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On a Two-Species Attraction–Repulsion Chemotaxis System with Nonlocal Terms

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Abstract
This paper deals with a two-species attraction–repulsion chemotaxis system

$$\begin{cases} u_t = d_1\Delta u - \xi_1\nabla\cdot(u\nabla v) + \chi_1\nabla\cdot(u\nabla z) + g_1(u,w), & (x,t)\in\Omega\times(0,\infty), \\ \tau v_t = d_2\Delta v + w - v, & (x,t)\in\Omega\times(0,\infty), \\ w_t = d_3\Delta w - \xi_2\nabla\cdot(w\nabla z) + \chi_2\nabla\cdot(w\nabla v) + g_2(u,w), & (x,t)\in\Omega\times(0,\infty), \\ \tau z_t = d_4\Delta z + u - z, & (x,t)\in\Omega\times(0,\infty) \end{cases}$$

under homogeneous Neumann boundary conditions in a smoothly bounded domain $\Omega\subset\mathbb{R}^n$ for $n\geq 1$, where $\tau\in\{0,1\}$, the parameters $d_i(i=1,2,3,4)$, ξ_j, $\chi_j(j=1,2)$ are positive and the kinetic terms $g_1(u,w)$, $g_2(u,w)$ satisfy

$$\begin{cases} g_1(u,w) = u\left(a_0 - a_1u - a_2w - a_3\int_\Omega u\mathrm{d}x - a_4\int_\Omega w\mathrm{d}x\right), \\ g_2(u,w) = w\left(b_0 - b_1u - b_2w - b_3\int_\Omega u\mathrm{d}x - b_4\int_\Omega w\mathrm{d}x\right) \end{cases}$$

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with $a_0, a_1, b_0, b_2 > 0, a_2, a_3, a_4, b_1, b_3, b_4 \in \mathbb{R}$. It is shown that under some suitable parameter conditions, the above system possesses a unique global and uniformly bounded solution in any spatial dimension. Moreover, we investigate the asymptotic stability of solutions under the locally intraspecific competition and globally interspecific cooperation. Finally, we present some numerical simulations, which not only support our analytically theoretical results, but also find some new and interesting phenomena.

Keywords Two-species · Attraction–repulsion · Boundedness · Stability · Nonlocal kinetics

Mathematics Subject Classification 35B35 · 35B40 · 35K15 · 35K55 · 92C17

1 Introduction

Chemotaxis is the biochemical process through which the directed movement of a cell or organism responses to the concentration gradient of a chemical signal. It plays a significant role in a wide range of biological applications, such as pattern formation Budrene and Berg (1991), wound healing Petter et al. (2003), embryonic development Li and Muneoka (1999) and blood vessel formation Chaplain and Logas (2005). The little movement is referred to as chemoattraction (i.e., positive chemotaxis) if the cells move toward the increasing signal concentration, whereas chemorepulsion (i.e., negative chemotaxis) whenever the cells move away from the direction of increased signal concentration (see Hillen and Painter (2009); Hazelbauer (1979)). In order to describe the aggregation of microglia observed in Alzhemer's disease, the single-species attraction–repulsion chemotaxis model was proposed by Luca et al. (2003). A particular core of this model is, in addition to random motion, that the cells can produce both chemoattractants and chemorepellents.

In this paper, we consider the following two-species attraction–repulsion chemotaxis system with nonlocal kinetics:

$$\begin{cases} u_t = d_1\Delta u - \xi_1\nabla\cdot(u\nabla v) + \chi_1\nabla\cdot(u\nabla z) + g_1(u,w), & (x,t)\in\Omega\times(0,\infty),\\ \tau v_t = d_2\Delta v + w - v, & (x,t)\in\Omega\times(0,\infty),\\ w_t = d_3\Delta w - \xi_2\nabla\cdot(w\nabla z) + \chi_2\nabla\cdot(w\nabla v) + g_2(u,w), & (x,t)\in\Omega\times(0,\infty),\\ \tau z_t = d_4\Delta z + u - z, & (x,t)\in\Omega\times(0,\infty),\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & (x,t)\in\partial\Omega\times(0,\infty),\\ (u,\tau v,w,\tau z)(x,0) = (u_0(x),\tau v_0(x),w_0(x),\tau z_0(x)), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a smoothly bounded domain, $\tau \in \{0, 1\}, d_i > 0 (i = 1, 2, 3, 4), \xi_j, \chi_j > 0 (j = 1, 2)$ and the kinetic terms g_1, g_2 satisfy

$$\begin{cases} g_1(u,w) = u\left(a_0 - a_1 u - a_2 w - a_3 \int_\Omega u\mathrm{d}x - a_4 \int_\Omega w\mathrm{d}x\right), \\ g_2(u,w) = w\left(b_0 - b_1 u - b_2 w - b_3 \int_\Omega u\mathrm{d}x - b_4 \int_\Omega w\mathrm{d}x\right) \end{cases} \tag{1.2}$$

with $a_0, a_1, b_0, b_2 > 0, a_2, a_3, a_4, b_1, b_3, b_4 \in \mathbb{R}$. Under the influences of random diffusion (i.e., $d_1\Delta u, d_2\Delta w$), chemoattractant (i.e., $-\xi_1\nabla\cdot(u\nabla v), -\xi_2\nabla\cdot(w\nabla z)$) and chemorepellent (i.e., $+\chi_1\nabla\cdot(u\nabla z), +\chi_2\nabla\cdot(w\nabla v)$), the movements of the two populations in system (1.1) become more complex. Furthermore, we also consider that both populations reproduce and compete (or cooperate) themselves, and mutually compete (or cooperate) with the other. These characteristics enable system (1.1) to describe more abundant and interesting biological phenomena between two species. Such as system (1.1) can be used to describe the differential chemotaxis driven cell sorting process (Ref. Painter (2009) and the references therein). Here, $u(x,t)$, $w(x,t)$ denote the densities of two different species, respectively, $v(x,t)$, $z(x,t)$ represent the concentrations of the chemicals produced by $w(x,t)$ and $u(x,t)$, separately. $d_i (i = 1,2,3,4)$ are the random diffusion coefficients for species and chemicals, ξ_1, ξ_2 represent the chemoattraction sensitivity coefficients, and χ_1, χ_2 refer to the chemorepulsion sensitivity coefficients. The parameters $a_0, b_0 > 0$ represent the intrinsic growth of species, and $a_1, b_2 > 0$ describe locally intraspecific competition. The parameters a_2 and b_1 describe the local influence of each species in interspecific population. Each subspecies globally competes if a_3 and b_4 are positive and globally cooperates if a_3 and b_4 are negative. When the coefficients a_4, b_3 represent the strength of nonlocal interspecific competition provided that $a_4, b_3 > 0$. When $a_4, b_3 < 0$, they represent the strength of nonlocal interspecific cooperation. The nonnegative initial data $(u_0, \tau v_0, w_0, \tau z_0)$ are nonnegative and satisfy

$$(u_0, \tau v_0, w_0, \tau z_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) \times C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega). \tag{1.3}$$

Now, we mention some previous contributions to understand the development of the system (1.1).

1.1 Single-Species Chemotaxis System

The well-known mathematical model was initially proposed by Keller and Segel (1970) to describe the aggregation of cellular slime molds Dictyostelium discoideum. Mimura and Tsujikawa (1996) firstly considered the cell proliferation and death in the well-known Keller–Segel model

$$\begin{cases} u_t = \Delta u - \xi\nabla\cdot(u\nabla v) + f(u), & (x,t) \in \Omega \times (0,\infty), \\ \tau v_t = \Delta v + u - v, & (x,t) \in \Omega \times (0,\infty), \end{cases} \tag{1.4}$$

where $\tau \in \{0,1\}$ and $f(u)$ describes the cell proliferation and death. A large number of results are found, which includes the global existence, boundedness and blow-up behavior of solutions under the chemoattraction condition $\xi > 0$ (see, e.g., Ishida et al.

(2014); Tello and Winkler (2007); Nagai (1995); Horstmann and Winkler (2005); Winkler (2010a, 2014a); Tao and Winkler (2012); Winkler (2010b); Burger et al. (2006); Winkler (2015); Kurt and Shen (2021); Winkler (2014b)). For the chemorepulsion case $\xi < 0$, there exist many interesting results about the global existence and boundedness of solutions (see, e.g., Tao (2013); Freitag (2018); Lin and Xiang (2021); Heihoff (2021); Hu and Zheng (2022a); Hu et al. (2022)).

