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Applications of Nijenhuis Geometry V: Geodesic Equivalence and Finite-Dimensional Reductions of Integrable Quasilinear Systems

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Abstract

We describe all metrics geodesically compatible with a gl-regular Nijenhuis operator L. The set of such metrics is large enough so that a generic local curve γ is a geodesic for a suitable metric g from this set. Next, we show that a certain evolutionary PDE system of hydrodynamic type constructed from L preserves the property of γ to be a g-geodesic. This implies that every metric g geodesically compatible with L gives us a finite-dimensional reduction of this PDE system. We show that its restriction onto the set of g-geodesics is naturally equivalent to the Poisson action of $\mathbb{R}^n$ on the cotangent bundle generated by the integrals coming from geodesic compatibility.

Keywords Nijenhuis operators · Geodesically equivalent metrics · Symmetries · Killing tensors · Integrals of motion · Systems of hydrodynamic type · Integrable systems · Finite-dimensional reduction

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1 Introduction

This work continues the research program started in Bolsinov et al. (2018, 2022a). The main object of study within this program are $(1, 1)$-tensor fields with vanishing Nijenhuis torsion known as *Nijenhuis operators*. They pop up in many, a priori unrelated, branches of mathematics, so it makes sense to develop a general theory of Nijenhuis operators and then to apply the results and methods obtained wherever these operators appear (e.g. in the theory of geodesically equivalent metrics as in the present paper). This approach treats a Nijenhuis operator as a *primary* object, even if it initially appeared as a *secondary* object in the study of another structure. An unexpected positive outcome of this change of perspective is that it reveals hidden relationships between different subjects. A demonstration of this phenomenon is the paper (Bolsinov et al. 2021b), showing that pencils of compatible Poisson brackets of hydrodynamic type are closely related to geodesically equivalent metrics of constant curvature. This allowed us to apply methods and results of a more developed theory of geodesically equivalent metrics to the theory of geometric Poisson structures. Another example is an unexpected relation between compatible geometric Poisson structures of type $\mathcal{P}_3 + \mathcal{P}_1$ (see Bolsinov et al. 2023a for details) and orthogonal separation of variables for spaces of constant curvature, which allowed us to obtain new results in both subjects by combining the relevant ideas and methods. In the theory of separation of variables, this led us in Bolsinov et al. (2022b) to a description of all orthogonal separating coordinates for pseudo-Riemannian spaces of constant curvature, solving a long-standing problem going back to Eisenhart (1934). On the other hand, in Bolsinov et al. (2023a) we have constructed all non-degenerate compatible Poisson structures of the type $\mathcal{P}_3 + \mathcal{P}_1$ such that $\mathcal{P}_3$ is Darboux–Poisson, which, in turn, led us to a construction of new integrable PDE systems in Bolsinov et al. (2023b).

In the present paper, the *partner structure* for a Nijenhuis operator L is a metric g which is geodesically compatible to it. In Theorem 1.3, we relate such metrics to the symmetries of the operator L. Combining this relation with results of Bolsinov et al. (2023d) , we describe all metrics geodesically compatible with a gl-regular Nijenhuis operator and, in particular, show that every such operator locally admits a geodesically compatible metric, see Theorem 1.2. Next, we consider an integrable PDE system of hydrodynamic type constructed from a gl-regular Nijenhuis operator L, which was studied (mostly, in the diagonal case) in many papers including (Bolsinov et al. 2023b; Ferapontov 1990; Lorenzoni and Magri 2005; Marciniak and Blaszak 2010). Theorem 1.4 shows that this system preserves the property of a curve to be a geodesic of any fixed geodesically compatible metric. In other words, every choice of a geodesically compatible partner for L on a manifold M gives us a finite-dimensional reduction of the system. Finally, Theorem 1.5 states that the corresponding finite-dimensional reduction is naturally equivalent to the Poisson action on $T^*\mathsf{M}$ generated by the commuting integrals coming from geodesic compatibility.

1.1 Definitions and Results

Two (pseudo)-Riemannian metrics g and $\bar{g}$ (of any, possibly different, signatures) are called *geodesically equivalent* if they share the same geodesics viewed as unparameterized curves. According to Bolsinov and Matveev (2003), a manifold endowed with a pair of such metrics carries a natural Nijenhuis structure defined by the operator

$$L = \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

Since $\bar{g}$ is uniquely reconstructed from L as $\bar{g} = \frac{1}{|\det L|} g L^{-1}$, the study of geodesically equivalent metrics reduces to the study of pairs (g, L) satisfying the following compatibility condition: a metric g and a Nijenhuis operator L are said to be *geodesically compatible*, if L is g-self-adjoint and the metric $\bar{g} = \frac{1}{|\det L|} g L^{-1}$ is geodesically equivalent to g.

Analytically, the geodesic compatibility condition is given by the PDE equation (Bolsinov and Matveev 2003; Sinjukov 1979)

$$\nabla_\eta L = \frac{1}{2}\big(\eta \otimes \mathrm{d}\,\mathrm{tr}\, L + (\eta \otimes \mathrm{d}\,\mathrm{tr}\, L)^*\big), \tag{1}$$

where η is an arbitrary vector field. Notice that this relation is linear in L. Our first result is an equivalent version of (1), which is linear both in L and in g and contains no covariant derivative.

Theorem 1.1 *An operator L and a metric g are geodesically compatible if and only if L is g-self-adjoint and the following relation holds*

$$\begin{aligned} &\mathcal{L}_{L\xi}\big(g(\eta, \xi)\big) - \mathcal{L}_\xi\big(g(\eta, L\xi)\big) - g\big(\eta, [L\xi, \xi]\big) + g\big([\eta, L\xi], \xi\big) \\ &\quad - g\big([\eta, \xi], L\xi\big) = g(\eta, \xi)\, \mathcal{L}_\xi\, \mathrm{tr}\, L, \end{aligned} \tag{2}$$

for any vector fields ξ and η. In local coordinates, (2) is equivalent to

$$g_{k\alpha} \frac{\partial L^\alpha_j}{\partial x^i} + \left(\frac{\partial g_{ik}}{\partial x^\alpha} - \frac{\partial g_{i\alpha}}{\partial x^k} \right) L^\alpha_j + (j \leftrightarrow k) \;=\; g_{ik} \frac{\partial\, \mathrm{tr}\, L}{\partial x^j} + (j \leftrightarrow k) \tag{3}$$

where $(j \leftrightarrow k)$ denotes the expression obtained by interchanging indices j and k.

