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On the Motion of a Nearly Incompressible Viscous Fluid Containing a Small Rigid Body

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Abstract

We consider the motion of a compressible viscous fluid containing a moving rigid body confined to a planar domain $\Omega \subset R^2$. The main result states that the influence of the body on the fluid is negligible if (i) the diameter of the body is small and (ii) the fluid is nearly incompressible (the low Mach number regime). The specific shape of the body as well as the boundary conditions on the fluid–body interface are irrelevant and collisions with the boundary $\partial\Omega$ are allowed. The rigid body motion may be enforced externally or governed solely by its interaction with the fluid.

Keywords Fluid–structure interaction · Compressible fluid · Small body motion · Low Mach number limit

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1 Introduction

There is a vast number of recent studies concerning the motion of a rigid body immersed in/or containing a compressible viscous fluid. We focus on the situation when the body is "small"; therefore, its influence on the fluid motion is expected to be negligible. By *small*, we mean that the body is contained in a ball with a small radius. The problem is mathematically more challenging in the case of planar ($2d$) flows, where even small objects may have large capacity.

The motion of a small object immersed in an inviscid (Euler) incompressible fluid is studied by Iftimie et al. (2003). Similar problems again in the framework of inviscid fluids have been considered by Glass et al. (2014), Glass et al. (2016). The asymptotic behavior of solutions of the incompressible Euler equations in the exterior of a single smooth obstacle when the obstacle becomes very thin tending to curve has been studied by Lacave (2009a).

In the context of viscous Newtonian fluids, the flow around a small rigid obstacle was studied by Iftimie et al. (2006). Lacave (2009b) studies the limit of a viscous fluid flow in the exterior of a thin obstacle shrinking to a curve. In the article (Feireisl et al. (2023)), we have established that the fluid flow is not influenced by the presence of the infinitely many bodies in the asymptotic limit.

Finally, let us mention results in planar domains, where the body does not influence the flow in the asymptotic limit. In Chipot et al. (2020), the authors considered two-dimensional "punctured periodic domain" with the periodic boundary conditions on the boundary of the domain and examine the behavior of solutions as the radius of the obstacle goes to zero. Lacave and Takahashi (2017) consider a single disk moving under the influence of a viscous fluid. They proved convergence toward the Navier–Stokes equations as the size of the solid tends to zero, its density is constant and the initial data small. Finally, He and Iftimie (2019) extend the above result to a general shape of the body and to the initial velocities not necessarily small.

To the best of our knowledge, the problem of negligibility of a small rigid body immersed in a planar viscous *compressible* fluid is completely open. Bravin and Nečasová (2023) addressed the problem in the $3d$ setting, where the capacity of the object in a suitable Sobolev norm is small enough but they need the restriction of adiabatic exponent $\gamma \geq 6$. Recently, in Feireisl et al. (2023), we can handle physically realistic adiabatic coefficient $\gamma > 3/2$ by proposing a new test function that need not vanish on the moving body, but only satisfy the rigid body motion constraint.

1.1 Problem Formulation

Neglecting completely the possible thermal effects as well as the external body forces, we consider the isentropic compressible fluid in the low Mach number regime governed by the following system of equations:

$$\partial_t \varrho + \mathrm{div}_x (\varrho \boldsymbol{u}) = 0, \tag{1.1}$$

$$\partial_t (\varrho \boldsymbol{u}) + \mathrm{div}_x (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{\varepsilon^{2m}} \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u}), \tag{1.2}$$

where the stress tensor is given by:

$$\mathbb{S}(\nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + \nabla_x^t \boldsymbol{u} - \mathrm{div}_x \boldsymbol{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \boldsymbol{u} \mathbb{I}, \ \mu > 0, \ \lambda \geq 0, \tag{1.3}$$

and the pressure

$$p = p(\varrho) = a\varrho^\gamma, \ \gamma > 1, \ a > 0. \tag{1.4}$$

The fluid is confined to a bounded planar domain $\Omega \subset R^2$ and the momentum equation (1.2) satisfied in

$$\Omega_{\varepsilon,t} = \Omega \setminus B_{\varepsilon,t}, \ t \in (0,T), \tag{1.5}$$

where

$$B_{\varepsilon,t} = \left\{ x \in R^2 \,\middle|\, |x - \boldsymbol{h}_\varepsilon(t)| \leq \varepsilon \right\}, \tag{1.6}$$

$$\boldsymbol{h}_\varepsilon \in W^{1,\infty}([0,T]; R^2), \ \varepsilon |\boldsymbol{h}'_\varepsilon(t)| \to 0 \text{ uniformly for a.a. } t \in (0,T) \text{ as } \varepsilon \to 0. \tag{1.7}$$

The ball $B_{\varepsilon,t}$ is the part of the plane containing the rigid object at the time t. Note carefully that, in general, we do not require $B_{\varepsilon,t} \subset \Omega$. Finally, we impose the no-slip boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0. \tag{1.8}$$

1.2 Main Results

Below, we formulate the main hypotheses imposed on the fluid motion. It is convenient to consider the density $\varrho = \varrho_\varepsilon$ as well as the velocity $\boldsymbol{u} = \boldsymbol{u}_\varepsilon$ to be defined on the whole physical space $(0,T) \times R^2$. Accordingly, we set

$$\begin{aligned} &\varrho = \varrho_\varepsilon(t,x) = \overline{\varrho} \ - \text{a positive constant whenever } x \in R^2 \setminus \Omega, \\ &\boldsymbol{u} = \boldsymbol{u}_\varepsilon(t,x) = 0 \text{ if } x \in R^2 \setminus \Omega. \end{aligned} \tag{1.9}$$

Throughout the whole text, we assume the following:

(H1)

$$\boldsymbol{h}_\varepsilon \in W^{1,\infty}([0,T]; R^2); \tag{1.10}$$

(H2) $(\varrho_\varepsilon, \boldsymbol{u}_\varepsilon)$, $\varrho_\varepsilon \geq 0$ is a weak renormalized solution of the equation of continuity (1.1), meaning

$$\int_0^T \int_{R^2} \left[\varrho_\varepsilon \partial_t \varphi + \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi \right] \mathrm{d}x \, \mathrm{d}t = -\int_{R^2} \varrho_{0,\varepsilon} \varphi(0,\cdot) \ \mathrm{d}x,$$

$$\int_0^T \int_{R^2} \left[b(\varrho_\varepsilon) \partial_t \varphi + b(\varrho_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi + \left(b(\varrho_\varepsilon) - b'(\varrho_\varepsilon)\varrho_\varepsilon \right) \mathrm{div}_x \boldsymbol{u}_\varepsilon \varphi \right] \mathrm{d}x \, \mathrm{d}t = -\int_{R^2} b(\varrho_{\varepsilon,0}) \varphi(0,\cdot) \ \mathrm{d}x, \tag{1.11}$$

for any $\varphi \in C_c^1([0,T)\times R^2)$ and any $b \in C^1[0,\infty)$, $b' \in C_c[0,\infty)$;
(H3) $(\varrho_\varepsilon, \boldsymbol{u}_\varepsilon)$ is a weak solution of the momentum equation (1.2) in the fluid domain $\cup_{t\in(0,T)}\Omega_{\varepsilon,t}$, meaning

