldots) \in \mathbf{S}^{d-1} \subset H^{(\text{out})}, \qquad w^{\pm} = g(e^{\pm}) \in \mathbf{S}^{d-1} \subset H^{(\text{in})},$$

$$b^{\pm} = (\pm 1, 0, 0, \ldots) \in \mathbf{S}^{d-1} \subset H^{(\text{in})},$$

and the following subsets

$$E(\delta) = \left\{(v, w) \in H^{(\text{in})}\colon\ \|v\| < \delta\right\},$$

$$F(\delta) = \left\{(v, w) \in H^{(\text{in})}\colon\ \min(\|w - w^{+}\|, \|w - w^{-}\|) < \delta\right\},$$

$$B(\delta) = E(\delta) \cap F(\delta).$$

Write $m(\cdot)$ as Lebesgue measure on $\mathbf{R}^d \times \mathbf{S}^{d-1}$. A *cuspoidal wedge* in $H^{(\text{in})}$ is defined for $\epsilon > 0$ by

$$V(\epsilon) = \left\{(v, w) \in H^{(\text{in})}\colon\ 1 - (\tau_1(v))^2 < \epsilon^2\right\}.$$

This consists of all points that are mapped to within ϵ of $e^{\pm}$ by the projection of the local map Ψ onto $\mathbf{S}^{d-1} \subset H^{(\text{out})}$. The subsets E and F and $V \cap E \cap F$ are shown schematically in Figure 3.

Theorem 2. *Suppose a closed homoclinic cycle $H(E)$ satisfying (H) has equilibria with real distinct eigenvalues and no transverse or radial eigenvalues. Suppose the sets of connections are of dimension $d \geq 2$. If*

$$\min_i \gamma_i = \gamma_d > \lambda_1 = \max_i \lambda_i,$$

then $H(E)$ is asymptotically stable and, moreover, generically the principal homoclinic cycle $H^p(E)$ is an essentially asymptotically stable Milnor attractor that is not Liapunov stable.

Proof. The asymptotic stability of H can be obtained by applying Theorem 1. Thus $\|v\| \to 0$ under iteration by f; we wish to prove that for an asymptotically full measure set of initial conditions, w will limit towards $w^{\pm}$. Define H^p to be the connections that cross $H^{(\text{in})}$ at $w = w^{\pm}$ (and therefore $H^{(\text{out})}$ at $v = e^{\pm}$).

Attraction of H^p is shown in Lemma 3. The fact that H^p is not Liapunov stable can be seen on observing that for any $\epsilon > 0$, $V(\epsilon)^c$ intersects all neighbourhoods of $(0, w^{\pm})$. □

In fact, the assumption that $H(E)$ is closed is not necessary, as long as C_{ij} is nonempty then C_{ij}^p is nonempty (this is closed because it is one-dimensional); this is because we argue local to connections in H^p.

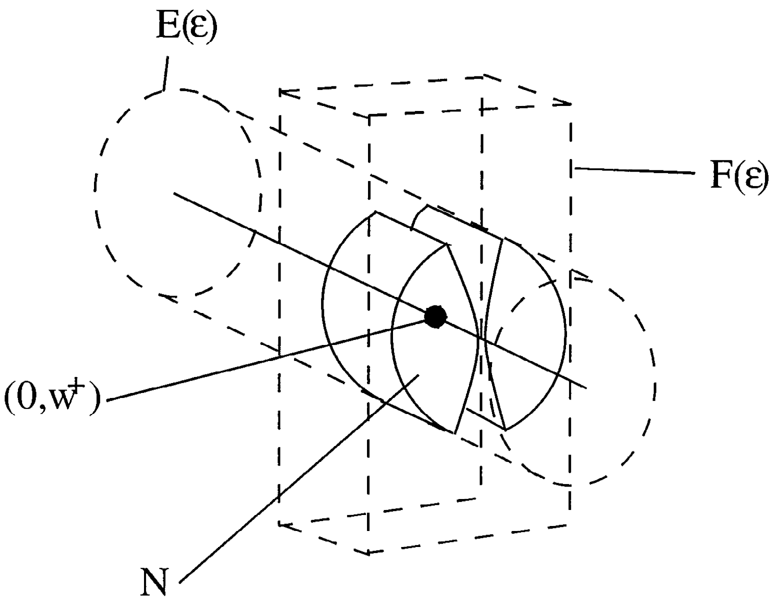


Fig. 3. This figure shows schematically the subsets $E(\epsilon)$ $F(\epsilon)$ and $N = V(\epsilon) \cap E(\epsilon) \cap F(\epsilon)$ in the section $H^{(\mathrm{in})} = \mathbf{R}^2 \times \mathbf{S}^1$; $\mathbf{S}^1$ is obtained by identifying the opposite faces of $E(\epsilon)$. The point $(0, w^+)$ is marked; this corresponds to the intersection of a principal connection with this section. The set of all connections is represented by the central line.

