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Global Well-Posedness and Asymptotic Behavior of the 3D MHD-Boussinesq Equations

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Abstract
In this paper, we study global well-posedness of the three-dimensional MHD-Boussinesq equations. The global existence of axisymmetric strong solutions to the MHD-Boussinesq equations in the presence of magnetic diffusion is shown by providing some smallness conditions only on the swirl component of velocity. As a by-product, long-time asymptotic behaviors are also presented.

Keywords MHD-Boussinesq equations · Global axisymmetric solutions · Asymptotic behavior

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1 Introduction and the Main Results

We are concerned with the following three-dimensional viscous incompressible MHD-Boussinesq equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p + (b \cdot \nabla)b + \rho e_3, \\ \partial_t b + (u \cdot \nabla)b = \eta \Delta b + (b \cdot \nabla)u, \\ \partial_t \rho + (u \cdot \nabla)\rho = \kappa \Delta \rho, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ (u, b, \rho)|_{t=0} = (u_0, b_0, \rho_0), \end{cases} \tag{1.1}$$

where $\nu \geq 0$, $\eta \geq 0$, and $\kappa \geq 0$ are the kinematic viscosity, magnetic diffusivity, and thermal diffusivity coefficients, respectively. $u = (u_1, u_2, u_3)(x, t)$, $b = (b_1, b_2, b_3)(x, t)$, $p = p(x, t)$, $\rho = \rho(x, t)$ with $x \in \mathbb{R}^3$, $t \geq 0$ are the unknown velocity field, magnetic field, pressure and the scalar temperature, respectively, e_3 is the unit vector in the x_3 direction. $u_0(x)$, $b_0(x)$ and $\rho_0(x)$ are the given initial conditions. Physically, the first equation describes the law of conservation of momentum in the presence of buoyancy, the second equation shows that the electromagnetic field is governed by the Maxwell's equations and the third one describes the temperature fluctuations around a constant state. For more physical background and numerical simulations, one can refer to Pratt et al. (2013), Schrinner et al. (2005, 2007), and references therein.

System (1.1) reduces to the Boussinesq equations if we set $b = 0$. Many efforts have been made to determine whether the Cauchy problem for the Boussinesq equations is well-posed. One can refer to Hou and Li (2005), Hmidi et al. (2010, 2011), Hmidi (2011), Larios et al. (2013), and references therein for the 2D problem. For 3D axisymmetric Boussinesq equations without swirl, Hmidi and Rousset (2010) proved the global well-posedness. Under the assumptions that the initial temperature ρ_0 does not intersect the z-axis and the orthogonal projection of the support of ρ_0 to the z-axis is compact, the global well-posedness was established in Abidi et al. (2011). If one assumes $\rho = 0$, then (1.1) reduces to the MHD equations. There have been lots of important progress on the well-posedness for the MHD equations. Duvaut and Lions (1972) (see also Sermange and Temam 1983) established the global existence of weak solutions and local well-posedness of strong solutions for the MHD equations in the classical Sobolev space $H^s(\mathbb{R}^3)$, $s \geq 3$. The global well-posedness for the MHD system was shown in Cai and Lei (2018) under the assumption that the initial velocity field and the displacement of the initial magnetic field from a nonzero constant are sufficiently small in certain weighted Sobolev spaces. In the axisymmetric setting, the global well-posedness of the 3D axisymmetric MHD equations was studied in Lei (2015) for a family of special axisymmetric initial data (u_0, b_0) with $u_0^\theta = b_0^r = b_0^z = 0$. Later, the global well-posedness of the 3D axisymmetric MHD equations with horizontal dissipation and vertical magnetic diffusion and vertical dissipation and vertical magnetic diffusion was established in Jiu and Liu (2015), Wang and Guo (2022), respectively. Moreover, strong axisymmetric solutions with only vertical dissipation on the velocity were proved to exist globally in Jiu et al. (2017). For the case of full dissipation and magnetic diffusion, the global small solutions to the

3D axisymmetric MHD equations were shown in Liu (2018) for axisymmetric initial data with $b_0^r = b_0^z = 0$.

For the full MHD-Boussinesq equations, there are also some works concentrated on the global well-posedness of weak and strong solutions. Bian and Gui (2016), Bian and Liu (2017) studied the global existence and uniqueness for the initial boundary value problem to the 2D stratified MHD-Boussinesq equations without smallness assumptions on the initial data. For the 3D case, Larios and Pei (2017) showed the local well-posedness in $H^3(\mathbb{R}^3)$. Liu et al. (2019) proved a global well-posedness result for large initial data for the MHD-Boussinesq equations with a nonlinear damping term. The investigation on global regularity of large axisymmetric solutions without swirl component u^θ was made in Bian and Pu (2020) under the assumption that the support of the initial thermal fluctuation is away from the z-axis and its projection on to the z-axis is compact. Later, this result was improved in Pan (2020) by removing the "support set" assumption on the initial data of the thermal fluctuation. Recently, Li (2022) established some critical conditions on the vorticity component ω^θ to guarantee the global regularity of the viscid or inviscid MHD-Boussinesq equations.

In this paper, we are interested in the global existence of axisymmetric strong solutions with swirl component of velocity and investigate the long-time behaviors of these solutions. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $r = \sqrt{x_1^2 + x_2^2}$. The cylindrical coordinate system (e_r, e_θ, e_z) is defined as:

$$\begin{aligned} e_r &= \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right) = (\cos\theta, \sin\theta, 0), \\ e_\theta &= \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right) = (-\sin\theta, \cos\theta, 0), \\ e_z &= (0, 0, 1). \end{aligned} \tag{1.2}$$

A scalar function f or a vector field $u = (u^r, u^\theta, u^z)$ is said to be axisymmetric if f, u^r, u^θ, u^z do not depend on θ :

$$u(x,t) = u^r(t,r,z)e_r + u^\theta(t,r,z)e_\theta + u^z(t,r,z)e_z.$$

Without loss of generality, one assumes that $\nu = 1$, $\eta = 1$, and $\kappa = 1$ in (1.1). The initial data (u_0, b_0, ρ_0) are assumed to be axisymmetric, and the initial magnetic field is supposed to only have the swirl component, i.e., $b_0(r,z) = b_0^\theta(r,z)e_\theta$. Since the initial data are axisymmetric, then the local strong solution to (1.1) is also axisymmetric. Moreover, by uniqueness of local classical solutions, it is clear that $b^r = b^z = 0$ for all later times if they vanish initially. Therefore, the aim of this paper is to establish a family of unique global solutions to (1.1) with the following structure

$$\begin{aligned} u(x,t) &= u^r(t,r,z)e_r + u^\theta(t,r,z)e_\theta + u^z(t,r,z)e_z, \\ b(x,t) &= b^\theta(t,r,z)e_\theta, \end{aligned} \tag{1.3}$$

instead of the general magnetic field. Note that the situation becomes much more difficult for general axisymmetric magnetic field. The main obstacle lies in the strong coupling effect between velocity and magnetic fields. Moreover, the general form will prevent us from obtaining some necessary a priori estimates, which are crucial in the analysis for the global solutions. Thus, in the axisymmetric setting (1.3), the MHD-Boussinesq equations (1.1) can be equivalently rewritten in the following form:

$$\begin{cases} \partial_t u^r + \left(\tilde{u}\cdot\nabla_{r,z}\right)u^r + \partial_r p = \left(\Delta_{r,z} - \frac{1}{r^2}\right)u^r + \frac{(u^\theta)^2}{r} - \frac{(b^\theta)^2}{r}, \\ \partial_t u^\theta + \left(\tilde{u}\cdot\nabla_{r,z}\right)u^\theta = \left(\Delta_{r,z} - \frac{1}{r^2}\right)u^\theta - \frac{u^r u^\theta}{r}, \\ \partial_t u^z + \left(\tilde{u}\cdot\nabla_{r,z}\right)u^z + \partial_z p = \Delta_{r,z}u^z + \rho, \\ \partial_t b^\theta + \left(\tilde{u}\cdot\nabla_{r,z}\right)b^\theta = \left(\Delta_{r,z} - \frac{1}{r^2}\right)b^\theta + \frac{u^r b^\theta}{r}, \\ \partial_t \rho + \left(\tilde{u}\cdot\nabla_{r,z}\right)\rho - \Delta_{r,z}\rho = 0, \\ \partial_r u^r + \frac{u^r}{r} + \partial_z u_z = 0, \\ (u^r, u^\theta, u^z, b^\theta, \rho)\big|_{t=0} = (u_0^r, u_0^\theta, u_0^z, b_0^\theta, \rho_0). \end{cases} \tag{1.4}$$

where

$$\tilde{u} = (u^r, u^z),\ \nabla_{r,z} = (\partial_r, \partial_z),\ \Delta_{r,z} = \partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r.$$

Then, the vorticity equations in the cylindrical coordinates can be written as:

$$\begin{cases} \frac{\widetilde{D}}{Dt}\omega^r - \left(\Delta_{r,z} - \frac{1}{r^2}\right)\omega^r = (\omega^r\partial_r + \omega^z\partial_z)u^r, \\ \frac{\widetilde{D}}{Dt}\omega^\theta - \left(\Delta_{r,z} - \frac{1}{r^2}\right)\omega^\theta = \frac{u^r}{r}\omega^\theta + \partial_z\frac{(u^\theta)^2}{r} - \partial_z\frac{(b^\theta)^2}{r} - \partial_r\rho, \\ \frac{\widetilde{D}}{Dt}\omega^z - \Delta_{r,z}\omega^z = (\omega^r\partial_r + \omega^z\partial_z)u^z, \\ \frac{\widetilde{D}}{Dt}j^r - \left(\Delta_{r,z} - \frac{1}{r^2}\right)j^r = \partial_z u^r\partial_r b^\theta + \partial_z u^z\partial_z b^\theta - \frac{u^r}{r}\partial_z b^\theta - \frac{b^\theta}{r}\partial_z u^r, \\ \frac{\widetilde{D}}{Dt}j^z - \Delta_{r,z}j^z = -\partial_r u^r\partial_r b^\theta - \partial_r u^z\partial_z b^\theta + \partial_r\left(\frac{u^r b^\theta}{r}\right), \end{cases} \tag{1.5}$$

where

$$\omega^r = -\partial_z u^\theta,\ \omega^\theta = \partial_z u^r - \partial_r u^z,\ \omega^z = \partial_r u^\theta + \frac{u^\theta}{r}, \tag{1.6}$$

$$j^r = -\partial_z b^\theta,\ j^z = \partial_r b^\theta + \frac{b^\theta}{r}, \tag{1.7}$$

and $\frac{\widetilde{D}}{Dt}$ is the convective derivative

$$\frac{\widetilde{D}}{Dt} = \partial_t + u^r\partial_r + u^z\partial_z.$$

Following the ideas of Majda and Bertozzi (2002), Lei and Zhang (2017), we introduce the following variables:

$$\Pi := \frac{b^\theta}{r},\ \Omega := \frac{\omega^\theta}{r},\ \Phi := \frac{\omega^r}{r},\ \Gamma := ru^\theta,\ \Lambda := \frac{u^\theta}{\sqrt{r}}.$$

Then, the equations of $(\Pi, \Omega, \Gamma, \Lambda)$ satisfy that

$$\begin{cases} \partial_t \Pi + \left(\tilde{u} \cdot \nabla_{r,z}\right) \Pi - (\Delta_{r,z} + \frac{2\partial_r}{r})\Pi = 0, \\ \partial_t \Omega + \left(\tilde{u} \cdot \nabla_{r,z}\right) \Omega - \left(\Delta_{r,z} + \frac{2\partial_r}{r}\right) \Omega = -\partial_z \Pi^2 - 2\frac{u^\theta}{r}\Phi - \frac{\partial_r \rho}{r}, \\ \partial_t \Gamma + \left(\tilde{u} \cdot \nabla_{r,z}\right) \Gamma - \left(\Delta_{r,z} - \frac{2\partial_r}{r}\right) \Gamma = 0, \\ \partial_t \Lambda + \left(\tilde{u} \cdot \nabla_{r,z}\right) \Lambda - \left(\Delta_{r,z} + \frac{\partial_r}{r} - \frac{3}{4r^2}\right) \Lambda = -\frac{3}{2}\frac{u^r}{r}\Lambda. \end{cases} \tag{1.8}$$

We state the main results as following.

