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Chaotic Dynamics at the Boundary of a Basin of Attraction via Non-transversal Intersections for a Non-global Smooth Diffeomorphism

Ernest Fontich[1,2] · Antonio Garijo[3] · Xavier Jarque[1,2]

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Abstract
In this paper, we give analytic proofs of the existence of transversal homoclinic points for a family of non-globally smooth diffeomorphisms having the origin as a fixed point which come out as a truncated map governing the local dynamics near a critical period three-cycle associated with the Secant map. Using Moser's version of Birkhoff–Smale's theorem, we prove that the boundary of the basin of attraction of the origin contains a Cantor-like invariant subset such that the restricted dynamics to it is conjugate to the full shift of N-symbols for any integer $N \geq 2$ or infinity.

Keywords Secant map · Basin of attraction · Stable and unstable manifold · Homoclinic connection · Periodic points · Symbolic dynamics

Mathematics Subject Classification 37D05 · 37D10 · 37C29

1 Introduction

The question of whether a dynamical system admits invariant subsets of the phase portrait in which the dynamics is *chaotic* goes back to the origins of this area of mathematics. Studying the existence, or not, of chaotic dynamics and determine the topology and the geometry of the subsets where this happens has become a classical problem. Nevertheless, since the question arises in so many distinct scenarios, there

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Extended author information available on the last page of the article

has been different approaches to this phenomena, including the use of non-equivalent mathematical definitions in order to capture the meaning of *chaos* in each particular case. Measuring chaos in high, or even infinite, dimensional Hamiltonian dynamical systems or doing so for one-dimensional interval dynamics requires to particularize the meaning of the word *chaos* to concrete mathematical definitions.

Nonetheless, once we agree on which dynamical properties characterize chaos (density of periodic points, transitivity, dense orbits, sensibility with respect to initial conditions, all at once,...), a common accepted approach to ensure chaotic dynamics is to show that, in certain dynamically invariant region(s) of our phase portrait, the dynamics is *conjugate* (that is, *equal* up to a homeomorphism) to the one of a model for which it is somehow easy to test the properties mentioned above.

The usual toy model is the dynamical system (Σ_N, σ), where Σ_N is the set of bi-infinite (or one-side) sequences of $N \geq 2$ symbols and σ is the *shift map*; see (Moser 2001). One can easily check that the system (Σ_N, σ) captures the dynamical properties proposed above. Since the conjugacy sends orbits of our dynamical systems to orbits of the shift map acting on the space of symbols, this methodology is also known as *symbolic dynamics*. To focus on the content of this paper and simplify the discussion, let us assume we have a discrete dynamical system in $\mathbb{R}^2$ generated by the iterates of a (smooth) map.

In any event, the difficult part to apply this strategy is to show that in some regions of the phase portrait our dynamics is conjugate to the dynamical system (Σ_N, σ). A major result in this direction goes back to the cornerstone ideas of S. Smale (Birkhoff–Smale's theorem) and J. Moser Moser (2001) who provide *checkable* (in some cases only numerically) dynamical conditions to ensure that a given dynamical system has a subset of the phase portrait whose dynamics is conjugated to the full-shift of an arbitrary number of symbols (even infinitely many). Roughly speaking they showed that if a smooth map has a transversal homoclinic intersection between the stable and unstable invariant manifolds of a hyperbolic saddle fixed point then, there is an invariant Cantor set whose restricted dynamics is conjugate to (Σ_N, σ).

Even though the results have been extremely helpful in many different contexts (and extended in many different directions), we emphasize that the hypotheses include three key ingredients: the *hyperbolicity* of the saddle point, the map is a *global diffeomorphism* and the *transversality* of the intersection of the invariant manifolds. The main goal of this paper is to address the presence of chaotic dynamics, for a concrete family of maps, under the lack of two of the conditions; the inverse map would not be globally smooth and in a first step we only can prove (analytically) that we have an intersection with a finite order contact.

Concretely, in this paper we consider the map

$$T_d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - (x+y)^d \\ y - 2(x+y)^d \end{pmatrix}, \tag{1.1}$$

with $d \geq 3$ being odd. Such map is a *truncated* expression of the third iterate of the (extended) Secant map applied to a polynomial $p(x)$ near a critical period three-cycle

$$(c, c) \mapsto (c, \infty) \mapsto (\infty, 0) \mapsto (c, c),$$

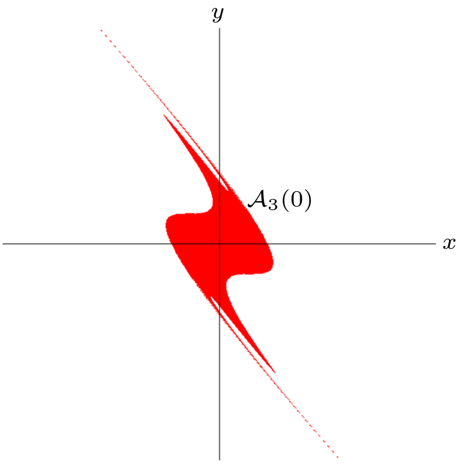

Fig. 1 The picture (in red) of the set $\mathcal{A}_3(0)$. Notice that according to Theorem 1.1 the red region is connected, simply connected and unbounded (Color figure online)

where $p'(c) = 0$ (but $p(c) \neq 0$). See (Bedford and Frigge 2018; Garijo and Jarque 2019, 2022; Fontich et al. 2024) for more details. For later discussions we point out here that T_d is a global homeomorphism, but it is not a global diffeomorphism since the inverse map, T_d^{-1}, is not smooth over the line $\{y = x\}$ (see (2.1) for its particular expression).

One can easily check that the origin of (1.1) is a fixed point and its basin of attraction

$$\mathcal{A}_d(0) = \{(x, y) \in \mathbb{R}^2 \mid T_d^n(x, y) \to (0, 0) \text{ as } n \to \infty\} \tag{1.2}$$

is not empty. In Fontich et al. (2024) we proved the following topological description of $\mathcal{A}_d(0)$ and further information about its boundary. We denote $p_0 = (0, 1)$ and $p_1 = (0, -1)$.

Theorem 1.1 *Let $d \geq 3$ odd. Then, $\mathcal{A}_d(0)$ is an open, simply connected, unbounded set. Moreover, $\partial \mathcal{A}_d(0)$ contains the stable manifold of the hyperbolic two-cycle $\{p_0, p_1\}$ lying in $\partial \mathcal{A}_d(0)$.*

The thesis of the above theorem glimpse the possible topological complexity of $\partial \mathcal{A}_d(0)$ (see Fig. 1). In fact, the main goal of this paper is to provide a better understanding of $\partial \mathcal{A}_d(0)$ by proving that, apart from the stable manifold of the hyperbolic two-cycle $\{p_0, p_1\}$ there is a Cantor subset of $\partial \mathcal{A}_d(0)$ where the dynamics is conjugated to the one of the shift of N symbols and so inhering all its chaotic dynamics.

In Fontich et al. (2024), we were able to describe and bound the shape of a piece of the unstable manifold of p_0 for T_d^2 (and it was a key point in the arguments to prove Theorem 1.1). In this paper, we mimic some of the arguments there to control the shape of a piece of the stable manifold of p_1. Using both constructions and a singular λ-Lemma (Rayskin 2003), we can ensure the existence of homoclinic (not necessarily linearly transversal) points for T_d^2.

Theorem A *Let $\{p_0, p_1\}$ be the hyperbolic two cycle lying in the boundary of $\partial \mathcal{A}_d(0)$. Then, the stable and unstable manifolds of p_1 (as well as p_0), as a fixed point for T_d^2, intersect at a homoclinic point.*

Going back to our previous arguments, if we want to apply Birkhoff–Smale's theorem, we need to prove the existence of transversal homoclinic points, so that Theorem A is not enough. In Churchill and Rod (1980), the authors are able to conclude transversal intersections under the presence of (topological) homoclinic intersections, but their map is area preserving, is a global smooth diffeomorphism and admits, in a sufficiently small neighbourhood the hyperbolic saddle, a concrete local normal form which provides a first integral.

In our case, we have not the previously mentioned normal form and since T_d^{-1} is not smooth over the line $\{y = x\}$ we cannot use that the globalization of the stable manifold of the 2-cycle $\{p_0, p_1\}$ by applying T_d^{-2} is analytic. In any event, inspired in the strategy proof in Churchill and Rod (1980), using alternative arguments to deal with our weaker conditions we are able to conclude the existence of transversal homoclinic intersections.

Theorem B *Let $\{p_0, p_1\}$ be the hyperbolic two cycle lying in the boundary of $\partial\mathcal{A}_d(0)$. Then, the stable and unstable manifolds of p_1 (as well as p_0), as a fixed point for T_d^2, intersect transversally.*

Although from the previous theorem, we have the existence of transversal homoclinic points we still cannot directly apply Birkhoff–Smale's theorem since the inverse map, T_d^{-1} map is not globally smooth. However, we can overcome this difficulty and prove the main result of this paper.

Theorem C *There exists an invariant Cantor set, contained in $\partial\mathcal{A}_{a,d}(0)$, where the dynamics of T_d^2 is conjugate to the full shift of N-symbols. In particular, $\partial\mathcal{A}_{a,d}(0)$ contains infinitely many periodic points with arbitrary high period.*

We emphasize that the theorems provide analytic proofs, rather than numerical evidence, of non-local properties of invariant manifolds for a family of maps. There are few cases where this has been done. For instance, in Fontich (1990), there is an analytical proof of the transversal intersection of the invariant manifolds for a wide range of a parameter for a class of maps which include the conservative Hénon map and the Chirikov standard map. In Gelfreich (1999), there is an analytical proof of the transversal intersection of the invariant manifolds of the standard map when the angle is exponentially small with respect to the parameter of the family. Also, in Delshams and Ramírez-Ros (1996) and Martín et al. (2011) they prove transversal intersection for the manifolds of close to integrable maps.

We organize the paper as follows. In Sect. 2, we summarize some preliminaries from Fontich et al. (2024) that we need in the proofs of the present paper, trying to make the present paper self-contained. In Sect. 3 we prove Theorem A, in Sect. 4 we prove Theorem B and finally in Sect. 5 we conclude the proof Theorem C.

2 Preliminaries

In this section, we collect some preliminary results about the map $T_d : \mathbb{R}^2 \to \mathbb{R}^2$, introduced in (1.1), for $d \geq 3$, an odd number. Everything was already introduced in

Fontich et al. (2024), but for the sake of completeness and easier reading, we include them here.

The map $T_d : \mathbb{R}^2 \to \mathbb{R}^2$ is a polynomial and a homeomorphism and its inverse map is real analytic in $\mathbb{R}^2 \setminus \{x = y\}$, but not differentiable on the line $\{x = y\}$. Its inverse is given by

$$T_d^{-1}(x, y) = \left(-2x + y + (x - y)^{1/d}\,,\, 2x - y\right). \tag{2.1}$$

Observe that $T_d^{-1}(x, x) = (-x, x)$ for all $x \in \mathbb{R}$. One can easily check that T_d has a unique two-cycle $\{p_0 = (0, 1), p_1 = (0, -1)\}$, i.e. $p_1 = T_d(p_0)$ and $p_0 = T_d(p_1)$. This two-cycle will play a fundamental role in the dynamics of T_d. Moreover, we have that

$$DT_d^2(p_0) = DT_d^2(p_1) = DT_d(p_0)DT_d(p_1) = \begin{pmatrix} 3d^2 - 2d & 3d^2 - 4d + 1 \\ 6d^2 - 2d & 6d^2 - 6d + 1 \end{pmatrix}. \tag{2.2}$$

A direct computation shows that the characteristic equation of $DT_d^2(p_0)$ is

$$p(\lambda) = \lambda^2 - (1 - 8d + 9d^2)\lambda + d^2 = 0$$

and the eigenvalues and eigenvectors are given by

$$\lambda_d^{\pm} = \frac{1}{2}\left(9d^2 - 8d + 1 \pm (3d - 1)\sqrt{9d^2 - 10d + 1}\right) \tag{2.3}$$

and

$$(1, m_d^{\pm}) = \left(1,\ \frac{4d}{1 - d \pm \sqrt{9d^2 - 10d + 1}}\right), \tag{2.4}$$

respectively.

On the one hand, it is easy to check that both eigenvalues are strictly positive. Moreover, λ_d^- is strictly decreasing and λ_d^+ is strictly increasing, with respect to the parameter d. We also have

$$\begin{aligned} &\lim_{d\to\infty} \lambda_d^- = 1/9 \quad \text{and} \quad 1/9 < \lambda_d^- \le \lambda_3^- = 29 - 8\sqrt{13} \approx 0.1556 \\ &\lim_{d\to\infty} \lambda_d^+ = \infty \quad \text{and} \quad \lambda_d^+ \ge \lambda_3^+ = 29 + 8\sqrt{13} \approx 57.8444. \end{aligned}$$

On the other hand, m_d^- is negative and strictly increasing while m_d^+ is positive and strictly decreasing (both with respect to the parameter d). We also have

$$\begin{aligned} &\lim_{d\to\infty} m_d^- = -1 \quad \text{and} \quad -1.3028 \approx \tfrac{-6}{1+\sqrt{13}} = m_3^- \le m_d^- < -1, \\ &\lim_{d\to\infty} m_d^+ = 2 \quad \text{and} \quad 2 < m_d^+ \le m_3^+ = \tfrac{6}{\sqrt{13}-1} \approx 2.3028. \end{aligned}$$

Therefore, the two cycle $\{p_0, p_1\}$ is hyperbolic of saddle type. We denote $W^s_{p_j}$, $W^u_{p_j}$ the stable and the unstable manifolds of the fixed points p_j for the map T_d^2, $j = 0, 1$. Similarly we denote by $W^s_{\mathrm{loc},\,p_j}$, $W^u_{\mathrm{loc},\,p_j}$ the corresponding local stable and unstable manifolds of some size δ that we do not make explicit in the notation. Actually, given some size $\delta > 0$,

$$W^s_{\mathrm{loc},\,p_j} = \{z \in \mathbb{R}^2 \mid T_d^{2k}(z) \in B_\delta(p_j) \quad \text{for all} \ \ k \geq 0\},$$

where $B_\delta(p_j)$ denotes the open ball centred at p_j with radius $\delta > 0$, for $j = 0, 1$. We define analogously $W^u_{\mathrm{loc},\,p_j}$ for T_d^{-2k}.