In the process of studying Alzhemer's disease, Luca et al. (2003) proposed the following attraction–repulsion chemotaxis model:

$$\begin{cases} u_t = \Delta u - \xi \nabla \cdot (u \nabla v) + \chi \nabla \cdot (u \nabla w), & (x,t) \in \Omega \times (0,\infty), \\ \tau v_t = \Delta v + \beta u - \alpha v, & (x,t) \in \Omega \times (0,\infty), \\ \tau w_t = \Delta w + \delta u - \gamma w, & (x,t) \in \Omega \times (0,\infty), \end{cases} \tag{1.5}$$

where $\tau \in \{0,1\}$ and all parameters $\chi, \xi, \alpha, \beta, \gamma, \delta$ are positive. Model (1.5) also be used to address the quorum effect in the chemotactic process (Ref. Painter and Hillen (2002)). There are a number of theoretical results about the attraction–repulsion system (1.5) in recent years. In the case of $\tau = 1$, Liu and Wang (2012) obtained the global existence and steady states of solutions in one-dimension domain. Tao and Wang (2013) proved that the solutions of system (1.5) are global and uniformly bounded in two-dimensional domain if the strength of the chemorepellent exceeds the chemoattractant (i.e., $\chi\delta > \xi\beta, \alpha = \gamma$). Under the similar case $\chi\delta > \xi\beta$, Jin (2015) proved that the global existence of classical solutions in two-dimensional domains and weak solution in three-dimensional domains with large initial mass $\int_\Omega u_0 \mathrm{d}x$. When $\xi\beta = \chi\delta$, Lin et al. (2015) proved that the solution is globally bounded and exponentially converges to the constant state in physical domains $\Omega \subset \mathbb{R}^n (n = 2, 3)$. In the case $\tau = 0$, Tao and Wang (2013) also obtained that the solutions of system (1.5) are globally bounded for $n \geq 2$ if the strength of the chemorepellent exceeds the chemoattractant. Moreover, in Tao and Wang (2013), the finite-time blow-up occurs at $x_0 \in \Omega$ under the conditions that $n = 2$ and $\int_\Omega u_0(x)|x - x_0|^2 \mathrm{d}x$ is small enough, provided that $\xi\beta > \chi\delta, \alpha = \gamma$ and $\int_\Omega u_0 \mathrm{d}x > \frac{8\pi}{\xi\beta - \chi\delta}$ hold. Whereafter, Espejo and Suzuki (2014) removed the condition $\alpha = \gamma$ when $\Omega = B(0, R) \subset \mathbb{R}^2$ is a ball. For more results, please refer to Li and Li (2016); Yu et al. (2017); Hsieh and Yu (2022); Chiyo and Yokota (2022a, b). However, there are few results about two-species attraction–repulsion chemotaxis system.

1.2 Two-Species Chemotaxis System

- **Two-species and one-signal chemotaxis system** To describe the movement of two species, the following two-species and one-stimuli chemotaxis system

$$\begin{cases} u_t = \Delta u - \xi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \\ v_t = \Delta v - \xi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), \\ \tau w_t = \Delta w + b_1 u + b_2 v - \beta w \end{cases} \tag{1.6}$$

was proposed by Tello and Winkler (2012), where $\tau \in \{0, 1\}$, $\beta, \xi_i, \mu_i, a_i, b_i > 0 (i = 1, 2)$. In the case of $\tau = 0$, Tello and Winkler (2012) obtained the global existence and asymptotic behavior of solutions when $a_1, a_2 \in [0, 1)$, $2(\xi_1 + \xi_2) + a_1\mu_2 < \mu_1$ and $2(\xi_1 + \xi_2) + a_2\mu_1 < \mu_2$. When $a_1 > 1 > a_2 \geq 0$, Stinner et al. (2014b) proved that the semi-trivial steady state is asymptotically stable. In the case of $\tau = 1$, Bai and Winkler (2016) derived the global existence of classical solutions if $n \leq 2$ and asymptotical behavior when the damping terms are suitably strong (i.e., μ_1 and μ_2 are large enough). More related interesting results can be found in Black et al. (2016); Mizukami and Yokota (2016); Negreanu and Tello (2014).

Furthermore, many scholars also studied chemotaxis models by nonlocal terms of integral type, such as Armstrong et al. (2006); Gerisch and Chaplain (2008). For more introductions of nonlocal terms, we can refer to Coville et al. (2008); Weinberger (1982, 2002); Kao et al. (2010). So far, the following chemotaxis system with nonlocal terms

$$\begin{cases} u_t = d_1\Delta u - \xi_1\nabla\cdot(u\nabla w) + u\big(a_0 - a_1u - a_2v - a_3\int_\Omega u\mathrm{d}x - a_4\int_\Omega v\mathrm{d}x\big), \\ v_t = d_2\Delta v - \xi_2\nabla\cdot(v\nabla w) + v\big(b_0 - b_1u - b_2v - b_3\int_\Omega u\mathrm{d}x - b_4\int_\Omega v\mathrm{d}x\big), \\ \tau w_t = d_3\Delta w + ku + lv + f - \lambda w \end{cases} \tag{1.7}$$

was studied by some authors, where $\tau \in \{0, 1\}$, $a_0, b_0, d_1, d_2, d_3, \lambda, k, l, a_i, b_i, \xi_i > 0 (i = 1, 2)$ and $a_3, a_4, b_3, b_4 \in \mathbb{R}$ and the forcing term $f \in C^{\alpha,\beta}_{x,t}\big(\bar\Omega \times [0, \infty)\big)(\alpha > 0, \beta \geq 1 + \frac{\alpha}{2})$ is uniformly bounded, which is introduced artificially by an external application. When $\tau = 0$ and $d_1 = d_2 = d_3 = 1$, Negreanu and Tello (2013) obtained the boundedness and asymptotic stabilization of the global solution for system (1.7) under some suitable assumptions. When $\tau = 0$ and $f \equiv 0$, Issa and Salako (2017) showed the global existence and asymptotical behavior of the nonnegative solutions. When $\tau = 1$ and $f \equiv 0$, Xu (Preprint) proved that the solution of system (1.7) is globally bounded and converges to the constant steady state. We refer the readers to Sherratt et al. (2009); Shen and Zhang (2012); Evje and Winkler (2020) for more interesting results.

- **Two-species and two-signal chemotaxis system** Some scholars considered the following chemotaxis system involving two species and two signals

$$\begin{cases} u_t = \Delta u - \xi_1\nabla\cdot(u\nabla v), & (x, t) \in \Omega \times (0, \infty), \\ 0 = \Delta v + w - v, & (x, t) \in \Omega \times (0, \infty), \\ w_t = \Delta w - \xi_2\nabla\cdot(w\nabla z), & (x, t) \in \Omega \times (0, \infty), \\ 0 = \Delta z + u - z, & (x, t) \in \Omega \times (0, \infty), \end{cases} \tag{1.8}$$

where $\xi_1, \xi_2 \in \{-1, 1\}, \tau \in \{0, 1\}$. In Tao and Winkler (2015), Tao and Winkler obtained the global boundedness and finite-time blow-up of solutions for (1.8) when $\tau = 0$. Particularly, when $n = 2, \xi_1 = \xi_2 = 1$ and the initial masses $\|u_0\|_{L^1(\Omega)}, \|w_0\|_{L^1(\Omega)}$ are small, system (1.8) has a globally bounded classical solution. Moreover, if the initial masses are large enough (i.e., $\min\big\{\|u_0\|_{L^1(\Omega)}, \|w_0\|_{L^1(\Omega)}\big\} > 4\pi$), the finite-time blow-up of solutions may occur in two-dimensional domains. Subsequently, Zheng (2017) generalized the results of Tao and Winkler (2015) to the quasilinear cases. For the fully parabolic case (i.e., $\tau = 1$), Li and Wang (2017) derived the global boundedness of solutions if the initial masses are suitably small. In recent years, considering the Lotka–Volterra-type kinetics, the following two-species chemotaxis–competition system with two chemicals