Theorem 1.1 *Assume axisymmetric initial data $(u_0, b_0, \rho_0) \in H^2(\mathbb{R}^3)$, u_0 and b_0 are divergence-free. Suppose that $\epsilon > 0$, $\Gamma_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $\Pi_0 \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ and $\nabla b_0 \in L^\infty(\mathbb{R}^3)$, there exists a sufficiently small constant $\delta > 0$, such that if*

$$(\|G_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4 + \|\Pi_0\|_{L^3}^2 \|\Pi_0\|_{L^2}^2 + \|\rho_0\|_{L^2}^2)^{\frac{1}{2}} \|\Gamma_0\|_{L^2} \|\Gamma_0\|_{L^\infty} \le \delta, \tag{1.9}$$

or

$$\Psi_0 \cdot \|\Gamma_0\|_{L^2} \sup_{t>0} \|\Gamma\|_{L^\infty(r \le \epsilon)} \le \delta, \tag{1.10}$$

where

$$\begin{aligned} \Psi_0 &:= \left(\|G_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4 + \frac{1}{\epsilon^4}\left(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + \|\rho_0\|_{L^2}^2\right) \|\Gamma_0\|_{L^\infty}^3 \right. \\ &\quad \left. + \|\Pi_0\|_{L^2}^2 \|\Pi_0\|_{L^3}^2 \right)^{\frac{1}{2}}, \\ G_0 &= \Omega_0 - \frac{1}{2}\rho_0. \end{aligned}$$

Then, there exists a global axisymmetric strong solution (u, b, ρ) to (1.1) with

$$(u, b, \rho) \in L^\infty([0, \infty); H^2) \cap L^2([0, \infty); H^3).$$

Remark 1.1 If (u, b, p, ρ) solves the system (1.1), then the same is true for the rescaled functions $(u_\lambda, b_\lambda, p_\lambda, \rho_\lambda)$ defined as

$$\begin{aligned} u_\lambda(x,t) &= \lambda u(\lambda x, \lambda^2 t), \quad b_\lambda(x,t) = \lambda b(\lambda x, \lambda^2 t), \\ p_\lambda(x,t) &= \lambda^2 p(\lambda x, \lambda^2 t), \quad \rho_\lambda(x,t) = \lambda^3 \rho(\lambda x, \lambda^2 t). \end{aligned}$$

However, the quantities in conditions (1.9) and (1.10) are not scaling invariant, since the L^2-norm of $\rho_\lambda(x, 0)$ is not conserved by the L^2-norm of $\rho(x, 0)$. It is not difficult to verify that these conditions are scaling invariant if ρ_0 is taken to be zero, i.e., they are scaling invariant for the standard MHD system.

The following result gives the long-time asymptotic behaviors of global solutions established in Theorem 1.1.

Theorem 1.2 *Under the same conditions of Theorem 1.1, if* $\rho_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$*, and* ρ_0 *satisfies*

$$\int_{\mathbb{R}^3} |\rho_0(x)||x|dx \leq \infty, \quad \int_{\mathbb{R}^3} \rho_0(x)dx = 0 \text{ and } \|\rho_0\|_{L^1} \leq \epsilon_0,$$

where ϵ_0 *is a small positive constant independent of the initial data, then*

$$\|\rho(t)\|_{L^2}^2 \leq C\langle t\rangle^{-\frac{5}{2}}.$$

In addition, if $u_0 \in L^{\frac{3}{2}}(\mathbb{R}^3)$*,* $\Gamma_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ *and* $\Pi_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$*, then the following decay estimates hold:*

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 \leq C\langle t\rangle^{-\frac{1}{2}}, \quad \|\Gamma\|_{L^2}^2 \leq C\langle t\rangle^{-\frac{3}{2}}, \quad \|\Pi\|_{L^2}^2 \leq C\langle t\rangle^{-\frac{3}{2}},\\
&\|u^\theta(t)\|_{L^2}^2 + \langle t\rangle\|\nabla(u^\theta e_\theta)(t)\|_{L^2}^2 + t\langle t\rangle(\|\partial_t u^\theta(t)\|_{L^2}^2 + \|\Delta(u^\theta e_\theta)(t)\|_{L^2}^2) \leq C\langle t\rangle^{-\frac{5}{2}},\\
&\|b^\theta(t)\|_{L^2}^2 + \langle t\rangle\|\nabla(b^\theta e_\theta)(t)\|_{L^2}^2 + t\langle t\rangle(\|\partial_t b^\theta(t)\|_{L^2}^2 + \|\Delta(b^\theta e_\theta)(t)\|_{L^2}^2) \leq C\langle t\rangle^{-\frac{5}{2}},
\end{aligned}$$

where $\langle t\rangle = \sqrt{1+t^2}$*.*

Remark 1.2 These decay estimates are optimal in the sense of heat semigroup in three dimensions, since we know that the optimal time decay of L^2 norm of solutions to the Cauchy problem of heat equations in 3D is $t^{-1/2}$ for any $L^{3/2}$ initial data, while it is in accordance with our decay estimates. Note that the swirl component of velocity and magnetic fields shares better decay estimates than u^r and u^z, since the additional condition on Γ_0 is imposed.

Besides, we would like to introduce the notations and conventions used in the sequel of this article. $X \lesssim Y$ means the existence of some constant $C > 0$ such that $X \leq CY$. We denote $\nabla_h = (\partial_{x_1}, \partial_{x_2})$, $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$, $\dot{H}^s$ denotes the homogeneous Sobolev space, equipped with the norm $\|f\|_{\dot{H}^s} = (\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}}$, and we also introduce the Banach space $L_T^{p,q}$, equipped with the norms

$$\|f\|_{L_T^{p,q}} = \begin{cases} \left(\int_0^T \|f(t)\|_{L^q}^p dt\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty,\\ \text{ess} \sup\limits_{t\in(0,T)} \|f(t)\|_{L^q}, & \text{if } p = \infty,\end{cases}$$

where

$$\|f\|_{L^q} = \begin{cases} \left(\int_{\mathbb{R}^3} |f(t,x)|^q dx\right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty,\\ \text{ess} \sup\limits_{x\in\mathbb{R}^3} |f(t,x)|, & \text{if } q = \infty.\end{cases}$$

The remaining of this paper is organized as follows: We prove Theorem 1.1 in Sect. 2 by establishing different levels of a priori estimates. The proof of Theorem 1.2 is given in Sect. 3.

2 Proof of Theorem 1.1

We give the outline of the proof for Theorem 1.1. To prove the global regularity, we introduce a quantity $\mathcal{A}(T) = \|\Omega\|^2_{L^\infty_T L^2} + \|\nabla\Omega\|^2_{L^2_T L^2}$ and then prove the bounds for $\|u\|_{L^\infty_T L^\infty}$ and $\|\nabla\omega\|_{L^4_T L^{12}}$ via the estimates of $\|\omega\|_{L^\infty_T L^4}$ and $\|\nabla\omega^2\|_{L^2_T L^2}$. The second step is to give the estimates for ∇u, ∇b, and $\nabla\rho$, which are different from the techniques used in Chen et al. (2017a). Here, the new strategy about the L^p_T-L^q_x estimates for parabolic version of singular integrals and potentials is applied. Then, we establish the higher-order estimates for the solution. Finally, the global regularity follows under the prescribed smallness conditions by closing the estimates for $\mathcal{A}(T)$. The proof is divided into 4 steps.

1. **Bound for** $\|\omega\|_{L^\infty_T L^4} + \|\nabla\omega^2\|_{L^2_T L^2}$

Now, we present some basic estimates, which depend on $\mathcal{A}(T)$, once the bound for $\mathcal{A}(T)$ is obtained, then some uniform bounds for vorticity immediately follow.

The first lemma gives some basic estimates for axisymmetric functions; one can refer to Chen et al. (2017a) for its detailed proof.

Lemma 2.1 *Assume u is the smooth axisymmetric solution to the Navier–Stokes equations and $\omega = \nabla \times u$, for some $T < \infty$, then we have*

$$\left\| \frac{u^r}{r} \right\|_{L^\infty} \le C \|\Omega\|^{\frac{1}{2}}_{L^2} \|\partial_z\Omega\|^{\frac{1}{2}}_{L^2} \le C \|\Omega\|^{\frac{1}{2}}_{L^2} \|\nabla\Omega\|^{\frac{1}{2}}_{L^2},$$

this implies that

$$\int_0^{\mathrm{T}} \left\| \frac{u^r}{r} \right\|^4_{L^\infty} dt \le C \left(\|\Omega\|^2_{L^\infty_T L^2} + \|\nabla\Omega\|^2_{L^2_T L^2} \right)^2 = C\mathcal{A}^2(T).$$

There exists a constant $C = C(q)$, such that for $\forall\ t \in [0, T]$ and $1 < q < \infty$,

$$\|\tilde{\nabla} u^r\|_{L^q} + \|\tilde{\nabla} u^z\|_{L^q} + \left\| \frac{u^r}{r} \right\|_{L^q} \le C \left\|\omega^\theta\right\|_{L^q},$$
$$\|\tilde{\nabla} u^\theta\|_{L^q} + \left\| \frac{u^\theta}{r} \right\|_{L^q} \le C\|\nabla u\|_{L^q}.$$

Lemma 2.2 *Assume $(u_0, b_0, \rho_0) \in H^2(\mathbb{R}^3)$. Let (u, b, ρ) be the corresponding axisymmetric solution of system (1.4) satisfying (1.3) on $[0, T)$, for some $T < \infty$, and then, we have*

$$\left\|b^\theta\right\|_{L_T^\infty L^\infty} \le C_1(T), \tag{2.1}$$

$$\|\Lambda\|_{L_T^\infty L^4}^4 + 3\|\nabla\Lambda^2\|_{L_T^2 L^2}^2 + 3\left\|\frac{u^\theta}{r}\right\|_{L_T^4 L^4}^4 \le C_2(T), \tag{2.2}$$

$$\|\Lambda\|_{L_T^\infty L^8}^8 + \|\nabla\Lambda^4\|_{L_T^2 L^2}^2 + \int_0^{\mathrm{T}}\int_{\mathbb{R}^3}\frac{\Lambda^8}{r^2}dxdt \le C_3(T), \tag{2.3}$$

where the constants $C_1(T)$, $C_2(T)$, $C_3(T)$ depend on the initial data, T, and $\mathcal{A}(T)$.

Proof Multiplying the b^θ equation of (1.4) by $|b^\theta|^{p-2}b^\theta$, $2 \le p < \infty$ and performing integration in space, one can get

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|b^\theta\|_{L^p}^p + \frac{4(p-1)}{p^2}\left\|\nabla|b^\theta|^{\frac{p}{2}}\right\|_{L^2}^2 = \int_{\mathbb{R}^3}\frac{u^r}{r}|b^\theta|^p\mathrm{d}x \le \left\|\frac{u^r}{r}\right\|_{L^\infty}\|b^\theta\|_{L^p}^p.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b^\theta\|_{L^p} \le \left\|\frac{u^r}{r}\right\|_{L^\infty}\|b^\theta\|_{L^p}.$$

The Gronwall's inequality implies

$$\left\|b^\theta\right\|_{L_T^\infty L^p} \le \left\|b_0^\theta\right\|_{L^p}\exp\left\{\int_0^{\mathrm{T}}\left\|\frac{u^r}{r}\right\|_{L^\infty}dt\right\}.$$

Taking $p \to +\infty$, from Lemma 2.1, one has

$$\left\|b^\theta\right\|_{L_T^\infty L^\infty} \le \left\|b_0^\theta\right\|_{L^p}\exp\left\{C\mathcal{A}^{\frac{1}{2}}(T)T^{\frac{3}{4}}\right\}.$$

Multiplying the Λ equation of (1.8) by Λ^3 and integrating the resulting equation over $\mathbb{R}^3$, one has

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\|_{L^4}^4 + \frac{3}{4}\|\nabla\Lambda^2\|_{L^2}^2 + \frac{3}{4}\left\|\frac{u^\theta}{r}\right\|_{L^4}^4 = \frac{3}{2}\int_{\mathbb{R}^3}\frac{u^r}{r}\Lambda^4\mathrm{d}x \le \frac{3}{2}\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4.$$

Using Gronwall's inequality and Lemma 2.1, we obtain

$$\begin{aligned}\|\Lambda\|_{L_T^\infty L^4}^4 + 3\|\nabla\Lambda^2\|_{L_T^2 L^2}^2 + 3\left\|\frac{u^\theta}{r}\right\|_{L_T^4 L^4}^4 &\le \|\Lambda_0\|_{L^4}^4\exp\left\{C\int_0^{\mathrm{T}}\left\|\frac{u^r}{r}\right\|_{L^\infty}dt\right\} \\ &\le C\|u_0\|_{H^2(\mathbb{R}^3)}^2\exp\left\{C\mathcal{A}^{\frac{1}{2}}(T)T^{\frac{3}{4}}\right\},\end{aligned}$$

where

$$\|\Lambda_0\|_{L^4}^4 \le \|u_0^\theta\|_{L^\infty}^2\left\|\frac{u_0^\theta}{r}\right\|_{L^2}^2 \le C\left(\left\|\nabla u_0^\theta\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla^2 u_0^\theta\right\|_{L^2}^{\frac{1}{2}}\right)^2\|\nabla u_0\|_{L^2}^2 \le C\|u_0\|_{H^2}^2.$$

Thus, we get (2.2).