We also denote

$$W^s := W^s_{\{p_0,p_1\}} \quad \text{and} \quad W^u := W^u_{\{p_0,p_1\}}$$

the global stable and unstable manifolds of the periodic orbit $\{p_0, p_1\}$, respectively. Since T_d is analytic on $\mathbb{R}^2$ and T_d^{-1} is analytic on $\mathbb{R}^2\backslash\{y = x\}$ the local versions of the invariant manifolds are analytic. Moreover, the (global) unstable manifold, obtained iterating by T_d the local one, is analytic and the (global) stable manifold, obtained iterating by T_d^{-1}, is analytic except at the preimages of the intersections of W^s with $\{y = x\}$.

When there is no confusion we use the simplified notation $\lambda^\pm := \lambda_d^\pm$ and $m^\pm := m_d^\pm$.

The triangle $\mathcal{D}$ and its images: $T_d(\mathcal{D})$ and $T_d^{-1}(\mathcal{D})$.

In Fontich et al. (2024, Section 5) we considered the triangle $\mathcal{D}$ of vertices

$$p_1 = (0, -1), \qquad \left(\frac{1}{m^+ + 1}, \frac{-1}{m^+ + 1}\right) \qquad \text{and} \qquad \left(\frac{1}{m^\star + 1}, \frac{-1}{m^\star + 1}\right),$$

where $m^\star = 7/2$, or equivalently,

$$\mathcal{D} = \{(t, -1 + mt) \,|\, t \in [0, 1/(m + 1)], m \in [m^+, m^\star]\}.$$

We also considered the sets $T_d(\mathcal{D})$ and $T_d^{-1}(\mathcal{D})$. We showed that the set $T_d(\mathcal{D})$ is bounded by the images of the sides of $\mathcal{D}$ given by the curves $\gamma_{m^+}(t)$, $\gamma_{m^\star}(t)$ where

$$\gamma_m(t) = T_d(t, -1 + mt) = (mt - 1 - ((m + 1)t - 1)^d, mt - 1 - 2((m+1)t - 1)^d), \tag{2.5}$$

for $0 \leq t \leq \frac{1}{m+1}$, and

$$\partial T_d(\mathcal{D}) \cap \{y = x\} = \left\{(t, t) \mid \frac{-1}{m^+ + 1} \leq t \leq \frac{-1}{m^\star + 1}\right\}.$$

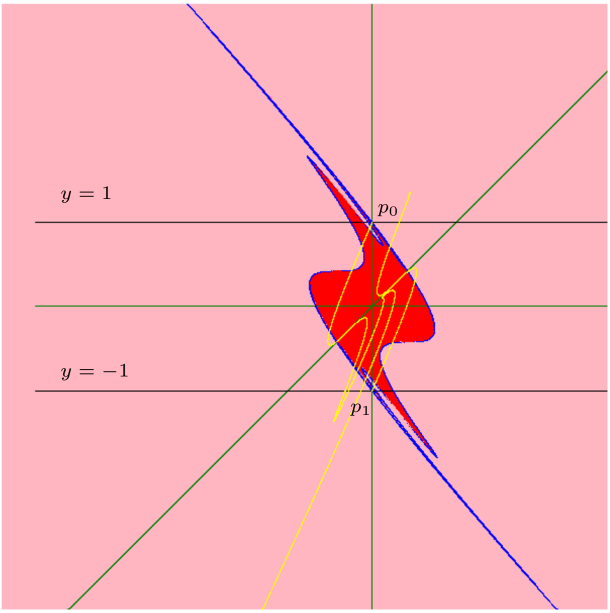

Fig. 2 This picture corresponds to $d = 3$. In red we plot the attracting basin $\mathcal{A}_3(0)$. In blue (respectively, yellow) we draw the stable (respectively, unstable) manifold of the two cycle $\{p_0, p_1\}$. The picture illustrates (numerically) the transversal intersections described in Theorem B. According to Theorem C, $\partial\mathcal{A}_d(0)$ contains the stable manifold of the two cycle (in blue) and a Cantor-set like with chaotic dynamics (Color figure online)

Finally, we claim that there is a (connected) piece of $W^u_{p_0} \cap \{y \leq 1\}$, tangent to the line $y = 1 + m^+ x$ at p_0, contained in $T_d(\mathcal{D})$ joining the point p_0 with some point in $\partial T_d(\mathcal{D}) \cap \{y = x\}$. We call left and right boundaries of $T_d(\mathcal{D})$ the curves $\gamma_{m^+}(t)$ and $\gamma_{m^\star}(t)$, respectively. See Fig. 3 (left). We do not include here the arguments used in Fontich et al. (2024, Lemma 5.4) to prove the claim but in the next section we mimic, including all computations, the ideas used in Fontich et al. (2024) for the case of $\widehat{\mathcal{D}}$, $T_d(\widehat{\mathcal{D}})$ and $T_d^{-1}(\widehat{\mathcal{D}})$.

3 Proof of Theorem A

To prove Theorem A, we first show the existence of an heteroclinic intersection for the map T_d. More precisely, we have the following statement.

Proposition 3.1 *Let $\{p_0, p_1\}$ be the hyperbolic two-cycle lying in the boundary of $\partial\mathcal{A}_d(0)$ (see Theorem 1.1). Then, the unstable manifold of p_0 and the stable manifold of p_1 intersect in a heteroclinic point.*

The idea is to show that $T_d(\mathcal{D})$ (Fig. 3, left) and $T_d^{-1}(\widehat{\mathcal{D}})$ ((Fig. 3, right) intersect in a suitable manner that forces the intersection of the invariant manifolds (Fig. 4). Since the proof of this proposition is quite long, we split it into several lemmas. We assume

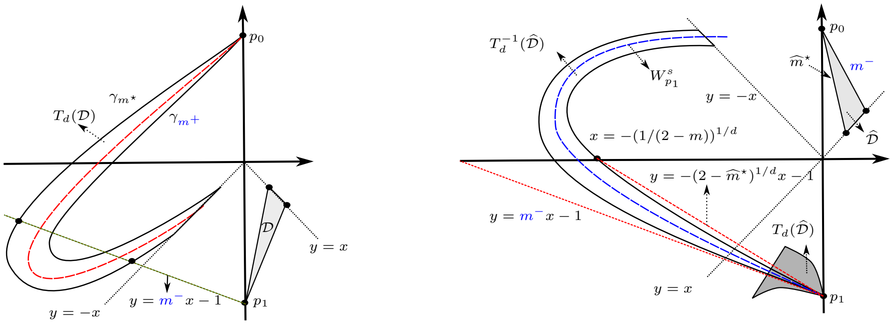

Fig. 3 Left: The triangle $\mathcal{D}$, its image $T_d(\mathcal{D})$ and (dashed, red) a piece of $W^u_{p_0}$ attached to p_0. Right: The triangle $\widehat{\mathcal{D}}$, its images $T_d(\mathcal{D})$ and $T_d^{-1}(\widehat{\mathcal{D}})$, and (dashed, blue) a piece of $W^s_{p_1}$ attached to p_1. We also add the relevant objects appearing in the proof of Proposition 3.1 and Theorem A (Color figure online)

all notation introduced in Sect. 2. In particular, we have described the construction provided in Fontich et al. (2024, Lemma 5.4) to localize the piece of the unstable manifold attached to p_0 inside $T_d(\mathcal{D})$. The first step is to make a similar construction to localize a piece of the stable manifold of p_1. Let

$$-\frac{3}{2} \le \widehat{m}^\star = \widehat{m}^\star_d := -1 - \frac{1}{d-1} = \frac{-d}{d-1} < m^- < -1, \tag{3.1}$$

where the inequalities follow from direct computations. We introduce the triangle $\widehat{\mathcal{D}}$ with vertices

$$p_0 = (0,1), \qquad \left(\frac{-1}{m^-+1}, \frac{-1}{m^-+1}\right) \qquad \text{and} \qquad \left(\frac{-1}{\widehat{m}^\star+1}, \frac{-1}{\widehat{m}^\star+1}\right),$$

or equivalently,

$$\widehat{\mathcal{D}} = \{(t, 1+mt) \mid t \in [0, 1/(1-m)],\ m \in [\widehat{m}^\star, m^-]\}.$$

As we did with the set $\mathcal{D}$ in Fontich et al. (2024, Lemma 5.3), we study the geometry of the sets $T_d(\widehat{\mathcal{D}})$ and $T_d^{-1}(\widehat{\mathcal{D}})$. From the properties of these sets, we will prove that there is a piece of $W^s_{p_1} \cap \{y \ge -1\}$ that is contained in $T_d^{-1}(\widehat{\mathcal{D}})$. Moreover, this piece joints p_1 with a point in $T_d^{-1}(\widehat{\mathcal{D}}) \cap \{y = -x\}$. See the right picture in Fig. 3. Then, using the geometry of the intersection of $T_d(\mathcal{D})$ and $T_d^{-1}(\widehat{\mathcal{D}})$ we will prove that $W^u_{p_0}$ and $W^s_{p_1}$ have to cross (topologically) in a heteroclinic intersection, proving Proposition 3.1. From this heteroclinic intersection, we will obtain a homoclinic intersection as claimed in Theorem A.

The preimage $T_d^{-1}(\widehat{\mathcal{D}})$. We denote by $\widehat{\Gamma}_m(t)$ the image by T_d^{-1} of the segment $\{(t, 1+mt) \mid t \in [0, 1/(1-m)]\}$. Thus,

$$\widehat{\Gamma}_m(t) = T_d^{-1}(t, 1+mt) =: (\widehat{\alpha}_m(t), \widehat{\beta}_m(t)), \tag{3.2}$$

where

$$\widehat{\alpha}_m(t) = (m-2)t + 1 + ((1-m)t-1)^{1/d} \quad \text{and} \quad \widehat{\beta}_m(t) = (2-m)t - 1.$$

We are interested in $\widehat{\Gamma}_m(t)$ for $m \in [\widehat{m}^\star, m^-]$. Note that the point on $\widehat{\mathcal{D}} \cap \{y = x\}$ corresponds to $t = 1/(1-m)$ and is mapped by T_d^{-1} to $(-1/(1-m), 1/(1-m))$ on the line $\{y = -x\}$. Taking derivatives, we have that

$$\widehat{\alpha}'_m(t) = m - 2 + \frac{1-m}{d}((1-m)t-1)^{(1-d)/d}, \ \ \widehat{\beta}'_m(t) = 2 - m > 0, \ \ \widehat{\alpha}''_m(t) > 0 \ \text{ and } \ \widehat{\beta}''_m(t) = 0.$$

A direct computation shows that $\widehat{\alpha}'_m(t) = 0$ if and only if $t = t_\pm$, where

$$t_\pm = \frac{1}{1-m}\left(1 \pm \left(\frac{1-m}{d(2-m)}\right)^{d/(d-1)}\right)$$

and $0 < t_- < \frac{1}{1-m} < t_+$, since, as $m < 0$, we have $0 < \frac{1-m}{d(2-m)} < 1$. It follows from these computations that $\widehat{\alpha}_m(t)$, $m \in [\widehat{m}^\star, m^-]$, has a unique minimum (in its domain) at $t_- \in (0, \frac{1}{1-m})$. Finally, $\widehat{\alpha}'_m(\frac{1}{1-m}) = \infty$ which means that when $\widehat{\Gamma}_m(t)$ meets $\{y = -x\}$, its tangent line is horizontal. See the right picture in Fig. 3. In other words the vectors $\widehat{\Gamma}'_m(\frac{1}{1-m})$ and $\widehat{\Gamma}'_m(t_-)$ are parallel to the lines $y = 0$ and $x = 0$, respectively.

Since $\widehat{\beta}_m(t)$ is invertible (linear), for any m we can represent the curve $\widehat{\Gamma}_m(t)$ as the graph of a function $x = g(y)$, $y \in [-1, 1/(1-m)]$ (remember that $1/(1-m) > 0$), by taking $g(y) = \widehat{\alpha}_m \circ \widehat{\beta}_m^{-1}(y)$. Since $\widehat{\beta}''_m(t) = 0$, we have that

$$\frac{\mathrm{d}g}{\mathrm{d}y}(y) = \left[\frac{\mathrm{d}\widehat{\alpha}_m}{\mathrm{d}t}\left(\frac{\mathrm{d}\widehat{\beta}_m}{dt}\right)^{-1}\right] \circ \widehat{\beta}_m^{-1}(y) \qquad \text{and}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}y^2}(y) = \frac{\mathrm{d}^2\widehat{\alpha}_m}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\widehat{\beta}_m}{\mathrm{d}t}\right)^{-2} \circ \widehat{\beta}_m^{-1}(y) > 0.$$

The convexity of g implies that the image of $\widehat{\Gamma}_m(t)$ is above its tangent line at $p_1 = (0, -1)$. In case $m = m^-$, this tangent line has slope m^- and it is the minimum slope for all $m \in [\widehat{m}^\star, m^-]$. Therefore, $T_d^{-1}(\widehat{\mathcal{D}})$ is above the line $y = m^- x - 1$.