$$\begin{cases} u_t = \Delta u - \xi_1 \nabla \cdot (u \nabla v) + \mu_1 u(1 - u - a_1 w), & (x,t) \in \Omega \times (0,\infty), \\ \tau v_t = \Delta v + \beta w - \alpha v, & (x,t) \in \Omega \times (0,\infty), \\ w_t = \Delta w - \xi_2 \nabla \cdot (w \nabla z) + \mu_2 w(1 - a_2 u - w), & (x,t) \in \Omega \times (0,\infty), \\ \tau z_t = \Delta z + \delta u - \gamma z, & (x,t) \in \Omega \times (0,\infty) \end{cases} \tag{1.9}$$

was studied by many authors, where $\tau \in \{0, 1\}$ and $\alpha, \beta, \gamma, \delta, \xi_i, \mu_i, a_i > 0 (i = 1, 2)$. When $\tau = 0$ and the production efficiency of the signals v, z is the same as the consumption (i.e., $\alpha = \beta = \gamma = \delta = 1$), Zheng and Mu (2017) derived the global boundedness of solutions under the conditions in two-dimensional domains. When $\tau = 0$ and $a_1, a_2 \in (0, 1)$, the global boundedness and large time behavior of solutions for system (1.9) were addressed in Zheng et al. (2018). Moreover, due to the positivity of Lotka–Volterra competition, Tu et al. (2018) proved the global boundedness and convergence rate of solutions for $n \geq 2$. Recently, Wang and Mu (2020) improved the previous conditions of Tu et al. (2018). On the other hand, when $\tau = 1$, Zheng and Mu (2017) showed that system (1.9) has a globally bounded classical solution for $n \geq 2$ provided that the chemotactic sensitivities are small enough (i.e., $\frac{\xi_2}{\mu_1} < \theta_0$, $\frac{\xi_1}{\mu_2} < \theta_0$ for some $\theta_0 > 0$). For more related contents, please refer to Liu and Dai (2022); Zhang et al. (2017); Zhang (2018); Zheng (2021). So far, to the best of our knowledge, no work has been done for the solution behavior when attraction–repulsion chemotaxis as well as nonlocal kinetic terms involving both species are present.

Recently, some scholars considered several special cases for system (1.1). For instance, when $\tau = 0$ and no kinetic terms (i.e., $g_1 = g_2 \equiv 0$) in (1.1), Liu and Dai (2021) proved that the cells aggregation occurs in finite time. Moreover, the solutions are globally bounded if the initial masses are small enough or $\chi_1 = \chi_2 = \xi_1 = \xi_2 > 0$ in Liu and Dai (2021). When the strength of the chemorepellent exceeds the chemoattractant (i.e., $\min\{\chi_1, \chi_2\} > \xi_1 + \xi_2$), Liu et al. (2022) proved that the system (1.1) has a unique globally bounded classical solution. Without nonlocal terms in (1.1), Zheng and Hu (Preprint) showed global boundedness and stability of the constant steady state. When g_1 and g_2 satisfy (1.2) with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j \in \mathbb{R} (j = 3, 4)$, Zheng et al. (2022) studied the fully parabolic two-species chemotaxis system with indirect signal production and proved that the solutions are globally bounded and converge to the constant steady state under the locally and globally competitive assumptions. Particularly, when g_1 and g_2 satisfy (1.2) with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j \in \mathbb{R} (j = 3, 4)$, Hu and Zheng (2022b) proved the boundedness and stabilization of global solutions for system (1.1) under the competitive case. However, considering only the intrinsic growth and locally intraspecific competition, to the best of our knowledge the literature does not provide any qualitative analysis on the solution behavior of the general two-species attraction–repulsion chemotaxis system with nonlocal terms. Hence, the above problems are responded affirmatively in this paper.

1.3 Main Results

In this paper, we mainly study the global boundedness of solutions for system (1.1) with attraction–repulsion and nonlocal terms. Moreover, we consider the global stability of solutions for system (1.1) under the locally intraspecific competition and globally interspecific cooperation case (i.e., $a_1, b_2 > 0, a_2, b_1, a_4, b_3 < 0$). For simplicity, we introduce a notation that is $(a)_- := \max\{0, -a\}$ for all $a \in \mathbb{R}$.

Firstly, we shall give the global boundedness of solutions for system (1.1) when $\tau = 0$ and $\tau = 1$, separately.

Theorem 1.1 *Let $\tau = 0$, $d_i > 0 (i = 1, 2, 3, 4)$, $\xi_j, \chi_j > 0 (j = 1, 2)$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain. Assume that g_1, g_2 satisfy (1.2) with $a_0, a_1, b_0, b_2 > 0, a_2, a_3, a_4, b_1, b_3, b_4 \in \mathbb{R}$, and that the nonnegative initial data (u_0, w_0) satisfy (1.3). Moreover, if the following assumptions hold:*

- $n = 1$, *the condition*

$$\begin{cases} a_1 > \frac{(a_2)_- + (b_1)_-}{2} + \left((a_3)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|, \\ b_2 > \frac{(a_2)_- + (b_1)_-}{2} + \left((b_4)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|; \end{cases} \tag{1.10}$$

- $n \geq 2$, *the condition* (1.10) *and*

$$\min\left\{\frac{\chi_1}{d_4} + a_1, \frac{\chi_2}{d_2} + b_2\right\} > \frac{\xi_1}{d_2} + \frac{\xi_2}{d_4} + (a_2)_- + (b_1)_-. \tag{1.11}$$

Then, system (1.1) *admits a unique global nonnegative classical solution (u, v, w, z) in $\Omega \times (0, \infty)$, which is uniformly bounded in the sense that there exists a constant $C > 0$ independent of t such that*

$$\begin{aligned} &||u(\cdot, t)||_{L^\infty(\Omega)} + ||v(\cdot, t)||_{L^\infty(\Omega)} + ||w(\cdot, t)||_{L^\infty(\Omega)} + ||z(\cdot, t)||_{L^\infty(\Omega)} \\ &\quad \leq C \quad for\ all\ t > 0. \end{aligned}$$

Remark 1.1 When $d_i = 1 (i = 1, 2, 3, 4)$ and $a_2 = a_3 = a_4 = b_1 = b_3 = b_4 = 0$ in (1.1), compared with the results of Theorem 1.1 in Liu and Dai (2022), it can be found that $\min\left\{\chi_1 + a_1, \chi_2 + b_2\right\} > \min\left\{a_1, b_2\right\} > \xi_1 + \xi_2$ under the conditions $\chi_1, \chi_2 > 0$ and $n \geq 2$, which implies that the repulsive mechanism is helpful for the global boundedness of solutions. When $n = 1$, $\chi_1 = \chi_2 = 0$ and $a_2 = b_1 = a_3 = a_4 = b_3 = b_4 = 0$, Theorem 1.1 only need the conditions $a_1, b_2 > 0$ in this paper, which removes the condition $\min\left\{a_1, b_2\right\} > \xi_1 + \xi_2$ in Liu and Dai (2022). If $a_2, b_1 > 0$, the conditions in this paper are same as in Hu and Zheng (2022b). Moreover, when $n = 2$, the small initial condition $\max\left\{\|u_0\|_{L^1(\Omega)}, \|w_0\|_{L^1(\Omega)}\right\} < \frac{4}{(\chi_1 + \chi_2 + \xi_1 + \xi_2)C_{GN}}$

in Liu and Dai (2021) can also be canceled under the effect of locally intraspecific competition.