Define $r = e^{-T}$ so that $v_i = r^{\lambda_i}\tau_i(v)$. Since $\tau \in \mathbf{S}^{d-1}$, we have

$$\begin{aligned} 1 - (\tau_1(v))^2 &= \sum_{i=2}^{d} \tau_i^2 = \sum_{i=2}^{d} (r^{-\lambda_i} v_i)^2 \\ &\leq r^{-2\lambda_2} \sum_{i=2}^{d} v_i^2 \\ &\leq v_1^{-2\frac{\lambda_2}{\lambda_1}} \sum_{i=2}^{d} v_i^2, \end{aligned}$$

where we have used the properties that $\lambda_i \leq \lambda_2$ and $v_1^2 e^{2\lambda_1 T} = v_1^2 r^{-2\lambda_1} \leq 1$.

For S a measurable subset of $\mathbf{R}^d \times \mathbf{S}^{d-1}$ we define

$$m_\delta(S) = \frac{m(S \cap E(\delta))}{m(E(\delta))}.$$

The following is contained in [19]; we include it here for completeness.

Lemma 2. *The set $V(\epsilon)$ has asymptotically full measure. More precisely, given $\epsilon > 0$ there is a $K(\epsilon)$ such that for all small enough δ we have*

$$m_\delta(V(\epsilon)^c) < K(\epsilon)\,\delta^{\xi-1},$$

where $\xi = \lambda_1/\lambda_2 > 1$. Thus $m_\delta(V(\epsilon)^c) \to 0$ as $\delta \to 0$.

Proof. Observe that

$$\begin{aligned} V(\epsilon)^c \cap E(\delta) &= \left\{(v,w) \in H^{(\text{in})} \colon \|v\| < \delta \quad \text{and} \quad 1-(\tau_1(v))^2 \geq \epsilon^2\right\} \\ &\subset \left\{(v,w) \colon \|v\| < \delta \quad \text{and} \quad \epsilon^2 (v_1)^{2\frac{\lambda_2}{\lambda_1}} \leq \sum_2^d v_i^2\right\} \\ &\subset \left\{(v,w) \colon \|v\| < \delta \quad \text{and} \quad v_1^2 \leq (\delta^2/\epsilon^2)^{\frac{\lambda_1}{\lambda_2}}\right\}, \end{aligned}$$

Thus

$$\begin{aligned} m(V(\epsilon)^c \cap E(\delta)) &\leq \int_{v_1=-\delta^\xi\epsilon^{-\xi}}^{\delta^\xi\epsilon^{-\xi}} \int_{v_2^2+\cdots+v_d^2=-\delta^2}^{\delta^2} dv \\ &= K\epsilon^{-\xi}\delta^\xi \frac{m(E(\delta))}{\delta}, \end{aligned}$$

for some $K > 0$, and the result follows. □

This implies that the set $V(\epsilon)$ has asymptotically full measure near the set of connections $\mathbf{S}^{d-1} \subset H^{(\text{in})}$.

Lemma 3. *For the system described in Theorem 2, H^p is generically an attractor.*

Proof. We show there are constants $\rho > 1$, $\beta > 0$, $K > 1$, and $\epsilon_0 > 0$ such that for all $\epsilon_0 > \epsilon > 0$ we have

$$\begin{aligned} &(a) \quad f(V(\epsilon)) \subset F(K\epsilon), \\ &(b) \quad f(E(\epsilon)) \subset E(\epsilon^\rho), \\ &(c) \quad f(V(\epsilon_0) \cap E(\epsilon) \cap F(\epsilon_0)) \subset V(\epsilon^{\rho\beta}). \end{aligned}$$

Recall that

$$(v'(v,w), w'(v,w)) = \left(A(\tau(v))(e^{-\gamma_1 T(v)}w_1, \ldots, e^{-\gamma_d T(v)}w_d), g(\tau(v))\right).$$

(a) Set $K = \max_\phi \left\|\frac{dg}{d\phi}\right\|$. Thus $1-(\tau_1(v))^2 < \epsilon^2$ implies that $\|g(\tau(v)) - w^\pm\| \leq K\epsilon$.

(b) This follows by observing that $\|v'\| < \max_\phi \|A(\phi)\|\|v\|^{\frac{\gamma_d}{\lambda_1}}$ and so for any $1 < \rho < \frac{\gamma_d}{\lambda_1}$ there is an ϵ_0 such that $\|v'\| < \|v\|^\rho$ if $\|v\| < \epsilon_0$.

(c) We define $v^\pm = A(e^\pm)b^\pm$ and then generically (due to Ha, Hd) we can assume that $w_1^\pm \neq 0$ and $v_1^\pm \neq 0$. Essentially we wish to show that the image of most points passing through $H^{(\text{out})}$ near $e^{\pm}$ hit $H^{(\text{in})}$ with a nonzero component in the v_1 direction, because they are nearly collinear with $v^{\pm}$.

We can find ϵ_0 such that $(v, w) \in E(\epsilon_0) \cap F(\epsilon_0)$ implies that

$$(e^{-\gamma_1 T(v)} w_1, \ldots, e^{-\gamma_d T(v)} w_d)$$

is arbitrarily close to a scalar multiple of $b^{\pm}$, as T is arbitrarily large for small ϵ_0 and $\gamma_1 > \gamma_i$ for $i > 1$.