Multiplying the Λ equation of (1.8) by Λ^7 and integrating the resulting equation over $\mathbb{R}^3$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\|_{L^8}^8 + \|\nabla\Lambda^4\|_{L^2}^2 + \int_{\mathbb{R}^3}\frac{\Lambda^8}{r^2}\mathrm{d}x \le C\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^8}^8 .$$

Using Gronwall's inequality and Lemma 2.1, one has

$$\begin{aligned}\|\Lambda\|_{L_T^\infty L^8}^8 + \|\nabla\Lambda^4\|_{L_T^2 L^2}^2 + \int_0^{\mathrm{T}}\int_{\mathbb{R}^3}\frac{\Lambda^8}{r^2}\mathrm{d}x dt &\le C\|\Lambda_0\|_{L^8}^8 \exp\left\{C\int_0^{\mathrm{T}}\left\|\frac{u^r}{r}\right\|_{L^\infty} dt\right\}\\ &\le C\|u_0\|_{H^2}^8 \exp\left\{C\mathcal{A}^{\frac{1}{2}}(T)T^{\frac{3}{4}}\right\},\end{aligned}$$

where

$$\begin{aligned}&\|\Lambda_0\|_{L^8}^8 \le \left\|u_0^\theta\right\|_{L^\infty}^4\left\|\frac{u_0^\theta}{r}\right\|_{L^4}^4\\ &\le \left(\left\|\nabla u_0^\theta\right\|_{L^2}^{\frac{1}{4}}\left\|\nabla^2 u_0^\theta\right\|_{L^2}^{\frac{3}{4}}\right)^4\left(\left\|\frac{u_0^\theta}{r}\right\|_{L^2}^{\frac{1}{4}}\left\|\nabla\frac{u_0^\theta}{r}\right\|_{L^2}^{\frac{3}{4}}\right)^4 \le C\|u_0\|_{H^2}^8.\end{aligned}$$

Therefore, we obtain (2.3).

The following lemma gives the estimates for components of vorticity.

Lemma 2.3 *Assume $(u_0, b_0, \rho_0) \in H^2(\mathbb{R}^3)$ and $\Pi_0 \in L^\infty(\mathbb{R}^3)$. Let (u, b) be the corresponding axisymmetric solution of system (1.4) satisfying (1.3) on $[0, T)$, for some $T < \infty$, then we have*

$$\left\|\omega^\theta\right\|_{L_T^\infty L^4}^4 + \left\|\nabla(\omega^\theta)^2\right\|_{L_T^2 L^2}^2 + \left\|\frac{\omega^\theta}{\sqrt{r}}\right\|_{L_T^4 L^4}^4 \le C(T), \tag{2.4}$$

$$\left\|\omega^\theta\right\|_{L_T^\infty L^2}^2 + \left\|\nabla\omega^\theta\right\|_{L_T^2 L^2}^2 + 2\left\|\frac{\omega^\theta}{r}\right\|_{L_T^2 L^2}^2 \le C(T), \tag{2.5}$$

$$\left\|\omega^r\right\|_{L_T^\infty L^4}^4 + \left\|\omega^z\right\|_{L_T^\infty L^4}^4 + \left\|\nabla(\omega^r)^2\right\|_{L_T^2 L^2}^2 + \left\|\nabla(\omega^z)^2\right\|_{L_T^2 L^2}^2 + \left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L_T^4 L^4}^4 \le C(T), \tag{2.6}$$

where the constants $C(T)$ depend on the initial data, T, and $\mathcal{A}(T)$.

Proof Multiplying (1.5) by $|\omega^\theta|^2\omega^\theta$ and integrating with respect to the space variable, it follows that

$$
\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^\theta\|_{L^4}^4+\frac{3}{4}\left\|\nabla(\omega^\theta)^2\right\|_{L^2}^2+\left\|\frac{\omega^\theta}{\sqrt{r}}\right\|_{L^4}^4\\
&=\int_{\mathbb{R}^3}\frac{u^r}{r}(\omega^\theta)^4\mathrm{d}x+\int_{\mathbb{R}^3}\partial_z\left(\frac{(u^\theta)^2}{r}\right)\cdot|\omega^\theta|^2\omega^\theta\mathrm{d}x-\int_{\mathbb{R}^3}\partial_z\left(\frac{(b^\theta)^2}{r}\right)\cdot|\omega^\theta|^2\omega^\theta\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\partial_r\rho\cdot|\omega^\theta|^2\omega^\theta\mathrm{d}x \qquad (2.7)\\
&:=A_1+A_2+A_3+A_4.
\end{aligned}
$$

For the first term A_1, it follows that

$$
A_1\le\left\|\frac{u^r}{r}\right\|_{L^\infty}\left\|\omega^\theta\right\|_{L^4}^4. \qquad (2.8)
$$

As for the second term A_2, by integrating by parts, we have

$$
\begin{aligned}
A_2&=-3\int_{\mathbb{R}^3}\frac{(u^\theta)^2}{r}\cdot(\omega^\theta)^2\cdot\partial_z\omega^\theta\mathrm{d}x=-\frac{3}{2}\int_{\mathbb{R}^3}\frac{(u^\theta)^2}{r}\cdot\omega^\theta\cdot\partial_z(\omega^\theta)^2\mathrm{d}x\\
&=-\frac{3}{2}\int_{\mathbb{R}^3}\left(\frac{u^\theta}{\sqrt{r}}\right)^2\cdot\omega^\theta\cdot\partial_z(\omega^\theta)^2\mathrm{d}x.
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
|A_2|&\le C\left\|\frac{u^\theta}{\sqrt{r}}\right\|_{L^8}^2\|\omega^\theta\|_{L^4}\left\|\partial_z(\omega^\theta)^2\right\|_{L^2}\\
&\le C\left\|\frac{u^\theta}{\sqrt{r}}\right\|_{L^8}^8+\|\omega^\theta\|_{L^4}^4+\frac{1}{8}\left\|\partial_z(\omega^\theta)^2\right\|_{L^2}^2. \qquad (2.9)
\end{aligned}
$$

For the third term A_3, by integration by parts, Hölder's inequality and Young's inequality, one has

$$
\begin{aligned}
A_3&=3\int_{\mathbb{R}^3}\frac{(b^\theta)^2}{r}\cdot(\omega^\theta)^2\cdot\partial_z\omega^\theta\mathrm{d}x=\frac{3}{2}\int_{\mathbb{R}^3}\frac{(b^\theta)^2}{r}\cdot\omega^\theta\cdot\partial_z(\omega^\theta)^2\mathrm{d}x\\
&\le\frac{3}{2}\|\Pi\|_{L^4}\|b^\theta\|_{L^\infty}\|\omega^\theta\|_{L^4}\left\|\partial_z(\omega^\theta)^2\right\|_{L^2}\\
&\le C\|\Pi_0\|_{L^4}^4\|b^\theta\|_{L^\infty}^4+\|\omega^\theta\|_{L^4}^4+\frac{1}{8}\left\|\partial_z(\omega^\theta)^2\right\|_{L^2}^2. \qquad (2.10)
\end{aligned}
$$

For the last term A_4, we have

$$
\begin{aligned}
A_4&=-2\pi\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r\rho(\omega^\theta)^3r\mathrm{d}r\mathrm{d}z\\
&=-2\pi\int_{-\infty}^{+\infty}\int_0^{+\infty}\rho\partial_r((\omega^\theta)^3r)\mathrm{d}r\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&= -2\pi \int_{-\infty}^{+\infty} \int_0^{+\infty} \rho(\omega^\theta)^2 \partial_r \omega^\theta r \mathrm{d}r \mathrm{d}z + \int_{\mathbb{R}^3} \rho \frac{(\omega^\theta)^3}{r} \mathrm{d}x \\
&\le C\|\rho\|_{L^\infty} \|\nabla(\omega^\theta)^2\|_{L^2} \|\omega\|_{L^2} + \|\rho\|_{L^\infty} \left\| \frac{\omega^\theta}{\sqrt{r}} \right\|_{L^4}^2 \|\omega^\theta\|_{L^2} \\
&\le C\|\rho\|_{L^\infty}^2 \|\omega^\theta\|_{L^2}^2 + \frac{1}{4}\|\nabla(\omega^\theta)^2\|_{L^2}^2 + \frac{1}{4} \left\| \frac{\omega^\theta}{\sqrt{r}} \right\|_{L^4}^4 .
\end{aligned} \tag{2.11}
$$

Inserting (2.8), (2.9), (2.10), and (2.11) into (2.7), one may conclude that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \|\omega^\theta\|_{L^4}^4 + \left\|\nabla(\omega^\theta)^2\right\|_{L^2}^2 + \left\| \frac{\omega^\theta}{\sqrt{r}} \right\|_{L^4}^4 \\
&\le \left\| \frac{u^r}{r} \right\|_{L^\infty} \|\omega^\theta\|_{L^4}^4 + C\,\|\Lambda\|_{L^8}^8 + C\,\|\omega^\theta\|_{L^4}^4 + C\|\Pi_0\|_{L^4}^4 \|b^\theta\|_{L^\infty}^4 + C\|\rho\|_{L^\infty}^2 \|\omega^\theta\|_{L^2}^2.
\end{aligned}
$$

Integrating with respect to time, applying the Gronwall's inequality, we obtain

$$
\begin{aligned}
&\|\omega^\theta\|_{L_T^\infty L^4}^4 + \left\|\nabla(\omega^\theta)^2\right\|_{L_T^2 L^2}^2 + 4\left\| \frac{\omega^\theta}{\sqrt{r}} \right\|_{L_T^4 L^4}^4 \\
&\le C\left(\|\omega_0^\theta\|_{L^4}^4 + \|\Lambda\|_{L_T^\infty L^8}^8 T + \|\Pi_0\|_{L^4}^4 \|b^\theta\|_{L_T^\infty L^\infty}^4 T + \|\rho\|_{L_T^\infty L^\infty}^2 \|\omega^\theta\|_{L_T^\infty L^2}^2 T \right) \\
&\quad \cdot \exp\left(C\int_0^{\mathrm{T}} \left\| \frac{u^r}{r} \right\|_{L^\infty} dt + CT \right) \\
&\le C(T),
\end{aligned}
$$

where $C(T)$ is a constant depending on the initial data, $\mathcal{A}(T)$ and T. Then, this gives (2.4).

Multiplying (1.5) by ω^θ and integrating with respect to space variable, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\omega^\theta\|_{L^2}^2 + \|\nabla\omega^\theta\|_{L^2}^2 + \left\| \frac{\omega^\theta}{r} \right\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} \left(\frac{\omega^\theta}{r} u^r \omega^\theta - \partial_z \omega^\theta \frac{\left(u^\theta\right)^2}{r} + \partial_z \omega^\theta \frac{\left(b^\theta\right)^2}{r} + \partial_r \rho \omega^\theta \right) \mathrm{d}x \\
&\le \left\| \frac{u^r}{r} \right\|_{L^\infty} \|\omega^\theta\|_{L^2}^2 + C\left\| \frac{u^\theta}{\sqrt{r}} \right\|_{L^4}^4 + \|b^\theta\|_{L^\infty}^2 \|\Pi\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \frac{1}{2}\left\| \frac{\omega^\theta}{r} \right\|_{L^2}^2 \\
&\quad + \frac{1}{2}\left(\|\partial_r \omega^\theta\|_{L^2}^2 + \|\partial_z \omega^\theta\|_{L^2}^2 \right).
\end{aligned}
$$

Thus,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\omega^\theta\right\|_{L^2}^2+\left\|\nabla\omega^\theta\right\|_{L^2}^2+\left\|\frac{\omega^\theta}{r}\right\|_{L^2}^2\\
&\le C\left\|\frac{u^\theta}{\sqrt{r}}\right\|_{L^4}^4+2\left\|\frac{u^r}{r}\right\|_{L^\infty}\left\|\omega^\theta\right\|_{L^2}^2+2\left\|\Pi_0\right\|_{L^2}^2\left\|b^\theta\right\|_{L^\infty}^2+2\|\rho\|_{L^2}^2.
\end{aligned}$$

Integrating with respect to time, applying the Gronwall's inequality, we have

$$\begin{aligned}
&\left\|\omega^\theta\right\|_{L_T^\infty L^2}^2+\left\|\nabla\omega^\theta\right\|_{L_T^2L^2}^2+2\left\|\frac{\omega^\theta}{r}\right\|_{L_T^2L^2}^2\\
&\le\left(\left\|\omega_0^\theta\right\|_{L^2}^2+2\|\rho\|_{L_T^\infty L^2}^2T+2\left\|\Pi_0\right\|_{L^2}^2\int_0^{\mathrm{T}}\left\|b^\theta\right\|_{L^\infty}^2dt\right)\exp\left\{\int_0^{\mathrm{T}}\left\|\frac{u^r}{r}\right\|_{L^\infty}dt\right\}\\
&\le\left(\left\|\omega_0^\theta\right\|_{L^2}^2+2\|\rho\|_{L_T^\infty L^2}^2T+2\left\|\Pi_0\right\|_{L^2}^2\left\|b^\theta\right\|_{L_T^\infty L^\infty}^2T\right)\exp\left\{\int_0^{\mathrm{T}}\left\|\frac{u^r}{r}\right\|_{L^\infty}dt\right\}\\
&\le C(T).
\end{aligned}$$

Then, this gives (2.5).