Theorem 1.2 *Let $\tau = 1, d_i > 0 (i = 1, 2, 3, 4), \xi_j, \chi_j > 0 (j = 1, 2)$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain. Assume that g_1, g_2 satisfy (1.2) with $a_0, a_1, b_0, b_2 > 0, a_2, a_3, a_4, b_1, b_3, b_4 \in \mathbb{R}$, and that the nonnegative initial data (u_0, v_0, w_0, z_0) satisfy (1.3). Moreover, suppose that the following conditions*

- $n = 1$ *and* (1.10);
- $n = 2$ *and*

$$\begin{cases} a_1 > \max\left\{\frac{(a_2)_- + (b_1)_-}{2} + \left((a_3)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|, \frac{2(a_2)_-}{3} + \frac{(b_1)_-}{3}\right\}, \\ b_2 > \max\left\{\frac{(a_2)_- + (b_1)_-}{2} + \left((b_4)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|, \frac{(a_2)_-}{3} + \frac{2(b_1)_-}{3}\right\}; \end{cases} \tag{1.12}$$

- $n \geq 3$ *and*

$$\begin{cases} a_1 > \max\left\{\frac{(a_2)_- + (b_1)_-}{2} + \left((a_3)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|, \xi_1 + \chi_1 + (a_2)_- + (b_1)_- + C_{S_1}(\chi_1 + \xi_2)\right\}, \\ b_2 > \max\left\{\frac{(a_2)_- + (b_1)_-}{2} + \left((b_4)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|, \xi_2 + \chi_2 + (a_2)_- + (b_1)_- + C_{S_2}(\chi_2 + \xi_1)\right\} \end{cases} \tag{1.13}$$

hold, where the positive constants C_{S_1} and C_{S_2} depend on the diffusion coefficients d_4 and d_2, respectively. Then, system (1.1) admits a unique global nonnegative classical solution (u, v, w, z) in $\Omega \times (0, \infty)$, which is uniformly bounded in the sense that there exists a constant $C > 0$ independent of t such that

$$\begin{aligned} &||u(\cdot, t)||_{L^\infty(\Omega)} + ||v(\cdot, t)||_{W^{1,\infty}(\Omega)} + ||w(\cdot, t)||_{L^\infty(\Omega)} \\ &+ ||z(\cdot, t)||_{W^{1,\infty}(\Omega)} \leq C \quad for\ all\ t > 0. \end{aligned}$$

Remark 1.2 When $n = 2, a_1, b_2 > 0$ and $\chi_1 = \chi_2 = a_3 = a_4 = b_3 = b_4 = 0$, Theorem 1.2 includes the result of Black (2017) in this paper. When $n \geq 3$ and $\chi_1 = \chi_2 = a_1 = a_3 = a_4 = b_2 = b_3 = b_4 = 0$, the result of Theorem 1.2 in Liu and Dai (2022) is covered by Theorem 1.2 in this paper. Particularly, for the case of $n \leq 2$ and $\chi_1 = \chi_2 = a_1 = a_3 = a_4 = b_2 = b_3 = b_4 = 0$, Theorem 1.2 only requires the conditions $a_1, b_2 > 0$ in this paper, which improves the conditions $a_1 > \xi_1 + \xi_2 C_S, b_2 > \xi_2 + \xi_1 C_S$ in Liu and Dai (2022). Moreover, when $n \leq 2$, the small initial condition in Liu et al. (2022) can also be removed under the effect of locally intraspecific competition.

Next we mainly focus on the asymptotic behavior of the global bounded solutions for system (1.1). When $a_0, b_0, a_1, b_2 > 0, a_2, a_4, b_1, b_3 < 0$ and $a_3, b_4 \in \mathbb{R}$, the locally intraspecific competition and globally interspecific cooperation cases are

$$\begin{aligned} a_1 &> (a_3)_-|\Omega|, \\ b_2 &> (b_4)_-|\Omega| \end{aligned} \tag{1.14}$$

and

$$\frac{a_1 + a_3|\Omega|}{b_1 + b_3|\Omega|} < \frac{a_2 + a_4|\Omega|}{b_2 + b_4|\Omega|} < 0 < \frac{a_0}{b_0}, \tag{1.15}$$

then the unique positive constant steady state (u_*, v_*, w_*, z_*) can be obtained as

$$\begin{aligned}
u_* &:= \frac{a_0\,(b_2 + b_4|\Omega|) - b_0\,(a_2 + a_4|\Omega|)}{(b_2 + b_4|\Omega|)\,(a_1 + a_3|\Omega|) - (a_2 + a_4|\Omega|)\,(b_1 + b_3|\Omega|)},\\
v_* &:= \frac{a_0\,(b_1 + b_3|\Omega|) - b_0\,(a_1 + a_3|\Omega|)}{(a_2 + a_4|\Omega|)\,(b_1 + b_3|\Omega|) - (b_2 + b_4|\Omega|)\,(a_1 + a_3|\Omega|)},\\
w_* &:= \frac{a_0\,(b_1 + b_3|\Omega|) - b_0\,(a_1 + a_3|\Omega|)}{(a_2 + a_4|\Omega|)\,(b_1 + b_3|\Omega|) - (b_2 + b_4|\Omega|)\,(a_1 + a_3|\Omega|)},\\
z_* &:= \frac{a_0\,(b_2 + b_4|\Omega|) - b_0\,(a_2 + a_4|\Omega|)}{(b_2 + b_4|\Omega|)\,(a_1 + a_3|\Omega|) - (a_2 + a_4|\Omega|)\,(b_1 + b_3|\Omega|)}.
\end{aligned} \tag{1.16}$$

Theorem 1.3 *Let the conditions in Theorem 1.1 and (1.14)–(1.15) hold. Suppose that $a_0, b_0, a_1, b_2 > 0$, $a_2, a_4, b_1, b_3 < 0$ and $a_3, b_4 \in \mathbb{R}$. Assume that system (1.1) admits a unique global classical solution (u, v, w, z) with the property*

$$\begin{aligned}
&\|u(\cdot,t)\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} + \|v(\cdot,t)\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])}\\
&+ \|w(\cdot,t)\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} + \|z(\cdot,t)\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} \le K
\end{aligned} \tag{1.17}$$

for all $t \ge 1$, where $K > 0$ and $\vartheta \in (0, 1)$. Furthermore, assume that there exist $\theta_1, \theta_2 \in (0, 1)$ such that

$$\begin{aligned}
a_1 > \max\Bigg\{ &\frac{u_*\chi_1^2}{4d_1d_4(1-\theta_1)} + \left((a_3)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|,\\
&\frac{w_*\xi_2^2}{4d_3d_4(1-\theta_1)} + \left((a_3)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|,\\
&\frac{(a_2 + b_1)^2}{2\theta_1\theta_2\Big[2b_2 - \big(2(b_4)_- + (b_3)_- + (a_4)_-\big)|\Omega|\Big]}\\
&\quad + \left((a_3)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|\Bigg\}
\end{aligned} \tag{1.18}$$

and

$$\begin{aligned}
b_2 > \max\Bigg\{ &\frac{u_*\xi_1^2}{4d_1d_2(1-\theta_2)} + \left((b_4)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|,\\
&\frac{w_*\chi_2^2}{4d_2d_3(1-\theta_2)} + \left((b_4)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|\Bigg\}.
\end{aligned} \tag{1.19}$$

Then, for some fixed time t_1, there exist $C > 0$ and $\kappa > 0$ such that

$$\begin{aligned}&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}+\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}+\|w(\cdot,t)-w_*\|_{L^\infty(\Omega)}\\&\quad+\|z(\cdot,t)-z_*\|_{L^\infty(\Omega)}\le Ce^{-\kappa t}\end{aligned} \tag{1.20}$$

for all $t > t_1$.

Theorem 1.4 *Let the conditions in Theorem 1.2 and (1.14)–(1.15) hold. Suppose that $a_0, b_0, a_1, b_2 > 0$, $a_2, a_4, b_1, b_3 < 0$ and $a_3, b_4 \in \mathbb{R}$. Assume that system (1.1) admits a unique global classical solution (u, v, w, z) with the property (1.17). Furthermore, assume that there exist $\theta_3, \theta_4 \in (0, 1)$ such that*

$$\begin{aligned}a_1 > \max\Bigg\{&\frac{1}{8d_4(1-\theta_3)}\left(\frac{\chi_1^2u_*}{d_1}+\frac{\xi_2^2w_*}{d_3}\right)+\left((a_3)_-+\frac{(a_4)_-+(b_3)_-}{2}\right)|\Omega|,\\&\frac{(a_2+b_1)^2}{2\theta_3\theta_4\Big[2b_2-\big(2(b_4)_-+(b_3)_-+(a_4)_-\big)|\Omega|\Big]}\\&+\left((a_3)_-+\frac{(a_4)_-+(b_3)_-}{2}\right)|\Omega|\Bigg\}\end{aligned} \tag{1.21}$$

and

$$b_2 > \frac{1}{8d_2(1-\theta_4)}\left(\frac{\xi_1^2u_*}{d_1}+\frac{\chi_2^2w_*}{d_3}\right)+\left((b_4)_-+\frac{(a_4)_-+(b_3)_-}{2}\right)|\Omega|. \tag{1.22}$$

Then, for some fixed time t_2, there exist $C > 0$ and $\lambda > 0$ such that

$$\begin{aligned}&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}+\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}\\&\quad+\|w(\cdot,t)-w_*\|_{L^\infty(\Omega)}+\|z(\cdot,t)-z_*\|_{L^\infty(\Omega)}\\&\le Ce^{-\lambda t}\end{aligned} \tag{1.23}$$

for all $t > t_2$.