By continuity of $A(\tau(v))(u)$ and the fact that $\tau(v)$ is close to $e^{\pm}$ and u is close to $b^{\pm}$, we can find a constant $K_1 > 0$ such that

$$v_1'^2 > K_1 \sum_{i=2}^{d} v_i'^2$$

for any $(v, w) \in V(\epsilon_0) \cap E(\epsilon_0) \cap F(\epsilon_0)$. Thus we have

$$\begin{aligned} 1 - \tau_1(v')^2 &= \sum_{i=2}^{d} \tau_i(v')^2 \\ &\le v_1'^{-2\frac{\lambda_2}{\lambda_1}} \sum_{2}^{d} v_i'^2 \\ &\le K_1^{-\frac{\lambda_2}{\lambda_1}} \|v'\|^{2\beta}, \end{aligned}$$

where $\beta = 1 - \frac{\lambda_2}{\lambda_1} > 0$. Therefore we can choose ϵ_0 small enough and use (b) to assure that if $(v, w) \in V(\epsilon_0) \cap E(\epsilon) \cap F(\epsilon_0)$ then

$$1 - \tau_1(v'(v, w))^2 < \epsilon^{2\rho\beta}.$$

Combining results (a), (b), and (c) means that for all $0 < \epsilon < \epsilon_0/K$,

$$f(V(\epsilon^{\rho}) \cap E(\epsilon^{\frac{\rho}{\beta}}) \cap F(K\epsilon)) \subset V(\epsilon^{\rho^2}) \cap E(\epsilon^{\frac{\rho^2}{\beta}}) \cap F(K\epsilon^{\rho}),$$

implying that, if we define

$$N_n = V(\epsilon^{\rho^{n+1}}) \cap E(\epsilon^{\frac{\rho^{n+1}}{\beta}}) \cap F(K\epsilon^{\rho^n}),$$

then for $n \ge 0$

$$f(N_n) \subset N_{n+1}.$$

This means that all points in N_0 converge uniformly to $(0, w^{\pm}) \in H^{(\text{in})}$. Lemma 2 implies that this set has asymptotically full measure near $(0, w^{\pm})$. □

Note that we have constructed an asymptotically full measure set that converges uniformly to $(0, w^{\pm})$. It may be that an open dense set of points converge *non*uniformly to $(0, w^{\pm})$: See the discussion in Section 5.

3.3. ω-Limit Sets for Complex Expanding Eigenvalues

In this case we do not have a general result comparable to that for real expanding eigenvalues, but make the following conjecture, a special case of Conjecture 1.

Conjecture 2. *Suppose we have a closed homoclinic cycle with two-dimensional connections satisfying (H). Suppose it has a complex pair of expanding eigenvalues* $\lambda \pm i\omega$, $\lambda > 0$, *and contracting eigenvalues with real parts all less than* $-\gamma < 0$. *If* $\gamma > \lambda$ *then the set is asymptotically stable, and generically, an asymptotically full measure set of nearby initial conditions have ω-limit sets that contain the whole of the two-dimensional set.*

Case of $\mathbf{SO}(2)$ **acting on the connections.** Suppose that the isotropy subgroups of the fixed points ξ_i contains a copy of $\mathbf{SO}(2)$ that acts on the set of connections nontrivially. Generically the expanding and contracting eigenvalues are complex, say $\lambda \pm i\omega$ and $-\gamma \pm i\rho$, and we consider the mapping (18). In this case A will be a rigid rotation and scaling (independent of ϕ) while $g(\phi)$ will be a uniform rotation, i.e.,

$$A = A_0 \begin{pmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix}, \qquad g(\phi) = \phi + \chi,$$

where $A_0 \neq 0$, ζ, and χ are real numbers. We write $(v_1, v_2) = (r\cos\psi, r\sin\psi)$ and the return mapping on $H^{(\mathrm{in})}$ has the form

$$\begin{aligned} r' &= A_0 r^{\frac{\gamma}{\lambda}}, \\ \psi' &= \theta - \frac{\rho}{\lambda}\ln r - \zeta, \\ \theta' &= \psi + \frac{\omega}{\lambda}\ln r. \end{aligned}$$

Thus, given a trajectory $(r_n, \psi_n, \theta_n) = f^n(r_0, \psi_0, \theta_0)$ and defining

$$\beta = \frac{\gamma}{\lambda} > 1,$$

we can write

$$r_n = A_0^{\sum_{k=0}^{n}\beta^k} r_0^{\beta^n} = A_0^{\frac{\beta^n-1}{\beta-1}} r_0^{\beta^n} = (A_0)^{\frac{-1}{\beta-1}} \left(A_0^{\frac{1}{(\beta-1)}} r_0 \right)^{\beta^n},$$

and we define K (independent of r_0) and s (linearly dependent on r_0) such that

$$r_n = K s^{\beta^n}.$$

Thus

$$\psi_n = \theta_{n-1} - \frac{\rho}{\lambda}(\ln K + \beta^n \ln s) - \zeta$$

and

$$\theta_n = \psi_{n-1} + \frac{\omega}{\lambda}(\ln K + \beta^n \ln s).$$

We can write

$$\theta_{n+1} = \theta_{n-1} + \left(\frac{\beta\omega - \rho}{\lambda}\right)\beta^n \ln s + \left(\frac{\omega - \rho}{\lambda}\ln K - \zeta\right).$$