Local description of geodesically compatible pairs (g, L) is known at algebraically generic points, near which the algebraic type of L does not change. For diagonalisable operators L, it follows from the classical work by Levi-Civita (1896), the general case was done in Bolsinov and Matveev (2015). However, for *singular* points, i.e. those at which the algebraic type of L changes (e.g. the eigenvalues of L collide), the description of Nijenhuis operators L admitting at least one geodesically equivalent partner g remains an open problem. Certain restrictions definitely exist. For instance, the operator $L = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ admits no geodesically compatible metric in the neighbourhood of the point $(x, y) = (0, 0)$. The next theorem shows that *regular* collisions of eigenvalues are always allowed.

Following (Bolsinov et al. 2023c), we say that an operator L is gl-*regular*, if its adjoint orbit $\mathcal{O}_L = \{PLP^{-1} \mid P \text{ is invertible}\}$ has maximal dimension. In simpler terms, this means that each eigenvalue λ of L admits only one linearly independent eigenvector (equivalently, only one λ-block in the Jordan normal form). If L is an operator on a manifold M, then its eigenvalues can still collide without violating the gl-regularity condition. In Nijenhuis geometry, scenarios of such collisions can be very different (see Bolsinov et al. 2023c). However, regardless of any particular scenario, we have the following general local result.

Theorem 1.2 *Let L be a* gl*-regular real-analytic Nijenhuis operator. Then (locally) there exists a pseudo-Riemannian metric g geodesically compatible with L. Moreover, such a metric g can be defined explicitly in terms of the second companion form of L.*

The construction of such a metric g is explained in Sect. 3, see Proposition 3.1 and formula (12). As already noticed, the statement of Theorem 1.2 was known only near algebraically generic points of L (e.g. Bolsinov and Matveev 2015), so the (local) existence of a geodesically compatible partner of L is a new result for singular points. Note that understanding the behaviour of geodesically equivalent metrics near singular points is fundamentally important for their global analysis on compact manifolds and also played a decisive role in proving the projective Lichnerowicz–Obata conjecture (Bolsinov et al. 2021a; Matveev 2007).

Remark 1.1 Let us emphasise that Theorem 1.2 is essentially local in the sense that a gl-regular Nijenhuis operator defined on a closed manifold M may not admit any geodesically compatible metric g on the whole of M (although locally such a metric can be found near each point). One of such examples is a complex structure J on a closed orientable surface M^2_g of genus $g \geq 2$. In dimension 2, J is gl-regular; however, it is known that M^2_g cannot carry non-proportional geodesically equivalent metrics.

Our third theorem generalises Sinjukov–Topalov hierarchy theorem (Bolsinov and Matveev 2011; Sinjukov 1979; Topalov 2000, 2003) and gives a complete description of geodesically compatible partners for gl-regular Nijenhuis operators in terms of their symmetries (both at algebraically generic and singular points).

Recall that an operator M is called a *symmetry* of L if these operators commute in algebraic sense, i.e. $LM = ML$, and the following relation holds for any vector field ξ:

$$M[L\xi, \xi] + L[\xi, M\xi] - [L\xi, M\xi] = 0. \tag{4}$$

This definition agrees with the concept of a symmetry in the context of evolutionary PDEs in mathematical physics (see, e.g. Fuchssteiner and Fokas 1981) and is equivalent to the fact that the evolutionary flows $u_t = L(u)u_x$ and $u_s = M(u)u_x$ commute.

If $LM = ML$ and

$$\langle L, M\rangle(\xi, \eta) \overset{\text{def}}{=} M[L\xi, \eta] + L[\xi, M\eta] - [L\xi, M\eta] - LM[\xi, \eta] = 0 \qquad \text{for all } \xi, \eta, \tag{5}$$

then M is called a *strong symmetry*[1] in Bolsinov et al. (2023d).

The definition of $\langle L, M\rangle$ in (5) is essentially due to Nijenhuis (1951, formula 3.9); it defines a (1, 2)-tensor field, provided that L and M commute. The l.h.s. of (4) is just $\langle L, M\rangle(\xi, \xi)$ so that (5) implies (4). Also notice that the Frölicher–Nijenhuis bracket of L and M can be written as $\langle L, M\rangle + \langle M, L\rangle$ and the Nijenhuis torsion of L coincides with $\langle L, L\rangle$ (up to sign).

Theorem 1.3 *Let L and g be geodesically compatible. Assume that M is g-self-adjoint and is a strong symmetry of L, then L and $gM := (g_{is}M^s_j)$ are geodesically compatible.*

Moreover, if L is gl*-regular, then every metric $\tilde{g}$ geodesically compatible with L is of the form $\tilde{g} = gM$, where M is a (strong)*[2] *symmetry of L.*

Remark 1.2 The first part of this theorem in a slightly different setting was proved by Sinjukov (1979) for $M = L$ and later reproved and applied by Topalov (2000, 2003), who also generalised the result for $M = L^{-1}$. The case $M = f(L)$ for a polynomial f immediately follows from the case $M = L$, and both Sinjukov and Topalov considered the case $M = L^k$ for $k \in \mathbb{N}$. For an arbitrary real analytic function, the results were generalised in Bolsinov and Matveev (2011), see also (Topalov 2008, Theorem 3). For diagonalisable gl-regular Nijenhuis operators L, every symmetry of L has the form $f(L)$ for some smooth function f so all metrics geodesically compatible with such an operator L form generalised Sinjukov–Topalov hierarchy in the terminology of Bolsinov and Matveev (2011). However, if gl-regular L contains non-trivial Jordan blocks, then there exist strong symmetries that cannot be presented in the form $f(L)$, see Bolsinov et al. (2023d).

For a given Nijenhuis operator L, we define the operator fields A_i by the following recursion relations

$$A_0 = \mathrm{Id}, \quad A_{i+1} = LA_i - \sigma_i \,\mathrm{Id}, \quad i = 0, \dots, n-1, \tag{6}$$

where functions σ_i are coefficients of the characteristic polynomial of L numerated as below:

$$\chi_L(\lambda) = \det(\lambda \,\mathrm{Id} - L) = \lambda^n - \sigma_1\lambda^{n-1} - \dots - \sigma_n. \tag{7}$$

Equivalently, the operators A_i can be defined from the matrix relation

$$\det(\lambda \,\mathrm{Id} - L) \cdot (\lambda \,\mathrm{Id} - L)^{-1} = \lambda^{n-1}A_0 + \lambda^{n-2}A_1 + \dots + \lambda A_{n-2} + A_{n-1}.$$

In the context of geodesically equivalent metrics, these operators can be characterised as (1, 1)-Killing tensors of any metric g geodesically compatible with L (see Bolsinov and Matveev 2003; Konyaev et al. 2024). Notice that they are not Nijenhuis.