Remark 1.3 We rewrite u_*, w_* as

$$u_* := \frac{a_0-\frac{b_0(a_2+a_4|\Omega|)}{(b_2+b_4|\Omega|)}}{(a_1+a_3|\Omega|)-\frac{(a_2+a_4|\Omega|)(b_1+b_3|\Omega|)}{(b_2+b_4|\Omega|)}},\quad w_* := \frac{\frac{a_0(b_1+b_3|\Omega|)}{(a_1+a_3|\Omega|)}-b_0}{\frac{(a_2+a_4|\Omega|)(b_1+b_3|\Omega|)}{(a_1+a_3|\Omega|)}-(b_2+b_4|\Omega|)},$$

then u_*, w_* are sufficiently close to zero as a_1, b_2 are large enough for fixed $a_0, a_2, a_3, a_4, b_0, b_1, b_3, b_4$. Hence, the assumptions (1.18) and (1.19) (or (1.21), (1.22)) make sense with suitably large a_1, b_2. Moreover, we only consider the stabilization of globally bounded solutions for system (1.1) under the case $a_0, b_0, a_1, b_2 > 0$, $a_2, a_4, b_1, b_3 < 0$ and $a_3, b_4 \in \mathbb{R}$ in this paper. For the case $a_i, b_i > 0 (i = 0, 1, 2, 3, 4)$, it follows from the same method of Zheng et al. (2022) that the stabilization of globally bounded solutions can also be derived.

Remark 1.4 It follows from the results in Theorems 1.1-1.4 and numerical simulations in Section 6 that it is not difficult to see that the size of bounded domain $\Omega \subset \mathbb{R}^n$ plays an important role in studying the global boundedness and asymptotic stability of solutions in this model. However, due to the appearance of the nonlocal kinetics, as far as we know, the model does not possess the rescaling invariance. What's more, the results and proofs of this paper are not invariant under the rescaling of the domain Ω. When the domain Ω is replaced by $\lambda\Omega$ ($\lambda > 0$), the parameter conditions of Theorems 1.1-1.4 shall be changed. Thus, how to keep the scale-invariant formulation, which is an interesting topic of mathematical analysis in future work. But in this paper, we pay more attention to the explanation of the biological field in a fixed and bounded domain Ω.

Mathematical challenges and main ideas. Our main ideas are inspired by two-species attraction–repulsion chemotaxis model in Liu and Dai (2022) and nonlocal chemotaxis model from Negreanu and Tello (2013). The model of this paper involves more complex cross-diffusion mechanisms, which leads to the mathematical difficulty of qualitative analysis of the solutions. As far as we know, the global boundedness of the problem is not difficult when two populations proliferate and compete according to the Lotka–Volterra-type kinetics (see, e.g., Bai and Winkler (2016); Stinner et al. (2014b); Tello and Winkler (2012)). However, compared with the well-understood situations of Lotka–Volterra-type kinetics, the present setting exposes its increased complexity through locally intraspecific competition and globally interspecific cooperation (or competition), as defined by

$$\begin{cases} g_1(u, w) = u\left(a_0 - a_1 u - a_2 w - a_3 \displaystyle\int_\Omega u \mathrm{d}x - a_4 \int_\Omega w \mathrm{d}x\right), \\ g_2(u, w) = w\left(b_0 - b_1 u - b_2 w - b_3 \displaystyle\int_\Omega u \mathrm{d}x - b_4 \int_\Omega w \mathrm{d}x\right) \end{cases}$$

with $a_0, a_1, b_0, b_2 > 0$ and $a_2, a_3, a_4, b_1, b_3, b_4 \in \mathbb{R}$. Thus, the method in (Bai and Winkler 2016, Lemma 2.6) is invalid for the fully parabolic case (i.e., $\tau = 1$) in proof of global boundedness. Therefore, we have to overcome some difficulties to prove it as follows.

Firstly, in investigating the globally bounded solution of (1.1), as pointed out in previous work (e.g., Chiyo and Yokota (2022a); Tello and Winkler (2012); Stinner et al. (2014b); Negreanu and Tello (2013)), the essential analytic obstacle is that the chemotaxis and positive kinetic terms in the first and third equations of (1.1). On the other hand, relying on our previous work in Hu et al. (2022); Hu and Zheng (2022a, b), we know that the repulsive and competitive mechanisms effectively prevent the occurrence of cells aggregation in the mathematical sense that the solutions blow up. When $\tau = 0$, by proceeding in a quite standard testing procedure of L^p-estimates, with the aids of the Agmon–Douglis–Nirenberg L^p-estimates for v, z, one can use the random diffusion (i.e., $d_1\Delta u, d_2\Delta w$), chemorepulsion (i.e., $+\chi_1\nabla \cdot (u\nabla z), +\chi_2\nabla \cdot (w\nabla v)$) and negative kinetic terms (i.e., $-a_1u^2, -b_2w^2$) to simultaneously estimate the chemotaxis and positive kinetic terms (see, e.g., Lemma 3.1 and Lemma 3.2). Hence, in any space dimension $n \geq 1$, we can improve the priori estimates of u, w

from L^1-estimates to L^p-estimates with any $p \geq 1$, so that we can deduce the uniform boundedness of u, v, w, z by the well-known elliptic maximum principle and Moser–Alikakos iteration in Alikakos (1979). When $\tau = 1$, we still need to overcome the above difficulties. In low-dimensional domains $n \leq 2$, one can use random diffusion terms, negative kinetic terms and the known regularities of v, z to effectively control the chemotaxis and positive kinetic terms, then we get L^2-estimates of u, w, which plays an important role for L^p-estimates of u, w (see Lemma 4.3 for L^2-estimates, Lemma 4.3 for L^q-estimates with any $q \geq 1$). And in high-dimensional domains $n \geq 3$, relying on the maximal Sobolev regularity ((Wang et al. 2018, Lemma 2.3)), we also obtain the L^p-estimates of u, w by proceeding in a quite standard testing procedure (see Lemma 4.5). Accordingly, in view of the standard parabolic regularity argument and the Moser–Alikakos iteration technique, we are able to derive the uniform boundedness of u, v, w, z for $\tau = 1$ in any spatial dimension.

Secondly, under the locally intraspecific competition and globally interspecific cooperation case, we mainly construct the following Lyapunov functionals to derive the globally asymptotic stabilization of coexistence steady states

$$E(t) := \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right) + \frac{\tau\rho_1}{2} \int_\Omega \left(v - v_*\right)^2 + \frac{\tau\rho_2}{2} \int_\Omega \left(z - z_*\right)^2$$

for $t > 0$ with $\tau \in \{0, 1\}$ and $\rho_1, \rho_2 > 0$, where (u_*, v_*, w_*, z_*) is given by (1.16). The cross-diffusion and nonlocal terms in the first and third equations of (1.1) are the main mathematical technical difficulty. To achieve our goal, we use the quadratic function instead of positive definite quadratic form to obtain the following energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq -C \int_\Omega \left((u - u_*)^2 + (v - v_*)^2 + (w - w_*)^2 + (z - z_*)^2\right)$$

for all $t > 0$ with some $C > 0$ provided that $a_1 > \bar{a}$ and $b_2 > \bar{b}$ for sufficiently large $\bar{a}, \bar{b}$ (see Lemma 5.2 and Lemma 5.5). Consequently, with the aids of boundedness for solutions and some delicate interpolations, we obtain the asymptotic behavior of solutions for (1.1) (see Lemma 5.4 and Lemma 5.7). Finally, some numerical simulations for model (1.1) are carried out with the help of the MATLAB, which verifies the analytically theoretical results and finds some new phenomena.

This paper is organized as follows. In Sect. 2, we give some preliminary lemmas and the local existence of solution for system (1.1). In Sect. 3 and Sect. 4, we study the global existence and boundedness of solutions for system (1.1), and prove Theorem 1.1 and Theorem 1.2. In Sect. 5, we study the asymptotic behavior of global solutions for system (1.1), and prove Theorem 1.3 and Theorem 1.4. In Sect. 6, we present numerical results to verify the analytically theoretical results. In addition, we let $u(\cdot, t) := u(x, t)$ and omit signs dx during integrating for concision throughout this paper.