Also, g has a unique minimum at $y_- = \widehat{\beta}_m(t_-)$. Moreover, $\widehat{\Gamma}_m(t)$ intersects $\{y = 0\}$ when $t = 1/(2-m)$ at the point $(x, y) = (\widehat{\alpha}_m(1/(2-m)), 0)$ with

$$\widehat{\alpha}_m(1/(2-m)) = -\left(\frac{1}{2-m}\right)^{1/d}.$$

Again, the convexity of the function g implies that its graph intersected with $\{y \leq 0\}$ is below the line

$$y = -(2-m)^{1/d} x - 1$$

and, in particular (see Fig. 3), taking $m = m^\star$ we conclude that $T_d^{-1}(\widehat{\mathcal{D}}) \cap \{y \leq 0\}$ is below

$$y = -(2 - \widehat{m}^\star)^{1/d} x - 1 = -\left(\frac{3d-2}{d-1}\right)^{1/d} x - 1. \tag{3.3}$$

The image $T_d(\widehat{\mathcal{D}})$. We notice that since $\{p_0, p_1\}$ is a two-cycle we have $T_d(p_0) = T_d^{-1}(p_0) = p_1$, so that $T_d(\widehat{\mathcal{D}})$ is attached to p_1 as it was the case of $T_d^{-1}(\widehat{\mathcal{D}})$.

We denote by $\widehat{\gamma}_m(t)$ the image by T_d of the segment $\{(t, 1+mt) \mid t \in [0, 1/(1-m)]\}$, with $m \in [\widehat{m}^\star, m^-]$. Hence, $\widehat{\gamma}_m(t) = T_d(t, 1+mt) =: (\widehat{x}_m(t), \widehat{y}_m(t))$ where

$$\widehat{x}_m(t) = mt + 1 - ((m+1)t+1)^d, \qquad \widehat{y}_m(t) = mt + 1 - 2((m+1)t+1)^d. \tag{3.4}$$

To simplify notation, we write $x(t) := \widehat{x}_m(t)$ and $y(t) := \widehat{y}_m(t)$ and $\gamma(t) = \widehat{\gamma}_m(t)$ unless it is strictly necessary to show the dependence in m. The derivatives are given by

$$\begin{aligned} x'(t) &= m - d(m+1)((m+1)t+1)^{d-1}, & y'(t) &= m - 2d(m+1)((m+1)t+1)^{d-1}, \\ x''(t) &= -d(d-1)(m+1)^2((m+1)t+1)^{d-2}, & y''(t) &= -2d(d-1)(m+1)^2((m+1)t+1)^{d-2}. \end{aligned}$$

Since $t \in [0, 1/(1-m)]$ and $m < -1$, we have the inequalities

$$0 < \frac{2}{1-m} = \frac{m+1}{1-m} + 1 < (m+1)t + 1 \leq 1.$$

Then, for $d \geq 3$ (odd), we have

$$x''(t) < 0 \qquad \text{and} \qquad y''(t) < 0.$$

Next lemma provides basic estimates on the parametrization $\gamma(t)$.

Lemma 3.2 *Let $m \in [\widehat{m}^\star, m^-]$ and $t \in [0, 1/(1-m)]$. The following conditions hold.*

(a) $x(0) = 0$, $x(\frac{1}{1-m}) < \frac{1}{1-m}[1 - \frac{2}{\sqrt{e}}] < 0$, $x(t) < 0$ *for* $t \neq 0$, *and* $y(t) < 0$.
(b) $x'(t) \leq 0$ *with* $x'(t) = 0$ *if and only if* $t = 0$ *and* $m = \widehat{m}^\star$.
(c) $y'(t) > 0$ *for* $m = m^\star$.

Proof The proof of the items follows from some computations based on the expressions of $x(t)$, $y(t)$ and their derivatives above.

Easily $x(0) = 0$. On the one hand, we have

$$\begin{aligned} x\left(\frac{1}{1-m}\right) = \frac{1}{1-m} - \left(\frac{2}{1-m}\right)^d &= \frac{1}{1-m}\left[1 - 2\left[\left(1 + \frac{1}{2(d-1)}\right)^{2(d-1)}\right]^{-1/2}\right] \\ &< \frac{1}{1-m}\left[1 - \frac{2}{\sqrt{e}}\right] < 0. \end{aligned}$$

On the other hand, $x'(0) = m - d(m+1) \leq (-1 - \frac{1}{d-1})(1-d) - d = 0$ where the equality only holds for $m = \widehat{m}^\star$ and $x''(t) < 0$ (see (3.1)). Hence, $x'(t) < 0$ (unless $t = 0$ and $m = \widehat{m}^\star$ where $x'_{m^\star}(0) = 0$) and so $x(t)$ is decreasing (and negative unless $t = 0$). Finally, we have $y(t) = x(t) - ((m+1)t + 1)^d < 0$. All together implies (a) and (b).

If $m = \widehat{m}^\star$, using (3.1) we have

$$\begin{aligned}
y'_{m^\star}(t) &= m^\star - 2d(m^\star + 1)((m^\star + 1)t + 1)^{d-1} \geq \frac{d}{d-1}\left[-1 + 2\left(\frac{-1}{d-1}t + 1\right)^{d-1}\right] \\
&\geq \frac{d}{d-1}\left[-1 + 2\left(1 - \frac{1}{2d-1}\right)^{d-1}\right] = \frac{d}{d-1}\left[-1 + 2\left(1 + \frac{1}{2(d-1)}\right)^{1-d}\right] \\
&> \frac{d}{d-1}\left[-1 + \frac{2}{\sqrt{e}}\right] > 0
\end{aligned}$$

that proves (c). ◻

Since $x'(t) < 0$, the function $x(t)$ is invertible. If $t = t(x)$ is the inverse map of $x(t)$, then (the image of) $\gamma(t)$ can be represented as the graph of the function $h(x) := h_m(x) := y \circ t(x)$. From its definition, the function h is smooth.

Lemma 3.3 *We have that $h(x) = y \circ t(x)$ is concave.*

Proof Taking derivatives we have

$$h'(x) = \frac{y'}{x'} \circ t(x) \qquad \text{and} \qquad h''(x) = \frac{1}{(x')^2}\left[y'' - x''\frac{y'}{x'}\right] \circ t(x).$$

From the expressions of x and y and their derivatives (see (3.4) and the derivatives below), we have $y'(t) = x'(t) - d(m+1)((m+1)t+1)^{d-1}$ and $y''(t) = 2x''(t)$. Therefore

$$h''(x) = \frac{x''}{(x')^2}\left[2 - \frac{y'}{x'}\right] \circ t(x) \quad \text{and} \quad 2 - \frac{y'}{x'} = 1 + \frac{d(m+1)((m+1)t+1)^{d-1}}{x'} > 1,$$

concluding $h''(x) < 0$ and hence h is concave (remember that the result is valid for all values of m in the range). ◻

Now we fix $m = \widehat{m}^\star$. We claim that (the image of) $\widehat{\gamma}_{\widehat{m}^\star}$ belongs to $\{x \leq 0, y \leq 0\}$ and it is above $T_d^{-1}(\widehat{\mathcal{D}})$. To check the claim, accordingly to the previous study of $T_d^{-1}(\widehat{\mathcal{D}})$ it is sufficient to check that $\widehat{\gamma}_{\widehat{m}^\star}$ is above the line $y = -(2 - \widehat{m}^\star)^{1/d}x - 1$ introduced in (3.3). Moreover, since (the image of) $\widehat{\gamma}_{\widehat{m}^\star}$ is the graph of a concave function it is enough to check that

$$y_{\widehat{m}^\star}\left(\frac{1}{1 - \widehat{m}^\star}\right) > -(2 - \widehat{m}^\star)^{1/d}\, x_{\widehat{m}^\star}\left(\frac{1}{1 - \widehat{m}^\star}\right) - 1.$$

This is equivalent to

$$\frac{1}{1-\widehat{m}^\star} - 2\left(\frac{2}{1-\widehat{m}^\star}\right)^d + (2-\widehat{m}^\star)^{1/d}\left[\frac{1}{1-\widehat{m}^\star} - \left(\frac{2}{1-\widehat{m}^\star}\right)^d\right] + 1 > 0.$$

If we substitute $\widehat{m}^\star = -1 - \frac{1}{d-1}$, the above inequality writes as

$$\begin{aligned} &1 - \left(\frac{2}{2+1/(d-1)}\right)^d + \left[\frac{1}{2+1/(d-1)} - \left(\frac{2}{2+1/(d-1)}\right)^d\right] \\ &\quad \times \left[1 + \left(3 + \frac{1}{d-1}\right)^{1/d}\right] > 0. \end{aligned} \tag{3.5}$$

On the one hand we have that

$$\left(\frac{2}{2+1/(d-1)}\right)^d < \left(\frac{1}{1+1/(2d)}\right)^d \leq \left(\frac{6}{7}\right)^3.$$

On the other hand we have that

$$\begin{aligned} &\frac{1}{2+1/(d-1)} - \left(\frac{2}{2+1/(d-1)}\right)^d \\ &\quad = \frac{1}{2+1/(d-1)}\left[1 - 2\left[\left(1 + \frac{1}{2(d-1)}\right)^{2(d-1)}\right]^{-1/2}\right] \\ &\quad \geq \frac{1}{2+1/(d-1)}\left[1 - 2\left(1+\frac{1}{4}\right)^{-2}\right] \geq \frac{-7}{50}, \end{aligned}$$

and

$$0 < 1 + \left(3 + \frac{1}{d-1}\right)^{1/d} < 1 + \left(3 + \frac{1}{2}\right)^{1/3} = 1 + (7/2)^{1/3}.$$

Hence, to prove (3.5) it is enough to check that

$$1 - \left(\frac{6}{7}\right)^3 - \frac{7}{50}\left(1 + \left(\frac{7}{2}\right)^{1/3}\right) \approx 0.02 > 0.$$

Moreover, we also claim that $\widehat{\gamma}_{m^-}$ is below $T_d^{-1}(\widehat{\mathcal{D}})$. This easily follows from the fact that, by the description of the preimage $T_d^{-1}(\widehat{\mathcal{D}})$, the left boundary of $T_d^{-1}(\widehat{\mathcal{D}})$ is the graph of a convex function $x = g(y)$ while $\widehat{\gamma}_{m^-}$ is the graph of a concave function $y = h_{m^-}(x)$ and both graphs are tangent at p_1.

It follows from lemmas above that we have a deep control on the *left* and *right* boundaries of $T_d(\widehat{\mathcal{D}})$, and their relative position with respect to the set $T_d^{-1}(\widehat{\mathcal{D}})$. See the right picture of Fig. 3. Now we close the argument by controlling the image of $\partial\widehat{\mathcal{D}} \cap \{y = x\}$.

Lemma 3.4 *The upper piece of the boundary of $T_d(\widehat{\mathcal{D}})$ is the image by T_d of the piece of the boundary $\left\{(t,t) \mid \frac{1}{1-\widehat{m}^\star} \le t \le \frac{1}{1-m^-}\right\}$ of $\widehat{\mathcal{D}}$. It can be represented as the graph of an increasing function and is contained in $\{x < 0,\ y < 0\}$.*

Proof We introduce

$$T_d(t,t) = (t-(2t)^d, t-2(2t)^d) =: (\xi(t), \eta(t)), \quad (1-\widehat{m}^\star)^{-1} \le t \le (1-m^-)^{-1}.$$

Taking first and second derivatives, we have

$$\begin{aligned} \xi'(t) &= 1-2d(2t)^{d-1}, \quad \eta'(t) = 1-4d(2t)^{d-1}, \\ \xi''(t) &= -4d(d-1)(2t)^{d-2} \text{ and } \eta''(t) = 2\xi''(t). \end{aligned}$$

First we check that, in the corresponding domain, $\xi'(t) < 0$ and $\eta'(t) < 0$. This follows from $\xi''(t) < 0,\ \eta''(t) < 0$ and

$$\xi'\left(\frac{1}{1-\widehat{m}^\star}\right) = 1-2d\left(\frac{2}{1-\widehat{m}^\star}\right)^{d-1} = 1-2d\frac{1}{\left(1+\frac{1}{2(d-1)}\right)^{d-1}} < 1-\frac{2d}{\sqrt{e}} < 0,$$

and

$$\eta'\left(\frac{1}{1-\widehat{m}^\star}\right) = \xi'\left(\frac{1}{1-\widehat{m}^\star}\right) - 2d\left(\frac{2}{1-\widehat{m}^\star}\right)^{d-1} < 0.$$

The condition $\xi'(t) < 0$ implies that $\xi(t)$ is invertible. Let $t = t(\xi)$ be its inverse function and $\eta = f(\xi) := \eta \circ t(\xi)$. The curve $T_d(t,t)$ is the graph of f and

$$f' = \frac{\eta'}{\xi'} \circ t(x) > 0.$$

Moreover, since

$$\begin{aligned} \xi\left(\frac{1}{1-\widehat{m}^\star}\right) &= \frac{1}{1-\widehat{m}^\star}\left(1-2\left(\frac{2}{1-\widehat{m}^\star}\right)^{d-1}\right) = \frac{1}{1-\widehat{m}^\star}\left(1-2\frac{1}{(1+\frac{1}{2(d-1)})^{d-1}}\right) \\ &< \frac{1}{1-\widehat{m}^\star}\left(1-\frac{2}{\sqrt{e}}\right) < 0, \end{aligned}$$

we have $\xi(t) < 0$ and $\eta(t) = \xi(t) - (2t)^d < 0$. ◻

Up to this point, we have completed the study of the geometry and relative positions of $T_d(\widehat{\mathcal{D}})$ and $T_d^{-1}(\widehat{\mathcal{D}})$ (see the right picture of Fig. 3). Next two lemmas show that there is a piece of $W^s_{p_1}$ attached to p_1, being tangent to $y = m^-x - 1$ at p_1, included in $T_d^{-1}(\widehat{\mathcal{D}})$ and connecting p_1 with a point in $\partial T_d^{-1}(\widehat{\mathcal{D}}) \cap \{y = -x\},\ x < 0$.