[1] Notice that this notion is different from *strong symmetry* introduced in Fuchssteiner and Fokas (1981) which is now commonly known as *recursion operator*.

[2] As proved in Bolsinov et al. (2023d), every symmetry of a gl-regular Nijenhuis operator is strong.

Consider the following system of quasilinear PDEs defined by these operators

$$\begin{aligned} u_{t_1} &= A_1 \, u_x, \\ &\dots \\ u_{t_{n-1}} &= A_{n-1} \, u_x, \end{aligned} \tag{8}$$

with $u^i = u^i(x, t_1, ..., t_{n-1})$ being unknown functions in n variables and $u = (u^1, \dots, u^n)^\top$.

The system (8) can be obtained within the framework of the general construction introduced by Lorenzoni and Magri (2005) and Magri (2003). In particular, it is consistent (in the real-analytic category) in the sense that for any initial curve $\gamma(x)$ there exists a solution $u = u(x, t_1, \dots, t_{n-1})$ such that $u(x, 0, \dots, 0) = \gamma(x)$. In the case of a diagonal Nijenhuis operator L, the corresponding system satisfies the semihamiltonicity condition of Tsarev (1991) and is weakly-nonlinear in the sense of Rozhdestvenskii and Sidorenko (1967). Such systems were studied and integrated in quadratures in the diagonal case by E. Ferapontov[3] (Ferapontov 1990, 1991; Ferapontov and Fordy 1997) and in Marciniak and Blaszak (2010), see also Blaszak and Sergyeyev (2009) and Blaszak and Ma (2003). The general, not necessarily diagonalisable, case was done in Bolsinov et al. (2023d).

The next portion of our results concerns finite-dimensional reductions of system (8). Various types of finite-dimensional reductions of infinite-dimensional nonlinear integrable systems have been investigated since the middle of 70s, see, e.g. Antonowicz et al. (1987), Bogojavlenskii and Novikov (1976), Hone (1996), Marciniak and Blaszak (2010), and Veselov (1980). Informally, a finite-dimensional reduction of an integrable PDE system is a subsystem of it, which is finite-dimensional and still integrable. The first of the following two theorems states that the set of geodesics of any metric geodesically compatible with a gl-regular Nijenhuis operator L is invariant with respect to the flow of (8). That is, by fixing a metric g geodesically compatible with L, we obtain a reduction of our infinite dimensional system to the set of g-geodesics, which can be naturally endowed with the structure of a smooth manifold of dimension $2n$.

Next, we show that the restriction of our system to the set of g-geodesics is equivalent, in the natural sense, to the Poisson action generated by the quadratic integrals of the geodesic flow which are closely related to the operators A_i (Bolsinov and Matveev 2003; Matveev and Topalov 1998). Namely, if g is geodesically compatible with L, then its geodesic flow (as a Hamiltonian system on $T^*\mathsf{M}$) admits n commuting first integrals $F_0, \dots, F_{n-1}$ of the form

$$F_i(u, p) = \tfrac{1}{2} \, g^{-1}(A_i^* p, p). \tag{9}$$

Theorem 1.4 *Consider any metric g geodesically compatible with L and take any geodesic $\gamma(x)$ of this metric. Let $u(x, t_1, \dots, t_{n-1})$ be the solution of (8) with the*

[3] Ferapontov's result is more general, he explicitly integrated all diagonal weakly-nonlinear semihamiltonian systems.

initial condition $u(x, 0, \dots, 0) = \gamma(x)$*. Then for any (sufficiently small)* $t_1, \dots, t_{n-1}$*, the curve* $x \mapsto u(x, t_1, \dots, t_{n-1})$ *is a geodesic of* g*.*

In other words, the evolutionary system corresponding to any of the equations from (8) sends geodesics of g to geodesics. Let us consider the space $\mathfrak{G}$ of all g-geodesics (viewed as parameterised curves). This set has a natural structure of a $2n$-dimensional manifold. By Theorem 1.4, system (8) defines a local action of $\mathbb{R}^n$ on $\mathfrak{G}$:

$$\Psi^{t_0,t_1,\dots,t_{n-1}} : \mathfrak{G} \to \mathfrak{G}, \qquad (t_0, t_1, \dots, t_{n-1}) \in \mathbb{R}^n.$$

More precisely, if $\gamma = \gamma(x) \in \mathfrak{G}$ is a g-geodesic, then we set $\Psi^{t_0,t_1,\dots,t_{n-1}}(\gamma)$ to be the g-geodesic $\tilde{\gamma}(x) = u(x + t_0, t_1, \dots, t_{n-1})$, where $u(x, t_1, \dots, t_{n-1})$ is the solution of (8) with the initial condition $u(x, 0, \dots, 0) = \gamma(x)$.

Theorem 1.5 *The action* Ψ *is conjugate to the Hamiltonian action of* $\mathbb{R}^n$ *on* $T^*\mathsf{M}$ *generated by the flows of the integrals* $F_0, \dots, F_{n-1}$ *defined by* (9)*. The conjugacy is given by* $\gamma \in \mathfrak{G} \mapsto (\gamma(0), g_{ij}\dot{\gamma}^i(0)) \in T^*\mathsf{M}$*.*

Remark 1.3 Strictly speaking, the action Ψ is *local*, since the solutions $u(x, t_1, \dots, t_{n-1})$ of (8) are, in general, defined only for small values of t_i and x from a certain (perhaps, small) interval. However, if the Hamiltonian flows of $F_0, \dots, F_{n-1}$ are complete, then Theorem 1.5 becomes global in the sense that the initial geodesic $\gamma(x)$ is defined for all $x \in \mathbb{R}$ and $(t_0, t_1, \dots, t_{n-1}) \in \mathbb{R}^n$ is arbitrary.

Let L be a gl-regular real-analytic Nijenhuis operator, then it follows from Bolsinov et al. (2023d) that for every curve γ with a cyclic velocity vector there exists a metric g geodesically compatible with L such that γ is a g-geodesic. Thus, the finite-dimensional reductions of (8) provided by Theorems 1.4 and 1.5 'cover' almost all (local) solutions of the Cauchy problem. Indeed, every generic initial curve belongs to a suitable set $\mathfrak{G}$ of geodesics from Theorem 1.5.