2 Preliminaries

In this section, we shall give several preliminary lemmas. Firstly, we state the local existence of solutions for system (1.1).

Lemma 2.1 *Let* $\tau \in \{0, 1\}, d_i > 0(i = 1, 2, 3, 4), \xi_j, \chi_j > 0(j = 1, 2)$ *and* $\Omega \subset \mathbb{R}^n (n \geq 1)$ *be a smoothly bounded domain. Assume that* g_1, g_2 *satisfy* (1.2)*. Suppose that the nonnegative initial data* $(u_0, \tau v_0, w_0, \tau z_0)$ *satisfy* (1.3)*. Then, there exist* $T_{\max} \in (0, \infty]$ *and uniquely determined nonnegative functions*

$$\begin{cases} u, w \in C^0\big(\bar{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\bar{\Omega} \times (0, T_{\max})\big), \\ v, z \in C^0\big(\bar{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\bar{\Omega} \times (0, T_{\max})\big) \cap L^{\infty}_{loc}\big([0, T_{\max}); W^{1,q}(\Omega)\big) \end{cases}$$

such that (u, v, w, z) *solves system* (1.1) *classically in* $\Omega \times (0, T_{\max})$*, where* $q > n$*. Moreover, if* $T_{\max} < \infty$*, then*

$$\limsup_{t \nearrow T_{\max}} \Big(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{2.1}$$

Proof By well-established methods involving the standard parabolic regularity theory and an appropriate fixed point framework, we can obtain the local existence and extensibility criterion (see Stinner et al. (2014b) and Winkler (2010a)). For the details of proof, please refer to Appendix A. □

Secondly, the following lemma gives a uniform L^1-bound for the solutions.

Lemma 2.2 *Let* (u, v, w, z) *be a solution for system* (1.1)*. Assume that the nonnegative initial data* $(u_0, \tau v_0, w_0, \tau z_0)$ *satisfy* (1.3) *with* $\tau \in \{0, 1\}$*. Suppose that the condition* (1.10) *holds. Then,*

$$\begin{aligned} &\|u(\cdot, t)\|_{L^1(\Omega)} + \|w(\cdot, t)\|_{L^1(\Omega)} \\ &\leq \max\left\{ \|u_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)}, 2|\Omega| \frac{\max\{a_0, b_0\}}{\min\{l_1, l_2\}} \right\} := M_0 \end{aligned} \tag{2.2}$$

and

$$\|v(\cdot, t)\|_{L^1(\Omega)} + \|z(\cdot, t)\|_{L^1(\Omega)} \leq \max\left\{ \|v_0\|_{L^1(\Omega)} + \|z_0\|_{L^1(\Omega)}, M_0 \right\} := m_0 \tag{2.3}$$

for all $t \in (0, T_{\max})$*, where*

$$\begin{cases} l_1 = a_1 - \frac{1}{2}\Big((a_2)_- + (b_1)_-\Big) - |\Omega|(a_3)_- - \frac{|\Omega|}{2}\Big((a_4)_- + (b_3)_-\Big), \\ l_2 = b_2 - \frac{1}{2}\Big((a_2)_- + (b_1)_-\Big) - |\Omega|(b_4)_- - \frac{|\Omega|}{2}\Big((a_4)_- + (b_3)_-\Big). \end{cases} \tag{2.4}$$

Proof Integrating the first equation in (1.1) and using Young's as well as Hölder's inequalities, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u =& \int_\Omega u\left(a_0 - a_1 u - a_2 w - a_3\int_\Omega u - a_4\int_\Omega w\right)\\
\le & a_0\int_\Omega u - \left(a_1 - \frac{(a_2)_-}{2}\right)\int_\Omega u^2 + \left((a_3)_- + \frac{(a_4)_-}{2}\right)\left(\int_\Omega u\right)^2\\
&+ \frac{(a_2)_-}{2}\int_\Omega w^2 + \frac{(a_4)_-}{2}\left(\int_\Omega w\right)^2\\
\le & a_0\int_\Omega u - \left(a_1 - \frac{(a_2)_-}{2} - \left((a_3)_- + \frac{(a_4)_-}{2}\right)|\Omega|\right)\int_\Omega u^2\\
&+ \left(\frac{(a_2)_-}{2} + \frac{|\Omega|(a_4)_-}{2}\right)\int_\Omega w^2
\end{aligned} \tag{2.5}
$$

for all $t \in (0, T_{\max})$. By the similar method to w-equation, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w \le & b_0\int_\Omega w - \left(b_2 - \frac{(b_1)_-}{2} - \left((b_4)_- + \frac{(b_3)_-}{2}\right)|\Omega|\right)\int_\Omega w^2\\
&+ \left(\frac{(b_1)_-}{2} + \frac{|\Omega|(b_3)_-}{2}\right)\int_\Omega u^2.
\end{aligned} \tag{2.6}
$$

Combining (2.5) with (2.6), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+w) \le & a_0\int_\Omega u - \left(a_1 - \frac{(a_2)_- + (b_1)_-}{2} - \left((a_3)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|\right)\int_\Omega u^2\\
&+ b_0\int_\Omega w - \left(b_2 - \frac{(a_2)_- + (b_1)_-}{2} - \left((b_4)_- + \frac{(a_4)_- + (b_3)_-}{2}\right)|\Omega|\right)\int_\Omega w^2.
\end{aligned} \tag{2.7}
$$

It follows from (1.10) that the constants l_1 and l_2 given by (2.4) are positive. Hence, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+w) \le & a_0\int_\Omega u - \frac{l_1}{|\Omega|}\left(\int_\Omega u\right)^2 + b_0\int_\Omega w - \frac{l_2}{|\Omega|}\left(\int_\Omega w\right)^2\\
\le & \max\{a_0, b_0\}\int_\Omega (u+w) - \frac{1}{2|\Omega|}\min\{l_1, l_2\}\left(\int_\Omega u + \int_\Omega w\right)^2.
\end{aligned} \tag{2.8}
$$

By the comparison argument of ODE, we have

$$
\int_\Omega (u+w) \le \max\left\{\int_\Omega (u_0 + w_0), 2|\Omega|\frac{\max\{a_0, b_0\}}{\min\{l_1, l_2\}}\right\} := M_0 \tag{2.9}
$$

for all $t \in (0, T_{\max})$.

When $\tau = 1$, integrating the second and fourth equations in (1.1), respectively, adding up them and using Young's inequality, we obtain

$$\|v\|_{L^1(\Omega)} + \|z\|_{L^1(\Omega)} \leq \max\left\{\|v_0\|_{L^1(\Omega)} + \|z_0\|_{L^1(\Omega)}, M_0\right\} := m_0 \tag{2.10}$$

for all $t \in (0, T_{\max})$.

When $\tau = 0$, by a straightforward computation and (2.9), we obtain from the second and fourth equations in (1.1) that

$$\|v\|_{L^1(\Omega)} + \|z\|_{L^1(\Omega)} \leq \|w\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \leq M_0 \tag{2.11}$$

for all $t \in (0, T_{\max})$. The proof of Lemma 2.2 is complete. □

Lemma 2.3 (see Friedman (1969)) *Let $p, k > 0$, $m \in [0, k)$ and $q, r \in [1, \infty]$. Then, for any $\Psi \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exists $C_{GN} = C(k, q, r, \Omega) > 0$ such that*

$$\|D^m\Psi\|_{L^p(\Omega)} \leq C_{GN}\|D^k\Psi\|^{\alpha}_{L^q(\Omega)}\|\Psi\|^{1-\alpha}_{L^r(\Omega)} + C_{GN}\|\Psi\|_{L^r(\Omega)}, \tag{2.12}$$

where α satisfies

$$\frac{1}{p} - \frac{m}{n} = \alpha\left(\frac{1}{q} - \frac{k}{n}\right) + \frac{1}{r}\left(1 - \alpha\right) \Leftrightarrow \alpha = \frac{\frac{1}{p} - \frac{m}{n} - \frac{1}{r}}{\frac{1}{q} - \frac{k}{n} - \frac{1}{r}} \in \left(\frac{m}{k}, 1\right)$$

and $D^k\psi$ is expressed as Fréchet derivative of order k.