$$\boldsymbol{u}_\varepsilon \in L^2(0,T; W_0^{1,2}(\Omega; R^2)), \tag{1.12}$$

and

$$\begin{aligned}\int_0^T \int_\Omega \Big[\varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} + \frac{1}{\varepsilon^{2m}} p(\varrho_\varepsilon)\mathrm{div}_x \boldsymbol{\varphi}\Big]\, \mathrm{d}x\, \mathrm{d}t \\ = \int_0^T \int_\Omega \mathbb{S}(\nabla_x \boldsymbol{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\, \mathrm{d}t - \int_\Omega \varrho_{\varepsilon,0}\boldsymbol{u}_{\varepsilon,0}\cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x\end{aligned} \tag{1.13}$$

for any $\boldsymbol{\varphi} \in C_c^1(\cup_{0\le t<T}\Omega_{\varepsilon,t}; R^2) \cap C_c^1([0,T)\times\Omega; R^2)$;
(H4) The energy inequality

$$\begin{aligned}&\int_\Omega \frac{1}{2}\varrho_\varepsilon |\boldsymbol{u}_\varepsilon|^2(\tau,\cdot)\,\mathrm{d}x + \frac{1}{\varepsilon^{2m}}\int_{\Omega_{\varepsilon,\tau}} \Big(P(\varrho_\varepsilon) - P'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - P(\overline{\varrho})\Big)(\tau,\cdot)\,\mathrm{d}x \\ &\quad + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \boldsymbol{u}_\varepsilon) : \nabla_x \boldsymbol{u}_\varepsilon\ \mathrm{d}x\,\mathrm{d}t \\ &\le \int_\Omega \frac{1}{2}\varrho_{\varepsilon,0}|\boldsymbol{u}_{\varepsilon,0}|^2\,\mathrm{d}x + \frac{1}{\varepsilon^{2m}}\int_{\Omega_{\mathcal{F},\varepsilon,0}} \Big(P(\varrho_{\varepsilon,0}) - P'(\overline{\varrho})(\varrho_{\varepsilon,0} - \overline{\varrho}) - P(\overline{\varrho})\Big)\,\mathrm{d}x\end{aligned} \tag{1.14}$$

holds for a.a. $\tau \in (0,T)$, for a certain constant $\overline{\varrho}$. The pressure potential $P(\varrho)$ is defined as

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho), \tag{1.15}$$

whence

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma. \tag{1.16}$$

In (1.14), $\Omega_{\varepsilon,0} \subset \Omega_{\mathcal{F},\varepsilon,0}$ and $\Omega_{\mathcal{F},\varepsilon,0}$ is the fluid domain at the initial time, meaning

$$\Omega_{\mathcal{F},\varepsilon,0} = \Omega \setminus \mathcal{S}_0,\ \mathcal{S}_0 \text{ is the initial position of the rigid body and } \mathcal{S}_0 \subset B_{\varepsilon,0}.$$

Remark 1.1 Let us mention that the specific form of the energy inequality (1.14) follows from Feireisl (2003, Lemma 3.2) and (5.18).

Remark 1.2 Observe that from (1.15), we have

$$P''(\varrho) = \frac{p'(\varrho)}{\varrho} \text{ for } \varrho > 0.$$

Thus, the function P is strictly convex, and consequently by using (1.16), we obtain

$$(P(\varrho_\varepsilon) - P'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho})) - P(\overline{\varrho}) \geq c(\overline{\varrho}) \begin{cases} (\varrho_\varepsilon - \overline{\varrho})^2 \text{ if } \varrho_\varepsilon \in (\frac{\overline{\varrho}}{2}, \overline{\varrho}), \\ 1 + \varrho_\varepsilon^\gamma \text{ otherwise }. \end{cases} \tag{1.17}$$

In particular, the second integral in (1.14) is non-negative.

Our main result reads as follows:

Theorem 1.3 *Let $\Omega \subset R^2$ be a bounded domain of class C^3. Let $(\varrho_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ satisfy the hypotheses* (H1)–(H4). *In addition, suppose*

$$\varrho_{\varepsilon,0} \geq 0 \text{ a.e. in } \Omega, \quad \frac{1}{\varepsilon^{2m}} \int_{\Omega_{\mathcal{F},\varepsilon,0}} \Big(P(\varrho_{\varepsilon,0}) - P'(\overline{\varrho})(\varrho_{\varepsilon,0} - \overline{\varrho}) - P(\overline{\varrho}) \Big) \, \mathrm{d}x \to 0, \tag{1.18}$$

where

$$\min \left\{ m; \frac{2m}{\gamma} \right\} > 3. \tag{1.19}$$

$$\boldsymbol{u}_{\varepsilon,0} \to \boldsymbol{u}_0 \text{ weakly in } L^2(\Omega; R^2), \int_\Omega \varrho_{\varepsilon,0} |\boldsymbol{u}_{\varepsilon,0}|^2 \, \mathrm{d}x \to \int_\Omega \overline{\varrho} |\boldsymbol{u}_0|^2 \, \mathrm{d}x \text{ as } \varepsilon \to 0,$$
$$\text{where } \boldsymbol{u}_0 \in W^{2,\infty}(\Omega), \ \mathrm{div}_x \boldsymbol{u}_0 = 0, \ \boldsymbol{u}_0|_{\partial\Omega} = 0; \tag{1.20}$$

$$\varepsilon |\boldsymbol{h}_\varepsilon'(t)| \to 0 \text{ uniformly for a.a. } t \in (0,T) \text{ as } \varepsilon \to 0. \tag{1.21}$$

Then,

$$\sup_{\tau \in [0,T]} \|\varrho_\varepsilon(\tau, \cdot) - \overline{\varrho}\|_{L^\gamma(\Omega_{\varepsilon,\tau})} \to 0 \text{ with } \gamma \text{ as in (1.4)}, \tag{1.22}$$

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ in } L^2(0,T; W_0^{1,2}(\Omega; R^2)) \tag{1.23}$$

as $\varepsilon \to 0$, where $\boldsymbol{u}$ is the (unique) classical solution of the incompressible Navier–Stokes system

$$\begin{aligned} \mathrm{div}_x \boldsymbol{u} &= 0, \\ \overline{\varrho} \partial_t \boldsymbol{u} + \overline{\varrho} \mathrm{div}_x (\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x \Pi &= \mu \Delta_x \boldsymbol{u}, \\ \boldsymbol{u}|_{\partial\Omega} &= 0, \\ \boldsymbol{u}(0, \cdot) &= \boldsymbol{u}_0 \end{aligned} \tag{1.24}$$

in $(0,T) \times \Omega$.

Remark 1.4 We want to point out that as observed by He and Iftimie (2021), assumption (1.21) holds for the fluid–structure interaction problem if the condition (5.21) satisfies. Observe that the condition (5.21) implies $\inf \varrho_\varepsilon^{\mathcal{S}} \to \infty$, where $\varrho_\varepsilon^{\mathcal{S}}$ is the density of the rigid body immersed in the fluid.