Similarly, using integration by parts, one has

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega^r\|_{L^4}^4+\|\omega^z\|_{L^4}^4)+\frac{3}{4}\|\nabla(\omega^r)^2\|_{L^2}^2+\frac{3}{4}\|\nabla(\omega^z)^2\|_{L^2}^2+\left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L^4}^4\\
&=\int_{\mathbb{R}^3}\omega^r\partial_r u^r|\omega^r|^2\omega^r\mathrm{d}x+\int_{\mathbb{R}^3}\omega^z\partial_z u^r|\omega^r|^2\omega^r\mathrm{d}x+\int_{\mathbb{R}^3}\omega^r\partial_r u^z|\omega^z|^2\omega^z\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\omega^z\partial_z u^z|\omega^z|^2\omega^z\mathrm{d}x\\
&:=B_1+B_2+B_3+B_4.
\end{aligned}$$

For the first term B_1, it follows that

$$\begin{aligned}
B_1&=2\pi\int_{-\infty}^{+\infty}u^r(\omega^r)^4|_{r=0}^{r=\infty}dz-2\pi\int_{-\infty}^{+\infty}\int_0^{+\infty}(4u^r(\omega^r)^3\cdot\partial_r\omega^r\cdot r+u^r\cdot(\omega^r)^4)\mathrm{d}r\mathrm{d}z\\
&\le 2\|u^r\|_{L^\infty}\|\omega^r\|_{L^4}^2\|\nabla(\omega^r)^2\|_{L^2}+\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\omega^r\|_{L^4}^4\\
&\le C\|u^r\|_{L^\infty}^2\|\omega^r\|_{L^4}^4+\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\omega^r\|_{L^4}^4+\frac{1}{8}\|\nabla(\omega^r)^2\|_{L^2}^2. \qquad (2.12)
\end{aligned}$$

For the second term B_2, one has

$$\begin{aligned}
B_2&=\int_{\mathbb{R}^3}\omega^z\partial_z u^r|\omega^r|^2\omega^r\mathrm{d}x=-2\pi\int_{-\infty}^{+\infty}\int_0^{+\infty}u^r\cdot\partial_z(|\omega^r|^2\cdot\omega^r\cdot\omega^z\cdot r)\mathrm{d}r\mathrm{d}x\\
&=-3\int_{\mathbb{R}^3}u^r\cdot(\omega^r)^2\partial_r\omega^r\cdot\omega^z\mathrm{d}x+2\pi\int_{-\infty}^{+\infty}\int_0^{+\infty}u^r\cdot(\omega^r)^3\cdot\partial_r(r\omega^r)\mathrm{d}r\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{3}{2}\int_{\mathbb{R}^3} u^r \cdot \partial_r(\omega^r)^2 \cdot \omega^r \cdot \omega^z \mathrm{d}x + \int_{\mathbb{R}^3} u^r \cdot (\omega^r)^2 \left(\frac{\omega^r}{\sqrt{r}}\right)^2 \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} u^r \cdot \nabla(\omega^r)^2 \cdot (\omega^r)^2 \mathrm{d}x \\
&\le \frac{3}{2}\|u^r\|_{L^\infty}\|\omega^r\|_{L^4}\|\omega^z\|_{L^4}\|\nabla(\omega^r)^2\|_{L^2} + \|u^r\|_{L^\infty}\|\omega^r\|_{L^4}^2 \left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L^4}^2 \\
&\quad + \|u^r\|_{L^\infty}\|\omega^r\|_{L^4}^2\|\nabla(\omega^r)^2\|_{L^2} \\
&\le C\|u^z\|_{L^\infty}^2(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4) + \frac{1}{2}\left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L^4}^4 + \frac{1}{8}\|\nabla(\omega^r)^2\|_{L^2}^2. \qquad (2.13)
\end{aligned}
$$

For the third term B_3, we have

$$
\begin{aligned}
B_3 &= \int_{\mathbb{R}^3} \omega^r \partial_r u^z |\omega^z|^2 \omega^z \mathrm{d}x - 2\pi \int_{-\infty}^{+\infty}\int_0^{+\infty} u^z \cdot \partial_r(\omega^r \cdot |\omega^z|^2 \cdot \omega^z r)\mathrm{d}r\mathrm{d}z \\
&= 2\pi \int_{-\infty}^{+\infty}\int_0^{+\infty} u^z \cdot \partial_z(r\omega^z) \cdot |\omega^z|^2 \cdot \omega^z \mathrm{d}r\mathrm{d}z - \frac{3}{2}\int_{\mathbb{R}^3} u^r \cdot \omega^r \cdot \omega^z \cdot \partial_r(\omega^z)^2 \mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^3} u^z(\omega^z)^2 \partial_z(\omega^z)^2 \mathrm{d}x - \frac{3}{2}\int_{\mathbb{R}^3} u^r \cdot \omega^r \cdot \omega^z \cdot \partial_r(\omega^z)^2 \mathrm{d}x \\
&\le C\|u^z\|_{L^\infty}\|\omega^z\|_{L^4}^2\|\nabla(\omega^z)^2\|_{L^2} + C\|u^z\|_{L^\infty}\|\omega^r\|_{L^4}\|\omega^z\|_{L^4}\|\nabla(\omega^z)^2\|_{L^2} \\
&\le C\|u^z\|_{L^\infty}^2\|\omega^z\|_{L^4}^4 + C\|u^z\|_{L^\infty}^2(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4) + \frac{1}{8}\|\nabla(\omega^z)^2\|_{L^2}^2 \\
&\le C\|u^z\|_{L^\infty}^2(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4) \\
&\quad + \frac{1}{8}\|\nabla(\omega^z)^2\|_{L^2}^2. \qquad (2.14)
\end{aligned}
$$

For the last term B_4, it follows that

$$
\begin{aligned}
B_4 &= -\int_{\mathbb{R}^3} u^z \partial_z(|\omega^z|^4)\mathrm{d}x = -2\int_{\mathbb{R}^3} u^z(\omega^z)^2 \cdot \partial_z(\omega^z)^2 \mathrm{d}x \\
&\le C\|u^z\|_{L^\infty}^2\|\omega^z\|_{L^4}^4 + \frac{1}{8}\|\nabla(\omega^z)^2\|_{L^2}^2. \qquad (2.15)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4) + \frac{1}{2}\|\nabla(\omega^r)^2\|_{L^2}^2 + \frac{1}{2}\|\nabla(\omega^z)^2\|_{L^2}^2 + \frac{1}{2}\left\|\frac{\omega^r}{\sqrt{r}}\right\|_{L^4}^4 \\
&\le C\left(\|u^r\|_{L^\infty}^2 + \|u^z\|_{L^\infty}^2 + \left\|\frac{u^r}{r}\right\|_{L^\infty}\right)(\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4). \qquad (2.16)
\end{aligned}
$$

The Gagliardo–Nirenberg's inequality and Lemma 2.1 give to

$$
\begin{aligned}
\int_0^{\mathrm{T}} \|(u^r, u^z)\|_{L^\infty}^2 dt &\leq C \int_0^{\mathrm{T}} \left(\|\nabla(u^r, u^z)\|_{L^2}^{\frac{1}{2}} \|\nabla^2(u^r, u^z)\|_{L^2}^{\frac{1}{2}} \right)^2 dt \\
&\leq C \int_0^{\mathrm{T}} \|\nabla u\|_{L^2} \left(\|\nabla \omega^\theta\|_{L^2} + \left\| \frac{\omega^\theta}{r} \right\|_{L^2} \right) dt \\
&\leq C \|\nabla u\|_{L_T^2 L^2} \left(\|\nabla \omega^\theta\|_{L_T^2 L^2} + \left\| \frac{\omega^\theta}{r} \right\|_{L_T^2 L^2} \right) \\
&\leq C \|u_0\|_{L^2} \left(\|\nabla \omega^\theta\|_{L_T^2 L^2} + \left\| \frac{\omega^\theta}{r} \right\|_{L_T^2 L^2} \right) \leq C(T).
\end{aligned}
\tag{2.17}
$$

Inserting (2.17) into (2.16), and by Gronwall's inequality, we conclude that

$$
\begin{aligned}
&\|\omega^r\|_{L_T^\infty L^4}^4 + \|\omega^z\|_{L_T^\infty L^4}^4 + 2\|\nabla(\omega^r)^2\|_{L_T^2 L^2}^2 + 2\|\nabla(\omega^z)^2\|_{L_T^2 L^2}^2 + 2 \left\| \frac{\omega^r}{\sqrt{r}} \right\|_{L_T^4 L^4}^4 \\
&\leq (\|\omega^r\|_{L^4}^4 + \|\omega^z\|_{L^4}^4) \exp\left\{ \int_0^{\mathrm{T}} \|(u^r, u^z)\|_{L^\infty}^2 dt + C \int_0^{\mathrm{T}} \left\| \frac{u^r}{r} \right\|_{L^\infty} dt \right\} \\
&\leq C(T),
\end{aligned}
$$

where $C(T)$ is a constant depending on the initial data, $\mathcal{A}(T)$ and T. Then, this gives (2.6), and from (2.4) and (2.6), we obtain that

$$
\|\omega\|_{L_T^\infty L^4}^4 + \|\nabla \omega^2\|_{L_T^\infty L^2} < \infty.
$$

2. **Estimates for ∇u, ∇b and $\nabla \rho$**

In the following, we focus on the estimates for ∇u, ∇b, and $\nabla \rho$.

Lemma 2.4 *Assume $(u_0, b_0, \rho_0) \in H^2(\mathbb{R}^3)$, $\Pi_0 \in L^\infty(\mathbb{R}^3)$ and $\nabla b_0 \in L^\infty(\mathbb{R}^3)$. Let (u, b) be the corresponding axisymmetric solution of system (1.4) satisfying (1.3) on $[0, T)$, for some $T < \infty$; then, we have*

$$
\begin{aligned}
\|\nabla u\|_{L_T^4 L^\infty} &\leq C(T), \\
\|\nabla b\|_{L_T^\infty L^\infty} &\leq C(T), \\
\|\nabla \rho\|_{L_T^\infty L^\infty} &\leq C(T),
\end{aligned}
$$

where the constants $C(T)$ depend on the initial data, T and $\mathcal{A}(T)$.

Proof Taking "Curl" operator to $(1.1)_1$, we can get

$$
\omega_t - \Delta \omega = -\nabla \times (\omega \times u) + \nabla \times (b \cdot \nabla b) + \nabla \times \rho e_3.
$$

Then, it follows that

$$
\omega = e^{t\Delta} \omega_0 - \int_0^t e^{(t-s)\Delta} (\nabla \times (\omega \times u) - \partial_z(\Pi b^\theta e_\theta) - \nabla \times \rho e_3) ds.
$$

Standard estimates Wahl (1982) show that

$$\begin{aligned}\|\nabla\omega\|_{L^4_TL^{12}} &\lesssim \|\omega\times u\|_{L^4_TL^{12}} + \|\Pi\cdot b^\theta\|_{L^4_TL^{12}} + \|\rho\|_{L^4_TL^{12}}\\
&\lesssim \|\omega\|_{L^4_TL^{12}}\|u\|_{L^\infty_TL^\infty} + \|\Pi_0\|_{L^{12}}\|b^\theta\|_{L^\infty_TL^\infty}T^{\frac14} + \|\rho\|_{L^4_TL^{12}}\\
&\lesssim \|\omega\|_{L^4_TL^{12}}\|u\|_{L^\infty_TL^\infty} + \|\Pi_0\|^{\frac56}_{L^\infty}\|\Pi_0\|^{\frac16}_{L^2}\|b^\theta\|_{L^\infty_TL^\infty}T^{\frac14} + \|\rho\|_{L^4_TL^{12}}\\
&\lesssim \|\omega\|_{L^4_TL^{12}}\|u\|_{L^\infty_TL^\infty} + \|\Pi_0\|^{\frac56}_{L^\infty}\|b_0\|^{\frac16}_{H^2}\|b^\theta\|_{L^\infty_TL^\infty}T^{\frac14} + \|\rho\|_{L^4_TL^{12}}.\end{aligned}$$