2 Proof of Theorem 1.1

Recall that L is assumed to be g-self-adjoint. We use the geodesic compatibility condition for g and L in the form (1). The operators in the left-hand side and the right-hand side of (1) are both g-self-adjoint, so that we can equivalently rewrite this relation as

$$g\big((\nabla_\eta L)\xi, \xi\big) = g(\eta, \xi)\, \mathcal{L}_\xi \operatorname{tr} L \quad \text{for all tangent vectors } \eta \text{ and } \xi. \tag{10}$$

Then, we have

$$\begin{aligned} g\big((\nabla_\eta L)\xi, \xi\big) &= g\big(\nabla_\eta(L\xi) - L\nabla_\eta\xi, \xi\big) = g\big([\eta, L\xi] + \nabla_{L\xi}\eta - L([\eta, \xi] + \nabla_\xi\eta), \xi\big) \\ &= g\big(\nabla_{L\xi}\eta, \xi\big) - g\big(\nabla_\xi\eta, L\xi\big) + g\big([\eta, L\xi], \xi\big) - g\big([\eta, \xi], L\xi\big) \\ &= \nabla_{L\xi}\big(g(\eta, \xi)\big) - g\big(\eta, \nabla_{L\xi}\xi\big) - \nabla_\xi\big(g(\eta, L\xi)\big) + g\big(\eta, \nabla_\xi L\xi\big) \\ &\quad + g\big([\eta, L\xi], \xi\big) - g\big([\eta, \xi], L\xi\big) \end{aligned}$$

$$
\begin{aligned}
&= \nabla_{L\xi}\big(g(\eta,\xi)\big) - \nabla_{\xi}\big(g(\eta, L\xi)\big) - g\big(\eta, \nabla_{L\xi}\xi - \nabla_{\xi} L\xi\big) \\
&\quad + g\big([\eta, L\xi], \xi\big) - g\big([\eta,\xi], L\xi\big) \\
&= \mathcal{L}_{L\xi}\big(g(\eta,\xi)\big) - \mathcal{L}_{\xi}\big(g(\eta, L\xi)\big) - g\big(\eta, [L\xi,\xi]\big) + g\big([\eta, L\xi],\xi\big) - g\big([\eta,\xi], L\xi\big),
\end{aligned}
$$

which leads to (2), as required.

In local coordinates, condition (1) takes the form:

$$
\frac{\partial L^m_j}{\partial x^i} + \Gamma^m_{i\alpha} L^\alpha_j - \Gamma^\alpha_{ij} L^m_\alpha = \frac{1}{2}(l_j \delta^m_i + g^{\alpha m} l_\alpha g_{ij}), \quad \text{where } l_j = \frac{\partial \operatorname{tr} L}{\partial x^j}.
$$

Multiplying by g_{km} (and summing over m) gives

$$
g_{km}\frac{\partial L^m_j}{\partial x^i} + \Gamma_{i\alpha,k} L^\alpha_j - g_{km}\Gamma^\alpha_{ij} L^m_\alpha = \frac{1}{2}(l_j g_{ik} + l_k g_{ij}).
$$

Using the fact that L is g-self-adjoint (i.e. $g_{km} L^m_\alpha = g_{\alpha m} L^m_k$), we get (after cosmetic changes of some summation indices)

$$
g_{k\alpha}\frac{\partial L^\alpha_j}{\partial x^i} + \Gamma_{i\alpha,k} L^\alpha_j - \Gamma_{ij,\alpha} L^\alpha_k = \frac{1}{2}(l_j g_{ik} + l_k g_{ij}).
$$

In more detail,

$$
\begin{aligned}
&g_{k\alpha}\frac{\partial L^\alpha_j}{\partial x^i} + \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^\alpha} + \frac{\partial g_{k\alpha}}{\partial x^i} - \frac{\partial g_{i\alpha}}{\partial x^k}\right) \\
&L^\alpha_j - \frac{1}{2}\left(\frac{\partial g_{i\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\alpha}\right) L^\alpha_k = \frac{1}{2}(l_j g_{ik} + l_k g_{ij}).
\end{aligned}
$$

Since L is g-self-adjoint, we can rewrite it as follows:

$$
\begin{aligned}
&\frac{1}{2} g_{k\alpha}\frac{\partial L^\alpha_j}{\partial x^i} + \frac{1}{2} g_{\alpha j}\frac{\partial L^\alpha_k}{\partial x^i} + \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^\alpha} - \frac{\partial g_{i\alpha}}{\partial x^k}\right) L^\alpha_j \\
&\quad + \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^\alpha} - \frac{\partial g_{i\alpha}}{\partial x^j}\right) L^\alpha_k = \frac{1}{2}(l_j g_{ik} + l_k g_{ij}),
\end{aligned} \tag{11}
$$

which makes it clear that the left-hand side and right-hand side of this relation are both symmetric in indices j and k. Up to the factor $\frac{1}{2}$, this relation coincides with (3), as needed.

3 Proof of Theorem 1.2

Let L be a gl-regular Nijenhuis operator. We start with a purely algebraic construction leading to a metric h that is geodesically compatible with L. This construction is performed in specific coordinates (and h will depend on the choice of such coordinates).

First, fix second companion coordinates $u^1, \dots, u^n$ of L so that

$$L = L_{\mathsf{comp2}} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \dots & \sigma_1 \end{pmatrix}.$$

Recall that for real-analytic gl-regular Nijenhuis operators such coordinates always exist (Bolsinov et al. 2023c)[4].

Let $p_1, \dots, p_n, u^1, \dots, u^n$ be the corresponding canonical coordinates on the cotangent bundle and consider the following algebraic identity

$$h_1 L^{n-1} + \dots + h_n \,\mathrm{Id} = \left(p_n L^{n-1} + \dots + p_1 \,\mathrm{Id}\right)^2. \tag{12}$$

The operator in the right-hand side commutes with L and therefore can be uniquely written as a linear combination of Id, $L, \dots, L^{n-1}$ [this is a characteristic properties of gl-regular operators (Bolsinov et al. 2023d)] so that the functions $h_1, \dots, h_n$ are uniquely defined. In fact, they are quadratic in $p_1, \dots, p_n$ and their coefficients are polynomials in σ_i's. Using the fact that $L = L_{\mathsf{comp2}}$, one can easily check that $h_n, \dots, h_1$ are the elements of the first row of the matrix $\left(p_n L^{n-1} + \dots + p_1 \,\mathrm{Id}\right)^2$.