Remark 1.5 The hypotheses (1.18), (1.20) correspond to the *well-prepared* data in the low Mach number limit, cf. Masmoudi (2000). Moreover, as $\boldsymbol{u}_0$ belongs to the class (1.20), the standard maximal regularity theory yields a strong solution of the Navier–Stokes system (1.24), unique in the class

$$\begin{aligned} &\boldsymbol{u} \in L^p(0,T; W^{2,p}(\Omega; R^2)),\ \partial_t \boldsymbol{u} \in L^p(0,T; L^p(\Omega; R^2)), \\ &\nabla_x \Pi \in L^p(0,T; L^p(\Omega; R^2)), \qquad 1 \le p < \infty \end{aligned} \tag{1.25}$$

see e.g., Gerhardt (1978), von Wahl (1977). The solution is classical in $(0,T)\times\Omega$ as a consequence of the interior regularity estimates.

The hypotheses of Theorem 1.3 are satisfied if $(\varrho_\varepsilon, \boldsymbol{u}_\varepsilon)$ is a weak solution of the fluid–structure interaction problem of a single rigid body immersed in a viscous compressible fluid in the sense of Feireisl (2003) (see also Desjardins and Esteban 2000) or if the motion of the body is prescribed as in Feireisl et al. (2013). A detailed proof is given in Appendix 5.

The remaining part of the paper is devoted to the proof of Theorem 1.3. Similarly to the purely incompressible setting studied by He and Iftimie (2021) (cf. Lacave and Takahashi 2017), the main problem is the rather weak estimate (1.21) that does not allow for a precise identification of the limit trajectory of the body. In addition, two new difficulties appear in the compressible regime:

- Possible fast oscillations of acoustic (gradient) component of the velocity that cannot be *a priori* excluded even for the well-prepared data because of the influence of the rigid body.
- Possible contacts of the body—intersection of the balls $B_{\varepsilon,t}$—with the outer boundary $\partial\Omega$.

To overcome the above-mentioned difficulties, we proceed as follows. In Sects. 2, 3, we identify the system of equations satisfied by the limit velocity $\boldsymbol{u}$. Due to the lack of information on $\partial_t \boldsymbol{u}_\varepsilon$, the limit of the convective term as well as the kinetic energy is described in terms of the corresponding Young measure. The limit $\boldsymbol{u}$ is therefore a generalized dissipative solution of the incompressible Navier–Stokes system in the sense of Abbatiello and Feireisl (2020). In particular, we adapt the approximation of the test functions introduced by He and Iftimie to the geometry of a bounded domain. Finally, in Sect. 4, apply the weak–strong uniqueness result proved in Abbatiello and Feireisl (2020) to conclude that the limit is, in fact, a strong solution of the Navier–Stokes system, whereas the associated Young measure reduces to a parametrized family of Dirac masses.

2 Identifying the Limit, the Equation of Continuity, Energy Balance

It follows from the hypotheses (1.18), (1.20) that the initial energy on the right-hand side of the energy inequality (1.14) is bounded uniformly for $\varepsilon \to 0$. Applying Korn–Poincaré inequality, we get, up to a suitable subsequence,

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega; R^2)). \tag{2.1}$$

Next, ϱ_ε satisfies the renormalized equation of continuity (1.11). Moreover, the energy inequality (1.14), the estimate (1.17) give $\|\varrho_\varepsilon(\tau,\cdot) - \overline{\varrho}\|_{(L^\gamma + L^2)(\Omega_{\varepsilon,\tau})} \to 0$ as $\varepsilon \to 0$, we combine this observation with the fact that $\Omega_{\varepsilon,t} \to \Omega$ as $\varepsilon \to 0$ and these yield

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } (0,T)\times\Omega \text{ in measure as } \varepsilon \to 0.$$

In particular, we may perform the limit in (1.11) obtaining

$$b'(\overline{\varrho})\overline{\varrho}\mathrm{div}_x \boldsymbol{u} = 0,$$

yielding

$$\mathrm{div}_x \boldsymbol{u} = 0. \tag{2.2}$$

Finally, using the hypotheses (1.20), (1.21) and the property of weak lower semi-continuity of convex functionals, we perform the limit in the energy inequality obtaining

$$\int_\Omega \frac{1}{2}\overline{\varrho}|\boldsymbol{u}|^2(\tau,\cdot)\,\mathrm{d}x + \mathfrak{E}(\tau) + \mu\int_0^\tau\int_\Omega \nabla_x\boldsymbol{u} : \nabla_x\boldsymbol{u}\ \mathrm{d}x\,\mathrm{d}t \le \int_\Omega \frac{1}{2}\overline{\varrho}|\boldsymbol{u}_0|^2\ \mathrm{d}x \tag{2.3}$$

for a.a. $\tau \in (0,T)$. Here, $\mathfrak{E}(\tau) \in L^\infty(0,T)$ is the so-called total energy defect defined as

$$\mathfrak{E}(\tau) = \liminf_{\varepsilon\to 0}\int_\Omega \frac{1}{2}\varrho_\varepsilon|\boldsymbol{u}_\varepsilon|^2(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \frac{1}{2}\overline{\varrho}|\boldsymbol{u}|^2(\tau,\cdot)\ \mathrm{d}x \ge 0 \text{ for a.a. } \tau \in (0,T). \tag{2.4}$$

3 Identifying the Limit, the Momentum Equation

The next and more delicate step is to perform the limit $\varepsilon \to 0$ in the momentum equation (1.2). To eliminate the singular pressure term, we consider the test functions

$$\boldsymbol{\varphi}_\varepsilon \in C^1_c(\cup_{0\le t<T}\Omega_{\varepsilon,t}; R^2) \cap C^1_c([0,T)\times\Omega; R^2),\ \mathrm{div}_x\boldsymbol{\varphi}_\varepsilon = 0. \tag{3.1}$$

Accordingly, the weak formulation (1.13) gives rise to

$$\int_0^T\int_\Omega \left[\varrho_\varepsilon\boldsymbol{u}_\varepsilon\cdot\partial_t\boldsymbol{\varphi}_\varepsilon + \varrho_\varepsilon\boldsymbol{u}_\varepsilon\otimes\boldsymbol{u}_\varepsilon : \nabla_x\boldsymbol{\varphi}_\varepsilon\right]\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \mathbb{S}(\nabla_x\boldsymbol{u}_\varepsilon) : \nabla_x\boldsymbol{\varphi}_\varepsilon\ \mathrm{d}x\,\mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon}\boldsymbol{u}_{0,\varepsilon}\cdot\boldsymbol{\varphi}_\varepsilon(0,\cdot)\ \mathrm{d}x. \tag{3.2}$$

3.1 Some Useful Estimates

Note that (3.2) is relevant only on the fluid part $\cup_{t\in[0,T]}\Omega_{\varepsilon,t}$, where the energy inequality (1.14) yields uniform bounds on the density. This motivates the following decomposition of any measurable functions v:

$$v = [v]_{\rm ess} + [v]_{\rm res},$$

where

$$[\ v]_{\rm ess} = v \mathbb{1}_{\frac{1}{2}\overline{\varrho} \leq v \leq 2\overline{\varrho}}.$$

Thanks to the energy inequality (1.14), we get

$$[\varrho_\varepsilon]_{\rm ess} \boldsymbol{u}_\varepsilon \text{ bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^q(\Omega)) \text{ for any } 1 \leq q < \infty. \tag{3.3}$$

Moreover, by the energy inequality,

$$[\varrho_\varepsilon]_{\rm ess} \to \overline{\varrho} \text{ in measure in } ((0,T)\times\Omega), \tag{3.4}$$

whence we conclude

$$[\varrho_\varepsilon]_{\rm ess} \boldsymbol{u}_\varepsilon \to \overline{\varrho}\boldsymbol{u} \text{ weakly -(*) in } L^\infty(0,T;L^2(\Omega;R^2)),$$
$$\text{and weakly in } L^2(0,T;L^q(\Omega;R^2)) \text{ for any } 1 \leq q < \infty. \tag{3.5}$$