Let $(x_0, y_0) \in T_d^{-1}(\widehat{\mathcal{D}})$. Then, we write $(x_{2k}, y_{2k}) := T_d^{2k}(x_0, y_0)$. The first lemma characterize the dynamics of points in $T_d^{-1}(\widehat{\mathcal{D}})$ whose all iterates under T_d^2 remain in $T_d^{-1}(\widehat{\mathcal{D}})$.

Lemma 3.5 *If $(x_0, y_0) \in T_d^{-1}(\widehat{\mathcal{D}})$ and $(x_{2k}, y_{2k}) \in T_d^{-1}(\widehat{\mathcal{D}})$ for all $k \geq 0$ then we have that $(x_{2k}, y_{2k}) \to p_1 = (0, -1)$ as $k \to \infty$.*

Proof First we note that $y_2 < 0$. Indeed, $(x_2, y_2) \in T_d(\widehat{\mathcal{D}})$ and by Lemma 3.4

$$\sup\left\{\eta(t) \mid \frac{1}{1-\widehat{m}^\star} \leq t \leq \frac{1}{1-m^-}\right\} = \eta\left(\frac{1}{1-\widehat{m}^\star}\right) < 0.$$

Moreover, the right boundary of $T_d(\widehat{\mathcal{D}})$ is given by $\widehat{\gamma}_{\widehat{m}^\star}(t) = (\widehat{x}_m(t), \widehat{y}_m(t))$ and, by Lemma 3.2(a), $\widehat{y}_{\widehat{m}^\star}(t) < \widehat{y}_{\widehat{m}^\star}(\frac{1}{1-\widehat{m}^\star}) = \eta(\frac{1}{1-\widehat{m}^\star}) < 0$.

Now, let (x_0, y_0) as in the statement with $y_0 < 0$. Using that $T_d^{-1}(\widehat{\mathcal{D}}) \cap \{y \leq 0\}$ is below the line $y = -(2-\widehat{m}^\star)^{1/d}x - 1$ we have that

$$x_0 < \frac{y_0 + 1}{-(2-\widehat{m}^\star)^{1/d}}. \tag{3.6}$$

First, we compute

$$(x_1, y_1) = T_d(x_0, y_0) = (y_0 - (x_0 + y_0)^d, y_0 - 2(x_0 + y_0)^d).$$

We observe that (x_1, y_1) belongs to the line $y = 2x - y_0$.

By the definition of $\widehat{\mathcal{D}}$, we have that x_1 is less than the first coordinate of the intersection $\{y = 2x - y_0\} \cap \{y = m^-x + 1\}$, i.e. $x_1 < \frac{1+y_0}{2-m^-}$. Moreover, using (3.6),

$$0 \leq x_1 < \frac{-(2-\widehat{m}^\star)^{1/d}}{2-m^-} x_0.$$

Next we bound

$$\left|\frac{-(2-\widehat{m}^\star)^{1/d}}{2-m^-}\right| < \frac{(3+\frac{1}{d-1})^{1/d}}{3} \leq \frac{1}{3}\left(\frac{7}{2}\right)^{1/3}.$$

Now we deal with the next iterate $(x_2, y_2) = (y_1 - (x_1 + y_1)^d, y_1 - 2(x_1 + y_1)^d)$. Since $(x_1, y_1) \in \widehat{\mathcal{D}}$, $0 < x_1 + y_1 \leq 1$ and $y_1 \geq \widehat{m}^\star x_1 + 1$ we conclude that

$$0 \geq x_2 = y_1 - (x_1 + y_1)^d \geq y_1 - 1 \geq \widehat{m}^\star x_1.$$

Consequently,

$$|x_2| \le |\widehat{m}^\star| x_1 \le \frac{3}{2}\frac{1}{3}\left(\frac{7}{2}\right)^{1/3} |x_0| \le \frac{4}{5}|x_0|.$$

Recursively, we obtain that $|x_{2k}| \le (\frac{4}{5})^k |x_0|$ and this implies $x_{2k} \to 0$, Since, by hypothesis, $(x_{2k}, y_{2k}) \in T_d^{-1}(\widehat{\mathcal{D}})$ for all $k \ge 0$ we conclude that $y_{2k} \to -1$. □

Lemma 3.6 *The set $T_d^{-1}(\widehat{\mathcal{D}})$ contains a piece of $W^s_{p_1}$ joining the point p_1 with a point in $T_d^{-1}(\widehat{\mathcal{D}}) \cap \{y = -x\}$.*

Proof We will use the same argument we have used in Fontich et al. (2024). Take I_0 any segment joining the right and left boundaries of $T_d^{-1}(\widehat{\mathcal{D}})$. By the previous lemmas, $T_d^2(I_0)$ is a curve contained in $T_d(\widehat{\mathcal{D}})$ joining its right and left boundaries which are outside $T_d^{-1}(\widehat{\mathcal{D}})$, thus it has to cross the right and left boundaries of $T_d^{-1}(\widehat{\mathcal{D}})$.

We define $I_1 = T_d^{-2}(T_d^2(I_0) \cap T_d^{-1}(\widehat{\mathcal{D}})) \subset I_0$ and, in general,

$$I_n = T_d^{-2n}(T_d^{2n}(I_{n-1}) \cap T_d^{-1}(\widehat{\mathcal{D}})) \subset I_{n-1}, \qquad n \ge 1.$$

Then, $\{I_n\}_{n\ge1}$ is a sequence of nested compact sets and $I_\infty := \bigcap_{n\ge1} I_n \ne \emptyset$. This set has the property that all points in I_∞ are such that all their iterates stay in $T_d^{-1}(\widehat{\mathcal{D}})$ and, by Lemma 3.5, converge to p_1. Therefore, $I_\infty = W^s_{p_1} \cap T_d^{-1}(\widehat{\mathcal{D}}) \cap I_0$. □

Proof of Proposition 3.1 We will see that the above description of the relative positions of $T_d(\mathcal{D})$ and $T_d^{-1}(\widehat{\mathcal{D}})$ (neighbourhoods of pieces of $W^u_{p_0}$ and $W^s_{p_1}$, respectively) implies a heteroclinic intersection between the stable manifold of p_1 and the unstable manifold of p_0. Unless it is necessary, we drop the dependence on the parameter m.

On the one hand, in Fontich et al. (2024, Lemma 5.4) it is proven that there is a connected piece of $W^u_{p_0}$ contained in $T_d(\mathcal{D})$ joining p_0 with some point in $\partial T_d(\mathcal{D}) \cap \{y = x\}$. On the other hand, the above lemmas show that there is a piece of $W^s_{p_1}$ contained in $T_d^{-1}(\widehat{\mathcal{D}})$ which joints $p_1 = (0, -1)$ with a point in $T_d^{-1}(\widehat{\mathcal{D}}) \cap \{y = -x\}$.

We claim that the line L given by $\{y = m^- x - 1\}$, tangent to the left boundary of $T_d^{-1}(\widehat{\mathcal{D}})$ at $(0, -1)$, intersects in two points the right boundary of $T_d(D)$ which is given by the curve $\gamma_m(t)$ in (2.5) with $m = m^\star = 7/2$. If we write $\gamma_{m^\star}(t) = (X(t), Y(t))$ we have

$$\begin{aligned} X(t) &= m^\star t - 1 - ((m^\star + 1)t - 1)^d, \\ Y(t) &= m^\star t - 1 - 2((m^\star + 1)t - 1)^d, \quad t \in \left[0, \frac{1}{m^\star + 1}\right]. \end{aligned}$$

See Fig. 4. To check the claim, recall that $\frac{-6}{\sqrt{13}+1} \le m^- < -1$. We consider the auxiliary function

$$\phi(t) = Y(t) - m^- X(t) + 1$$

$$= (2 - m^-)[(1 - (m^\star + 1)t)^d + m^\star t - 1] - m^\star t + 2, \qquad t \in \left[0, \frac{1}{m^\star + 1}\right],$$

which measures whether $\gamma_{m^\star}(t)$ is below, above or on the line L. We have

$$\phi(0) = 2 > 0, \qquad \phi(\frac{1}{m^\star + 1}) = \frac{m^- + m^\star}{m^\star + 1} >, \qquad \text{and}$$
$$\phi''(t) = (2 - m^-)d(d-1)(m^\star + 1)^2(1 - (m^\star + 1)t)^{d-2} > 0, \qquad t \in \left[0, \frac{1}{m^\star + 1}\right].$$

Accordingly, in order to see that ϕ has two zeros in its domain it is enough to show that there is a point t_1 in $(0, \frac{1}{1+m^\star})$ such that $\phi(t_1) < 0$. We take $t_1 = 1/8$ and, using that $m^- > -4/3$, we have

$$\phi(1/8) = (2 - m^-)\left(\left(\frac{7}{16}\right)^d - \frac{9}{16}\right) + \frac{25}{16} < \frac{10}{3}\left(\frac{7^3}{16^3} - \frac{9}{16}\right) + \frac{25}{16} < 0.$$

Therefore, $W^u_{p_0}$ has to cross L.

Next we claim that $T_d^{-1}(\widehat{\mathcal{D}}) \cap \{y = 0\}$ is a segment $[a_-, a_+] \times \{0\}$ with $a_+ < -3/5$. To see this claim we look for the intersection of the right and left boundaries of $T_d^{-1}(\widehat{D})$, given by $\Gamma_{\widehat{m}^\star}(t) = (\widehat{\alpha}_{\widehat{m}^\star}(t), \widehat{\beta}_{\widehat{m}^\star}(t))$ and $\Gamma_{m^-}(t) = (\widehat{\alpha}_{m^-}(t), \widehat{\beta}_{m^-}(t))$, respectively, with $\{y = 0\}$. We recall that $\widehat{m}^\star = -1 - 1/(d-1)$ and

$$\widehat{\alpha}_{\widehat{m}^\star}(t) = (\widehat{m}^\star - 2)t + 1 + ((1 - \widehat{m}^\star)t - 1)^{1/d},$$
$$\widehat{\beta}_{\widehat{m}^\star}(t) = (2 - \widehat{m}^\star)t - 1, \qquad t \in \left[0, \frac{1}{1 - \widehat{m}^\star}\right].$$

The value $t = t_2$ such that $\widehat{\beta}_{\widehat{m}^\star}(t) = 0$ is $t_2 = \frac{1}{2 - \widehat{m}^\star} \in \left[0, \frac{1}{1 - \widehat{m}^\star}\right]$, and

$$a_+ = \widehat{\alpha}_{\widehat{m}^\star}(t_2) = -\left(\frac{1}{2 - \widehat{m}^\star}\right)^{1/d} = -\left(\frac{1}{3 + 1/(d-1)}\right)^{1/d} \leq -\left(\frac{2}{7}\right)^{1/3} < -\frac{3}{5}.$$

In the same way, denoting t_3 the value such that $\widehat{\beta}_{m^-}(t_3) = 0$, we obtain $a_- = -(2 - \widehat{m}^\star)^{-1/d} < a_+$

Putting together the information of the two previous claims we get that when $y = 0$, $W^u_{p_0}$ is to the right of $W^s_{p_1}$ and that there exists some $y = y^0 < 0$ for which $\gamma_{m^\star}$ is to the left of L and therefore $W^u_{p_0}$ has to be at the left of $W^s_{p_1}$. This finish the proof of the proposition. See Fig. 4. □

Proof of Theorem A Since d is odd, the map T_d is symmetric with respect to $(x, y) \mapsto (-x, -y)$. Proposition 3.1 provides a (maybe non-transversal) heteroclinic point q in $T_d^{-1}(\widehat{\mathcal{D}}) \cap T_d(\mathcal{D})$. In any case at this point the manifolds cross each other. Therefore $\overline{q} = -q$ is also a heteroclinic point. By symmetry, at the point $\overline{q}$ the unstable manifold of p_1 intersects the stable manifold of p_0. We know that the unstable manifold is analytic. The stable manifold is analytic in a neighbourhood of q since the globalization of the local manifold has not meet $\{y = x\}$ yet. Since the manifolds do not coincide, they have a finite order contact.

Since we do not know if the intersection is transversal, we cannot apply the λ-Lemma of Palis in Palis (1969). However, we can apply the singular λ-Lemma in Rayskin (2003). In the two dimensional case, it asserts that the iteration of a disc in the unstable manifold accumulates in a C^1 manner to the unstable manifold of p_0, except for an arbitrarily small neighbourhood of p_0.

Now consider a piece of the connected component of the unstable manifold of p_0 in $T_d^{-1}(\widehat{\mathcal{D}}) \cap T(\mathcal{D})$ joining two points of the upper and lower boundaries of $T^{-1}(\widehat{\mathcal{D}})$, respectively. Then, by the singular λ-Lemma, the unstable manifold of p_1 will have discs arbitrary C^1-close to the unstable manifold of p_0 and therefore the discs will be in $T_d^{-1}(\widehat{\mathcal{D}}) \cap T_d(\mathcal{D})$.