Setting

$$K_1 = \left(\frac{\beta\omega - \rho}{\lambda}\right)\beta \ln s, \qquad K_2 = \left(\frac{\omega - \rho}{\lambda}\ln K - \zeta\right),$$

we get an expression for even terms of θ_n as

$$\theta_{2k} = \theta_0 + \sum_{l=1}^{k}(K_1(\beta^2)^{(l-1)} + K_2),$$

and on evaluating the sum, we obtain

$$\theta_{2k} = \left(\frac{K_1}{\beta^2 - 1} + \theta_0\right) + K_2 k + \frac{K_1}{\beta^2 - 1}(\beta^2)^k. \tag{10}$$

Note that the dominant term is exponential as long as $K_1 \neq 0$—in other words, as long as

$$\frac{\gamma}{\rho} \neq \frac{\lambda}{\omega}$$

—and corresponds to an assumption that the pitch of the contracting and expanding spirals near the fixed point are not equal and opposite.

We investigate the sequence (10) by means of the following proposition.

Proposition 1. *For given $\lambda > 1$ and almost all C_1, C_2, and C_3, the sequence*

$$\{C_1 + C_2 n + C_3\lambda^n\colon\ n \in \mathbf{N}\}$$

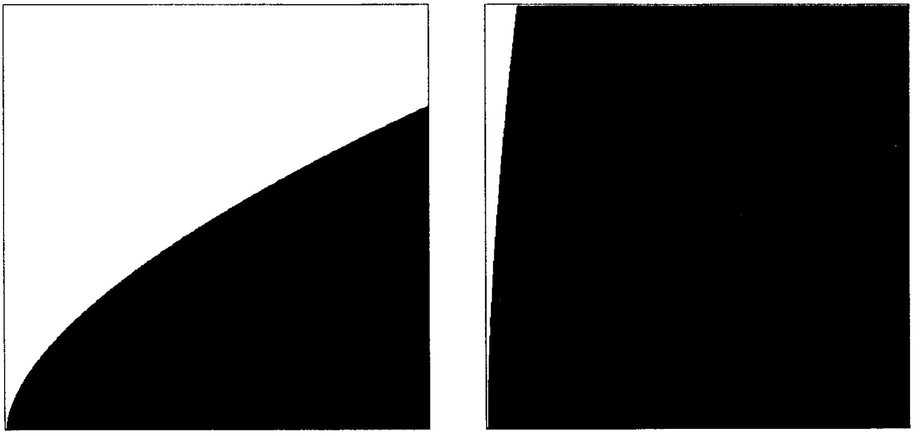
is dense in $[0, 1]$. Moreover, if $C_2 = 0$ then for almost all C_1 and C_3 this sequence is uniformly distributed with respect to Lebesgue measure.

Proof. Fix any C_1, C_2, and C_3 nonzero and consider the sequence $s_k = C_1 + C_2 k + C_3\lambda^k$. Since this is unbounded, the result follows directly from Corollary 2.4 in Melbourne and Stewart [20]. The second part is a restatement of a theorem quoted in Cornfeld et al. [5, p. 164]. □

From this, we obtain

Theorem 3. *Suppose the hypotheses of Conjecture 2 hold. Suppose additionally that the vector field on each connection commutes with a nontrivial action of* $\mathbf{SO}(2)$ *that fixes the end points. Then the results of Conjecture 2 hold; in particular, the cycle is a minimal attractor.*

Without an assumption of $\mathbf{SO}(2)$ symmetry, we expect that the sequence of return angles θ_k is almost always uniformly distributed with respect to a Lebesgue *equivalent* measure.



(i) (ii)

Fig. 4. The black region shows all initial points for the real contracting, real expanding case that stay uniformly close to $\theta = \theta^{\pm}$ under the action of the map (7). The square represents (v_1, v_2) within (i) $[0, 0.1]^2$, (ii) $[0, 10^{-3}]^2$.

4. Examples

We consider examples illustrating dynamics near closed homoclinic cycles with two-dimensional connecting sets. We report on simulations of approximate return maps near such homoclinic cycles. We also consider an example described by Kirk and Silber [15] which shows the necessity of our assumption (Ha).

4.1. Simulations of the Approximate Return Map

We have investigated the mappings (8) and (9) by examining individual trajectories as well as by scanning through the phase space to plot out the basins of attraction.

This is a much easier way to examine multiple approaches to the fixed point than direct simulation of an ODE, because for the ODE there is an 'asymptotic slowing down' with each approach to the fixed point. We present examples where

$$A(\phi) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad g(\phi) = \phi + 1,$$

and use a Newton method to solve the equation for $T(v)$. Figure 4 shows the basin of attraction of H^p in the (v_1, v_2) plane at $\theta = \theta^-$. Black points correspond to initial conditions that remain within 0.3 of $\theta^{\pm}$. In this case, the expanding eigenvalues are $\lambda_1 = 0.5$ and $\lambda_2 = 0.3$, while the contracting ones are $-\gamma_1 = -0.8$ and $-\gamma_2 = -0.6$. In agreement with Theorem 2, the measure of the basin of attraction limits to full measure on approaching the connection. Figure 5 shows the same plots but for complex contracting eigenvalues $-0.6 \pm i$ and real expanding eigenvalues 0.3 and 0.5. Note that the structure of the basin complement is much more complicated than in the real contracting case,