In addition, we also have

$$\varrho_\varepsilon \boldsymbol{u}_\varepsilon = (\varrho_\varepsilon - \overline{\varrho})\boldsymbol{u}_\varepsilon + \overline{\varrho}\boldsymbol{u}_\varepsilon,$$

where thanks to the energy inequality (1.14) and estimate (1.17),

$$\int_{\Omega_{\varepsilon,\tau}} |\varrho_\varepsilon - \overline{\varrho}||\boldsymbol{u}_\varepsilon| \ {\rm d}x \lesssim \|\varrho_\varepsilon(\tau,\cdot) - \overline{\varrho}\|_{(L^\gamma + L^2)(\Omega_{\varepsilon,\tau})} \|\boldsymbol{u}_\varepsilon\|_{W_0^{1,2}(\Omega;R^2)} \lesssim$$
$$\varepsilon^{\min\{m,\frac{2m}{\gamma}\}} \|\boldsymbol{u}_\varepsilon(\tau,\cdot)\|_{W_0^{1,2}(\Omega;R^2)} \tag{3.6}$$

for any $\tau \in [0,T]$.

Similarly,

$$[\varrho_\varepsilon]_{\rm ess} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon \text{ is bounded in } L^1(0,T;L^q(\Omega;R^{d\times d})) \cap L^\infty(0,T;L^1(\Omega;R^{d\times d}))$$
$$\text{for any } 1 \leq q < \infty; \tag{3.7}$$

whence, by interpolation,

$$[\varrho_\varepsilon]_{\rm ess} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon \to \overline{\varrho \boldsymbol{u} \otimes \boldsymbol{u}} \text{ weakly in } L^r((0,T;L^2(\Omega;R^2)) \text{ for some } r > 1. \tag{3.8}$$

The tensor $\overline{\varrho \boldsymbol{u} \otimes \boldsymbol{u}} \in R^{d\times d}_{\rm sym}$ is positively semi-definite and

$$\overline{\varrho \boldsymbol{u} \otimes \boldsymbol{u}} - \overline{\varrho}\boldsymbol{u} \otimes \boldsymbol{u} \geq 0. \tag{3.9}$$

Indeed, for any $\boldsymbol{d} \in R^d$:

$$\left[\overline{\varrho \boldsymbol{u} \otimes \boldsymbol{u}} - \overline{\varrho} \boldsymbol{u} \otimes \boldsymbol{u}\right] : (\boldsymbol{d} \otimes \boldsymbol{d}) = \lim_{\varepsilon \to 0} |\sqrt{[\varrho_\varepsilon]_{\rm ess}} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{d}|^2 - |\sqrt{\overline{\varrho}} \boldsymbol{u} \cdot \boldsymbol{d}|^2.$$

Thus, the desired conclusion (3.9) follows from (2.1), (3.4) and weak lower semi-continuity of convex functions. Finally, as

$$[\varrho_\varepsilon]_{\rm ess} |\boldsymbol{u}_\varepsilon|^2 \leq \varrho_\varepsilon |\boldsymbol{u}_\varepsilon|^2,$$

we get

$$0 \leq \int_\Omega {\rm trace}\left[\overline{\varrho \boldsymbol{u} \otimes \boldsymbol{u}} - \overline{\varrho} \boldsymbol{u} \otimes \boldsymbol{u}\right] \, {\rm d}x \leq 2\mathfrak{E}, \tag{3.10}$$

where $\mathfrak{E}$ is the total energy defect appearing on the left-hand side of the energy inequality (2.3).

As for the residual components, we deduce from the energy inequality

$$\int_{\Omega_{\varepsilon,\tau}} [\varrho_\varepsilon]^\gamma_{\rm res}(\tau, \cdot) \, {\rm d}x \lesssim \varepsilon^{2m}, \ 0 \leq \tau \leq T. \tag{3.11}$$

Consequently, by Hölder's inequality,

$$\int_{\Omega_{\varepsilon;\tau}} [\varrho_\varepsilon]_{\rm res} |\boldsymbol{u}_\varepsilon| \, {\rm d}x \lesssim \varepsilon^{\frac{2m}{\gamma}} \|\boldsymbol{u}_\varepsilon(\tau, \cdot)\|_{L^q(\Omega; R^d)}, \ \frac{1}{\gamma} + \frac{1}{q} = 1, \tag{3.12}$$

and, similarly,

$$\int_{\Omega_{\varepsilon;\tau}} [\varrho_\varepsilon]_{\rm res} |\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon| \, {\rm d}x \lesssim \varepsilon^{\frac{2m}{\gamma}} \|\boldsymbol{u}_\varepsilon(\tau, \cdot)\|^2_{L^q(\Omega; R^d)}, \ \frac{1}{\gamma} + \frac{2}{q} = 1 \tag{3.13}$$

for a.a. $\tau \in (0, T)$.

3.2 Constructing a Suitable Class of Test Functions

Our goal is to approximate a test function

$$\boldsymbol{\varphi} \in C^\infty_c([0, T] \times \Omega; R^2), \ {\rm div}_x \boldsymbol{\varphi} = 0,$$

by a suitable family of admissible test functions $(\boldsymbol{\varphi}_\varepsilon)_{\varepsilon > 0}$ in (3.2).

The test function are obtained following the construction of He and Iftimie (2019, 2021), specifically,

$$\widetilde{\boldsymbol{\varphi}}_\varepsilon = \nabla^\perp_x (\eta^\varepsilon (x - \boldsymbol{h}_\varepsilon(t)) \Psi_\varepsilon),$$

with the potential Ψ_ε,

$$\nabla^\perp_x \Psi_\varepsilon = \boldsymbol{\varphi} \text{ normalized as } \Psi_\varepsilon(t, \boldsymbol{h}_\varepsilon(t)) = 0.$$

The cut-off functions η_ε near the disk $D(\boldsymbol{h}_\varepsilon(t), \varepsilon)$ are smooth and satisfy the following properties (see He and Iftimie 2019, Lemma 3):

$$|\eta_\varepsilon| \leq 1,\, \eta_\varepsilon(y) = 0 \text{ if } |y| \leq \varepsilon,\, \eta_\varepsilon(y) = 1 \text{ if } |y| \geq \alpha(\varepsilon)\varepsilon, \tag{3.14}$$

$$|\nabla_x \eta_\varepsilon(y)| \lesssim \frac{1}{\varepsilon} \frac{1}{\log(\alpha(\varepsilon))},\ |\nabla_x^2 \eta_\varepsilon(y)| \lesssim \frac{1}{\varepsilon^2}. \tag{3.15}$$

where $\alpha(\varepsilon)$ is chosen in such a way that

$$\alpha(\varepsilon) \to \infty,\ \alpha(\varepsilon)\varepsilon(1 + |\boldsymbol{h}'_\varepsilon(t)|) \to 0 \text{ as } \varepsilon \to 0. \tag{3.16}$$