Finally, using the same argument as in the end of the proof of the first part of Theorem A these discs should have an intersection with the stable manifold of p_1, thus providing the desired homoclinic point. □

4 Proof of Theorem B

In the previous section, we have proven the existence of homoclinic points associated with the stable and unstable of the cycle $\{p_0, p_1\}$. Using this fact, in this section we demonstrate that stable and the unstable manifolds of p_1 intersect in a transverse homoclinic point. Our approach is inspired in the work of Churchill and Rod (1980). However, there is an important difference. In Churchill and Rod (1980) the authors deal with analytic area preserving maps and can use tools as the Birkhoff normal form, while our map is not area preserving and it is not an analytic diffeomorphism. Our presentation uses the special structure of the map and the fact that we can linearize the map T_d^2 around p_1 which a C^∞ conjugation.

For the point $p_1 = (0, -1)$, we will denote by

$$W^s_{\mathrm{loc}} := W^s_{\mathrm{loc},p_1}, \qquad W^u_{\mathrm{loc}} := W^u_{\mathrm{loc},p_1}, \qquad W^s := W^s_{p_1} \qquad \text{and} \qquad W^u := W^u_{p_1}$$

the local stable, local unstable, global stable and global unstable manifolds associated with p_1 for the map T_d^2, respectively. The size of the local manifolds will be as small as we need.

We split the proof of Theorem B into several lemmas. Given $z \in \mathbb{R}^2$ we let $B_\varepsilon(z)$ be the open ball centred at z and radius $\varepsilon > 0$.

Lemma 4.1 *Let $\varepsilon > 0$ be small enough. Then, there exist two points q_s and q_u in $B_\varepsilon(p_1)$ such that*

$$q_s \in W^s_{\mathrm{loc}} \cap W^u \qquad \textit{and} \qquad q_u \in W^u_{\mathrm{loc}} \cap W^s.$$

Moreover, there exist analytic local parametrizations of W^s around q_u and of W^u around q_s given by $\{\phi^s(u) \mid u \in (-\delta, \delta)\}$ with $\phi^s(0) = q_u$ and $\{\phi^u(u) \mid u \in (-\delta, \delta)\}$ with $\phi^u(0) = q_s$ for some $\delta > 0$ small.

Since the manifolds do not coincide, the above intersections (at the points q_s and q_u) have finite order contact.

Fix $\varepsilon_1 > 0$ small enough such that $B_{\varepsilon_1}(q_s) \subset B_\varepsilon(p_1)$ and $B_{\varepsilon_1}(q_u) \subset B_\varepsilon(p_1)$. We denote by $\widehat{W}^u$ the piece of $W^u \subset B_{\varepsilon_1}(q_s)$ and by $\widetilde{W}^s$ the piece of $W^s \subset B_{\varepsilon_1}(q_u)$.

Proof Fix $\varepsilon > 0$ small enough and consider local manifolds W^s_{loc}, W^u_{loc} contained in $B_\varepsilon(p_1)$. Let $q \in W^s \cap W^u$ be the point determined by the topological transversal intersection of the stable and the unstable manifolds of p_1 for the map T_d^2 given by Theorem A. By iterating forward this point by T_d^2 and $(T_d^2)^{-1}$ we obtain the existence of q_s and q_u in $B_\varepsilon(p_1)$, respectively. Moreover, since W^s_{loc} and W^u are analytic we have that $W^s_{\text{loc}} \cap W^u$ intersect with finite order contact (otherwise they would coincide). Then, there exists ϕ^u as claimed. By construction, there exists $n_0 > 0$ such that

$$T_d^{-n_0}(q_s) = q_u.$$

According to the previous arguments if

$$T_d^{-j}(q_s) \cap \{y = x\} = \emptyset, \qquad j = 1, \ldots, n_0 - 1, \tag{4.1}$$

then $W^u_{\text{loc}} \cap W^s$ intersect at q_u, W^s is analytic in a neighbourhood of q_u and the intersection has a finite order contact and the lemma follows. Now, we consider the case that there exists a finite sequence of natural numbers $0 < j_1 < j_2 < \cdots < j_\ell < n_0$, $1 \le \ell < n_0$, such that

$$T_d^{-j_k}(q_s) \cap \{y = x\} =: \mathbf{r}_k \in \mathbb{R}^2, \qquad k = 1, \ldots, \ell. \tag{4.2}$$

Note that $\mathbf{r}_k = (r_k, r_k)$, $r_k \in \mathbb{R}$, and hence, $T_d^{-1}(r_k, r_k) = (-r_k, r_k)$. First, we deal with $\mathbf{r}_1$, the first time the globalization of W^s_{loc} meets $\{y = x\}$ so that W^s is analytic from p_1 to this point. Thus, near $\mathbf{r}_1$ the stable manifold W^s, is analytic and can be parametrized as

$$\phi(t) = \left(r_1 + t^{\alpha_1}(a_1 + f_1(t)), r_1 + t^{\beta_1}(b_1 + g_1(t))\right), \qquad |t| < \delta_1,$$

where $\alpha_1, \beta_1 \in \mathbb{N}$, $a_1, b_1 \in \mathbb{R}\backslash\{0\}$, $f_1(t), g_1(t)$ are analytic, satisfy $f_1(0) = 0$ and $g_1(0) = 0$ and $\delta_1 > 0$ is small enough. Since $W^s \not\subset \{y = x\}$ we have

$$t^{\alpha_1}(a_1 + f_1(t)) - t^{\beta_1}(b_1 + g_1(t)) \not\equiv 0.$$

Using the expression of T_d^{-1} (see Eq. 2.1) we have

$$T_d^{-1}(\phi(t)) = \begin{pmatrix} -r_1 - 2t^{\alpha_1}(a_1 + f_1(t)) + t^{\beta_1}(b_1 + g_1(t)) + t^{\gamma_1/d}(1 + O(t))^{1/d} \\ r_1 + 2t^{\alpha_1}(a_1 + f_1(t)) - t^{\beta_1}(b_1 + g_1(t)) \end{pmatrix},$$

where $\gamma_1 \geq \min\{\alpha_1, \beta_1\}$. Since d is odd we can reparametrize the curve $\phi(t)$ using the new parameter $u = t^{1/d}$ to obtain $\widehat{\phi}(u) = \phi(u^d)$ analytic and

$$T_d^{-1}\left(\widehat{\phi}(u)\right) = T_d^{-1}\left(\phi(u^d)\right) = \begin{pmatrix} -r_1 + O(u^{\widehat{\alpha}_1}) \\ r_1 + O(u^{\widehat{\beta}_1}) \end{pmatrix},$$

with $\widehat{\alpha}_1, \widehat{\beta}_1 \in \mathbb{N}$.

We conclude thus that W^s admits an analytic parametrization in a sufficiently small neighbourhood of $T_d^{-1}(r_1, r_1) = (-r_1, r_1)$. Repeating the same procedure a finite number of times it is clear that W^s intersects W^u_{loc} with finite order contact at the point q_u. □

The translation

$$\mathcal{T} : (\widehat{x}, \widehat{y}) \mapsto (x, y) = (\widehat{x}, \widehat{y} - 1)$$

moves p_1 to the origin. For simplicity, we write the new coordinates again as (x, y). Observe that (in the new coordinates) $T_d^2(0, 0) = (0, 0)$ and that

$$DT_d^2(0, 0) = \begin{pmatrix} 3d^2 - 2d & 3d^2 - 4d + 1 \\ 6d^2 - 2d & 6d^2 - 6d + 1 \end{pmatrix}. \tag{4.3}$$

The eigenvalues and eigenvectors are given in (2.3) and (2.4), respectively.

We will denote

$$\lambda := \lambda_d^+ > 1, \qquad \mu := \lambda_d^- < 1, \qquad m_\lambda := m_d^+ \qquad \text{and} \qquad m_\mu := m_d^- \tag{4.4}$$

(we drop the dependence on d unless it is strictly necessary). We recall from Sect. 2 that

$$\lambda > 57, \quad 1/9 < \mu < 0.1556, \quad 2 < m_\lambda < 2.3028 \quad \text{and} \quad -1.3027 < m_\mu < -1. \tag{4.5}$$

We parametrize the local stable and unstable manifolds associated with the origin by the x-variable so that the expressions can be written as $y = \Psi^s(x)$ and $y = \Psi^u(x)$, respectively. We obviously have

$$\frac{\mathrm{d}\Psi^s}{\mathrm{d}x}(x)|_{x=0} = m_\mu \qquad \text{and} \qquad \frac{\mathrm{d}\Psi^u}{\mathrm{d}x}(x)|_{x=0} = m_\lambda. \tag{4.6}$$

Next step is to introduce local analytic coordinates $(\widehat{\xi}, \widehat{\eta})$ around $(0, 0)$ so that the expression of the local stable and unstable manifolds would be $\widehat{\eta} = 0$ and $\widehat{\xi} = 0$, respectively.

Lemma 4.2 *We consider the local change of variables*

$$(x, y) \mapsto (\widehat{\xi}, \widehat{\eta}) = \Theta(x, y) := (y - \Psi^u(x), y - \Psi^s(x)).$$

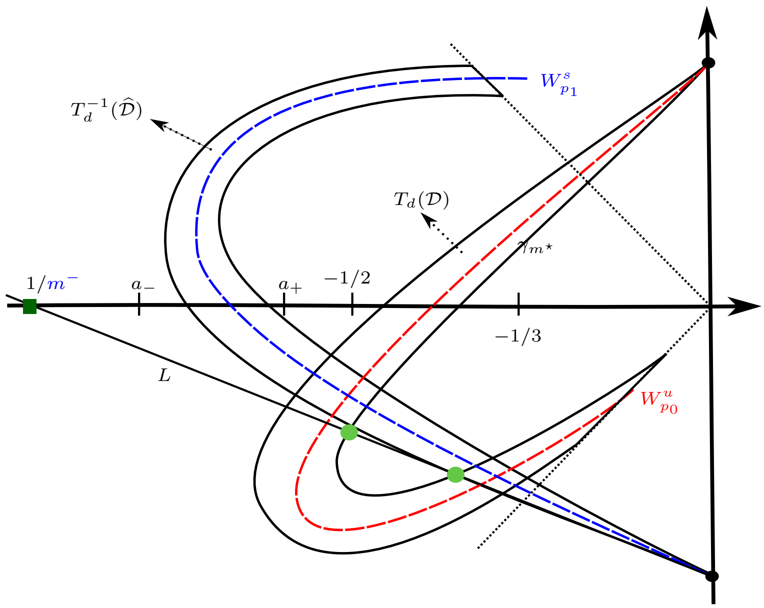

Fig. 4 Sketch of the arguments providing the (topological, not necessarily transversal as it is shown in the picture) intersection of the stable and unstable manifold of the hyperbolic two-cycle $\{p_0, p_1\}$ (Theorem A). The green dots indicates the two intersections between $\gamma_{m^\star}(t)$ and the line $L := \{y = m^- x - 1\}$ (Color figure online)

Then, for $\varepsilon > 0$ *small enough the local expression of* T_d^2 *in* $B_\varepsilon(0,0)$ *is given by*

$$\mathcal{F}(\widehat{\xi}, \widehat{\eta}) = \mathcal{L}(\widehat{\xi}, \widehat{\eta}) + \mathcal{N}(\widehat{\xi}, \widehat{\eta}),$$

where

$$\mathcal{L}(\widehat{\xi}, \widehat{\eta}) = (\lambda\widehat{\xi}, \mu\widehat{\eta}), \quad \mathcal{N}(0,0) = (0,0) \quad \textit{and} \quad D\mathcal{N}(0,0) = 0.$$

Moreover, the local change of coordinates, $(\widehat{\xi}, \widehat{\eta}) = \Theta(x, y)$*, is analytic.*

Proof We claim that the new variables define a local change of coordinates around the origin. Indeed, since $\Psi^s(x)$ and $\Psi^u(x)$ are analytic, by the inverse function theorem we only need to check that

$$D\Theta(0,0) = \begin{pmatrix} -\frac{\partial \Psi^u}{\partial x}(x)|_{x=0} & 1 \\ -\frac{\partial \Psi^s}{\partial x}(x)|_{x=0} & 1 \end{pmatrix} = \begin{pmatrix} -m_\mu & 1 \\ -m_\lambda & 1 \end{pmatrix}$$

is non-singular and this is a direct consequence of (4.5). Clearly Θ is a local analytic diffeomorphism. See Fig. 5. □

We will see next that $\mathcal{F}$ is $\mathcal{C}^\infty$-conjugate to its linear part $\mathcal{L}$. For this, we will apply Sternberg's Theorem Sternberg (1958, Theorem 1). The following lemma checks a key hypothesis of that theorem.

Lemma 4.3 *The eigenvalues $\lambda = \lambda_d^+ > 1$ and $\mu = \lambda_d^- < 1$ of the linear part $\mathcal{L}$ of $\mathcal{F}$ at $(0,0)$ given in (4.4) and (2.3) are non-resonant.*

Proof We first note that $\lambda\mu = d^2$ (the determinant of $DT_d^2(0,0)$). It is easy to check by induction that

$$\lambda^k = a_k + b_k\sqrt{\Delta} \qquad \text{and} \qquad \mu^k = a_k - b_k\sqrt{\Delta}, \qquad k \geq 1,$$

and

$$\lambda^{-k} = \frac{1}{g^k}\left(a_k' - b_k'\sqrt{\Delta}\right) \qquad \text{and} \qquad \mu^{-k} = \frac{1}{g^k}\left(a_k' + b_k'\sqrt{\Delta}\right), \qquad k \geq 1,$$

where $a_k, b_k, a_k', b_k' \in \mathbb{N}$, $g = 12d^2(6d^2 - 4d + 1) \in \mathbb{N}$ and $\Delta = 9d^2 - 10d + 1$ is not a perfect square. This means that $\lambda^k, \mu^k \in \mathbb{R}\backslash\mathbb{Q}$ for all $k \neq 0$.