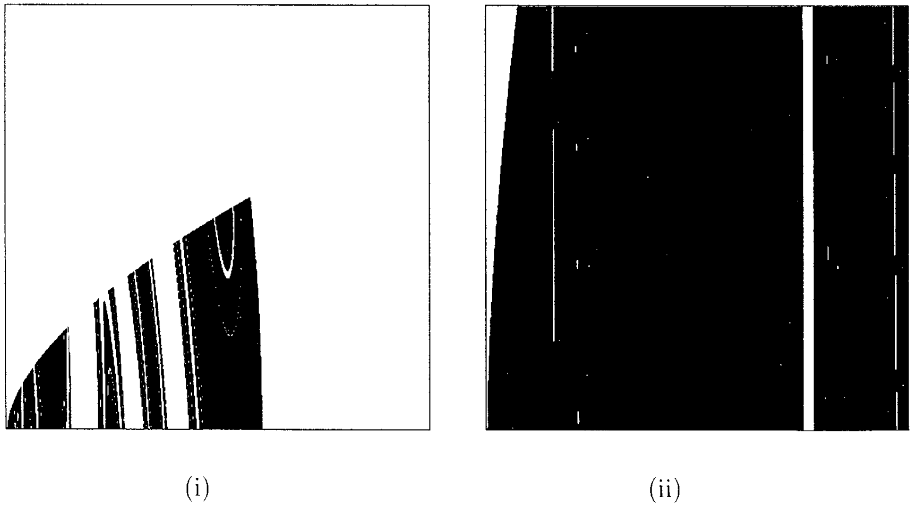


Fig. 5. The black region shows all initial points for the complex contracting, real expanding case that stay uniformly close to $\theta = \theta^{\pm}$. The square represents (v_1, v_2) within (i) $[0, 0.1]^2$, (ii) $[0, 10^{-3}]$.

and we conjecture it may even be 'riddled' [1], [2]. Nonetheless, the images suggest that the measure of the basin is asymptotically full near H^p.[3]

4.2. A Vector Field with $\mathbf{Z}_2^4$ *Symmetry*

As another example, we consider a heteroclinic chain investigated by Kirk and Silber [15]. This is a pair of competing cycles on four equilibria for a flow on $\mathbf{R}^4$ equivariant under $\mathbf{Z}_2^4$ acting by reflections in all the coordinate hyperplanes. One cycle is of the form

$$\xi_1 \to \xi_2 \to \xi_3 \to \xi_1,$$

and the other is

$$\xi_1 \to \xi_2 \to \xi_4 \to \xi_1,$$

where ξ_i has nonzero components only in the ith component. The equilibria have unstable manifolds of dimension one except for ξ_2 that has dimension two, and at this point the expanding eigenvalue in the ξ_3 direction is assumed to be stronger than that in the ξ_4 direction. In Figure 6 we illustrate some of the main features of this cycle. (a) shows the first cycle in the invariant space $x_4 = 0$ while (b) shows the second cycle in the invariant space $x_3 = 0$. (c) gives the simplest possible dynamics in the plane $x_1 = x_2 = 0$ consistent with the description; note that there is an extra saddle point labelled η. Thus the simplest closed heteroclinic chain containing the two cycles is (d), with some one-dimensional and some two-dimensional connections. (This is still a simplification in

[3] We have prepared a short video showing the structure of the basin of attraction, available on request to P. Ashwin (please include a blank VHS videocassette with any request).

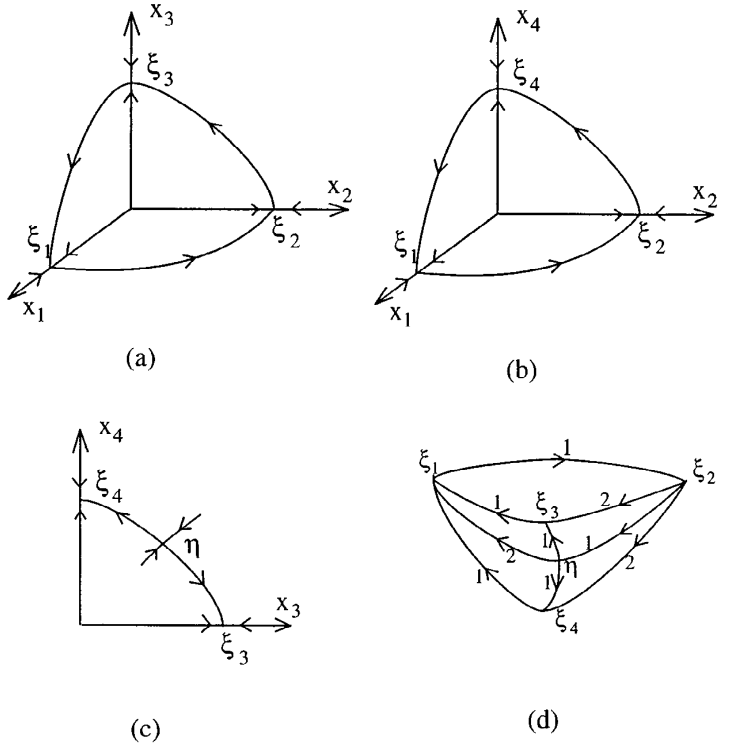


Fig. 6. (a), (b) Sketch of the two competing cycles in Kirk and Silber's example. (c) Simplest consistent dynamics in the $x_1 = x_2 = 0$ invariant plane involves addition of an extra equilibrium, η, with dim $W^u(\eta) = 2$. (d) Simplest consistent closed heteroclinic chain. The numbers on the connections indicate the dimensions of the connecting sets.