The statement of Theorem 1.2 follows from

Proposition 3.1 *The quadratic function $h_1(u, p)$ defines a non-degenerate (contravariant) metric which is geodesically compatible with $L = L_{\mathsf{comp2}}$.*

To prove this proposition, we need to verify three conditions:

(i) the quadratic form $h_1(u, p) = \sum h_1^{\alpha\beta}(u) p_\alpha p_\beta$ is non-degenerate,
(ii) L is h_1-self-adjoint,
(iii) L and h_1 satisfy the geodesic compatibility condition.

Lemma 3.1 *The contravariant quadratic form h_i is non-degenerate, i.e.* $\det\left(h_1^{\alpha\beta}(u)\right) \neq 0$.

Proof We write the r.h.s. of (12) as

$$\left(p_n L^{n-1} + \dots + p_1 \,\mathrm{Id}\right)^2 = \sum_{i=0}^{2n-1} \left(\sum_{k=1}^{i+1} p_k p_{i-k+2}\right) L^i.$$

Hence, from (12) we get

$$\frac{\partial^2 h_1}{\partial p_i \partial p_j} L^{n-1} + \dots + \frac{\partial^2 h_n}{\partial p_i \partial p_j} \,\mathrm{Id} = L^{i+j-2}.$$

[4] It is an open question whether such coordinates exist for any *smooth* gl-regular Nijenhuis operator L. By Bolsinov et al. (2023c), it is the case if L is algebraically generic, i.e. its eigenvalues have constant multiplicities.

This immediately implies that nonzero terms of $\left(h_1^{\alpha\beta}\right)$ are the only ones with $\alpha+\beta \geq n+1$. Moreover, one can see that $h_1^{\alpha\beta} = 1$ for $\alpha+\beta = n+1$. In other words, the matrix of h_1 has the form

$$\left(h_1^{\alpha\beta}\right) = \begin{pmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & * \\ 0 & \dots & 1 & * & * \\ & \iddots & & & \\ 1 & \dots & * & * & * \end{pmatrix},$$

and is obviously non-degenerate, as stated. □

The second condition (ii) is algebraic and can be checked directly by using standard matrix algebra. We, however, will derive it differently. For our next construction, we will need an operator field $\widehat{L}$, which can be understood as a prolongation of L to the cotangent bundle. In the given second companion coordinates (p, u), we set

$$\widehat{L} = \begin{pmatrix} L^\top & 0 \\ 0 & L \end{pmatrix}.$$

The next Lemma provides some differential identities, which are crucial for our construction.

Lemma 3.2 *The functions h_i from (12) satisfy the following identities*

$$\begin{aligned} \widehat{L}^* \mathrm{d}h_i &= \sigma_i \mathrm{d}h_1 + \mathrm{d}h_{i+1}, \quad i = 1, \dots, n-1, \\ \widehat{L}^* \mathrm{d}h_n &= \sigma_n \mathrm{d}h_1 \end{aligned} \tag{13}$$

Proof If we consider p_i to be (scalar) parameters, then the l.h.s. of (12) is a symmetry of L. This implies [see Bolsinov et al. (2023d, Lemma 2.3)] that

$$\begin{aligned} L^* \bar{\mathrm{d}}h_i &= \sigma_i \bar{\mathrm{d}}h_1 + \bar{\mathrm{d}}h_{i+1}, \quad i = 1, \dots, n-1, \\ L^* \bar{\mathrm{d}}h_n &= \sigma_n \bar{\mathrm{d}}h_1. \end{aligned} \tag{14}$$

Here $\bar{\mathrm{d}}h_i = \frac{\partial h_i}{\partial u^1}\mathrm{d}u^1 + \dots + \frac{\partial h_i}{\partial u^n}\mathrm{d}u^n$, that is, *one half* of the differential $\mathrm{d}h_i$ on the cotangent bundle. Differentiating (12) in p_i, we get

$$\frac{\partial h_1}{\partial p_i} L^{n-1} + \dots + \frac{\partial h_n}{\partial p_i}\,\mathrm{Id} = 2L^{i-1}\Big(p_n L^{n-1} + \dots + p_1\,\mathrm{Id}\Big), \quad i = 1, \dots, n.$$

This implies (using the Cayley–Hamilton theorem)

$$\begin{aligned}\frac{\partial h_1}{\partial p_{i+1}}L^{n-1}+\cdots+\frac{\partial h_n}{\partial p_{i+1}}\,\mathrm{Id}&=L\Big(\frac{\partial h_1}{\partial p_i}L^{n-1}+\cdots+\frac{\partial h_n}{\partial p_i}\,\mathrm{Id}\Big)\\ &=\frac{\partial h_1}{\partial p_i}L^{n}+\cdots+\frac{\partial h_n}{\partial p_i}L=\Big(\frac{\partial h_2}{\partial p_i}+\sigma_1\frac{\partial h_1}{\partial p_i}\Big)L^{n-1}+\cdots\\ &\quad+\Big(\frac{\partial h_n}{\partial p_i}+\sigma_{n-1}\frac{\partial h_1}{\partial p_i}\Big)L+\sigma_n\frac{\partial h_1}{\partial p_i}\,\mathrm{Id}\,.\end{aligned}$$

As L is gl-regular, the coefficients in front of the powers of L coincide, and we arrive to the system of $n-1$ matrix equations

$$\frac{\partial h}{\partial p_{i+1}}=L_{\mathsf{comp1}}\frac{\partial h}{\partial p_i},\quad i=1,\ldots,n-1. \tag{15}$$

Here $\frac{\partial h}{\partial p_i}$ is the column-vector $\left(\frac{\partial h_1}{\partial p_i},\ldots,\frac{\partial h_n}{\partial p_i}\right)^\top$ and L_{comp1} denotes the first companion form of L (obtained from L_{comp2} by transposition w.r.t. the anti-diagonal). Notice that (15) can be rewritten in equivalent form $\frac{\partial h}{\partial p_{i+1}}=(L_{\mathsf{comp1}})^i\frac{\partial h}{\partial p_1}$. In particular, $\frac{\partial h}{\partial p_n}=(L_{\mathrm{comp1}})^{n-1}\frac{\partial h}{\partial p_1}$ implying

$$L_{\mathsf{comp1}}\frac{\partial h}{\partial p_n}=L^n\frac{\partial h}{\partial p_1}=(\sigma_1L^{n-1}+\cdots+\sigma_n\mathrm{Id})\frac{\partial h}{\partial p_1}=\sigma_1\frac{\partial h}{\partial p_n}+\cdots+\sigma_n\frac{\partial h}{\partial p_1}.$$