Lemma 2.4 (see (Temam 1997, Chapter III, Lemma 5.1)) *Let $\Phi(t) \geq 0$ satisfy*

$$\begin{cases} \Phi'(t) + k_1 y \Phi^{\theta}(t) \leq k_2, & t > 0, \\ \Phi(0) = \Phi_0 \end{cases} \tag{2.13}$$

for $\Phi_0 \geq 0$ with some constants $k_1, k_2 > 0$ and $\theta > 0$. Then,

$$\Phi(t) \leq \max\left\{\Phi_0, \left(\frac{k_2}{k_1}\right)^{\frac{1}{\theta}}\right\} \quad \textit{for all} \quad t > 0.$$

Lemma 2.5 (see (Wang et al. 2018, Lemma 2.3)) *Let $0 \leq t_0 < T_{\max} \leq \infty$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain. Assume that $u_0 \in W^{2,p}(\Omega) (p > n)$ with $\partial_\nu u_0 = 0$ on $\partial\Omega$. Then, for each $d > 0$ and $h \in L^p\big((0, T_{\max}); L^p(\Omega)\big)$, the system*

$$\begin{cases} u_t = d\Delta u - u + h, & (x, t) \in \Omega \times (0, T_{\max}), \\ \dfrac{\partial u}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, T_{\max}), \\ u(\cdot, 0) = u_0, & x \in \Omega \end{cases} \tag{2.14}$$

has a unique solution $u \in W^{1,p}\big([0,T_{\max}); L^p(\Omega)\big) \cap L^p\big([0,T_{\max}); W^{2,p}(\Omega)\big)$. *Moreover, if* $u(\cdot,t_0) \in W^{2,p}(\Omega)$ *with* $\frac{\partial u(\cdot,t_0)}{\partial \nu} = 0$ *on* $\partial\Omega$, *then there exists* $C_S = C_S(d) > 0$ *such that*

$$\begin{aligned}
&\int_{t_0}^{t} e^{ps} \int_\Omega |\Delta u(\cdot,s)|^p ds \\
\leq & C_S \int_{t_0}^{t} e^{ps} \int_\Omega |h(\cdot,s)|^p ds + C_S e^{pt_0}\Big(\|u(\cdot,t_0)\|^p_{L^p(\Omega)} + \|\Delta u(\cdot,t_0)\|^p_{L^p(\Omega)} \Big)
\end{aligned} \tag{2.15}$$

for any $t \in (t_0, T_{\max})$.

3 Boundedness for $\tau = 0$

The aim of this section is to show the global boundedness of solution to (1.1) with $\tau = 0$ and prove Theorem 1.1.

Lemma 3.1 *Let* $\tau = 0$, $d_i > 0 (i = 1,2,3,4)$, $\xi_j, \chi_j > 0 (j = 1,2)$ *and* $\Omega \subset \mathbb{R}$ *be a smoothly bounded domain. Suppose that* (1.10) *holds. Then, there exists* $C_1 > 0$ *such that*

$$\|u(\cdot,t)\|_{L^2(\Omega)} + \|w(\cdot,t)\|_{L^2(\Omega)} \leq C_1 \tag{3.1}$$

for all $t \in (0, T_{\max})$.

Proof Multiplying the first equation in (1.1) by $2u$ and integrating by parts over Ω, we derive from Young's inequality and Lemma 2.2 that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 + \int_\Omega u^2 =& -2d_1\int_\Omega |\nabla u|^2 - \frac{\xi_1}{d_2}\int_\Omega u^2(v-w) + \frac{\chi_1}{d_4}\int_\Omega u^2(z-u) \\
&+ 2\int_\Omega u^2\Big(a_0 - a_1 u - a_2 w - a_3\int_\Omega u - a_4\int_\Omega w\Big) + \int_\Omega u^2 \\
\leq& -2d_1\int_\Omega |\nabla u|^2 + \Big(\frac{\xi_1}{d_2} + 2(a_2)_-\Big)\int_\Omega u^2 w + \frac{\chi_1}{d_4}\int_\Omega u^2 z \\
&+ \Big(2a_0 + 1 + 4a_5 M_0\Big)\int_\Omega u^2 - 2a_1\int_\Omega u^3 \\
\leq& -2d_1\int_\Omega |\nabla u|^2 + \Big(\frac{\xi_1}{d_2} + 2(a_2)_- + \frac{\chi_1}{d_4}\Big)\int_\Omega u^3 \\
&+ \Big(\frac{\xi_1}{d_2} + 2(a_2)_-\Big)\int_\Omega w^3 + \frac{\chi_1}{d_4}\int_\Omega z^3 + C_2
\end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max})$, where $a_5 := \max\{(a_3)_-, (a_4)_-\}$ and $C_2 := \frac{(2a_0+1+4a_5M_0)^3}{27a_1^2}|\Omega|$. Similarly, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^2 + \int_\Omega w^2 \le & -2d_3\int_\Omega |\nabla w|^2 + \left(\frac{\xi_2}{d_4} + 2(b_1)_- + \frac{\chi_2}{d_2}\right)\int_\Omega w^3 \\ & + \left(\frac{\xi_2}{d_4} + 2(b_1)_-\right)\int_\Omega u^3 + \frac{\chi_2}{d_2}\int_\Omega v^3 + C_3\end{aligned} \tag{3.3}$$

for all $t \in (0, T_{\max})$, where $b_5 := \max\{(b_3)_-, (b_4)_-\}$ and $C_3 := \frac{(b_0+1+4b_5M_0)^3}{27b_2^2}|\Omega|$.

In view of (3.2) and (3.3), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^2 + \int_\Omega w^2\right) + \left(\int_\Omega u^2 + \int_\Omega w^2\right) \\ &\le -2d_1\int_\Omega |\nabla u|^2 + \left(\frac{\xi_1}{d_2} + \frac{\xi_2}{d_4} + \frac{\chi_1}{d_4} + 2(a_2)_- + 2(b_1)_-\right)\int_\Omega u^3 + \frac{\chi_1}{d_4}\int_\Omega z^3 \\ &\quad -2d_3\int_\Omega |\nabla w|^2 + \left(\frac{\xi_1}{d_2} + \frac{\xi_2}{d_4} + \frac{\chi_2}{d_2} + 2(a_2)_- + 2(b_1)_-\right)\int_\Omega w^3 + \frac{\chi_2}{d_2}\int_\Omega v^3 + C_2 + C_3\end{aligned} \tag{3.4}$$

for all $t \in (0, T_{\max})$.

Since $0 = d_2\Delta v - v + w$ and $0 = d_4\Delta z - z + u$, it follows from Young's inequality that

$$\int_\Omega v^3 \le \int_\Omega w^3 \tag{3.5}$$

and

$$\int_\Omega z^3 \le \int_\Omega u^3 \tag{3.6}$$

for all $t \in (0, T_{\max})$. By substituting (3.5), (3.6) into (3.4), we derive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^2 + \int_\Omega w^2\right) + \left(\int_\Omega u^2 + \int_\Omega w^2\right) \\ &\le -2d_1\int_\Omega |\nabla u|^2 + \left(\frac{\xi_1}{d_2} + \frac{\xi_2}{d_4} + \frac{2\chi_1}{d_4} + 2(a_2)_- + 2(b_1)_-\right)\int_\Omega u^3 \\ &\quad -2d_3\int_\Omega |\nabla w|^2 + \left(\frac{\xi_1}{d_2} + \frac{\xi_2}{d_4} + \frac{2\chi_2}{d_2} + 2(a_2)_- + 2(b_1)_-\right)\int_\Omega w^3 + C_4\end{aligned} \tag{3.7}$$

for all $t \in (0, T_{\max})$ with $C_4 = C_2 + C_3 > 0$.