Since

$$\|\omega\|_{L^{12}} = \|\omega^2\|^{\frac12}_{L^6} \le \|\nabla\omega^2\|^{\frac12}_{L^2},$$

then

$$\begin{aligned}\|\nabla\omega\|_{L^4_TL^{12}} &\lesssim \|\nabla\omega^2\|^{\frac12}_{L^2_TL^2}\|u\|_{L^\infty_TL^\infty} + \|\rho\|_{L^4_TL^{12}}\\
&\quad + \|\Pi_0\|^{\frac56}_{L^\infty}\|b_0\|^{\frac16}_{H^2}\|b^\theta\|_{L^\infty_TL^\infty}T^{\frac14}.\end{aligned} \tag{2.18}$$

On the other hand, by the Gagliardo–Nirenberg inequality, we obtain

$$\|\nabla u\|_{L^\infty} \le C\|\nabla u\|^{\frac12}_{L^4}\|\nabla^2 u\|^{\frac12}_{L^{12}},$$

then

$$\begin{aligned}\|\nabla u\|^4_{L^4_TL^\infty} &\le C\|\nabla u\|^2_{L^\infty_TL^4}\|\nabla^2u\|^2_{L^2_TL^{12}} \le C\|\omega\|^2_{L^\infty_TL^4}\|\nabla\omega\|^2_{L^2_TL^{12}}\\
&\le C\|\omega\|^2_{L^\infty_TL^4}\|\nabla\omega\|^2_{L^4_TL^{12}}T^{\frac12}.\end{aligned} \tag{2.19}$$

Combining (2.18) and (2.19) together, one has

$$\begin{aligned}&\|\nabla u\|^4_{L^4_TL^\infty}\\
&\lesssim \|\omega\|^2_{L^\infty_TL^4}\left(\|\nabla\omega^2\|_{L^2_TL^2}\|u\|^2_{L^\infty_TL^\infty} + \|\Pi_0\|^{\frac53}_{L^\infty}\|b_0\|^{\frac13}_{H^2}\|b^\theta\|^2_{L^\infty_TL^\infty} + \|\rho_0\|^2_{L^4_TL^{12}}\right)T^{\frac12}.\end{aligned} \tag{2.20}$$

Using the Gagliardo–Nirenberg inequality, Young's inequality, and Lemma 2.3, one obtains that

$$\begin{aligned}\|u\|_{L^\infty_TL^\infty} &\le C(\|u\|_{L^\infty_TL^2} + \|\omega\|_{L^\infty_TL^4})\\
&\le C(\|u_0\|_{L^2} + \|\omega^r\|_{L^\infty_TL^4} + \|\omega^\theta\|_{L^\infty_TL^4} + \|\omega^z\|_{L^\infty_TL^4})\\
&\le C(T).\end{aligned} \tag{2.21}$$

Therefore, it follows from (2.20) that

$$\|\nabla u\|_{L_T^4 L^\infty}^4 \le C(T). \tag{2.22}$$

Then, taking "∇" operator to $(1.4)_4$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla b + u\cdot\nabla\nabla b - \Delta\nabla b = -\nabla u\cdot\nabla b + \frac{u^r}{r}\nabla b + \nabla u^r\Pi - \frac{u^r}{r}\Pi e_r. \tag{2.23}$$

Multiplying the above equation by $|\nabla b|^{p-2}\nabla b$ and then integrating the resulting equation over $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla b\|_{L^p}^p + \frac{4(p-1)}{p^2}\left\|\nabla|b|^{\frac{p}{2}}\right\|_{L^2}^2 &\le \|\nabla u\|_{L^\infty}\|\nabla b\|_{L^p}^p + \left\|\frac{u^r}{r}\right\|_{L^\infty}\|\nabla b\|_{L^p}^p\\ &\quad + \left(\|\nabla u^r\|_{L^\infty} + \left\|\frac{u^r}{r}\right\|_{L^\infty}\right)\|\Pi\|_{L^p}\|\nabla b\|_{L^p}^{p-1},\end{aligned}$$

applying Gronwall's inequality and taking $p\to\infty$, we have

$$\begin{aligned}\|\nabla b\|_{L_T^\infty L^\infty} &\le \left\{\|\nabla b_0\|_{L^\infty} + \int_0^{\mathrm{T}}\left(\|\nabla u\|_{L^\infty} + \left\|\frac{u^r}{r}\right\|_{L^\infty}\right)\|\Pi\|_{L^\infty}dt\right\}\\ &\quad\cdot\exp\left\{\int_0^{\mathrm{T}}\left(\|\nabla u\|_{L^\infty} + \left\|\frac{u^r}{r}\right\|_{L^\infty}\right)dt\right\}\\ &\le \left\{\|\nabla b_0\|_{L^\infty} + \|\Pi_0\|_{L^\infty}\int_0^{\mathrm{T}}\left(\|\nabla u\|_{L^\infty} + \left\|\frac{u^r}{r}\right\|_{L^\infty}\right)dt\right\}\\ &\quad\cdot\exp\left\{\int_0^{\mathrm{T}}\left(\|\nabla u\|_{L^\infty} + \left\|\frac{u^r}{r}\right\|_{L^\infty}\right)dt\right\}\\ &\le \left\{\|\nabla b_0\|_{L^\infty} + \|\Pi_0\|_{L^\infty}\left(\|\nabla u\|_{L_T^4 L^\infty}T^{\frac{3}{4}} + \mathcal{A}^{\frac{1}{2}}(T)T^{\frac{3}{4}}\right)\right\}\\ &\quad\cdot\exp\left(\|\nabla u\|_{L_T^4 L^\infty}T^{\frac{3}{4}} + \mathcal{A}^{\frac{1}{2}}(T)T^{\frac{3}{4}}\right).\end{aligned} \tag{2.24}$$

Using (2.1) and (2.22), there holds

$$\|\nabla b\|_{L_T^\infty L^\infty} \le C(T).$$

Similar techniques used to the third equation of (1.1) yield

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^p}^p + \frac{4(p-1)}{p^2}\left\|\nabla|\rho|^{\frac{p}{2}}\right\|_{L^2}^2 \le \|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^p}^p, \tag{2.25}$$

by Gronwall's inequality, and taking $p\to\infty$, one has

$$
\begin{aligned}
\|\nabla\rho\|_{L^\infty_T L^\infty} &\le \|\nabla\rho_0\|_{L^\infty}\exp\left\{\int_0^{\mathrm{T}}\|\nabla u\|_{L^\infty}dt\right\}\\
&\le \|\nabla\rho_0\|_{L^\infty}\exp\left\{\left(\int_0^{\mathrm{T}}\|\nabla u\|_{L^\infty}^4dt\right)^{\frac14}\left(\int_0^{\mathrm{T}}1^{\frac43}dt\right)^{\frac34}\right\}\\
&\le \|\nabla\rho_0\|_{L^\infty}\exp\left\{\|\nabla u\|_{L^4_T L^\infty}T^{\frac34}\right\}.
\end{aligned}
$$

It follows from (2.22) that

$$
\|\nabla\rho\|_{L^\infty_T L^\infty} \le C(T).
$$

3. $H^2(\mathbb{R}^3)$ **estimates of** (u, b, ρ)

The following lemma shows that the boundedness of $\mathcal{A}(T)$ guarantees the smoothness of axisymmetric solutions to (1.4).

Lemma 2.5 *Assume* $(u_0, b_0, \rho_0) \in H^2(\mathbb{R}^3)$*,* $\Pi_0 \in L^\infty(\mathbb{R}^3)$ *and* $\nabla b_0 \in L^\infty(\mathbb{R}^3)$*. If*

$$
\mathcal{A}(T) = \|\Omega\|^2_{L^\infty_T L^2} + \|\nabla\Omega\|^2_{L^2_T L^2} < \infty,
$$

for some $0 < T < \infty$*, then the corresponding solution of system (1.4) remains smooth on* $[0, T]$*.*

Proof In the following, applying "Δ" operator to (1.1) and then taking the inner product, we have

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|^2_{L^2} + \|\Delta b\|^2_{L^2} + \|\Delta\rho\|^2_{L^2}) + \|\nabla^3 u\|^2_{L^2} + \|\nabla^3 b\|^2_{L^2} + \|\nabla^3\rho\|^2_{L^2}\\
&= -\int_{\mathbb{R}^3}\Delta u\cdot\Delta(u\cdot\nabla u)\mathrm{d}x + \int_{\mathbb{R}^3}\Delta u\cdot\Delta(b\cdot\nabla b)\mathrm{d}x + \int_{\mathbb{R}^3}\Delta u\cdot\Delta\rho e_3\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^3}\Delta b\cdot\Delta(u\cdot\nabla b)\mathrm{d}x + \int_{\mathbb{R}^3}\Delta b\cdot\Delta(b\cdot\nabla u)\mathrm{d}x - \int_{\mathbb{R}^3}\Delta\rho\cdot\Delta(u\cdot\nabla\rho)\mathrm{d}x\\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
$$

For the first term I_1, one has

$$
\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3}\Delta u\cdot(\Delta u\cdot\nabla u)\mathrm{d}x - \int_{\mathbb{R}^3}\Delta u\cdot(u\cdot\nabla\Delta u)\mathrm{d}x - 2\int_{\mathbb{R}^3}\Delta u\cdot(\nabla u\cdot\nabla^2 u)\mathrm{d}x\\
&\le 3\|\nabla u\|_{L^\infty}\|\Delta u\|^2_{L^2} + \|u\|_{L^\infty}\|\Delta u\|_{L^2}\|\Delta\nabla u\|_{L^2}\\
&\le 3\|\nabla u\|_{L^\infty}\|\Delta u\|^2_{L^2} + C\|u\|^2_{L^\infty}\|\Delta u\|^2_{L^2} + \frac18\|\Delta\nabla u\|^2_{L^2}.
\end{aligned}
$$

For the second term I_2, utilizing the integration by parts and the fact div $b = 0$ give

$$
I_2 = \int_{\mathbb{R}^3}\Delta u\cdot(\Delta b\cdot\nabla b)\mathrm{d}x + \int_{\mathbb{R}^3}\Delta u\cdot(b\cdot\nabla\Delta b)\mathrm{d}x + 2\int_{\mathbb{R}^3}\Delta u\cdot(\nabla b\cdot\nabla^2 b)\mathrm{d}x
$$

$$\begin{aligned}
&\leq 3\|\nabla b\|_{L^\infty}\|\Delta b\|_{L^2}\|\Delta u\|_{L^2}+\|b\|_{L^\infty}\|\Delta b\|_{L^2}\|\Delta\nabla u\|_{L^2}\\
&\leq C\|\nabla b\|_{L^\infty}(\|\Delta b\|_{L^2}^2+\|\Delta u\|_{L^2}^2)+C\|b\|_{L^\infty}^2\|\Delta b\|_{L^2}^2+\frac{1}{8}\|\Delta\nabla u\|_{L^2}^2.
\end{aligned}$$

The third term I_3 can be estimated as following

$$I_3=\int_{\mathbb{R}^3}\Delta u\cdot\Delta\rho e_3\mathrm{d}x\leq C\|\Delta u\|_{L^2}\|\Delta\rho\|_{L^2}.$$

The fourth term I_4 can be estimated as follows:

$$I_4=-\int_{\mathbb{R}^3}\Delta b\cdot(\Delta u\cdot\nabla b)\mathrm{d}x-\int_{\mathbb{R}^3}\Delta b\cdot(u\cdot\Delta\nabla b)\mathrm{d}x-2\int_{\mathbb{R}^3}\Delta b\cdot(\nabla u\cdot\nabla^2 b)\mathrm{d}x.$$

Integrating by parts and taking the divergence-free of u into account, we see that

$$\int_{\mathbb{R}^3}\Delta b\cdot(u\cdot\Delta\nabla b)\mathrm{d}x=0.$$

Thus,

$$I_4\leq\|\nabla b\|_{L^\infty}\|\Delta u\|_{L^2}\|\Delta b\|_{L^2}\leq\|\nabla b\|_{L^\infty}(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2).$$

The term I_5 is similar to I_1; one obtains that

$$\begin{aligned}
I_5&=\int_{\mathbb{R}^3}\Delta b\cdot(\Delta b\cdot\nabla u)\mathrm{d}x+\int_{\mathbb{R}^3}\Delta b\cdot(b\cdot\nabla\Delta u)\mathrm{d}x+2\int_{\mathbb{R}^3}\Delta b\cdot(\nabla b\cdot\Delta u)\mathrm{d}x\\
&\leq\|\nabla u\|_{L^\infty}\|\Delta b\|_{L^2}^2+\|b\|_{L^\infty}\|\Delta b\|_{L^2}\|\nabla\Delta u\|_{L^2}+\|\nabla b\|_{L^\infty}\|\Delta b\|_{L^2}\|\Delta u\|_{L^2}\\
&\leq\|\nabla u\|_{L^\infty}\|\Delta b\|_{L^2}^2+C\|b\|_{L^\infty}^2\|\Delta b\|_{L^2}^2+\frac{1}{8}\|\nabla\Delta u\|_{L^2}\\
&+\|\nabla b\|_{L^\infty}(\|\Delta b\|_{L^2}^2+\|\Delta u\|_{L^2}^2).
\end{aligned}$$