that all of the $-\xi_i$ are also involved in the closure of the unstable manifolds of this set). Note that the connections ξ_2 to ξ_3 involve a one-dimensional connecting set in the plane $x_1 = x_4 = 0$ and a two-dimensional connecting set in the invariant three-space $x_1 = 0$; thus (Ha) is broken.

Kirk and Silber have shown that in a robust way, one can find such cycles where both of the one-dimensional cycles are simultaneously attracting; this suggests that (Ha) is a necessary assumption for Conjecture 1.

5. Discussion

We have shown how some of the behaviour seen in the example of Section 1.1 can be explained theoretically, in particular how we can robustly have a continuum of connections that are asymptotically stable, while only a discrete set of them are attractors. We have also shown some ideas of what one might expect for more general cases. A crucial hypothesis is (Ha), that all connections between a given pair of equilibria lie within a single isotropy type. There are also genericity assumptions that can be thought of as requiring that the principal connections are not at inclination or orbit flip bifurcations. We have also indicated that closedness of a heteroclinic chain, which is automatic for one-dimensional connections, is not automatic for higher dimensional cases, and this needs to be properly addressed.

For homoclinic cycles that are asymptotically stable, the most positive expanding eigenvalues seem to determine the asymptotic behaviour of most initial nearby points; if this is a single real eigenvalue, we get a one-dimensional attractor, whereas if it is a complex pair, the rotation causes a trajectory to be thrown out in many directions in this eigenspace on successive approaches to the fixed point. What is possibly surprising is that the dimension and complexity or otherwise of the weakest contracting eigenspace appears not to be important for the asymptotic behaviour.

This paper poses a variety of questions related to attractors for systems containing robust heteroclinic chains, and we recall some of them here:

1. Prove or find counterexamples to Conjecture 1 and the special case, Conjecture 2.
2. Account for the effect of transverse and radial eigenvalues.
3. Understand heteroclinic chains that break hypothesis (H) in various ways.
4. Understand what happens in the case that the eigenvalues are real and multiple. This can happen if, for example, there is an irreducible nontrivial group action that commutes with the flow on the connecting orbits.

The notion of stability we have used can very likely be strengthened to say that, at least in the case of a closed homoclinic cycle, a full measure set of initial points converge to the principal homoclinic cycle H^p. This may even be open and dense, but because we do not have Liapunov stability, it would have nonuniform convergence of points in the basin to H^p.

The dimension of the phase space needs to be reasonably large (greater than three) to observe this behaviour; hence, it cannot be seen for very simple group actions at generic bifurcations. However for higher dimensional systems (e.g., parabolic PDEs) we expect these structures to be characteristically present, even if the group is relatively simple, for example in higher dimensional (reducible) representations of $\mathbf{O}(2)$.

Appendix A: Derivation of the Local Return Map for $d = 2$

We use the notation defined in Section 3. The form of the mapping between the neighbourhoods of adjacent equilibria is independent of the dynamics near the equilibria. Observe that $\Phi\colon H^{(\mathrm{out})} \to H^{(\mathrm{in})}$ can be written (to linear order in w_1, w_2) as

$$\Phi(\phi, w_1, w_2) = (v_1', v_2', \theta'),$$

where

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = A(\phi) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$
$$\theta' = g(\phi).$$

A and g are smoothly dependent on $\phi \in \mathbf{S}^1$ for closed cycles; A: $\mathbf{S}^1 \to GL(\mathbf{R}^2)$ and g: $\mathbf{S}^1 \to \mathbf{S}^1$. Note that, in particular, the matrix norm of A is uniformly bounded in ϕ. Moreover, continuity of A and U imply that its image lies within one connected component of the image space; $\det(A)$ lies on one side of zero.

Case (i): Real expanding and real contracting eigenvalues. A local change of coordinates gives

$$M_e = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad M_c = \begin{pmatrix} -\gamma_1 & 0 \\ 0 & -\gamma_2 \end{pmatrix},$$

where the constants λ_i and γ_i are real and positive. We will assume that $\lambda_1 > \lambda_2 > 0$ and $\gamma_1 > \gamma_2 > 0$. The flow for the initial value problem for (4) is

$$\begin{aligned} v_i(t) &= e^{\lambda_i t} v_i(0), \\ w_i(t) &= e^{-\gamma_i t} w_i(0). \end{aligned} \tag{11}$$

The 'time of flight' from $H^{(\text{in})}$ to $H^{(\text{out})}$ is given by T such that $v_1^2 e^{2\lambda_1 T} + v_2^2 e^{2\lambda_2 T} = 1$, giving a unique solution $T(v) > 0$. In what follows we shall make use of the estimates

$$\ln \|v\|^{-\frac{1}{\lambda_1}} \le T(v) \le \ln \|v\|^{-\frac{1}{\lambda_2}}. \tag{12}$$