Together with this additional equation, we can write (15) in matrix form:

$$L_{\mathsf{comp1}}\left(\frac{\partial h}{\partial p}\right)=\left(\frac{\partial h}{\partial p}\right)L_{\mathsf{comp2}}^\top,$$

where $\left(\frac{\partial h}{\partial p}\right)$ is the Jacobi matrix of $h_1,\ldots,h_n$ w.r.t. $p_1,\ldots,p_n$. Introducing $\tilde{\mathrm{d}}h_i=\left(\frac{\partial h_i}{\partial p_1},\ldots,\frac{\partial h_i}{\partial p_n}\right)$ and recalling that in our coordinates $L=L_{\mathsf{comp2}}$, we can finally rewrite this system as

$$\begin{aligned}\tilde{\mathrm{d}}h_i\,L^\top&=\sigma_i\tilde{\mathrm{d}}h_1+\tilde{\mathrm{d}}h_{i+1},\quad i=1,\ldots,n-1,\\ \tilde{\mathrm{d}}h_n\,L^\top&=\sigma_n\tilde{\mathrm{d}}h_1.\end{aligned}\tag{16}$$

Gathering (16) and (14), we get (13) from the statement of the Lemma. □

Corollary 3.1 *L is h_1-self-adjoint.*

Proof By saying that L is self-adjoint w.r.t. the contravariant metric h_1, we mean that $h_1(L^*\alpha,\beta)=h_1(\alpha,L^*\beta)$ for any 1-forms (co-vectors) α and β. In local coordinates, this condition means that the tensor $h_1^{ks}L_s^j$ is symmetric in indices k and j. This is, of course, equivalent to the fact L is self-adjoint w.r.t. the covariant metric h_1^{-1}.

For $i = 1$, relations (16) give $\tilde{\mathrm{d}}h_1\, L^\top = \sigma_1 \tilde{\mathrm{d}}h_1 + \tilde{\mathrm{d}}h_2$. In coordinates,

$$h_1^{ks} L_s^j = \sigma_1 h_1^{kj} + h_2^{kj} \tag{17}$$

and we see that the l.h.s. is indeed symmetric in k and j, since h_1 and h_2 in the r.h.s. are both symmetric by construction. This completes the proof. □

The standard geodesic compatibility condition (1) is not very convenient in our current setting as we deal with a *contravariant* metric h_1. However, there is another elegant Benenti condition which is quite suitable for our purposes, see Bolsinov and Matveev (2003, Definition 1 and Theorem 1). Namely, h_1 and L will be geodesically compatible if the quadratic functions

$$H = \frac{1}{2} h_1(p, u) = \frac{1}{2} h_1^{ij}(u) p_i p_j \quad \text{and} \quad F = h_1^{ks} L_s^j p_k p_j \tag{18}$$

satisfy the following commutation relation w.r.t. the standard Poisson structure on $T^*\mathsf{M}$

$$\{H, F\} = 2H \cdot \left(\tfrac{\partial\, \mathrm{tr}\, L}{\partial u^q} h_1^{\alpha q} p_\alpha \right). \tag{19}$$

To verify this relation, we need the following algebraic lemma.

Lemma 3.3 *Let $\mathcal{P}$ be a skew-symmetric form on a vector space V^n and L be an $\mathcal{P}$-symmetric operator, that is, $\mathcal{P}(L\xi, \eta) = \mathcal{P}(\xi, L\eta)$ for all vectors ξ, η. Then for all integer $p, q \geq 0$ and arbitrary ξ, one has*

$$\mathcal{P}(L^p \xi, L^q \xi) = 0.$$

Proof The above symmetry condition implies that the form $\mathcal{P}_L(\xi, \eta) = \mathcal{P}(L\xi, \eta)$ is skew-symmetric. Now, if $p + q = 2k$, then due to L being $\mathcal{P}$-symmetric, we get $\mathcal{P}(L^p\xi, L^q\xi) = \mathcal{P}(L^k\xi, L^k\xi) = 0$. Similarly for $p + q = 2k + 1$, we get $\mathcal{P}(L^p\xi, L^q\xi) = \mathcal{P}(L^{k+1}\xi, L^k\xi) = \mathcal{P}_L(L^k\xi, L^k\xi) = 0$ as $\mathcal{P}_L$ is skew-symmetric. Lemma is proved. □

Now notice that by construction $\widehat{L}^*$ is $\mathcal{P}$-symmetric with respect to the Poisson structure $\mathcal{P} = \Omega^{-1}$. On the other hand, relations (13) imply that $\mathrm{d}h_1, \dots, \mathrm{d}h_n$ belong to the subspace spanned by $\left(\widehat{L}^*\right)^k \mathrm{d}h_1$ ($k = 0, 1, \dots$). By Lemma 3.3, this subspace is isotropic w.r.t. $\mathcal{P}$, which means that

$$\{h_i, h_j\} = 0,$$

where $\{\,,\,\}$ in the standard Poisson bracket on the cotangent bundle.

To finish the proof, it remains to notice that relation (17) means that the second function F from (18) can be written as $F = h_2 + \sigma_1 h_1$, where $\sigma_1 = \mathrm{tr}\, L$. Due to Poisson commutativity of h_1 and h_2, we have

$$\{H, F\} = \{\tfrac{1}{2}h_1, h_2 + \sigma_1 h_1\} = \tfrac{1}{2}h_1 \cdot \{h_1, \sigma_1\} = H \cdot \{h_1, \mathrm{tr}\, L\} = 2H \cdot \left(\frac{\partial\, \mathrm{tr}\, L}{\partial u^q} h_1^{\alpha q} p_\alpha\right),$$

which coincides with (19). This completes the verification of conditions (i), (ii), and (iii) and hence the proof of Theorem 1.2.

4 Proof of Theorem 1.3

Let L and g be geodesically compatible. We start with the first statement of Theorem 1.3 and consider a g-self-adjoint operator M which is a strong symmetry of L. We need to show that the geodesic compatibility condition still holds if we replace g with $\tilde{g} = gM$ (i.e. $\tilde{g}(\eta, \xi) = g(M\eta, \xi)$).