As shown in He and Iftimie (2019, Lemma 5), the functions $\widetilde{\boldsymbol{\varphi}}_\varepsilon$ enjoy the following properties:

$$\widetilde{\boldsymbol{\varphi}}_\varepsilon,\ \nabla_x \widetilde{\boldsymbol{\varphi}}_\varepsilon \in C_c(([0, T] \times R^d) \setminus \cup_{t\in[0,T]} B_{\varepsilon,t}),\ \partial_t \widetilde{\boldsymbol{\varphi}}_\varepsilon \in L^\infty((0, T) \times R^2; R^2), \tag{3.17}$$

$$\text{dist}[\boldsymbol{h}_\varepsilon(\tau); \partial\Omega] > \varepsilon\alpha(\varepsilon) \ \Rightarrow\ \widetilde{\boldsymbol{\varphi}}_\varepsilon(\tau, \cdot)|_{\partial\Omega} = 0, \tag{3.18}$$

$$\widetilde{\boldsymbol{\varphi}}_\varepsilon \to \boldsymbol{\varphi} \text{ strongly in } L^\infty(0, T; W^{1,2}(R^2; R^2)) \text{ as } \varepsilon \to 0. \tag{3.19}$$

Unfortunately, the functions $\widetilde{\boldsymbol{\varphi}}_\varepsilon$ do not vanish on $\partial\Omega$ unless $\text{dist}[\boldsymbol{h}(t); \partial\Omega] > \varepsilon\alpha(\varepsilon)$. To remedy this, we consider a convex combination

$$\boldsymbol{\varphi}_\varepsilon = \chi_\varepsilon(t)\widetilde{\boldsymbol{\varphi}}_\varepsilon + (1 - \chi_\varepsilon(t))\boldsymbol{\varphi} \text{ for suitable } 0 \leq \chi_\varepsilon(t) \leq 1,\ \chi_\varepsilon \in W^{1,\infty}(0, T).$$

First observe that similarly to $\boldsymbol{\varphi}_\varepsilon$,

$$\|\chi_\varepsilon(t)\widetilde{\boldsymbol{\varphi}}_\varepsilon + (1 - \chi_\varepsilon)\boldsymbol{\varphi}\|_{L^\infty(0,T;W^{1,2}(\Omega;R^2))} \lesssim 1,$$

and

$$\begin{aligned}\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\varphi} &= \Big(\chi_\varepsilon(t)\widetilde{\boldsymbol{\varphi}}_\varepsilon + (1 - \chi_\varepsilon)\boldsymbol{\varphi}\Big) \\ &\quad -\boldsymbol{\varphi} = \chi_\varepsilon(\widetilde{\boldsymbol{\varphi}}_\varepsilon - \boldsymbol{\varphi}) \to 0 \text{ in } L^\infty\Big(0, T; W^{1,2}(\Omega; R^2)\Big) \text{ as } \varepsilon \to 0.\end{aligned} \tag{3.20}$$

Next, we compute the approximation error in the time derivative

$$\partial_t\Big(\chi_\varepsilon(t)\widetilde{\boldsymbol{\varphi}}_\varepsilon + (1 - \chi_\varepsilon)\boldsymbol{\varphi}\Big) - \partial_t\boldsymbol{\varphi} = \chi_\varepsilon(t)(\partial_t\widetilde{\boldsymbol{\varphi}}_\varepsilon - \partial_t\boldsymbol{\varphi}) + \chi'_\varepsilon(t)(\widetilde{\boldsymbol{\varphi}}_\varepsilon - \boldsymbol{\varphi}),$$

where the former error term

$$\chi_\varepsilon(t)(\partial_t\widetilde{\boldsymbol{\varphi}}_\varepsilon - \partial_t\boldsymbol{\varphi})$$

can be controlled in $W^{-1,2}$ exactly as in He and Iftimie (2021) since χ is independent of x. As for the latter, we have

$$\chi_\varepsilon'(t)(\widetilde{\boldsymbol{\varphi}}_\varepsilon - \boldsymbol{\varphi}) = \chi_\varepsilon'(t)\nabla_x^\perp\Big([\eta_\varepsilon(x-\boldsymbol{h}(t))-1]\Psi_\varepsilon\Big)$$
$$= \nabla_x^\perp\Big[\chi_\varepsilon'(t)\Big([\eta_\varepsilon(x-\boldsymbol{h}(t))-1]\Psi_\varepsilon\Big)\Big],$$

where, in accordance with (3.14),

$$\|\chi_\varepsilon'(t)[\eta_\varepsilon(x-\boldsymbol{h}(t))-1]\Psi_\varepsilon\|_{L^2(\Omega)}^2 \lesssim |\chi_\varepsilon'(t)|^2\varepsilon^2\alpha^2(\varepsilon). \tag{3.21}$$

Thus if

$$|\chi_\varepsilon'(t)| \lesssim |\boldsymbol{h}_\varepsilon'(t)|, \tag{3.22}$$

the latter error vanishes in $W^{-1,2}$ for $\varepsilon \to 0$ as a consequence of (3.16).

For $\delta > 0$ fixed, let $\boldsymbol{\varphi} \in C^1([0,T) \times \Omega)$ be given such that

$$\boldsymbol{\varphi}(t,x) = 0 \text{ whenever } \mathrm{dist}[x,\partial\Omega] \le 2\delta. \tag{3.23}$$

Finally, we choose

$$\chi_\varepsilon(t) = H_\delta\Big(\mathrm{dist}[\boldsymbol{h}_\varepsilon(t);\partial\Omega]\Big),\ 0 \le H_\delta \le 1,\ H_\delta(z)$$
$$= 0 \text{ for } z \le \frac{\delta}{2},\ H_\delta(z) = 1 \text{ for } z \ge \delta,$$

where H_δ is a Lipschitz function. We claim that the test functions

$$\boldsymbol{\varphi}_\varepsilon = \chi_\varepsilon(t)\widetilde{\boldsymbol{\varphi}}_\varepsilon + (1-\chi_\varepsilon(t))\boldsymbol{\varphi}$$

vanish both on the boundary $\partial\Omega$ and on the balls $B_{\varepsilon,t}$, $t \in [0,T]$. First, by construction, the function

$$\chi_\varepsilon\widetilde{\boldsymbol{\varphi}}_\varepsilon$$

vanishes on $B_{\varepsilon,t}$ for any $t \in [0,T]$. Moreover, if $\chi_\varepsilon > 0$, then, in view of (3.16),

$$\mathrm{dist}[\boldsymbol{h}_\varepsilon(t),\partial\Omega] > \frac{\delta}{2} > \varepsilon\alpha(\varepsilon) \text{ for } \varepsilon \text{ small enough.}$$

It follows from (3.18) that $\chi_\varepsilon\boldsymbol{\varphi}_\varepsilon|_{\partial\Omega} = 0$.

Second, obviously $(1-\chi_\varepsilon)\boldsymbol{\varphi}|_{\partial\Omega} = 0$. Next, if $\chi_\varepsilon < 1$, we have $\mathrm{dist}[\boldsymbol{h}_\varepsilon(t);\partial\Omega] < \delta$. Thus, in view of (3.23), $(1-\chi_\varepsilon)\boldsymbol{\varphi}(t,\cdot)|_{B_{\varepsilon,t}} = 0$ as soon as $\varepsilon < \delta$.

3.3 Asymptotic Limit

The function $\boldsymbol{\varphi}_\varepsilon$ constructed in Sect. 3.2 represents a legitimate test function for the momentum balance (3.2). Our goal is to perform the limit $\varepsilon \to 0$.