There are two possible types of resonances:

$$\lambda = \lambda^n\mu^m \qquad \text{with} \qquad n, m \geq 0 \qquad \text{and} \quad n + m \geq 2, \tag{4.7}$$

and

$$\mu = \lambda^n\mu^m \qquad \text{with} \qquad n, m \geq 0 \qquad \text{and} \quad n + m \geq 2. \tag{4.8}$$

We deal with (4.7). We rewrite it as

$$1 = \lambda^{n-m-1}(\lambda\mu)^m = \lambda^{n-m-1}d^{2m}. \tag{4.9}$$

We distinguish two cases: (a) $n \neq m + 1$ and (b) $n = m + 1$.

In case (a), since $\lambda^{n-m-1} \in \mathbb{R}\backslash\mathbb{Q}$ and $d \in \mathbb{N}$ the previous equality is impossible. In case (b), m cannot be 0. Then, $(\lambda\mu)^m = d^{2m} \geq 9$ so that (4.9) is also impossible.

Concerning resonances of the form (4.8) the argument is completely analogous. □

Theorem 1 in Sternberg (1958) provides a $\mathcal{C}^\infty$ local change of coordinates conjugating $\mathcal{F}$ to its linear part $\mathcal{L}$. From it we will obtain a near the identity conjugation.

Lemma 4.4 *There is a conjugacy from $\mathcal{F}$ to its linear part $\mathcal{L}$ at the origin of the form*

$$(\xi, \eta) := \Phi(\widehat{\xi}, \widehat{\eta}) = \begin{pmatrix} \widehat{\xi}(1 + \phi_1(\widehat{\xi}, \widehat{\eta})) \\ \widehat{\eta}(1 + \phi_2(\widehat{\xi}, \widehat{\eta})) \end{pmatrix}, \tag{4.10}$$

where $\phi_j(\widehat{\xi}, \widehat{\eta})$, $j = 1, 2$, are $\mathcal{C}^\infty$ functions defined in a sufficiently small neighbourhood of the origin with $\phi_j(0,0) = (0,0)$, $j = 1, 2$.

Proof Let $\widehat{\Phi}(\widehat{\xi}, \widehat{\eta})$ be the $\mathcal{C}^\infty$ local conjugacy given by Sternberg's Theorem. Consequently, $\widehat{\Phi}$ should send the stable and unstable manifolds of $\mathcal{F}$ to the corresponding ones of $\mathcal{L}$, which in this case means that it preserves the axes. Writing $\widehat{\Phi} = (\widehat{\Phi}_1, \widehat{\Phi}_2)$, this is translated into the conditions $\widehat{\Phi}_1(0, \widehat{\eta}) = 0$ and $\widehat{\Phi}_2(\widehat{\xi}, 0) = 0$. Then,

$$\widehat{\Phi}_1(\widehat{\xi}, \widehat{\eta}) = \widehat{\Phi}_1(0, \widehat{\eta}) + \int_0^1 \partial_\xi \widehat{\Phi}_1(t\widehat{\xi}, \widehat{\eta})\, \xi\, dt = \widehat{\xi}(\alpha + \widehat{\phi}_1(\widehat{\xi}, \widehat{\eta}))$$

with $\widehat{\phi}_1(0, 0) = 0$, and analogously $\widehat{\Phi}_2(\widehat{\xi}, \widehat{\eta}) = \widehat{\eta}(\beta + \widehat{\phi}_2(\widehat{\xi}, \widehat{\eta}))$ with $\phi_2(0, 0) = 0$. Since $\widehat{\Phi}$ is a diffeomorphism, $\alpha\beta \neq 0$. We write $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. We claim that $\Phi := A^{-1}\widehat{\Phi}$ is also a conjugation from $\mathcal{F}$ to $\mathcal{L}$. Indeed, since A commutes with $\mathcal{L}$,

$$\Phi\mathcal{F} = A^{-1}\widehat{\Phi}\mathcal{F} = A^{-1}\mathcal{L}\widehat{\Phi} = \mathcal{L}A^{-1}\widehat{\Phi} = \mathcal{L}\Phi.$$

Moreover, Φ is of the form given in (4.10). See Fig. 5. □

In Lemma 4.1, we have proven the existence of the points

$$q_s \in W^s_{\text{loc}} \cap \widehat{W}^u \qquad \text{and} \qquad q_u \in W^u_{\text{loc}} \cap \widetilde{W}^s.$$

Then, we can use the changes of coordinates introduced in the previous lemmas to transport those curves to a neighbourhood of the origin. Denote by $\gamma_1(t)$ and $\gamma_2(t)$ the parametrizations of $(\Phi \circ \Theta \circ \mathcal{T})(\widehat{W}^u)$ and $(\Phi \circ \Theta \circ \mathcal{T})(\widetilde{W}^s)$, respectively. We focus on the pieces of $\gamma_1(t)$ and $\gamma_2(t)$ in the first quadrant. Without loss of generality we can assume that these pieces are parametrized by $t \geq 0$.

Lemma 4.5 *The curves $\gamma_1(t)$ and $\gamma_2(t)$ intersect the coordinate axes $\{\xi = 0\}$ and $\{\eta = 0\}$ at points $\widehat{q_s} = (0, \eta_0)$ and $\widehat{q_u} = (\xi_0, 0)$ and have a finite order contact there, respectively. Moreover, for t small enough we have that $\gamma_j(t)$, $j = 1, 2$, admit the following parametrization*

$$\begin{aligned} \gamma_1(t) &= (t^{\ell_1}(a_1 + g_1(t)), \eta_0 + t), \\ \gamma_2(t) &= (\xi_0 + t^{\ell_2}(a_2 + g_2(t)), t^{\ell_3}(a_3 + g_3(t))), \end{aligned} \tag{4.11}$$

where $\ell_j \geq 1$ for $j = 1, 2, 3$, $a_1a_2a_3 \neq 0$, $\xi_0, \eta_0 > 0$, and $g_j(t)$ are C^∞ functions with $g_j(0) = 0$, for $j = 1, 2, 3$. See Fig. 6.

Proof The lemma follows from the fact that q_s and q_u are points of finite order contact between the stable and the unstable manifolds of p_1 and the change of coordinates we have used is $\mathcal{C}^\infty$. □

We want to show that, for k large enough, $\gamma_2(t)$ and $\mathcal{L}^k(\gamma_1(t))$ intersect transversally. This framework is quite close to the one in Churchill and Rod (1980, Theorem 1.1) but in their case the linear map admits the function $H(x, y) = xy$ as a first integral, which is not our case. Then, we provide a proof in our case to get the same conclusion.

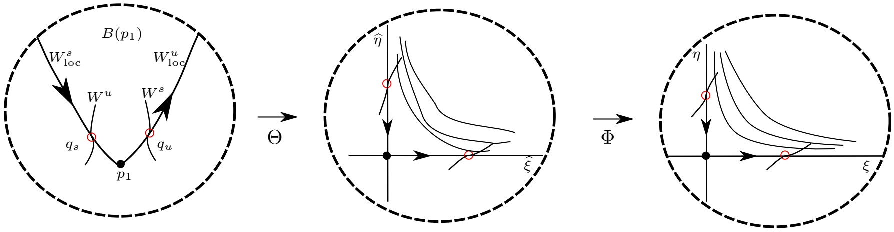

Fig. 5 The changes of coordinates corresponding to Lemma 4.2 and Lemma 4.4. In fact Θ in this figure includes a primer change of coordinates to move p_1 to the origin (Color figure online)

Given $\lambda > 1$ and $0 < \mu < 1$ introduced in (4.4) and (2.3) and we consider the auxiliary interpolation map

$$\mathcal{L}^\tau(\xi,\eta) = \begin{pmatrix} \lambda^\tau & 0 \\ 0 & \mu^\tau \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad \tau > 0.$$

We also consider the first quadrant $Q = \{(\xi,\eta) \mid \ \xi \geq 0, \eta \geq 0\}$ and $H : Q \to Q$ defined by

$$H(\xi,\eta) = \xi^{\log\mu^{-1}}\eta^{\log\lambda}. \tag{4.12}$$

It is continuous and real analytic in the interior of Q.

Lemma 4.6 *The function H is a first integral of $\mathcal{L}^\tau$, $\tau > 0$, in Q.*

Proof To prove the lemma, we compute

$$H\left(\mathcal{L}^\tau(\xi,\eta)\right) = H\left(\lambda^\tau\xi,\mu^\tau\eta\right) = \lambda^{\tau\log\mu^{-1}}\mu^{\tau\log\lambda}\xi^{\log\mu^{-1}}\eta^{\log\lambda} = H(\xi,\eta).$$

□

Next step is to show that there exist reparametrizations $t = \sigma_j(s)$, $j = 1, 2$, of the curves $\gamma_j(t)$, $j = 1, 2$, which have a useful property.

Lemma 4.7 *There exist continuous reparametrizations $\widetilde{\gamma}_j(s) = \gamma_j(\sigma_j(s))$, $s \in [0, s_0)$, of the curves $\gamma_j(t)$ given by $t = \sigma_j(s)$, $j = 1, 2$, that are C^∞ in $(0, s_0)$ and they satisfy*

$$H\left(\widetilde{\gamma}_j(s)\right) = s, \qquad s \in (0, s_0)$$

for some $s_0 > 0$ small enough.

Proof For $j = 1$ we impose the condition $H(\gamma_1(t)) = s$ to obtain $t := \sigma_1(s)$. Using (4.11) and (4.12) we have

$$\begin{aligned} H(\gamma_1(t)) = s \iff & t^{\ell_1\log\mu^{-1}}(a_1 + g_1(t))^{\log\mu^{-1}}(\eta_0 + t)^{\log\lambda} = s \\ \iff & G_1(t) := t(a_1 + g_1(t))^{1/\ell_1}(\eta_0 + t)^{\log\lambda/(\ell_1\log\mu^{-1})} \\ & = s^{1/\left(\ell_1\log\mu^{-1}\right)}. \end{aligned} \tag{4.13}$$

We have $G_1(0) = 0$, $c_1 := G_1'(0) = a_1^{1/\ell_1} \eta_0^{\log\lambda/(\ell_1 \log\mu^{-1})} \neq 0$.

Consequently, by the inverse function theorem, G_1 is locally invertible and we can write

$$t = \sigma_1(s) := G_1^{-1}(s^{1/(\ell_1 \log\mu^{-1})}) \tag{4.14}$$

for $|s| < s_0$ for some $s_0 > 0$ small.

For $j = 2$, arguing as above, we have

$$H(\gamma_2(t)) = s \iff G_2(t) := t(a_3 + g_3(t))^{1/\ell_3}$$
$$\left(\xi_0 + t^{\ell_2}(a_2 + g_2(t))\right)^{\log\mu^{-1}/(\ell_3 \log\lambda)} = s^{1/(\ell_3 \log\lambda)}.$$

Analogous computations imply that $c_2 := G_2'(0) = a_3^{1/\ell_3} \xi_0^{\log\mu^{-1}/(\ell_3 \log\lambda)} \neq 0$ and

$$t = \sigma_2(s) := G_2^{-1}(s^{1/(\ell_3 \log\lambda)}). \tag{4.15}$$

□

The following lemma establishes a relation between $\mathcal{L}^\tau(\widetilde{\gamma}_1(s))$ and $\widetilde{\gamma}_2(s)$.

Lemma 4.8 *Let $s_0 > 0$ be small enough. Then, there exists a C^∞ function $\tau(s)$, $s \in (0, s_0)$, such that*

$$\mathcal{L}^{\tau(s)}(\widetilde{\gamma}_1(s)) = \widetilde{\gamma}_2(s), \qquad s \in (0, s_0).$$

Moreover

$$\lim_{s\to 0^+} \tau(s) = \infty. \tag{4.16}$$

In particular, there exist k_0 in $\mathbb{N}$ and a sequence of positive values $\{s_k\}_{k\geq k_0}$, such that $s_k \to 0$ and $\tau(s_k) = k$ for every $k \geq k_0$.