We use this condition in the form (2). We have

$$\begin{aligned}
&\mathcal{L}_{L\xi}\big(\tilde{g}(\eta, \xi)\big) - \mathcal{L}_{\xi}\big(\tilde{g}(\eta, L\xi)\big) + \tilde{g}\big(\eta, [\xi, L\xi]\big) + \tilde{g}\big([\eta, L\xi], \xi\big) - \tilde{g}\big(L[\eta, \xi], \xi\big) \\
&\quad = \mathcal{L}_{L\xi}\big(g(M\eta, \xi)\big) - \mathcal{L}_{\xi}\big(g(M\eta, L\xi)\big) + g\big(M\eta, [\xi, L\xi]\big) + g\big(M[\eta, L\xi] - ML[\eta, \xi], \xi\big) \\
&\quad = \mathcal{L}_{L\xi}\big(g(M\eta, \xi)\big) - \mathcal{L}_{\xi}\big(g(M\eta, L\xi)\big) + g\big(M\eta, [\xi, L\xi]\big) + g\big([M\eta, L\xi] - L[M\eta, \xi], \xi\big) \\
&\qquad + g\big(M[\eta, L\xi] - ML[\eta, \xi] - [M\eta, L\xi] + L[M\eta, \xi], \xi\big) \\
&\quad = g\big((\nabla_{M\eta}L)\xi, \xi\big) + g\big(\langle M, L\rangle(\eta, \xi), \xi\big) = g\big((\nabla_{M\eta}L)\xi, \xi\big) + 0 \\
&\quad = g(M\eta, \xi)\,\mathcal{L}_{\xi}\operatorname{tr} L = \tilde{g}(\eta, \xi)\,\mathcal{L}_{\xi}\operatorname{tr} L,
\end{aligned}$$

as required. This proves the first statement of Theorem 1.3.

Remark 4.1 This computation also leads to the following conclusion. Let g and L be geodesically compatible. Then a metric $\tilde{g} = gM$ is geodesically compatible with L if and only if $g\big(\langle M, L\rangle(\eta, \xi), \xi\big)$ vanishes for all η, ξ.

Now assume that L is gl-regular. To prove the second statement of Theorem 1.3, we need the following algebraic fact.

Lemma 4.1 *Let operators L and M be g-self-adjoint. Then L is self-adjoint with respect to the metric $\tilde{g} = gM$ if and only if L and M commute (i.e. $ML - LM = 0$).*

Proof The statement immediately follows from the identity

$$\tilde{g}(L\xi, \eta) - \tilde{g}(\xi, L\eta) = g(ML\xi, \eta) - g(M\xi, L\eta) = g\big((ML - LM)\xi, \eta\big).$$

Since g is non-degenerate, the vanishing of the l.h.s. is equivalent to $ML - LM = 0$, as stated. □

Now let L be a gl-regular Nijenhuis operator which is geodesically compatible with a metric g (notice that such a metric always exists by Theorem 1.2).

Let $\tilde{g}$ be another metric geodesically compatible with L. Recall that L is $\tilde{g}$-self-adjoint by definition. Define M to be the operator field that relates these two metrics, that is, $\tilde{g} = gM$ so that M is automatically g-self-adjoint. By Lemma 4.1, $ML - LM = 0$ and, thus, $M = f_1 L^{n-1} + \cdots + f_n \operatorname{Id}$ for some smooth functions $f_1, \ldots, f_n$. We introduce the following tensor of type $(1, 2)$ which we treat as a vector-valued bilinear form:

$$T_M = \mathrm{d} f_n \otimes L^{n-1} + \cdots + \mathrm{d} f_n \otimes \operatorname{Id}$$

By definition,

$$LT_M(\xi, \eta) = T_M(\xi, L\eta). \tag{20}$$

As L is g-self-adjoint, we have

$$g\big(T_M(\xi, \eta), \zeta\big) = g\big(\eta, T_M(\xi, \zeta)\big) \tag{21}$$

for all vectors ξ, η, ζ. By straightforward computation, using the fact that $\langle L^i, L\rangle = 0$, we get

$$\begin{aligned} \langle M, L\rangle(\eta, \xi) &= L[M\eta, \xi] + M[\eta, L\xi] - LM[\eta, \xi] - [M\eta, L\xi] \\ &= T_M(\xi, L\eta) - T_M(L\xi, \eta). \end{aligned} \tag{22}$$

From (20), (21) and (22) we obtain

$$\begin{aligned} &g\big(\langle M, L\rangle(\eta, \xi), \xi\big) \\ &\quad = g\big(T_M(\xi, L\eta), \xi\big) - g\big(T_M(L\xi, \eta), \xi\big) = g\big(L\eta, T_M(\xi, \xi)\big) - g\big(\eta, T_M(L\xi, \xi)\big) \\ &\quad = g\big(\eta, LT_M(\xi, \xi)\big) - g\big(\eta, T_M(L\xi, \xi)\big) = g\big(\eta, T_M(\xi, L\xi) - T_M(L\xi, \xi)\big) \\ &\quad = -g\big(\eta, \langle M, L\rangle(\xi, \xi)\big) \end{aligned} \tag{23}$$

By Remark 4.1, $\tilde{g} = gM$ is geodesically compatible with L if and only if

$$g\big(\langle M, L\rangle(\eta, \xi), \xi\big) = 0,$$

for all ξ, η. In view of (23), this implies that $\langle M, L\rangle(\xi, \xi) = 0$, i.e. M is a symmetry of L. Since L is gl-regular, the symmetry M is strong (Bolsinov et al. 2023d, Theorem 1.2), as required.

5 Proof of Theorems 1.4 and 1.5

Recall that a symmetric $(0, 2)$-tensor $A = A_{ij}$ is called a Killing tensor for a (pseudo)-Riemannian metric g if it satisfies the following condition:

$$\nabla_k A_{ij} + \nabla_i A_{jk} + \nabla_j A_{ki} = 0,$$

where ∇ denote the Levi-Civita connection of g. We will also refer to the operator $A^i_j = g^{is} A_{sj}$ obtained from $A = A_{ij}$ by raising index as Killing $(1, 1)$-tensor. Recall that the equivalent definition for Killing $(1, 1)$-tensors is as follows: $A = A^i_j$ is Killing $(1, 1)$-tensor for a metric g if A is g-self-adjoint and the Hamiltonians

$$H(u, p) = \frac{1}{2} g^{ij}(u) p_i p_j \quad \text{and} \quad F(u, p) = \frac{1}{2} A^i_\alpha g^{\alpha j}(u) p_i p_j \tag{24}$$

commute on T^*M w.r.t. the canonical Poisson structure.

Theorems 1.4 and 1.5 are based on the following general statement that establishes a natural relation between quadratic Killing tensors of (pseudo)-Riemannian metrics and solutions of some quasilinear systems. Its special case is Marciniak and Blaszak (2010, Remark 2 and Proposition 3), see also Blaszak and Sergyeyev (2009) and Blaszak and Ma (2003).