Step 1: Viscous term. In view of hypothesis (1.20), (2.1), and (2.2), it follows from (3.20) that

$$\int_0^T \int_\Omega \mathbb{S}(\nabla_x \boldsymbol{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \, \mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon} \boldsymbol{u}_{0,\varepsilon} \cdot \boldsymbol{\varphi}_\varepsilon(0,\cdot) \ \mathrm{d}x \\ \to \mu \int_0^T \int_\Omega \nabla_x \boldsymbol{u} : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t - \int_\Omega \overline{\varrho} \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x \tag{3.24}$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \Omega; R^d)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

Step 2: Convective term. We can write

$$\int_0^T \int_\Omega \varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon : \nabla_x \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega [\varrho_\varepsilon]_{\mathrm{ess}} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon : \nabla_x \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \, \mathrm{d}t \\ + \int_0^T \int_\Omega [\varrho_\varepsilon]_{\mathrm{res}} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon : \nabla_x \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \, \mathrm{d}t$$

We use (3.8) to obtain

$$\int_0^T \int_\Omega [\varrho_\varepsilon]_{\mathrm{ess}} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon : \nabla_x \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \overline{\varrho} \boldsymbol{u} \otimes \boldsymbol{u} : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t \\ + \int_0^T \int_\Omega \left(\overline{\varrho \boldsymbol{u} \otimes \boldsymbol{u}} - \overline{\varrho} \boldsymbol{u} \otimes \boldsymbol{u} \right) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t. \tag{3.25}$$

Step 3: Time derivative. Using the same arguments as in He and Iftimie (2021) combined with (3.21), we get

$$\int_\Omega \overline{\varrho} \boldsymbol{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \lesssim \|\boldsymbol{u}_\varepsilon\|_{W_0^{1,2}(\Omega; R^2)} \|\partial_t \boldsymbol{\varphi}_\varepsilon\|_{W^{-1,2}(\Omega; R^2)} \to 0 \text{ in } L^2(0,T). \tag{3.26}$$

Step 4: Remaining terms. The final step is to show

$$\int_0^T \int_{\Omega_{\varepsilon,t}} (\varrho_\varepsilon - \overline{\varrho}) \boldsymbol{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \, \mathrm{d}t \to 0, \\ \int_0^T \int_{\Omega_{\varepsilon,t}} [\varrho_\varepsilon]_{\mathrm{res}} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon : \nabla_x \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \, \mathrm{d}t \to 0. \tag{3.27}$$

A direct manipulation reveals

$$\|\nabla_x \boldsymbol{\varphi}_\varepsilon\|_{L^\infty((0,T)\times\Omega; R^{2\times 2})} \lesssim \|\nabla^2 \eta_\varepsilon\|_{L^\infty(R^2)} + 1,$$

$$\|\partial_t \boldsymbol{\varphi}_\varepsilon\|_{L^\infty((0,T)\times\Omega;R^{2\times 2})} \lesssim (1 + |h'_\varepsilon(t)|)(\|\nabla^2 \eta_\varepsilon\|_{L^\infty(R^2)} + 1). \tag{3.28}$$

Consequently, in view of (3.15) and (3.6), (3.13), the desired conclusion (3.27) follows as soon as

$$\min\left\{m; \frac{2m}{\gamma}\right\} > 3. \tag{3.29}$$

4 Proof of the Main Result

Summarizing the results obtained in the preceding section, we may infer that limit velocity

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega;R^2)) \cap L^2(0,T;W_0^{1,2}(\Omega;R^2))$$

solves the following problem:

$$\begin{aligned} \mathrm{div}_x \boldsymbol{u} = 0,\ \boldsymbol{u}|_{\partial\Omega} = 0; \\ \int_0^T \int_\Omega \Big[\overline{\varrho}\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \overline{\varrho}\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t =& \mu \int_0^T\int_\Omega \nabla_x\boldsymbol{u} : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t \\ &- \int_\Omega \overline{\varrho}\boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x \\ &- \int_0^T\int_\Omega \mathfrak{R} : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.1}$$

for any $\boldsymbol{\varphi} \in C_c^1([0,T)\times\Omega)$;

$$\int_\Omega \frac{1}{2}\overline{\varrho}|\boldsymbol{u}|^2(\tau,\cdot)\,\mathrm{d}x + \mathfrak{E}(\tau) + \mu\int_0^\tau\int_\Omega |\nabla_x\boldsymbol{u}|\ \mathrm{d}x\,\mathrm{d}t \le \int_\Omega \frac{1}{2}\overline{\varrho}|\boldsymbol{u}_0|^2\ \mathrm{d}x \tag{4.2}$$

for a.a. $\tau \in (0,T)$. Here, the tensor $\mathfrak{R} = \overline{\varrho\boldsymbol{u}\otimes\boldsymbol{u}} - \overline{\varrho}\boldsymbol{u}\otimes\boldsymbol{u}$ is positively semi-definite and satisfies (3.10), specifically

$$0 \le \int_\Omega \mathrm{trace}[\mathfrak{R}]\ \mathrm{d}x \le 2\mathfrak{E} \text{ for a.a. } \tau \in (0,T). \tag{4.3}$$

Consequently, the limit function $\boldsymbol{u}$ is a dissipative solution of the Navier–Stokes system (1.24) in the sense of Abbatiello and Feireisl (2020). As the initial velocity is regular, the same problem admits a strong solution in the class (1.25). Thus, applying the weak–strong uniqueness result (Abbatiello and Feireisl 2020, Theorem 2.6. and Remark 2.5), we conclude that $\boldsymbol{u}$ coincides with the strong solution of (1.24).

Finally, as the strong solution satisfies the energy equality, its follows from (4.2) that $\mathfrak{E} = 0$, and

$$\int_0^T\int_\Omega \mathbb{S}(\nabla_x\boldsymbol{u}_\varepsilon) : \nabla_x\boldsymbol{u}_\varepsilon\ \mathrm{d}x\,\mathrm{d}t \to \mu\int_0^T\int_\Omega |\nabla_x\boldsymbol{u}|^2\ \mathrm{d}x$$

yielding the strong convergence claimed in (1.23).

Theorem 1.3 is proved.

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Declarations

Conflict of interest The authors declare that there are no conflicts of interest.