Proof Let $s_0 > 0$ be as in Lemma 4.7. We use the following notation

$$\widetilde{\gamma}_1(s) = (\widetilde{\xi}_1(s), \widetilde{\eta}_1(s)) \qquad \text{and} \qquad \widetilde{\gamma}_2(s) = (\widetilde{\xi}_2(s), \widetilde{\eta}_2(s)).$$

We define

$$\tau(s) := \frac{1}{\log\lambda} \log\frac{\widetilde{\xi}_2(s)}{\widetilde{\xi}_1(s)}, \qquad s \in (0, s_0)$$

and therefore

$$\frac{\lambda^{\tau(s)}\widetilde{\xi}_1(s)}{\widetilde{\xi}_2(s)} = 1. \tag{4.17}$$

Clearly, $\tau(s)$ is C^∞ in $(0, s_0)$. Next, we will check that $\lambda^{\tau(s)}\widetilde{\eta}_1(s) = \widetilde{\eta}_2(s)$. Using that H is a first integral and Lemma 4.7 we have that

$$H(\lambda^\tau \widetilde{\xi}_1(s), \mu^\tau \widetilde{\eta}_1(s)) = H(\widetilde{\xi}_1(s), \widetilde{\eta}_1(s)) = s = H(\widetilde{\xi}_2(s), \widetilde{\eta}_2(s))$$

from which we deduce that

$$\left(\frac{\lambda^\tau \widetilde{\xi}_1(s)}{\widetilde{\xi}_2(s)}\right)^{\log \mu^{-1}} = \left(\frac{\widetilde{\eta}_2(s)}{\mu^\tau \widetilde{\eta}_1(s)}\right)^{\log \lambda}.$$

Taking $\tau = \tau(s)$, from (4.17)) we conclude

$$\mu^{\tau(s)}\widetilde{\eta}_1(s) = \widetilde{\eta}_2(s), \tag{4.18}$$

i.e. $\mathcal{L}^\tau(\widetilde{\gamma}_1(s)) = \widetilde{\gamma}_2(s)$ as desired. Of course,

$$\lim_{s\to 0^+} \tau(s) = \frac{1}{\log \lambda} \log \lim_{s\to 0^+} \frac{\widetilde{\xi}_2(s)}{\widetilde{\xi}_1(s)} = \infty, \tag{4.19}$$

since $\lim_{s\to 0^+} \widetilde{\xi}_2(s) = \xi_0 >$ and $\lim_{s\to 0^+} \widetilde{\xi}_1(s) = 0$. Using and the fact that $\tau(s)$ is a C^∞ function in its domain we get from Bolzano's theorem that there exist k_0 such that for every $k \geq k_0$ there exists s_k such that $t(s_k) = k$, and the lemma follows. □

The lemma above shows that the curves $\mathcal{L}^k(\widetilde{\gamma}_1(s))$ and $\widetilde{\gamma}_2(s)$ intersect at the values $s = s_k$. Next lemma shows that these intersections, for k large enough, are transversal.

Lemma 4.9 *Let k be large enough. The following limits hold.*

$$\lim_{s\to 0^+} \frac{\lambda^k \frac{d\widetilde{\xi}_1}{ds}(s)}{\frac{d\widetilde{\xi}_2}{ds}(s)} = \infty \quad \text{and} \quad \lim_{s\to 0^+} \frac{\mu^k \frac{d\widetilde{\eta}_1}{ds}(s)}{\frac{d\widetilde{\eta}_2}{ds}(s)} = 0. \tag{4.20}$$

In particular, for k large enough, the curves $\mathcal{L}^k(\widetilde{\gamma}_1(s))$ and $\widetilde{\gamma}_2(s)$ intersect transversally at the values $s = s_k$.

Proof To prove the lemma we first make some computations. From the proof of Lemma 4.7, and equations (4.14) and (4.15), we deduce the following asymptotic behaviours (as $s \to 0$)

$$\begin{aligned}
\sigma_1(s) &= c_1^{-1} s^{1/(\ell_1 \log \mu^{-1})} + \ldots, & \sigma_1'(s) &= c_1^{-1} \frac{1}{\ell_1 \log \mu^{-1}} s^{1/(\ell_1 \log \mu^{-1})} s^{-1} + \ldots, \\
\sigma_2(s) &= c_2^{-1} s^{1/(\ell_3 \log \lambda)} + \ldots, & \sigma_2'(s) &= c_2^{-1} \frac{1}{\ell_3 \log \lambda} s^{1/(\ell_3 \log \lambda)} s^{-1} + \ldots,
\end{aligned} \tag{4.21}$$

where

$$c_1 = a_1^{1/\ell_1} \eta_0^{\log \lambda/(\ell_1 \log \mu^{-1})} \quad \text{and} \quad c_2 = a_3^{1/\ell_3} \xi_0^{\log \mu^{-1}/(\ell_3 \log \lambda)}$$

are nonzero. Hence, using (4.11), we can compute the following expressions for some of the terms of the numerator and denominator in (4.20). On the one hand, we have

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\xi}_1}{\mathrm{d}s}(s) &= \frac{\mathrm{d}(\xi_1 \circ \sigma_1)}{\mathrm{d}s}(s) = \sigma_1^{\ell_1-1}(s)\left[\ell_1(a_1 + g_1(\sigma_1(s))) + \sigma_1(s)g_1'(\sigma_1(s))\right]\sigma_1'(s)\\ &= \ell_1 a_1 \sigma_1^{\ell_1-1}(s)\sigma_1'(s) + \cdots = \frac{a_1}{c_1^{\ell_1}\log\mu^{-1}} s^{1/\log\mu^{-1}} s^{-1} + \ldots,\\ \frac{\mathrm{d}\widetilde{\xi}_2}{\mathrm{d}s}(s) &= \frac{\mathrm{d}(\xi_2 \circ \sigma_2)}{\mathrm{d}s}(s) = \sigma_2^{\ell_2-1}(s)\left[\ell_2(a_2 + g_2(\sigma_2(s))) + \sigma_2(s)g_2'(\sigma_2(s))\right]\sigma_2'(s)\\ &= \ell_2 a_2 \sigma_2^{\ell_2-1}(s)\sigma_2'(s) + \cdots = \frac{\ell_2 a_2}{\ell_3 c_2^{\ell_2}\log\lambda} s^{\ell_2/(\ell_3\log\lambda)} s^{-1} + \ldots.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\eta}_1}{\mathrm{d}s}(s) &= \frac{\mathrm{d}(\eta_1 \circ \sigma_1)}{\mathrm{d}s}(s) = \sigma_1'(s) = c_1^{-1}\frac{1}{\ell_1\log\mu^{-1}} s^{1/(\ell_1\log\mu^{-1})} s^{-1} + \ldots,\\ \frac{\mathrm{d}\widetilde{\eta}_2}{\mathrm{d}s}(s) &= \frac{\mathrm{d}(\eta_2 \circ \sigma_2)}{\mathrm{d}s}(s) = \sigma_2^{\ell_3-1}(s)\left[\ell_3(a_3 + g_3(\sigma_2(s))) + \sigma_2(s)g_3'(\sigma_2(s))\right]\sigma_2'(s)\\ &= \ell_3 a_3 \sigma_2^{\ell_3-1}(s)\sigma_2'(s) + \cdots = \frac{a_3}{c_2^{\ell_3}\log\lambda} s^{1/\log\lambda} s^{-1} + \ldots.\end{aligned}$$

From (4.17) to (4.18) evaluated at the values of $s = s_k$ corresponding to $\tau(s_k) = k$, we can also conclude that

$$\lambda^k = \frac{\xi_2(\sigma_2(s_k))}{\xi_1(\sigma_1(s_k))} = \frac{\xi_0}{a_1}\sigma_1^{-\ell_1}(s_k) + \cdots = \frac{\xi_0 c_1^{\ell_1}}{a_1} s^{-1/\log\mu^{-1}} + \ldots$$

and

$$\mu^k = \frac{\eta_2(\sigma_2(s_k))}{\eta_1(\sigma_1(s_k))} = \frac{a_3}{\eta_0}\sigma_2^{\ell_3}(s_k) + \cdots = \frac{a_3 c_2^{-\ell_3}}{\eta_0} s_k^{1/\log\lambda} + \ldots.$$

All together allows us to compute the limits of the statement

$$\begin{aligned}\lim_{s\to 0^+} \lambda^k \frac{\frac{\mathrm{d}\widetilde{\xi}_1}{\mathrm{d}s}(s)}{\frac{\mathrm{d}\widetilde{\xi}_2}{\mathrm{d}s}(s)} &= \lim_{s\to 0^+} \frac{\xi_0 \ell_3 \log\lambda}{\ell_2 a_2 c_2^{-1} r_2^{\ell_2-1}\log\mu^{-1}} s^{-\ell_2/(\ell_3\log\lambda)} = \infty\\ \lim_{s\to 0^+} \mu^k \frac{\frac{\mathrm{d}\widetilde{\eta}_1}{\mathrm{d}s}(s)}{\frac{\mathrm{d}\widetilde{\eta}_2}{\mathrm{d}s}(s)} &= \lim_{s\to 0^+} \frac{c_1^{-1}\log\lambda}{\eta_0 \ell_1 \log\mu^{-1}} s^{1/(\ell_1\log\mu^{-1})} = 0.\end{aligned}$$

□

Proof (End of the proof of Theorem B) Let $k \geq k_0$ satisfy the conditions of the previous lemmas. We consider the sequence of points

$$\left\{\widetilde{\gamma}_j(s_k) = \left(\widetilde{\xi}_j(s_k), \widetilde{\eta}_j(s_k)\right)\right\}_{k\geq k_0}, \qquad j = 1, 2. \tag{4.22}$$

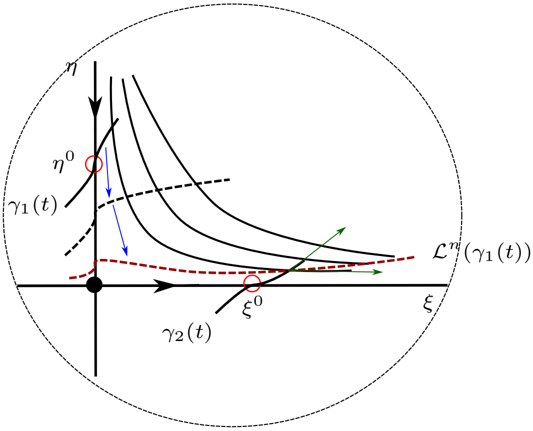

Fig. 6 Sketch of the situation described in Lemma 4.5 (Color figure online)

From Lemma 4.9, $\mathcal{L}^k\,(\widetilde{\gamma}_1(s_k))$ and $\widetilde{\gamma}_2(s_k)$ intersect transversally. It follows from (4.11) and the lemmas above than

$$\widetilde{\gamma}_1(s_k) \to (0, \eta_0) \qquad \text{and} \qquad \widetilde{\gamma}_2(s_k) \to (\xi_0, 0). \tag{4.23}$$

We introduce the following notation

$$\begin{aligned} \widehat{\mathbf{z}}_k &:= \left(\mathcal{T}^{-1} \circ \Theta^{-1} \circ \Psi^{-1}\right)(\widetilde{\gamma}_1(s_k)) \in \widehat{W}^u \qquad \text{and} \\ \widetilde{\mathbf{z}}_k &:= \left(\mathcal{T}^{-1} \circ \Theta^{-1} \circ \Psi^{-1}\right)(\widetilde{\gamma}_2(s_k)) \in \widetilde{W}^s, \end{aligned} \tag{4.24}$$

and note that Lemma 4.8 implies that

$$\widetilde{\mathbf{z}}_k = T_d^{2k}\,(\widehat{\mathbf{z}}_k)\,, \qquad k \geq k_0. \tag{4.25}$$

To conclude the proof of Theorem B, we argue as follows. We know that $\{\widehat{\mathbf{z}}_k\}_{k\geq k_0} \in W^u$ and $\{\widetilde{\mathbf{z}}_k\}_{k\geq k_0} \in W^s$. Also the stable and unstable manifolds are invariant sets for the map T_d^2. Finally, Lemma 4.9, definition (4.24) and equation (4.25) imply that $\{\widehat{\mathbf{z}}_k\}_{k\geq k_0}$ and $\{\widetilde{\mathbf{z}}_k\}_{k\geq k_0}$ correspond to transversal intersections of the stable and unstable manifolds of p_1 accumulating to the points q_s and q_u, respectively. See Fig. 7. □

5 Proof of Theorem C

The proof of Theorem C is based on Moser's version of Birkhoff–Smale theorem; concretely we will apply Theorem 3.7 in Moser (2001) in our setup. The key difficulty comes from the fact that, in our case, T_d^{-1} is not differentiable on the line $\{y = x\}$. Therefore, we need to make sure that the construction in Moser (2001) can be made so that we only have to deal with our map and its inverse in a domain that does not meet the line $\{y = x\}$. However, notice that both the stable and the unstable manifolds of the two-cycle $\{p_0, p_1\}$ cross the line $y = x$. See Fig. 2.

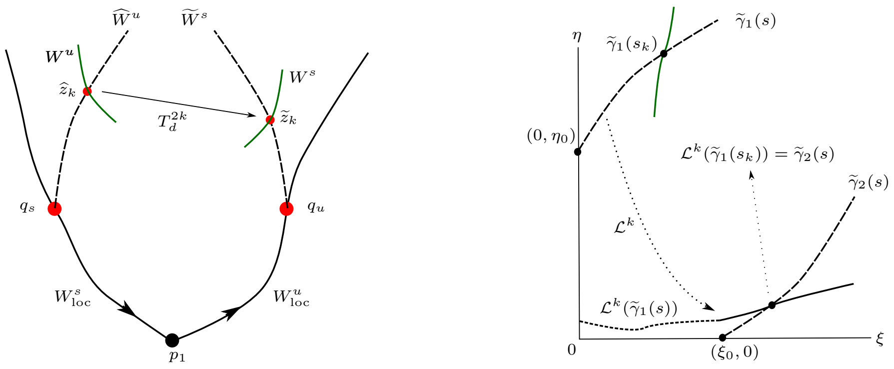

Fig. 7 Sketch of the proof of Theorem B with the points $\widehat{\mathbf{z}}_k$ and $\widetilde{\mathbf{z}}_k$ being transversal intersections of the stable and unstable manifolds of p_1 (for T_d^2) (Color figure online)

Remark 5.1 It follows from Lemma 4.1 that W^s as well as W^u intersect the line $\{y = x\}$ at isolated points. In other words, finite length pieces of W^s and W^u only contain finitely many intersections with $\{y = x\}$.

Let U be a neighbourhood of p_1 as in Lemma 4.4 where we can take local coordinates for which p_1 is located at $(0, 0)$ and the stable and the unstable manifolds of $(0, 0)$ are the vertical and horizontal axes, respectively. Assume also that $U \cap \{y = x\} = \emptyset$.

In the following items, we summarize notation and facts of the constructions we have made in the previous section that will be important in the proof of Theorem C. See Fig. 8.