Proposition 5.1 *Let $A = A^i_j$ be a Killing $(1,1)$-tensor for a Riemannian metric g. Consider the Hamiltonian $\mathbb{R}^2$-action on $T^*\mathsf{M}$ generated by the Poisson commuting Hamiltonians (24):*

$$\Phi^{x,t} = \Phi^x_H \circ \Phi^t_F : T^*\mathsf{M} \to T^*\mathsf{M}.$$

Then for any initial condition (u_0, p_0), the function $u(t,x)$ defined from the relation $\Phi^{x,t}(u_0, p_0) = \big(u(x,t), p(x,t)\big)$ is a solution of the quasilinear system

$$u_t = A(u)u_x. \tag{25}$$

Proof By definition, the functions $\big(u(x,t), p(x,t)\big)$ define the natural parametrisation of the Φ-orbit of the point (u_0, p_0). In particular, $\big(u(x,t_c), p(x,t_c)\big)$ for a fixed $t = t_c$ is a solution of the Hamiltonian system generated by H and, similarly, $\big(u(x_c,t), p(x_c,t)\big)$ for a fixed $x = x_c$ is a solution of the Hamiltonian system generated by F. Hence at each point (x,t), we have

$$\frac{\partial u^i}{\partial x} = \frac{\partial H}{\partial p_i} = g^{ij} p_j \quad \text{and similarly} \quad \frac{\partial u^i}{\partial t} = \frac{\partial F}{\partial p_i} = A^i_\alpha g^{\alpha j} p_j,$$

which immediately implies $\frac{\partial u^i}{\partial t} = A^i_\alpha \frac{\partial u^\alpha}{\partial x}$ or, shortly, $u_t = A(u)u_x$, as required. □

This proposition can be naturally generalised to the case when g admits several *commuting* Killing tensors. Indeed, consider a (pseudo)-Riemannian metric g and g-self-adjoint operators $A_0 = \mathrm{Id}, A_1, \dots, A_{n-1}$ such that the quadratic functions $F_i(x,p) = \frac{1}{2} g^{-1}(A_i^* p, p)$, $i = 0, 1, \dots, n-1$, pairwise commute on $T^*\mathsf{M}$ w.r.t. the canonical Poisson structure (in particular, each of A_i is a Killing $(1,1)$-tensor for g). These functions generate a Hamiltonian $\mathbb{R}^n$-action on $T^*\mathsf{M}$:

$$\Phi^{(x,t_1,\dots,t_k)} : T^*\mathsf{M} \to T^*\mathsf{M}, \quad \Phi^{(x,t_1,\dots,t_k)} = \Phi^x_{F_0} \circ \Phi^{t_1}_{F_1} \circ \dots \circ \Phi^{t_k}_{F_k},$$

where $\Phi^t_{F_i}$ denotes the Hamiltonian flow generated by F_i.

Corollary 5.1 *In the above setting, let*

$$\big(u(x,t_1,\dots,t_{n-1}), p(x,t_1,\dots,t_{n-1})\big) = \Phi^{(x,t_1,\dots,t_{n-1})}(u_0, p_0)$$

be an orbit of this action. Then, $u(x, t_1, \dots, t_{n-1})$ is a solution of the system of quasi-linear equations

$$\begin{aligned} u_{t_1} &= A_1(u)u_x, \\ u_{t_2} &= A_2(u)u_x, \\ &\dots \\ u_{t_{n-1}} &= A_{n-1}(u)u_x. \end{aligned} \tag{26}$$

The statement of Theorem 1.4 follows from this Corollary and the fact that the operators A_i from system (8) are commuting Killing $(1, 1)$-tensors for any metric g that is geodesically compatible with L (Bolsinov and Matveev 2003, Corollary 1). Indeed, if $\gamma(x) = (u^1(x), \dots, u^n(x))$ is a geodesic of g, then the curve $(u(x), p(x))$ with $p_i(x) = g_{ij}(u)\dot{u}^j(x)$ is an orbit of the Hamiltonian flow $\Phi^x_{F_0}$ on $T^*\mathsf{M}$ generated by $F_0 = H = \frac{1}{2}g^{ij}(u)p_i p_j$. This $\mathbb{R}^1$-orbit can be naturally included into an $\mathbb{R}^n$-orbit of the action $\Phi^{(x,t_1,\dots,t_{n-1})}$ so that $\big(u(x, 0, \dots, 0), p(x, 0, \dots, 0)\big) = (u(x), p(x))$.

Now for any fixed $t_1, \dots, t_{n-1}$, the curve $\big(u(x, t_1, \dots, t_{n-1}), p(x, t_1, \dots, t_{n-1})\big)$ parametrised by x is still an orbit of the Hamiltonian flow $\Phi^x_{F_0}$. Since $F_0 = H$ is the Hamiltonian of the geodesic flow of g, the curve $u(x, t_1, \dots, t_{n-1})$ is a g-geodesic (for fixed $t_1, \dots, t_{n-1}$). On the other hand, by Corollary 5.1, $u(x, t_1, \dots, t_{n-1})$ is the solution of 1.4 with the initial condition $u(x, 0, \dots, 0) = \gamma(x)$. This completes the proof of Theorem 1.4.

Thus, the evolutionary flows $u_{t_i} = A_i(u)u_x$ naturally act on the space of g-geodesics. Since every geodesic is uniquely defined by its initial condition $(u(0), p(0)) \in T^*\mathsf{M}$, we can naturally identify the space $\mathfrak{G}$ of all (parametrised) geodesics with the cotangent bundle $T^*\mathsf{M}$ by setting

$$\gamma \in \mathfrak{G} \mapsto (u(0), p(0)) \in T^*\mathsf{M}, \tag{27}$$

where $u(0) = \gamma(0)$ and $p_i(0) = g_{ij}(u(0))\dot{u}^j$.

To prove Theorem 1.5, it remains to compare the action $\Psi^{t_0,\dots,t_{n-1}}$ on $\mathfrak{G}$ and the Hamiltonian action $\Phi^{t_0,\dots,t_{n-1}}$ on $T^*\mathsf{M}$. We have

$$\Psi^{t_0,\dots,t_{n-1}}(\gamma(x)) = u(x + t_0, t_1, \dots, t_{n-1}) \quad \text{and} \quad \Phi^{t_0,\dots,t_{n-1}}(u(0), p(0)) = (u(t_0, \dots, t_{n-1}), p(t_0, \dots, t_{n-1})).$$

Taking into account that $p_i(t_0, \dots, t_{n-1}) = g_{ij} \frac{\mathrm{d}}{\mathrm{d}x}|_{x=0} u^j(t_0 + x, t_1, \dots, t_{n-1})$, we see that the map (27) indeed conjugates the actions Ψ and Φ. This completes the proof of Theorem 1.5.

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