Appendix

Our main result (Theorem 1.3) is valid whenever $(\varrho_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ satisfy the hypotheses (H1) – (H4) along with the conditions (1.18)–(1.23). These hypotheses (see (1.10)–(1.14)) are satisfied if $(\varrho_\varepsilon, \boldsymbol{u}_\varepsilon)$ is a weak solution of the fluid–structure interaction problem of a single rigid body immersed in a viscous compressible fluid in the sense of Feireisl (2003) (see also Desjardins and Esteban 2000) or if the motion of the body is prescribed as in Feireisl et al. (2013). Let the rigid body $\mathcal{S}_\varepsilon(t)$ be a regular, bounded domain and moving inside $\Omega \subset R^2$. The motion of the rigid body is governed by the balance equations for linear and angular momentum. We assume that the fluid domain $\mathcal{F}_\varepsilon(t) = \Omega \backslash \overline{\mathcal{S}_\varepsilon(t)}$ is filled with a viscous isentropic compressible fluid. Initially, the domain of the rigid body is given by $\mathcal{S}_{\varepsilon,0}$ included in the ball $B_{\varepsilon,0}$, and $\mathcal{F}_{\varepsilon,0}$ is the domain of the fluid. Let h_ε be the position of the center of mass and β_ε be the angle of rotation of the rigid body. The solid domain at time t is given by

$$\mathcal{S}_\varepsilon(t) = h_\varepsilon(t) + \mathcal{R}_{\beta_\varepsilon}(t)\mathcal{S}_{\varepsilon,0},$$

where $\mathcal{R}_{\beta_\varepsilon}$ is the rotation matrix, defined by

$$\mathcal{R}_{\beta_\varepsilon} = \begin{pmatrix} \cos\beta_\varepsilon & -\sin\beta_\varepsilon \\ \sin\beta_\varepsilon & \cos\beta_\varepsilon \end{pmatrix}.$$

The evolution of this fluid–structure system can be described by the following equations:

$$\frac{\partial {\varrho_\varepsilon}^{\mathcal{F}}}{\partial t} + \operatorname{div}\left({\varrho_\varepsilon}^{\mathcal{F}} {\boldsymbol{u}_\varepsilon}^{\mathcal{F}}\right) = 0, \quad t \in (0,T), \ x \in \mathcal{F}_\varepsilon(t), \tag{5.1}$$

$$\frac{\partial}{\partial t}\left(\varrho_\varepsilon{}^{\mathcal{F}}\boldsymbol{u}_\varepsilon{}^{\mathcal{F}}\right)+\operatorname{div}\left(\rho_\varepsilon^{\mathcal{F}}\boldsymbol{u}_\varepsilon{}^{\mathcal{F}}\otimes\boldsymbol{u}_\varepsilon{}^{\mathcal{F}}\right)-\operatorname{div}\mathbb{S}\left(\nabla_x\boldsymbol{u}_\varepsilon{}^{\mathcal{F}}\right)+\frac{1}{\varepsilon^{2m}}\nabla p^{\mathcal{F}}=0,$$
$$t\in(0,T),\ x\in\mathcal{F}_\varepsilon(t), \tag{5.2}$$

$$m_\varepsilon h_\varepsilon''(t)=-\int_{\partial\mathcal{S}_\varepsilon(t)}\left(\mathbb{S}(\nabla_x\boldsymbol{u}_\varepsilon{}^{\mathcal{F}})-\frac{1}{\varepsilon^{2m}}p_\varepsilon^{\mathcal{F}}\mathbb{I}\right)\nu_\varepsilon\,d\Gamma,\quad \text{in } (0,T), \tag{5.3}$$

$$J_\varepsilon\beta_\varepsilon''(t)=-\int_{\partial\mathcal{S}_\varepsilon(t)}\left(\mathbb{S}(\nabla_x\boldsymbol{u}_\varepsilon{}^{\mathcal{F}})-\frac{1}{\varepsilon^{2m}}p_\varepsilon^{\mathcal{F}}\mathbb{I}\right)\nu_\varepsilon\cdot(x-h_\varepsilon(t))^\perp\,d\Gamma,\quad \text{in } (0,T), \tag{5.4}$$

the boundary conditions

$$\boldsymbol{u}_\varepsilon{}^{\mathcal{F}}=h_\varepsilon'(t)+\beta_\varepsilon'(t)(x-h_\varepsilon(t))^\perp,\qquad \text{for } t\in(0,T),\ x\in\partial\mathcal{S}_\varepsilon(t), \tag{5.5}$$
$$\boldsymbol{u}_\varepsilon{}^{\mathcal{F}}=0,\qquad \text{on } (t,x)\in(0,T)\times\partial\Omega, \tag{5.6}$$

and the initial conditions

$$\varrho_\varepsilon{}^{\mathcal{F}}(0,x)=\varrho_{\mathcal{F}_0}(x),\quad (\varrho_\varepsilon{}^{\mathcal{F}}\boldsymbol{u}_\varepsilon{}^{\mathcal{F}})(0,x)=q_{\mathcal{F}_0}(x),\quad \forall\, x\in\mathcal{F}_{\varepsilon,0}, \tag{5.7}$$
$$h_\varepsilon(0)=0,\quad h_\varepsilon'(0)=\ell_0,\quad \beta_\varepsilon(0)=0,\quad \beta_\varepsilon'(0)=\omega_0. \tag{5.8}$$

In the above, the outward unit normal to $\partial\mathcal{F}_\varepsilon(t)$ is denoted by $\nu_\varepsilon(t,x)$. For all $x=(x_1,x_2)\in R^2$, we denote by $x^\perp$, the vector $(-x_2,x_1)$. Moreover, the constants m_ε and J_ε are the mass and the moment of inertia of the rigid body.

We want to state the existence result of the fluid-rigid body interaction system (5.1)–(5.8). To do so, we extend the density and the velocity in the following way:

$$\varrho_\varepsilon(t,x)=\begin{cases}\varrho_\varepsilon{}^{\mathcal{F}}(t,x), & x\in\mathcal{F}_\varepsilon(t),\\ \varrho_\varepsilon{}^{\mathcal{S}}(t,x), & x\in\mathcal{S}_\varepsilon(t),\\ \overline{\varrho}, & x\in R^2\setminus\Omega,\end{cases}$$
$$\boldsymbol{u}_\varepsilon(t,x)=\begin{cases}\boldsymbol{u}_\varepsilon{}^{\mathcal{F}}(t,x), & x\in\mathcal{F}_\varepsilon(t),\\ h_\varepsilon'(t)+\beta_\varepsilon'(t)(x-h_\varepsilon(t))^\perp, & x\in\mathcal{S}_\varepsilon(t),\\ 0, & x\in R^2\setminus\Omega.\end{cases} \tag{5.9}$$

$$\varrho_{\varepsilon,0}(x)=\begin{cases}\varrho_{\mathcal{F}_0}(x), & x\in\mathcal{F}_{\varepsilon,0},\\ \varrho_\varepsilon^{\mathcal{S}}(0,x), & x\in\mathcal{S}_{\varepsilon,0},\\ \overline{\varrho}, & x\in R^2\setminus\Omega,\end{cases}$$
$$q_{\varepsilon,0}(x)=\begin{cases}q_{\mathcal{F}_0}, & x\in\mathcal{F}_{\varepsilon,0},\\ \varrho_\varepsilon^{\mathcal{S}}(0,x)(\ell_0+\omega_0x^\perp), & x\in\mathcal{S}_{\varepsilon,0},\\ 0, & x\in R^2\setminus\Omega.\end{cases} \tag{5.10}$$

We have the following existence result for system (5.1)–(5.8) following (Feireisl 2003, Theorem 4.1):

Theorem 4.1 *Let $\Omega \subset R^2$ be a bounded domain, and the pressure $p^{\mathcal{F}}$ be given by the isentropic constitutive law*

$$p^{\mathcal{F}} = p(\varrho^{\mathcal{F}}) = a(\varrho^{\mathcal{F}})^{\gamma}, \ \gamma > 1, \ a > 0.$$