(a) Let q be the homoclinic point given in Theorem A and $q_s = T_d^{2\alpha}(q) \in W^s_{\rm loc} \cap W^u$ and $q_u = T_d^{-2\beta}(q) \in W^u_{\rm loc} \cap W^s$, $\alpha, \beta \in \mathbb{N}$, be the points given in Lemma 4.1. Let $\widehat{W}^u \subset W^u$ and $\widetilde{W}^s \subset W^s$ introduced after the statement of Lemma 4.1. Then, $q_s \in W^s_{\rm loc} \cap \widehat{W}^u$ and $q_u \in W^u_{\rm loc} \cap \widetilde{W}^s$. Moreover, taking $n_0 = \alpha + \beta$ we have that

$$T_d^{2n_0}(q_u) = q_s.$$

(b) Let $\widehat{\mathbf{z}}_k \in \widehat{W}^u$ and $\widetilde{\mathbf{z}}_k \in \widetilde{W}^s$ be the points introduced in (4.24). We have that $\widetilde{\mathbf{z}}_k$ and $\widehat{\mathbf{z}}_k$ are transversal homoclinic points,

$$\lim_{k\to\infty} \hat{\mathbf{z}}_k = q_s \qquad \text{and} \qquad \lim_{k\to\infty} \tilde{\mathbf{z}}_k = q_u.$$

and

$$T_d^{2k}(\widehat{\mathbf{z}}_k) = \widetilde{\mathbf{z}}_k, \qquad T_d^{2j}(\widehat{\mathbf{z}}_k) \in U \quad \text{for all} \quad j = 1, \ldots, k.$$

(c) Let $\widehat{W}^s = T_d^{2n_0}(\widetilde{W}^s)$ and $\widetilde{W}^u = T_d^{-2n_0}(\widehat{W}^u)$.

(d) We consider the points

$$\widehat{\mathbf{w}}_k := T_d^{2n_0}(\widetilde{\mathbf{z}}_k) \in \widehat{W}^s \cap W^s_{\rm loc} \cap W^u \quad \text{and} \quad \widetilde{\mathbf{w}}_k := T_d^{-2n_0}(\widehat{\mathbf{z}}_k) \in \widetilde{W}^u \cap W^u_{\rm loc} \cap W^s.$$

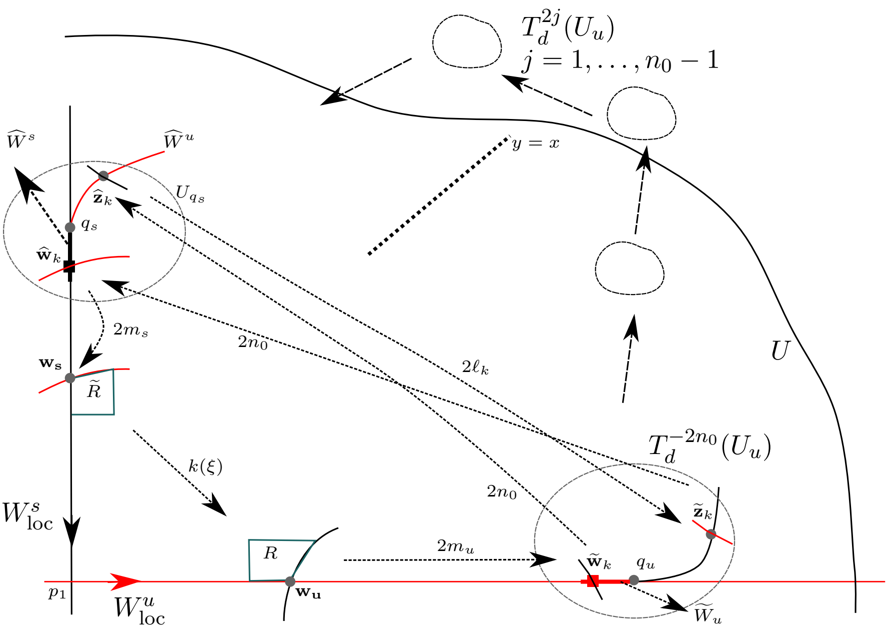

Fig. 8 The illustration of all items a–f (Color figure online)

As a consequence of the previous items, we have

$$T_d^{2n_0+2k+2n_0}(\widetilde{\mathbf{w}}_k) = \widehat{\mathbf{w}}_k.$$

(e) For any $m_u \geq 1$ and $m_s \geq 1$, if we write,

$$\mathbf{w}_u := T_d^{-2m_u}(\widetilde{\mathbf{w}}_k) \in W^u_{\text{loc}} \quad \text{and} \quad \mathbf{w}_s := T_d^{2m_s}(\widehat{\mathbf{w}}_k) \in W^s_{\text{loc}},$$

we have that

$$T_d^{2m_s+2n_0+2k+2n_0+2m_u}(\mathbf{w}_u){=}\mathbf{w}_s \quad \text{or} \quad T_d^{-2m_s-2n_0-2k-2n_0-2m_u}(\mathbf{w}_s){=}\mathbf{w}_u. \quad (5.1)$$

In particular, considering m_u and m_s as large as necessary we know that the corresponding points $\mathbf{w}_u$ and $\mathbf{w}_s$ are as close as needed to the point p_1, and, by the λ-Lemma (Palis 1969) they are transverse homoclinic points with tangent vectors close to the tangent vectors of the local manifolds.

Proof of Theorem C Let $U_u \subset U$ be a neighbourhood of q_u. Assume it is sufficiently small so that $U_s := T_d^{2n_0}(U_u)$ is contained in U. Clearly, U_s is a neighbourhood of q_s. From items (b) and (d), there exists $k_0 > 0$ such that $\widehat{\mathbf{z}}_k, \widehat{\mathbf{w}}_k \in U_s$ and $\widetilde{\mathbf{z}}_k, \widetilde{\mathbf{w}}_k \in U_u$ for all $k \geq k_0$.

Let k be large enough and let V be a small neighbourhood of $\mathbf{w_u}$ (suitable size will be decided later on). Then, $T_d^{2m_u}(V)$ is a neighbourhood of $\widetilde{\mathbf{w}}_k \subset U_u \subset U$ and, if we denote

$$m_0 := m_u + n_0 + k_0 + n_0 + m_s,$$

then $T_d^{2m_0}(V)$ is a neighbourhood of $\mathbf{w}_s$.

Let $R \subset V$ be a pseudo-rectangle in *the first quadrant* attached to $\mathbf{w}_u$ whose boundaries are given by pieces of W^s and W^u and the others are just straight lines (parallel to the tangent lines of W^s and W^u at $\mathbf{w}_u$).

Define

$$\widetilde{R} := T_d^{2m_0}(R). \tag{5.2}$$

If we iterate $\widetilde{R}$ by T_d^2, while staying in U where the dynamics is C^∞ conjugate to the one of the linearization at p_0, we eventually meet R.

Following Moser we introduce the *transversal map* $\widetilde{\phi}$. Given $\xi \in R$ we consider a number of iterates bigger that m_0 of T_d^2. By construction $T_d^{2m_0}(\xi) \in \widetilde{R}$. Next we consider $k = k(\xi) > m_0$ to be the smallest integer such that $T_d^{2k}(T_d^{2m_0}(\xi)) \in R$ and $T_d^{2j}(T_d^{2m_0}(\xi)) \in U$, $1 \leq j \leq k$, if it exists. We denote by $\mathcal{D}$ the set of $\xi \in R$ such that $k(\xi)$ exists and we define $\widetilde{\phi} : \mathcal{D} \subset R \to R$ by

$$\widetilde{\phi}(\xi) = T_d^{2m_0+2k(\xi)}(\xi), \qquad \xi \in \mathcal{D}. \tag{5.3}$$

To apply Moser's theorem, we should check that the restriction of T_d to $\bigcup_{k=1}^{2m_0} T_d^k(\mathcal{D})$ is a $\mathcal{C}^\infty$ diffeomorphism or equivalently that $\bigcup_{k=1}^{2m_0} T_d^k(\mathcal{D}) \cap \{y = x\} = \emptyset$.

That is, we should prove that, by choosing R small enough the points travelling from R to itself would not meet the line $\{y = x\}$, where T_d^{-1} is not smooth.

Since $\xi \in \mathcal{D} \subset R \subset V$ and V is a small neighbourhood of $\mathbf{w}_u$ the iterates of $\xi \in \mathcal{D}$ will travel following the orbit of $\mathbf{w}_u$

$$\mathbf{w}_u \xmapsto{T_d^{2m_u}} \widetilde{\mathbf{w}}_k \xmapsto{T_d^{2n_0}} \widehat{\mathbf{z}}_k \xmapsto{T_d^{2k}} \widetilde{\mathbf{z}}_k \xmapsto{T_d^{2n_0}} \widehat{\mathbf{w}}_k \xmapsto{T_d^{2m_s}} \mathbf{w}_s$$

until they arrive to $\widetilde{R}$. Then, by item (b), from $\widetilde{R}$ to R the iterates will stay in U. Hence, the only iterates of points $q \in \mathcal{D} \subset R$ which might fall in the line $\{y = x\}$ are the $2n_0$ iterates needed to go from $\widetilde{\mathbf{w}}_k$ to $\widehat{\mathbf{z}}_k$ and the $2n_0$ ones from $\widetilde{\mathbf{z}}_k$ to $\widehat{\mathbf{w}}_k$. To finish the argument, we distinguish two cases.

Case 1. The finite set $\{T_d^j(q_u) \mid 0 \leq j \leq 2n_0\}$ does not intersect $\{y = x\}$. By continuity there exists a sufficiently small open neighbourhood U_u of q_u such that the open set

$$\mathcal{U}_u := \bigcup_{j=0}^{2n_0} T_d^j(U_u)$$

does not intersect $\{y = x\}$ either. Of course, by items (b) and (d) there are infinitely many points of the sequences $\{\widetilde{\mathbf{w}}_k\}$, $\{\widetilde{\mathbf{z}}_k\}$ belonging to $\mathcal{U}_u$.

Choose V (the neighbourhood of $\mathbf{w}_u$ above) small enough and k large enough such that $T_d^{2m_u}(V) \subset U_u$ and such that $T_d^{2j}(V)$ belong to U, for $j = 0, \ldots, m_u$. Choose $R \subset V$ and define $\widetilde{R}$ as in (5.2). By construction, the map $\widetilde{\phi}$ is well defined and $\mathcal{C}^\infty$, it has an inverse $\widetilde{\phi}^{-1} : R \cap \widetilde{\phi}(R)$ which is also $\mathcal{C}^\infty$ since no iterates of T_d or T_d^{-1} in the definition of $\widetilde{\phi}$ (see (5.3)) intersect $\{y = x\}$.

Case 2. The set $\{T_d^j(q_u) \mid 1 \le j \le 2n_0\}$ intersects $\{y = x\}$. Let $\{T_d^{\ell_j}(q_u) \mid j = 1, \ldots, r\}$, for $0 < \ell_1 < \ldots < \ell_r < 2n_0$, for some $r \ge 1$, be the intersection. We recall that the stable and unstable manifolds of the point p_1 have discrete intersection with $\{y = x\}$. Again, items (b) and (d) imply that we can choose k_0 large enough such that there exist two small open neighbourhoods U_s of q_s and U_u of q_u such that for all $k \ge k_0$ we have that

$$\{T_d^j(U_u \setminus \{q_u\}) \mid j = 0, \ldots, 2n_0\} \cap \{y = x\} = \emptyset \qquad \text{and} \qquad \widetilde{\mathbf{w}}_k, \widetilde{\mathbf{z}}_k \in U_u.$$

We are in the same situation as in the previous case. Therefore, choosing V (the neighbourhood of $\mathbf{w}_u$ above) small enough and k large enough we obtain the regularity claim for $\widetilde{\phi}$ and $\widetilde{\phi}^{-1}$.

Now, Theorem 3.7 in Moser (2001) implies that there is a Cantor set $\mathcal{I}$ contained in R and a homeomorphism from $\mathcal{I}$ to the space of sequences of N symbols ($2 \le N \le \infty$) which conjugates $\widetilde{\phi}$ with the Bernoulli shift and, as a consequence, there is a dense set $\mathcal{P}$ of periodic orbits of $\widetilde{\phi}$, and therefore of T_d^2 and T_d in $\mathcal{I}$.

Moreover, Theorem 3.8 in Moser (2001) implies that there is a dense subset $\mathcal{H}$ of homoclinic points to p_1 in $\mathcal{I}$. We recall from Theorem A (b) of Fontich et al. (2024) that $W^s_{p_1} \subset \partial\Omega$. Finally, since $\mathcal{H} \subset W^s_{p_1}$,

$$\mathcal{P} \subset \overline{\mathcal{P}} = \mathcal{I} = \overline{\mathcal{H}} \subset \overline{\partial\Omega} = \partial\Omega,$$

that is, the boundary of Ω has infinitely many periodic orbits with arbitrary high period. □

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Authors and Affiliations

Ernest Fontich[1,2] · Antonio Garijo[3] · Xavier Jarque[1,2]

✉ Xavier Jarque
xavier.jarque@ub.edu

Ernest Fontich
fontich@ub.edu

Antonio Garijo
antonio.garijo@urv.cat

1 Departament de Matemàtiques i Informàtica, Universitat de Barcelona (UB), Gran Via de les Corts Catalanes 585, 08007 Barcelona, Catalonia, Spain

2 Centre de Recerca Matemàtica (CRM), Edifici C, Campus Bellaterra, 08193 Barcelona, Catalonia, Spain

3 Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili (URV), Campus Sescelades, Av. Països Catalans 26, Edifici E4, 43007 Tarragona, Catalonia, Spain