The last term I_6 is similar to I_4; we have

$$I_6\leq C\|\nabla\rho\|_{L^\infty}\|\Delta u\|_{L^2}\|\Delta\rho\|_{L^2}\leq C\|\nabla\rho\|_{L^\infty}(\|\Delta u\|_{L^2}^2+\|\Delta\rho\|_{L^2}^2).\tag{2.26}$$

Combining the above estimates, it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2+\|\Delta\rho\|_{L^2}^2)+\|\Delta\nabla u\|_{L^2}^2+\|\Delta\nabla b\|_{L^2}^2+\|\Delta\nabla\rho\|_{L^2}^2\\
&\leq C(\|\nabla u\|_{L^\infty}+\|\nabla b\|_{L^\infty}+\|\nabla\rho\|_{L^\infty})(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)+C\|\nabla u\|_{L^\infty}^2\|\Delta u\|_{L^2}^2\\
&\quad+C\|\nabla b\|_{L^\infty}^2\|\Delta b\|_{L^2}^2+C\|\nabla\rho\|_{L^\infty}^2\|\Delta\rho\|_{L^2}^2.
\end{aligned}$$

Thus, it follows from Lemmas 2.2 and 2.4, (2.21), Gronwall's inequality, and thanks to $\mathcal{A}(T) \leq \infty$, one has

$$\begin{aligned}
&\|\Delta u\|_{L_T^\infty L^2}^2 + \|\Delta b\|_{L_T^\infty L^2}^2 + \|\Delta \rho\|_{L_T^\infty L^2}^2 + \|\nabla^3 u\|_{L_T^2 L^2}^2 + \|\nabla^3 b\|_{L_T^2 L^2}^2 + \|\nabla^3 \rho\|_{L_T^2 L^2}^2 \\
&\lesssim \exp\left\{\int_0^{\mathrm{T}} (\|u\|_{L^\infty}^2 + \|b\|_{L^\infty}^2 + \|\rho\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} + \|\nabla \rho\|_{L^\infty}) dt\right\} \\
&\leq C(T).
\end{aligned}$$

Moreover, together with the basic energy estimates for (1.1), we conclude that

$$\begin{aligned}
\|u\|_{L_T^\infty H^2} + \|u\|_{L_T^2 H^3} \leq \infty, \\
\|b\|_{L_T^\infty H^2} + \|b\|_{L_T^2 H^3} \leq \infty, \\
\|\rho\|_{L_T^\infty H^2} + \|\rho\|_{L_T^2 H^3} \leq \infty.
\end{aligned}$$

Therefore, the proof of Lemma 2.5 is complete.

4. Contradiction argument

Let (u, b, ρ) be the axisymmetric local strong solution to the MHD-Boussinesq equations on $[0, T^*)$ with the axisymmetric initial data (u_0, b_0, ρ_0), where T^* is the lifespan. Next, we will prove $T^* = \infty$ by contradiction. Note that (Ω, ρ) satisfies

$$\begin{cases}
\partial_t \Omega + (u \cdot \nabla)\Omega - (\Delta + \frac{2}{r}\partial_r)\Omega = -\partial_z \Pi^2 - 2\frac{u^\theta}{r}\Phi - \frac{\partial_r \rho}{r} \\
\partial_t \rho + (u \cdot \nabla)\rho - (\Delta + \frac{2}{r}\partial_r)\rho = -\frac{2}{r}\partial_r \rho
\end{cases} \tag{2.27}$$

Let $G = \Omega - \frac{1}{2}\rho$; one has

$$\partial_t G + (u \cdot \nabla) G - (\Delta + \frac{2}{r}\partial_r) G = -\partial_z \left(\frac{b^\theta}{r}\right)^2 - 2\frac{u^\theta}{r}\Phi.$$

Using energy estimates and integration by parts, note that the boundary term should be dealt with by applying the methods introduced in Leonardi et al. (1999); Neustupa and Pokorny (2001), which can help to avoid the singularity coming from the change of variables on the z-axis. Thus, one has for any $t \in [0, T^*)$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|G\|_{L^2}^2 + \|\nabla G\|_{L^2}^2 + \int_{-\infty}^{+\infty} |G(t, r=0, z)|^2 dz \\
&= -\int_{\mathbb{R}^3} \partial_z \Pi^2 G \mathrm{d}x + \int_{\mathbb{R}^3} \partial_z \left(\frac{u^\theta}{r}\right)^2 G \mathrm{d}x \leq \|\Pi\|_{L^4}^2 \|\partial_z G\|_{L^2} + \left\|\frac{u^\theta}{r}\right\|_{L^4}^2 \|\partial_z G\|_{L^2} \\
&\leq 4\|\Pi\|_{L^4}^4 + 4\left\|\frac{u^\theta}{r}\right\|_{L^4}^4 + \frac{1}{2}\|\nabla G\|_{L^2}^2.
\end{aligned} \tag{2.28}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|G\|_{L^2}^2+\|\nabla G\|_{L^2}^2\le 8\|\Pi\|_{L^4}^4+8\left\|\frac{u^\theta}{r}\right\|_{L^4}^4. \tag{2.29}$$

In the following, we estimate $\left\|\frac{u^\theta}{r}\right\|_{L^4}$. Firstly, the equation for Λ reads

$$\partial_t\Lambda+u\cdot\nabla\Lambda-\left(\Delta+\frac{\partial_r}{r}-\frac{3}{4}\cdot\frac{1}{r^2}\right)\Lambda=-\frac{3}{2}\frac{u^r}{r}\Lambda. \tag{2.30}$$

Multiplying both sides of (2.30) by Λ^3 and integrating the resulting equation over $\mathbb{R}^3$ yield

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\|_{L^4}^4+\frac{3}{4}\|\nabla\Lambda^2\|_{L^2}^2+\frac{3}{4}\left\|\frac{u^\theta}{r}\right\|_{L^4}^4=\frac{3}{2}\int_{\mathbb{R}^3}\frac{u^r}{r}\Lambda^4\mathrm{d}x\le\frac{3}{2}\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4.$$

Hence,

$$4\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\|_{L^4}^4+12\|\nabla\Lambda^2\|_{L^2}^2+12\left\|\frac{u^\theta}{r}\right\|_{L^4}^4\le 24\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4. \tag{2.31}$$

Combining (2.29) and (2.31) leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|G\|_{L^2}^2+4\|\Lambda\|_{L^4}^4)+2\|\nabla G\|_{L^2}^2+12\|\nabla\Lambda^2\|_{L^2}^2+4\left\|\frac{u^\theta}{r}\right\|_{L^4}^4\\ \le&24\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4}^4+8\|\Pi\|_{L^4}^4.\end{aligned} \tag{2.32}$$

We estimate the right-hand-side term $\|\frac{u^r}{r}\|_{L^\infty}\|\Lambda\|_{L^4}^4$; then, one will see that with the smallness condition (1.9) in hand, $\|\frac{u^r}{r}\|_{L^\infty}\|\Lambda\|_{L^4}^4$ can be absorbed by the left-hand side of (2.32). By virtue of Lemma 2.1, it follows that

$$\left\|\frac{u^r}{r}\right\|_{L^\infty}\le C\left\|\frac{\omega^\theta}{r}\right\|_{L^2}^{\frac{1}{2}}\left\|\partial_z\frac{\omega^\theta}{r}\right\|_{L^2}^{\frac{1}{2}}\le C\|\Omega\|_{L^2}^{\frac{1}{2}}\|\partial_z\Omega\|_{L^2}^{\frac{1}{2}}. \tag{2.33}$$

Using the Hölder's inequality, it is obvious to see

$$\begin{aligned}\|\Lambda\|_{L^4}^4&=\int_{\mathbb{R}^3}\frac{(u^\theta)^4}{r^2}\mathrm{d}x=\int_{\mathbb{R}^3}\left(\frac{u^\theta}{r}\right)^3(ru^\theta)\mathrm{d}x\\&\le\left\|\frac{u^\theta}{r}\right\|_{L^4}^3\|\Gamma\|_{L^4}\le\left\|\frac{u^\theta}{r}\right\|_{L^4}^3\|\Gamma\|_{L^2}^{\frac{1}{2}}\|\Gamma\|_{L^\infty}^{\frac{1}{2}}\end{aligned}$$

$$\leq \left\| \frac{u^\theta}{r} \right\|_{L^4}^3 \|\Gamma_0\|_{L^2}^{\frac{1}{2}} \|\Gamma_0\|_{L^\infty}^{\frac{1}{2}}. \tag{2.34}$$

Inserting (2.33) and (2.34) into (2.32), we can obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|G\|_{L^2}^2 + 4\|\Lambda\|_{L^4}^4\right) + \|\nabla G\|_{L^2}^2 + 12\|\nabla\Lambda^2\|_{L^2}^2 + 4\left\|\frac{u^\theta}{r}\right\|_{L^4}^4 \\ &\leq C\|\Omega\|_{L^2}^{\frac{1}{2}}\|\partial_z\Omega\|_{L^2}^{\frac{1}{2}}\left\|\left(\frac{u^\theta}{r}\right)^2\right\|_{L^4}^{\frac{3}{2}}\|\Gamma\|_{L^4} + 8\|\Pi\|_{L^4}^4 \\ &\leq C\|\Omega|_{L^2}^{\frac{1}{2}}\|\Gamma\|_{L^4}\left(\|\partial_z\Omega\|_{L^2}^2 + \left\|\frac{u^\theta}{r}\right\|_{L^4}^4\right) + 8\|\Pi\|_{L^4}^4 \\ &\leq C\|\Omega\|_{L^2}^{\frac{1}{2}}\|\Gamma_0\|_{L^2}^{\frac{1}{2}}\|\Gamma_0\|_{L^\infty}^{\frac{1}{2}}\left(\|\nabla G\|_{L^2}^2 + \left\|\frac{u^\theta}{r}\right\|_{L^4}^4 + \|\nabla\rho\|_{L^2}^2\right) + 8\|\Pi\|_{L^4}^4. \end{aligned} \tag{2.35}$$

We now define a finite time T_0 as

$$\sup\left\{t > 0 \middle| \|G(t,\cdot)\|_{L^2}^2 + \|\nabla G\|_{L_t^2 L^2}^2 + 4\,\|\Lambda(t,\cdot)\|_{L^4}^4 \leq 2\delta_0\right\} := T_0 < \infty, \tag{2.36}$$

where

$$\delta_0 := \|G_0\|_{L^2}^2 + 4\,\|\Lambda_0\|_{L^4}^4 + C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2 + \|\rho_0\|_{L^2}^2.$$

Indeed, for any $0 \leq t < T_0$, we can obtain

$$\|\Omega(t) = G(t) + \frac{1}{2}\rho(t)\|_{L^2}^2 \leq 4(\|G_0\|_2^2 + \|\Lambda_0\|_{L^4}^4) + \|\rho_0\|_{L^2}^2.$$

Considering the equation for Π gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi + u\cdot\nabla\Pi - \left(\Delta + 2\frac{\partial_r}{r}\right)\Pi = 0.$$

It is not difficult to get for $2 \leq p \leq \infty$ that

$$\|\Pi(t,\cdot)\|_{L^p} \leq \|\Pi_0\|_{L^p}.$$

On the other hand, one has the following uniform estimate

$$\int_0^{\mathrm{T}} \|\Pi\|_{L^4}^4 \mathrm{d}t \leq \int_0^{\mathrm{T}} \left(\|\Pi\|_{L^3}^{\frac{1}{2}}\|\Pi\|_{L^6}^{\frac{1}{2}}\right)^4 \mathrm{d}t \leq \|\Pi_0\|_{L^3}^2 \int_0^{\mathrm{T}} \|\Pi\|_{L^6}^2 \mathrm{d}t$$

$$\leq \|\Pi_0\|_{L^3}^2 \int_0^T \|\nabla \Pi\|_{L^2}^2 \mathrm{d}t \leq C\|\Pi_0\|_{L^3}^2 \|\Pi_0\|_{L^2}^2. \tag{2.37}$$

Integrating (2.35) with respect to time variable over $[0, T_0)$, one has:

$$\begin{aligned}
&\|G\|_{L^\infty_{T_0}L^2}^2 + 4\|\Lambda\|_{L^\infty_{T_0}L^4}^4 + 2\|\nabla G\|_{L^2_{T_0}L^2}^2 + 12\|\nabla \Lambda^2\|_{L^2_{T_0}L^2}^2 + 4\left\|\frac{u^\theta}{r}\right\|_{L^4_{T_0}L^4}^4 \\
&\leq \|G_0\|_{L^2}^2 + C\left(\|G_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4 + C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2 + \|\rho_0\|_{L^2}^2\right)^{\frac{1}{4}} \\
&\times \|\Gamma_0\|_{L^2}^{\frac{1}{2}}\|\Gamma_0\|_{L^\infty}^{\frac{1}{2}}\left(\|\nabla G\|_{L^2L^2}^2 + \left\|\frac{u^\theta}{r}\right\|_{L^4_{T_0}L^4}^4\right) + 4\|\Lambda_0\|_{L^4}^4 + C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2.
\end{aligned}$$