Let the initial data $(\varrho_{\varepsilon,0}, q_{\varepsilon,0})$ be defined by (5.10) satisfying

$$\varrho_{\varepsilon,0} \in L^{\gamma}(\Omega), \quad \varrho_{\varepsilon,0} \geq 0 \text{ a.e. in } \Omega, \tag{5.11}$$

$$q_{\mathcal{F}_0} \mathbb{1}_{\{\rho_{\mathcal{F}_0}=0\}} = 0 \text{ a.e. in } \Omega, \quad \frac{|q_{\mathcal{F}_0}|^2}{\rho_{\mathcal{F}_0}} \mathbb{1}_{\{\rho_{\mathcal{F}_0}>0\}} \in L^1(\Omega). \tag{5.12}$$

Then, system (5.1)–(5.8) admit a variational solution $(\varrho_{\varepsilon},)$ in the following sense:

$$\varrho_{\varepsilon} \geq 0, \quad \varrho_{\varepsilon} \in L^{\infty}(0, T; L^{\gamma}(\Omega)), \quad \boldsymbol{u}_{\varepsilon} \in L^2(0, T; W_0^{1,2}(\Omega; R^2)), \tag{5.13}$$

$$\boldsymbol{u}_{\varepsilon} = h'_{\varepsilon}(t) + \beta'_{\varepsilon}(t)(x - h(t))^{\perp} \text{ in } \mathcal{S}_{\varepsilon}(t), \tag{5.14}$$

$$\int_0^T \int_{R^2} \left[\varrho_{\varepsilon} \frac{\partial \phi}{\partial t} + (\varrho_{\varepsilon} \boldsymbol{u}_{\varepsilon}) \cdot \nabla \phi \right] dx\, dt = 0, \tag{5.15}$$

$$\int_0^T \int_{R^2} \left[b(\varrho_{\varepsilon}) \frac{\partial \phi}{\partial t} + (b(\varrho_{\varepsilon}) \boldsymbol{u}_{\varepsilon}) \cdot \nabla \phi + \big(b(\varrho_{\varepsilon}) - b'(\varrho_{\varepsilon})\varrho_{\varepsilon}\big) \operatorname{div} \boldsymbol{u}_{\varepsilon}\, \phi \right] dx\, dt = 0, \tag{5.16}$$

for any $\phi \in C_c^1([0, T) \times R^2)$ and any $b \in C^1[0, \infty)$, $b' \in C_c[0, \infty)$;

$$\int_0^T \int_{R^2} \left[(\varrho_{\varepsilon} \boldsymbol{u}_{\varepsilon}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + (\varrho_{\varepsilon} \boldsymbol{u}_{\varepsilon} \otimes \boldsymbol{u}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi} + \frac{1}{\varepsilon^{2m}} a \varrho_{\varepsilon}{}^{\gamma} \operatorname{div} \boldsymbol{\varphi} \right] dx\, dt = \int_0^T \int_{R^2} \mathbb{S}(\nabla_x \boldsymbol{u}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi}\, dx\, dt, \tag{5.17}$$

for any $\boldsymbol{\varphi} \in C_c^{\infty}((0, T) \times \Omega)$, with $\mathbb{D}(\boldsymbol{\varphi}) = 0$ in a neighborhood of $\mathcal{S}_{\varepsilon}(t)$ where $\mathbb{D}\boldsymbol{\varphi} = \frac{1}{2}\left(\nabla_x \boldsymbol{\varphi} + \nabla_x^t \boldsymbol{\varphi}\right)$;

The following energy inequality holds for a.e. $t \in [0, T]$:

$$\begin{aligned} &\int_{\Omega} \frac{1}{2} \varrho_{\varepsilon} |\boldsymbol{u}_{\varepsilon}|^2(\tau, \cdot)\, \mathrm{d}x + \int_{\Omega} \frac{1}{\varepsilon^{2m}} \Big(P(\varrho_{\varepsilon}) - P'(\overline{\varrho})(\varrho_{\varepsilon} - \overline{\varrho}) - P(\overline{\varrho}) \Big)(\tau, \cdot)\, \mathrm{d}x \\ &\quad + \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \boldsymbol{u}_{\varepsilon}) : \nabla_x \boldsymbol{u}_{\varepsilon} \ \mathrm{d}x\, \mathrm{d}t \\ &\leq \int_{\{\varrho_{\varepsilon,0}>0\}} \frac{1}{2} \frac{|q_{\varepsilon,0}|^2}{\varrho_{\varepsilon,0}}\, \mathrm{d}x + \frac{1}{\varepsilon^{2m}} \int_{\Omega} \Big(P(\varrho_{\varepsilon,0}) - P'(\overline{\varrho})(\varrho_{\varepsilon,0} - \overline{\varrho}) - P(\overline{\varrho}) \Big)\, \mathrm{d}x, \end{aligned} \tag{5.18}$$

where P is the pressure potential

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^{\gamma}.$$

We can verify the hypotheses (H1)–(H4) and apply Theorem 1.3 under certain conditions to obtain the following result in the framework of fluid-rigid body interaction:

Theorem 4.2 *Let $\Omega \subset R^2$ be a bounded domain of class C^3 and (ϱ_0, q_0) satisfy (5.11)–(5.12). Assume that $\mathcal{S}_{\varepsilon,0} \subset B_{\varepsilon,0}$,*

- $$\frac{1}{\varepsilon^{2m}} \int_{\Omega_{\varepsilon,0}} \Big(P(\varrho_{\varepsilon,0}) - P'(\overline{\varrho})(\varrho_{\varepsilon,0} - \overline{\varrho}) - P(\overline{\varrho}) \Big) \, \mathrm{d}x \to 0, \text{ where } \min\left\{m; \frac{2m}{\gamma}\right\} > 3. \tag{5.19}$$
- $$\int_{\{\varrho_{\varepsilon,0}>0\}} \frac{1}{2} \frac{|q_{\varepsilon,0}|^2}{\varrho_{\varepsilon,0}} \, \mathrm{d}x \to \int_{\Omega} \overline{\varrho} |\boldsymbol{u}_0|^2 \, \mathrm{d}x \text{ as } \varepsilon \to 0, \text{ where } \boldsymbol{u}_0 \in W^{2,\infty}(\Omega), \quad \mathrm{div}_x \boldsymbol{u}_0 = 0, \ \boldsymbol{u}_0|_{\partial\Omega} = 0. \tag{5.20}$$
- *The mass m_ε verifies that* $$\frac{m_\varepsilon}{\varepsilon^2} \to \infty \text{ as } \varepsilon \to 0. \tag{5.21}$$

Then,

$$\sup_{\tau \in [0,T]} \|\varrho_\varepsilon(\tau, \cdot) - \overline{\varrho}\|_{(L^2 + L^\gamma)(\Omega)} \to 0, \tag{5.22}$$

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ in } L^2(0, T; W_0^{1,2}(\Omega; R^2)) \tag{5.23}$$

as $\varepsilon \to 0$, where $\boldsymbol{u}$ is the (unique) classical solution of the incompressible Navier–Stokes system

$$\begin{aligned} \mathrm{div}_x \boldsymbol{u} &= 0, \\ \overline{\varrho}\partial_t \boldsymbol{u} + \overline{\varrho}\mathrm{div}_x(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x \Pi &= \mu \Delta_x \boldsymbol{u}, \\ \boldsymbol{u}|_{\partial\Omega} &= 0, \\ \boldsymbol{u}(0, \cdot) &= \boldsymbol{u}_0 \end{aligned} \tag{5.24}$$

in $(0, T) \times \Omega$.

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