Thus the mapping Ψ: $H^{(\text{in})} \to H^{(\text{out})}$ can be written as

$$\Psi(v_1, v_2, \theta) = (\tau, w_1, w_2),$$

where

$$\begin{aligned} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \begin{pmatrix} \cos\theta e^{-\gamma_1 T} \\ \sin\theta e^{-\gamma_2 T} \end{pmatrix}, \\ \tau(v) &= \arg \begin{pmatrix} v_1 e^{\lambda_1 T} \\ v_2 e^{\lambda_2 T} \end{pmatrix} = \arg \begin{pmatrix} v_1 \\ v_2 e^{(\lambda_2 - \lambda_1)T} \end{pmatrix}, \end{aligned}$$

where arg refers to the angle the vector makes to 1-axis, measured in the positive direction, i.e.,

$$\arg \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \theta$$

for all positive r. The return map f has the form

$$f(v_1, v_2, \theta) = (v_1', v_2', \theta'),$$

where

$$\begin{aligned} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} &= A(\tau(v)) \begin{pmatrix} \cos\theta e^{-\gamma_1 T(v)} \\ \sin\theta e^{-\gamma_2 T(v)} \end{pmatrix}, \\ \theta' &= g(\tau(v)). \end{aligned} \tag{13}$$

Case (ii): Real expanding and complex contracting eigenvalues. Suppose that

$$M_e = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad M_c = \begin{pmatrix} -\gamma & \omega \\ -\omega & -\gamma \end{pmatrix},$$

with real constants $\lambda_1 > \lambda_2 > 0$, $\gamma > 0$, and $\omega \neq 0$. As before, we define $T(v) > 0$ to be the solution of $v_1^2 e^{2\lambda_1 T} + v_2^2 e^{2\lambda_2 T} = 1$ and the flow is given by

$$\begin{aligned} v_1(t) &= e^{\lambda_1 t} v_1(0), \\ v_2(t) &= e^{\lambda_2 t} v_2(0), \\ w_1(t) &= e^{-\gamma t}(w_1(0)\cos\omega t + w_2(0)\sin\omega t), \\ w_2(t) &= e^{-\gamma t}(w_2(0)\cos\omega t - w_1(0)\sin\omega t). \end{aligned}$$

We can write the local map $\Psi(v_1, v_2, \theta) = (\tau, w_1, w_2)$ where

$$\tau(v) = \arg\begin{pmatrix} v_1 e^{\lambda_1 T} \\ v_2 e^{\lambda_2 T} \end{pmatrix},$$

and

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = e^{-\gamma T}\begin{pmatrix} \cos\theta\cos\omega T + \sin\theta\sin\omega T \\ \sin\theta\cos\omega T - \cos\theta\sin\omega T \end{pmatrix} = e^{-\gamma T}\begin{pmatrix} \cos(\theta - \omega T) \\ \sin(\theta - \omega T) \end{pmatrix}.$$

This gives a return map f of the form $f(v_1, v_2, \theta) = (v_1', v_2', \theta')$ where

$$\begin{aligned} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} &= e^{-\gamma T(v)} A(\tau(v)) \begin{pmatrix} \cos(\theta - \omega T(v)) \\ \sin(\theta - \omega T(v)) \end{pmatrix}, \\ \theta' &= g(\tau(v)). \end{aligned} \tag{14}$$

Case (iii): Complex expanding and real contracting eigenvalues. We perform a local change of coordinates such that

$$M_e = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix}, \qquad M_c = \begin{pmatrix} -\gamma_1 & 0 \\ 0 & -\gamma_2 \end{pmatrix},$$

where the constants satisfy $\omega \neq 0$, $\lambda > 0$, and $\gamma_1 > \gamma_2 > 0$.

It is a simple matter to find the flow for the initial value problem for (4), resulting in

$$\begin{aligned} v_1(t) &= e^{\lambda t}(v_1(0)\cos\omega t + v_2(0)\sin\omega t), \\ v_2(t) &= e^{\lambda t}(v_2(0)\cos\omega t - v_1(0)\sin\omega t), \\ w_1(t) &= e^{-\gamma_1 t} w_1(0), \\ w_2(t) &= e^{-\gamma_2 t} w_2(0). \end{aligned} \tag{15}$$

The flow (15) induces a map from $H^{(\text{in})}$ to $H^{(\text{out})}$ which can be written as

$$\Psi(v_1, v_2, \theta) = (\tau(v), w_1, w_2).$$

This can be found by first computing the time of flight $T > 0$ such that:

$$(v_1^2 + v_2^2)e^{2\lambda T} = 1,$$

implying that

$$T = -\frac{1}{2\lambda}\ln(v_1^2 + v_2^2) = \ln \|v\|^{-\frac{1}{\lambda}},$$

where $v^2 = v_1^2 + v_2^2$. Thus we can write

$$\Psi(v_1, v_2, \theta) = (\tau(v), \|v\|^{\frac{\gamma_1}{\lambda}} \cos\theta, \|v\|^{\frac{\gamma_2}{\lambda}} \sin\theta),$$

where

$$\tau(v) = \arg\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \ln \|v\|^{\frac{\omega}{\lambda}}.$$