By condition (1.9) in Theorem 1.1, one has

$$\|\Gamma_0\|_{L^\infty} \leq \delta\left(\|G_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4 + C\|\Pi_0\|_{L^2}^2\|\Pi_0\|_{L^3}^2 + \|\rho_0\|_{L^2}^2\right)^{-\frac{1}{2}}\|\Gamma_0\|_{L^2}^{-1},$$

when the positive constant δ is small enough, such that

$$C\left(\|G_0\|_{L^2}^2 + \|\Lambda_0\|_{L^4}^4 + C\|\Pi_0\|_{L^2}^2\|\Pi_0\|_{L^3}^2 + \|\rho_0\|_{L^2}^2\right)^{\frac{1}{4}}\|\Gamma_0\|_{L^2}^{\frac{1}{2}}\|\Gamma_0\|_{L^\infty}^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}} \leq \frac{1}{2}.$$

Therefore, we conclude that

$$\begin{aligned}
&\|G\|_{L^\infty_{T_0}L^2}^2 + 4\|\Lambda\|_{L^\infty_{T_0}L^4}^4 + 2\|\nabla G\|_{L^2_{T_0}L^2}^2 \\
&\leq \|G_0\|_{L^2}^2 + 4\|\Lambda_0\|_{L^2}^2 + C\|\Pi_0\|_{L^2}^2\|\Pi_0\|_{L^3}^2 + \|\rho_0\|_{L^2}^2,
\end{aligned}$$

This contradicts the definition of (2.36). In the following, multiplying the ρ equation of (2.27) by ρ $(2 \leq p \leq \infty)$ and integrating over $\mathbb{R}^3$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p + \frac{4(p-1)}{p^2}\left\|\nabla|\rho|^{\frac{p}{2}}\right\|_{L^2}^2 = 0,$$

taking $p = 2$ and integrating the above inequality in time variable over $[0, T_0)$ yields

$$\|\rho\|_{L^\infty_{T_0}L^2}^2 + 2\|\nabla\rho\|_{L^2_{T_0}L^2}^2 \leq \|\rho_0\|_{L^2}^2.$$

Therefore, the global existence of axisymmetric strong solutions follows by Lemma 2.5 (see also Theorem 2.5.5 in Zheng 2004), so we completed the proof of the first case of Theorem 1.1.

Next, we deal with (2.32) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|G\|_{L^2}^2 + 4\|\Lambda\|_{L^4}^4\right) + 2\|\nabla G\|_{L^2}^2 + 12\|\nabla\Lambda^2\|_{L^2}^2 + 4\left\|\frac{u^\theta}{r}\right\|_{L^4}^4$$

$$
\begin{aligned}
&\le 24\left\|\frac{u^r}{r}\right\|_{L^\infty}\|\Lambda\|_{L^4(r\le\epsilon)}^4+24\int_{r\ge\epsilon}\left|\frac{u^r}{r}\Lambda^4\right|\mathrm{d}x+8\|\Pi\|_{L^4}^4\\
&\le C\|\Omega\|_{L^2}^{\frac12}\|\partial_z\Omega\|_{L^2}^{\frac12}\left\|\frac{u^\theta}{r}\right\|_{L^4}^3\|\Gamma\|_{L^2}^{\frac12}\|\Gamma\|_{L^\infty(r\le\epsilon)}^{\frac12}\\
&\quad+\frac{1}{\epsilon^4}\left\|\frac{u^r}{r}\right\|_{L^2}\left\|\frac{u^\theta}{r}\right\|_{L^2}\|\Gamma\|_{L^\infty(r\ge\epsilon)}^3+8\|\Pi\|_{L^4}^4\\
&\le C\|\Omega\|_{L^2}^{\frac12}\|\Gamma\|_{L^2}^{\frac12}\|\Gamma\|_{L^\infty(r\le\epsilon)}^{\frac12}\left(\|\nabla G\|_{L^2}^2+\left\|\frac{u^\theta}{r}\right\|_{L^4}^4+\|\nabla\rho\|_{L^2}^2\right)\\
&\quad+\frac{1}{\epsilon^4}\left\|\frac{u^r}{r}\right\|_{L^2}\left\|\frac{u^\theta}{r}\right\|_{L^2}\|\Gamma\|_{L^\infty(r\ge\epsilon)}^3+8\|\Pi\|_{L^4}^4.
\end{aligned}\tag{2.38}
$$

Let’s define

$$
\sup\left\{t>0\Big|\|G(t,\cdot)\|_{L^2}^2+4\|\Lambda(t,\cdot)\|_{L^4}^4\le 2\Psi_0^2\right\}:=T_1.\tag{2.39}
$$

Integrating (2.38) with respect to time variable over $[0,T_1)$ yields

$$
\begin{aligned}
&\|G\|_{L_{T_1}^\infty L^2}^2+4\|\Lambda\|_{L_{T_1}^\infty L^4}^4+\|\nabla G\|_{L_{T_1}^2L^2}^2+12\|\nabla\Lambda^2\|_{L_{T_1}^2L^2}^2+4\left\|\frac{u^\theta}{r}\right\|_{L_{T_1}^4L^4}^4\\
&\le C\|\Omega\|_{L_{T_1}^\infty L^2}^{\frac12}\|\Gamma_0\|_{L^2}^{\frac12}\sup_{t\in(0,T_1)}\|\Gamma\|_{L^\infty(r\le\epsilon)}^{\frac12}\left(\|\nabla G\|_{L_{T_1}^2L^2}^2+\left\|\frac{u^\theta}{r}\right\|_{L_{T_1}^4L^4}^4+\|\nabla\rho\|_{L^2}^2\right)\\
&\quad+\frac{1}{\epsilon^4}\left(\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2+\|\rho_0\|_{L^2}^2\right)\|\Gamma_0\|_{L^\infty}^3\\
&\quad+C\|\Pi_0\|_{L^3}^2\|\Pi_0\|_{L^2}^2+\|G_0\|_{L^2}^2+4\,\|\Lambda_0\|_{L^4}^4.
\end{aligned}
$$

By condition (1.10) and (2.39), we obtain

$$
\|G\|_{L_{T_1}^\infty L^2}^2+4\|\Lambda\|_{L_{T_1}^\infty L^4}^4\le\Psi_0^2,
$$

similar as the first case, one can conclude that the axisymmetric strong solutions exists globally. Therefore, the proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

In this section, we are devoted to the proof of Theorem 1.2. To this end, we first give an estimate of the global decay of $\|u(x,t)\|_{L^2}$ (see 3.1), then establish decay estimates for the swirl components of velocity and magnetic fields, and find out that the swirl components decay faster for a class of initial data. Following the ideas of Brandolese and Schonbek (2012), Chen et al. (2017b), Liu and Han (2020) and using the Fourier splitting method in Schonbek (1985), one can obtain the following estimates for the

MHD-Boussinesq equations with slight modifications of those for the Boussinesq equations in Fang et al. (2018), and we are not going to repeat it here.

If $\rho_0 \in L^1 \cap L^p$ for any $p \in [1, \infty)$, then

$$\|\rho(t)\|_{L^p} \leq C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}.$$

Furthermore, if ρ_0 satisfies

$$\int_{\mathbb{R}^3} |\rho_0(x)||x|\mathrm{d}x \leq \infty, \quad \int_{\mathbb{R}^3} \rho_0(x)\mathrm{d}x = 0 \ \text{ and } \ \|\rho_0\|_{L^1} \leq \epsilon_0,$$

where ϵ_0 is a small positive constant independent of the initial data, then

$$\|\rho(t)\|_{L^2}^2 \leq C\langle t\rangle^{-\frac{5}{2}}.$$

Moreover, for $u_0 \in L^{\frac{3}{2}}$ and $b_0 \in L^2$, one can deduce the following decay estimates, whose proof is very similar to the one in Fang et al. (2018), and we also skip the details here.

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\rho(t)\|_{L^2}^2 + \langle t\rangle\|\nabla u(t)\|_{L^2}^2 + \langle t\rangle\|\nabla b(t)\|_{L^2}^2 \\ &\quad + \langle t\rangle\|\nabla\rho(t)\|_{L^2}^2 + t\langle t\rangle\|(\partial_t u, \Delta u)\|_{L^2}^2 + t\langle t\rangle\|(\partial_t b, \Delta b)\|_{L^2}^2 \\ &\quad + t\langle t\rangle\|(\partial_t\rho, \Delta\rho)\|_{L^2}^2 \leq C\langle t\rangle^{-\frac{1}{2}}. \end{aligned} \tag{3.1}$$

Next, we focus on the decay estimates for the components which don't appear in the Boussinesq equations.

- Decay estimates for $\|\Gamma\|_{L^2}^2$ and $\|\Pi\|_{L^2}^2$

For $\Gamma_0 \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, where $p \in [1, \infty)$, one has

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma_0\|_{L^p}. \tag{3.2}$$

Moreover, multiplying the Γ equation of (1.8) by Γ and integrating the resulting equation over $\mathbb{R}^3$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma\|_{L^2}^2 + \|\nabla\Gamma\|_{L^2}^2 = 0. \tag{3.3}$$

By the Sobolev embedding theorem, we obtain

$$\|\Gamma\|_{L^2} \leq C\|\Gamma\|_{L^1}^{\frac{2}{5}}\|\nabla\Gamma\|_{L^2}^{\frac{3}{5}} \leq C\|\Gamma_0\|_{L^1}^{\frac{2}{5}}\|\nabla\Gamma\|_{L^2}^{\frac{3}{5}} \leq C\|\nabla\Gamma\|_{L^2}^{\frac{3}{5}}. \tag{3.4}$$

From (3.3) and (3.4), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma\|_{L^2}^2 \leq C(\|\Gamma\|_{L^2}^2)^{\frac{5}{3}},$$

and

$$\|\Gamma\|_{L^2}^2 \le C\langle t\rangle^{-\frac{3}{2}}.$$

Similarly, for $\Pi_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with $p \in [1, \infty)$, we can obtain the following decay estimate

$$\|\Pi\|_{L^2}^2 \le C\langle t\rangle^{-\frac{3}{2}}.$$

- Decay estimates for $\|u^\theta\|_{L^2}^2$ and $\|b^\theta\|_{L^2}^2$

Multiplying $(1.4)_2$ and $(1.4)_4$ by u^θ, b^θ, respectively, and applying Lemma 2.1 and the decay estimates in (3.1), one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\theta\|_{L^2}^2 + \|\nabla u^\theta\|_{L^2}^2 + \left\|\frac{u^\theta}{r}\right\|_{L^2}^2 &\le \left\|\frac{u^r}{r}\right\|_{L^2}^4 \|u^\theta\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^4\|u^\theta\|_{L^2}^2 \\ &\le C\langle t\rangle^{-3}\langle t\rangle^{-\frac{1}{2}} \le C\langle t\rangle^{-\frac{7}{2}}.\end{aligned} \tag{3.5}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b^\theta\|_{L^2}^2 + \|\nabla b^\theta\|_{L^2}^2 + \|\Pi\|_{L^2}^2 \le \left\|\frac{u^r}{r}\right\|_{L^2}^4 \|b^\theta\|_{L^2}^2 \le C\langle t\rangle^{-\frac{7}{2}}. \tag{3.6}$$

Set $S(t) = \{x \mid r \le g(t)^{-1}\}$, $g(t) = \sqrt{\alpha}(1+t)^{-\frac{1}{2}}$, $\alpha \ge \frac{5}{2}$. It follows from (3.5) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\theta\|_{L^2}^2 + g^2(t)\|u^\theta\|_{L^2}^2 &\le \frac{\mathrm{d}}{\mathrm{d}t}\|u^\theta\|_{L^2}^2 + \int_{S(t)}\left|\frac{u^\theta}{r}\right|^2 \mathrm{d}x + g^2(t)\int_{S^c(t)}\frac{|ru^\theta|^2}{r^2}\mathrm{d}x \\ &\le C\langle t\rangle^{-\frac{7}{2}} + g^4(t)\|ru^\theta\|_{L^2}^2 \le C\langle t\rangle^{-\frac{7}{2}}.\end{aligned}$$

Then,

$$e^{\int_0^t g^2(\tau)\mathrm{d}\tau}\|u^\theta(t)\|_{L^2}^2 \le \|u_0^\theta\|_{L^2}^2 + C\int_0^t e^{\int_0^\tau g^2(s)\mathrm{d}s}\langle\tau\rangle^{-\frac{7}{2}}\mathrm{d}\tau.$$

Since $e^{\int_0^t g^2(\tau)\mathrm{d}\tau} \approx \langle t\rangle^\alpha$ and $\alpha > \frac{5}{2}$, one has

$$\|u^\theta(t)\|_{L^2}^2 \le C\langle t\rangle^{-\frac{5}{2}}. \tag{3.7}$$

Similarly,

$$\|b^\theta(t)\|_{L^2}^2 \le C\langle t\rangle^{-\frac{5}{2}}. \tag{3.8}$$

- Decay estimates for $\|\nabla(u^\theta e_\theta)\|_{L^2}^2$ and $\|\nabla(b^\theta e_\theta)\|_{L^2}^2$

Indeed, one has

$$\|\nabla(u^\theta e_\theta)\|_{L^2}^2 = \|\nabla u^\theta\|_{L^2}^2 + \left\|\frac{u^\theta}{r}\right\|_{L^2}^2 = \|\omega^r\|_{L^2}^2 + \|\omega^z\|_{L^2}^2,$$
$$\|\nabla(b^\theta e_\theta)\|_{L^2}^2 = \|\nabla b^\theta\|_{L^2}^2 + \|\Pi\|_{L^2}^2 = \|j^r\|_{L^2}^2 + \|j^z\|_{L^2}^2,$$

and

$$\Delta(u^\theta e_\theta) = \left(\Delta - \frac{1}{r^2}\right) u^\theta e_\theta, \quad \left(\Delta - \frac{1}{r^2}\right) u^\theta = \partial_r \omega^z - \partial_z \omega^r,$$
$$\Delta(b^\theta e_\theta) = \left(\Delta - \frac{1}{r^2}\right) b^\theta e_\theta, \quad \left(\Delta - \frac{1}{r^2}\right) b^\theta = \partial_r j^z - \partial_z j^r.$$

Using (1.7) and the equation for b^θ, one has

$$\begin{aligned}\left\|\left(\Delta - \frac{1}{r^2}\right) b^\theta\right\|_{L^2} &\leq \|\partial_t b^\theta\|_{L^2} + \|(u^r j^r - u^z j^r)\|_{L^2} \\ &\leq 2\|\partial_t b^\theta\|_{L^2} + C\|\nabla u\|_{L^2}^2 (\|j^r\|_{L^2} + \|j^z\|_{L^2}).\end{aligned} \tag{3.9}$$

Integrating (3.6) over time interval $\left[\frac{t}{2}, t\right]$, using Gronwall's inequality and (3.8), we obtain

$$\begin{aligned}&\|b^\theta(t)\|_{L^2}^2 + \int_{\frac{t}{2}}^t \left(\|\nabla b^\theta(\tau)\|_{L^2}^2 + \left\|\frac{b^\theta(\tau)}{r}\right\|_{L^2}^2\right) \mathrm{d}\tau \\ &\leq C \left\|b^\theta\left(\frac{t}{2}\right)\right\|_{L^2}^2 \exp C \left(\int_{\frac{t}{2}}^t \|\nabla u(\tau)\|_{L^2}^4\right) \mathrm{d}\tau \\ &\leq C \left\|b^\theta\left(\frac{t}{2}\right)\right\|_{L^2}^2 \leq C\langle t\rangle^{-\frac{5}{2}}.\end{aligned} \tag{3.10}$$

Multiplying $(1.4)_4$ by $\partial_t b^\theta$ and integrating by parts lead to:

$$\begin{aligned}&\|\partial_t b^\theta\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla b^\theta\|_{L^2}^2 + \left\|\frac{b^\theta}{r}\right\|_{L^2}^2\right) = -\int_{\mathbb{R}^3} (u^r j^z - u^z j^r) b_t^\theta \mathrm{d}x \\ &\leq C\|\partial_t b^\theta\|_{L^2} \|\nabla u\|_{L^2} \left(\|\nabla j^z\|_{L^2}^{\frac{1}{2}} \|j^z\|_{L^2}^{\frac{1}{2}} + \|\nabla j^r\|_{L^2}^{\frac{1}{2}} \|j^r\|_{L^2}^{\frac{1}{2}}\right) \\ &\leq \frac{1}{4}\left(\|\partial_t b^\theta\|_{L^2}^2 + \|\nabla j^r\|_{L^2}^2 + \|\nabla j^z\|_{L^2}^2\right) \\ &\quad + C\|\nabla u\|_{L^2}^4 \left(\|j^r\|_{L^2}^2 + \|j^z\|_{L^2}^2\right),\end{aligned} \tag{3.11}$$

and we also have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|j^r\|_{L^2}^2+\|j^z\|_{L^2}^2\right)+\|\nabla j^r\|_{L^2}^2+\|\nabla j^z\|_{L^2}^2+\left\|\frac{j^r}{r}\right\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(j^r\partial_r+j^z\partial_z)u^r j^r+(j^r\partial_r+j^z\partial_z)u^z j^z\mathrm{d}x\\
&\le\frac{1}{4}\left(\|\nabla(j^r,j^z)\|_{L^2}^2+\left\|\frac{j^r}{r}\right\|_{L^2}^2\right)+C\|\nabla u\|_{L^2}^4\left(\|j^r\|_{L^2}^2+\|j^z\|_{L^2}^2\right).
\end{aligned}\tag{3.12}$$

Set $f_1(t)=\|j^r(t)\|_{L^2}^2+\|j^z(t)\|_{L^2}^2$, from (3.10), it satisfies that

$$\int_{\frac{t}{2}}^t f_1(\tau)\mathrm{d}\tau\le C\langle t\rangle^{-\frac{5}{2}}.\tag{3.13}$$

Combining (3.9), (3.11), and (3.12), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}f_1(t)+\left\|\left(\Delta-\frac{1}{r^2}\right)b^\theta(t)\right\|_{L^2}^2+\|\partial_t b^\theta\|_{L^2}^2\le C\|\nabla u(t)\|_{L^2}^4 f_1(t).$$

Multiplying the above inequality by $(t-s)$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}((t-s)f_1(t))\le f_1(t)+C\|\nabla u\|_{L^2}^4(t-s)f_1(t),\tag{3.14}$$

and applying Gronwall's inequality gives

$$(t-s)f_1(t)\le\int_s^t f_1(\tau)\mathrm{d}\tau\exp\left(C\int_s^t\|\nabla u(\tau)\|_{L^2}^4\tau\right)\le C\int_s^t f_1(\tau)\mathrm{d}\tau.$$

Choosing $s=\frac{t}{2}$, from (3.13) we have

$$f_1(t)\le Ct^{-1}\int_{\frac{t}{2}}^t f_1(\tau)\mathrm{d}\tau\exp\left(C\int_{\frac{t}{2}}^t\|\nabla u(\tau)\|_{L^2}^4\mathrm{d}\tau\right)\le Ct^{-1}\langle t\rangle^{-\frac{5}{2}}.$$

Therefore, there holds

$$\|\nabla(b^\theta e_\theta)(t)\|_{L^2}^2=f_1(t)\le C\langle t\rangle^{-\frac{7}{2}}.$$

Similarly, we can obtain

$$\|\nabla(u^\theta e_\theta)(t)\|_{L^2}^2\le C\langle t\rangle^{-\frac{7}{2}}.$$

- Decay estimates for $\|\partial_t u^\theta\|_{L^2}^2 + \left\|\left(\Delta - \frac{1}{r^2}\right) u^\theta\right\|_{L^2}^2$ and $\|\partial_t b^\theta\|_{L^2}^2 + \left\|\left(\Delta - \frac{1}{r^2}\right) b^\theta\right\|_{L^2}^2$

Applying Gronwall's inequality to (3.14) over $\left[\frac{t}{2}, t\right]$, we have

$$\begin{aligned} & f_1(t) + \int_{\frac{t}{2}}^t \left(\left\|\left(\Delta - \frac{1}{r^2}\right) b^\theta\right\|_{L^2}^2 + \|\partial_t b^\theta\|_{L^2}^2\right) \mathrm{d}\tau \\ & \quad \le C f_1\left(\frac{t}{2}\right) \exp\left(C \int_{\frac{t}{2}}^t \|\nabla u\|_{L^2}^4 \mathrm{d}\tau\right) \le C f_1\left(\frac{t}{2}\right) \le C\langle t\rangle^{-\frac{7}{2}}. \end{aligned} \tag{3.15}$$

Taking the time derivative to $(1.4)_4$, one has

$$\partial_{tt} b^\theta + u \cdot \nabla \partial_t b^\theta - \left(\Delta - \frac{1}{r^2}\right) \partial_t b^\theta = -\partial_t b \cdot \nabla b^\theta - \partial_t \left(\frac{u^r b^\theta}{r}\right).$$

Taking L^2 inner product of the above equation with $\partial_t b^\theta$, and using incompressibility condition, we have

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t b^\theta\|_{L^2}^2 + \|\nabla \partial_t b^\theta\|_{L^2}^2 + \left\|\frac{\partial_t b^\theta}{r}\right\|_{L^2}^2 \\ & = \int_{\mathbb{R}^3} \left(-\partial_t u^r j^z \partial_t b^\theta + \partial_t u^z j^r \partial_t b^\theta - \frac{u^r}{r}(\partial_t b^\theta)^2\right) \mathrm{d}x \\ & \le \|\partial_t u\|_{L^2}^2 \left(\|j^z\|_{L^3}^2 + \|j^r\|_{L^3}^2\right) + C\|\partial_t b^\theta\|_{L^2}^2 \|\nabla u\|_{L^2}^4 \\ & \quad + \frac{1}{2}\left\|\frac{\partial_t b^\theta}{r}\right\|_{L^2}^2 + \frac{1}{2}\|\nabla \partial_t b^\theta\|_{L^2}^2. \end{aligned} \tag{3.16}$$

Next, it follows that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t b^\theta\|_{L^2}^2 + \|\nabla \partial_t b^\theta\|_{L^2}^2 + \left\|\frac{\partial_t b^\theta}{r}\right\|_{L^2}^2 \le C\|\partial_t b^\theta\|_{L^2}^2 \|\nabla u\|_{L^2}^4 \\ & \quad + C\|\partial_t u\|_{L^2}^2 \left(\|\nabla b\|_{L^2}\|\partial_t b^\theta\|_{L^2} + \|\nabla u\|_{L^2}^4\right). \end{aligned}$$

Multiplying the above inequality by $(t-s)$ and using Gronwall's inequality on $[s, t]$, we obtain

$$\begin{aligned} & (t-s)\|\partial_t b^\theta(t)\|_{L^2}^2 \\ & \quad \le C\left(\int_s^t \|\partial_t b^\theta\|_{L^2}^2 + (\tau - s)\left(\|\nabla u\|_{L^2}^4 + \|\nabla b\|_{L^2}\|\partial_t b^\theta\|_{L^2}\right)\|\partial_t b\|_{L^2}^2 \mathrm{d}\tau\right) \\ & \quad \cdot \exp\left(C\int_s^t \|\nabla u\|_{L^2}^4 \mathrm{d}\tau\right). \end{aligned} \tag{3.17}$$

Taking $s = \frac{t}{2}$ and applying (3.15), one has

$$t\|\partial_t b^\theta(t)\|_{L^2}^2 \le C\left(\int_s^t \|\partial_t b^\theta(\tau)\|_{L^2}^2 + (\tau - s)\|b_t\|_{L^2}^2(\|\nabla b\|_{L^2}\|\partial_t b^\theta\|_{L^2} + \|\nabla u\|_{L^2}^4)\mathrm{d}\tau\right)$$
$$\le C\left(\langle t\rangle^{-\frac{7}{2}} + t \sup_{\tau\in[s,t]}(\|\nabla u\|_{L^2}^4 + \|\nabla b\|_{L^2}\|\partial_t b^\theta\|_{L^2})\int_s^t (\tau - s)\|\partial_t b\|_{L^2}^2\mathrm{d}\tau\right)$$
$$\le C\left(\langle t\rangle^{-\frac{7}{2}} + (\langle t\rangle^{-5} + \langle t\rangle^{-\frac{5}{4}}t^{-\frac{1}{2}}\langle t\rangle^{-\frac{5}{4}})\langle t\rangle^{-\frac{3}{2}}\right) \le C\langle t\rangle^{-\frac{7}{2}},$$

and from (3.9), there holds

$$\|\partial_t b^\theta(t)\|_{L^2}^2 + \left\|\left(\Delta - \frac{1}{r^2}\right)b^\theta(t)\right\|_{L^2}^2 \le Ct^{-1}\langle t\rangle^{-\frac{7}{2}}, \quad \forall t > 0.$$

Similarly, we can also obtain

$$\|\partial_t u^\theta(t)\|_{L^2}^2 + \left\|\left(\Delta - \frac{1}{r^2}\right)u^\theta(t)\right\|_{L^2}^2 \le Ct^{-1}\langle t\rangle^{-\frac{7}{2}}, \quad \forall t > 0.$$

Therefore, we complete the proof of Theorem 1.2.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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