This means that we can write the action of the return map as $f(v_1, v_2, \theta) = (v_1', v_2', \theta')$ where

$$\begin{aligned} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} &= A(\tau(v))\begin{pmatrix} \cos\theta\|v\|^{\frac{\gamma_1}{\lambda}} \\ \sin\theta\|v\|^{\frac{\gamma_2}{\lambda}} \end{pmatrix}, \\ \theta' &= g(\tau(v)). \end{aligned} \tag{16}$$

Case (iv): Complex expanding and complex contracting eigenvalues. Finally, consider the case where a local change of coordinates gives linear parts

$$M_e = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix}, \qquad M_c = \begin{pmatrix} -\gamma & \rho \\ -\rho & -\gamma \end{pmatrix},$$

with real constants $\lambda > 0$, $\omega \neq 0$, $\gamma > 0$, and $\rho \neq 0$. Solving (4) gives the flow

$$\begin{aligned} v_1(t) &= e^{\lambda t}(v_1(0)\cos\omega t + v_2(0)\sin\omega t), \\ v_2(t) &= e^{\lambda t}(v_2(0)\cos\omega t - v_1(0)\sin\omega t), \\ w_1(t) &= e^{-\gamma_1 t}(w_1(0)\cos\rho t + w_2(0)\sin\rho t), \\ w_2(t) &= e^{-\gamma_2 t}(w_2(0)\cos\rho t - w_1(0)\sin\rho t). \end{aligned} \tag{17}$$

The map from $H^{(\text{in})}$ to $H^{(\text{out})}$ induced by this flow is

$$\Psi(v_1, v_2, \theta) = (\tau, w_1, w_2).$$

This can be found by first computing the time of flight $T > 0$ such that

$$(v_1^2 + v_2^2)e^{2\lambda T} = 1,$$

implying that

$$T = \frac{1}{2\lambda}\ln(v_1^2 + v_2^2) = \ln r^{\frac{1}{\lambda}},$$

with $r^2 = v_1^2 + v_2^2$. Thus we can write

$$\Psi(v_1, v_2, \theta) = (\tau, w_1, w_2),$$

where

$$\tau = \arg\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \ln \|v\|^{\frac{\omega}{\lambda}},$$

and

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \|v\|^{\frac{\gamma}{\lambda}} \begin{pmatrix} \cos(\theta - \frac{\rho}{\lambda}\ln\|v\|) \\ \sin(\theta - \frac{\rho}{\lambda}\ln\|v\|) \end{pmatrix}.$$

This implies that the return map f has the form $f(v_1, v_2, \theta) = (v_1', v_2', \theta')$ where

$$\begin{aligned} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} &= \|v\|^{\frac{\gamma}{\lambda}} A(\tau(v)) \begin{pmatrix} \cos(\theta - \frac{\rho}{\lambda}\ln\|v\|) \\ \sin(\theta - \frac{\rho}{\lambda}\ln\|v\|) \end{pmatrix}, \\ \theta' &= g(\tau(v)). \end{aligned} \tag{18}$$

Appendix B: Derivation of the Local Return Map for Arbitrary *d* and Distinct Real Eigenvalues

In the case of distinct real eigenvalues one can simplify the analysis of Appendix A. Writing $H^{(\mathrm{in})}$ as in (5) observe that $\Phi\colon H^{(\mathrm{out})} \to H^{(\mathrm{in})}$ can be written (to linear order in w_1, w_2) as

$$\Phi(v, w) = (v', w'),$$

where

$$\begin{aligned} v' &= A(v)w, \\ w' &= g(v), \end{aligned}$$

and A and g are (by virtue of hyperbolicity and closedness) smoothly dependent on $\phi \in \mathbf{S}^{d-1}$ with $A\colon \mathbf{S}^{d-1} \to GL(\mathbf{R}^d)$ and $g\colon \mathbf{S}^{d-1} \to \mathbf{S}^{d-1}$ a smooth diffeomorphism. Define $T\colon \mathbf{R}^d \setminus \{0\} \to \mathbf{R}^+$ such that

$$\sum_j v_j^2 e^{2\lambda_j T(v)} = 1,$$

and $\tau\colon \mathbf{R}^d \setminus \{0\} \to \mathbf{S}^{d-1}$ by

$$\tau_i(v) = e^{\lambda_i T(v)} v_i.$$

Note that $\|\tau(v)\| = 1$ and also

$$\ln\|v\|^{-\frac{1}{\lambda_1}} \le T \le \ln\|v\|^{-\frac{1}{\lambda_d}}. \tag{19}$$

The return map $f\colon H^{(\mathrm{in})} \to H^{(\mathrm{in})}$ can be written

$$f(v, w) = (v', w'),$$

where

$$\begin{aligned} v' &= A(\tau(v))(e^{-\gamma_1 T(v)} w_1, \ldots, e^{-\gamma_d T(v)} w_d), \\ w' &= g(\tau(v)). \end{aligned} \tag{20}$$

Acknowledgments

We thank Mike Field, Martin Krupa, Ian Melbourne, Mark Pollicott, and the referees for helpful comments. PA was supported by EU HCM fellowship ERBCHBCT930503.

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