We next estimate the terms $\left(\frac{\xi_1}{d_2} + \frac{\xi_2}{d_4} + \frac{2\chi_1}{d_4} + 2(a_2)_- + 2(b_1)_-\right)\int_\Omega u^3$ and $\left(\frac{\xi_1}{d_2} + \frac{\xi_2}{d_4} + \frac{2\chi_2}{d_2} + 2(a_2)_- + 2(b_1)_-\right)\int_\Omega w^3$. By using Lemma 2.2 and Lemma 2.3($n = 1$), there exist some positive constants C_5, C_6 such that

$$\begin{aligned}
&\left(\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}+\frac{2\chi_1}{d_4}+2(a_2)_-+2(b_1)_-\right)\int_\Omega u^3\\
=&\left(\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}+\frac{2\chi_1}{d_4}+2(a_2)_-+2(b_1)_-\right)\|u\|^3_{L^3(\Omega)}\\
\leq& C_5\left(\|\nabla u\|^{\frac{4}{3}}_{L^2(\Omega)}\|u\|^{\frac{5}{3}}_{L^1(\Omega)}+\|u\|^3_{L^1(\Omega)}\right)\\
=&C_5\left(\int_\Omega|\nabla u|^2\right)^{\frac{2}{3}}\left(\int_\Omega u\right)^{\frac{5}{3}}+C_5\left(\int_\Omega u\right)^3\\
\leq& C_6\left(\int_\Omega|\nabla u|^2\right)^{\frac{2}{3}}+C_6
\end{aligned}\tag{3.8}$$

for all $t\in(0,T_{\max})$. It follows from Young's inequality with (3.8) that

$$\begin{aligned}
\left(\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}+\frac{2\chi_1}{d_4}+2(a_2)_-+2(b_1)_-\right)\int_\Omega u^3 &\leq C_6\left(\int_\Omega|\nabla u|^2\right)^{\frac{2}{3}}+C_6\\
&\leq 2d_1\int_\Omega|\nabla u|^2+C_7
\end{aligned}\tag{3.9}$$

for all $t\in(0,T_{\max})$ with $C_7>0$. Similarly, we obtain

$$\left(\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}+\frac{2\chi_2}{d_2}+2(a_2)_-+2(b_1)_-\right)\int_\Omega w^3\leq 2d_3\int_\Omega|\nabla w|^2+C_8 \tag{3.10}$$

for all $t\in(0,T_{\max})$ with some $C_8>0$.

As a consequence of (3.7), (3.9) and (3.10), we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^2+\int_\Omega w^2\right)+\left(\int_\Omega u^2+\int_\Omega w^2\right)\leq C_7+C_8 \tag{3.11}$$

for all $t\in(0,T_{\max})$. According to Lemma 2.4, we get

$$\int_\Omega u^2+\int_\Omega w^2\leq\max\left\{\int_\Omega u_0^2+\int_\Omega w_0^2, C_7+C_8\right\}$$

for all $t\in(0,T_{\max})$. The proof of Lemma 3.1 is complete. □

Lemma 3.2 *Let $\tau=0$, $d_i>0 (i=1,2,3,4)$, $\xi_j,\chi_j>0 (j=1,2)$ and $\Omega\subset\mathbb{R}^n (n\geq 2)$ be a smoothly bounded domain. Suppose that (1.10) and (1.11) hold. Then, for some $p>p_0$, there exists $C_9>0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)}+\|w(\cdot,t)\|_{L^p(\Omega)}\leq C_9 \tag{3.12}$$

for all $t\in(0,T_{\max})$, where

$$p_0:=\max\left\{\frac{n}{2},\frac{\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}-\frac{\chi_1}{d_4}}{\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}+(a_2)_-+(b_1)_--\frac{\chi_1}{d_4}-a_1},\frac{\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}-\frac{\chi_2}{d_2}}{\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}+(a_2)_-+(b_1)_--\frac{\chi_2}{d_2}-b_2}\right\}.$$

Proof For some $p > p_0 \geq 1$, multiplying the first equation in (1.1) by pu^{p-1} and integrating by parts over Ω, we deduce from Young's inequality and Lemma 2.2 that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \int_\Omega u^p =& -\frac{4d_1(p-1)}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 - \frac{\xi_1(p-1)}{d_2}\int_\Omega u^p(v-w)\\
&+\frac{\chi_1(p-1)}{d_4}\int_\Omega u^p(z-u)\\
&+p\int_\Omega u^p\left(a_0 - a_1u - a_2w - a_3\int_\Omega u - a_4\int_\Omega w\right) + \int_\Omega u^p\\
\leq& -\frac{4d_1(p-1)}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + \left(\frac{\xi_1(p-1)}{d_2} - a_2p\right)\int_\Omega u^p w\\
&+\frac{\chi_1(p-1)}{d_4}\int_\Omega u^p z\\
&+\Big((a_0+2a_5M_0)p+1\Big)\int_\Omega u^p - \left(a_1p + \frac{\chi_1(p-1)}{d_4}\right)\int_\Omega u^{p+1}
\end{aligned}
\tag{3.13}
$$

for all $t \in (0, T_{\max})$, where $a_5 := \max\left\{(a_3)_-, (a_4)_-\right\}$. By applying Lemma 2.2, Lemma 2.3 and Young's inequality, there exists a positive constant C_{10} such that

$$
\begin{aligned}
\Big((a_0+2a_5M_0)p+1\Big)\int_\Omega u^p =& \Big((a_0+2a_5M_0)p+1\Big)\left\|u^{\frac{p}{2}}\right\|^2_{L^2(\Omega)}\\
\leq& C_{10}\left(\left\|\nabla u^{\frac{p}{2}}\right\|^{2\alpha_1}_{L^2(\Omega)}\left\|u^{\frac{p}{2}}\right\|^{2-2\alpha_1}_{L^{\frac{2}{p}}(\Omega)} + \left\|u^{\frac{p}{2}}\right\|^2_{L^{\frac{2}{p}}(\Omega)}\right)\\
\leq& C_{10}M_0^{p(1-\alpha_1)}\left(\int_\Omega |\nabla u^{\frac{p}{2}}|^2\right)^{\alpha_1} + C_{10}M_0^p\\
\leq& \frac{4d_1(p-1)}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + C_{11}
\end{aligned}
\tag{3.14}
$$

for all $t \in (0, T_{\max})$ with $C_{11} = (1-\alpha_1)C_{10}^{\frac{1}{1-\alpha_1}}\left(\frac{p\alpha_1}{4d_1(p-1)}\right)^{\frac{\alpha_1}{1-\alpha_1}} + C_{10}M_0^p > 0$, where $\alpha_1 = \frac{\frac{np}{2} - \frac{n}{2}}{1 - \frac{n}{2} + \frac{np}{2}} \in (0, 1)$ due to $p > 1$. By inserting (3.14) into (3.13), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \int_\Omega u^p \leq& \left(\frac{\xi_1(p-1)}{d_2} - a_2p\right)\int_\Omega u^p w + \frac{\chi_1(p-1)}{d_4}\int_\Omega u^p z\\
&- \left(a_1p + \frac{\chi_1(p-1)}{d_4}\right)\int_\Omega u^{p+1} + C_{11}
\end{aligned}
\tag{3.15}
$$

for all $t \in (0, T_{\max})$.

Similarly, there is a positive constant C_{12} such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^p+\int_\Omega w^p\le\left(\frac{\xi_2(p-1)}{d_4}-b_1p\right)\int_\Omega w^pu+\frac{\chi_2(p-1)}{d_2}\int_\Omega w^pv\\-\left(b_2p+\frac{\chi_2(p-1)}{d_2}\right)\int_\Omega w^{p+1}+C_{12}\end{aligned}\tag{3.16}$$

for all $t\in(0,T_{\max})$.

It follows from (1.11) that

$$p>\max\left\{\frac{\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}-\frac{\chi_1}{d_4}}{\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}+(a_2)_-+(b_1)_--\frac{\chi_1}{d_4}-a_1},\frac{\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}-\frac{\chi_2}{d_2}}{\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}+(a_2)_-+(b_1)_--\frac{\chi_2}{d_2}-b_2}\right\},$$

which implies that the constants $\sigma_1:=\Big(a_1-(a_2)_--(b_1)_-\Big)p-\left(\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}-\frac{\chi_1}{d_4}\right)(p-1)$ and $\sigma_2:=\Big(b_2-(a_2)_--(b_1)_-\Big)p-\left(\frac{\xi_1}{d_2}+\frac{\xi_2}{d_4}-\frac{\chi_2}{d_2}\right)(p-1)$ are positive. By adding up (3.15) and (3.16), and using Young's inequality, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p+\int_\Omega w^p\right)+\left(\int_\Omega u^p+\int_\Omega w^p\right)\\&\le\left(\frac{\xi_1(p-1)}{d_2}-a_2p\right)\int_\Omega u^pw+\frac{\chi_1(p-1)}{d_4}\int_\Omega u^pz-\left(a_1p+\frac{\chi_1(p-1)}{d_4}\right)\int_\Omega u^{p+1}\\&\quad+\left(\frac{\xi_2(p-1)}{d_4}-b_1p\right)\int_\Omega w^pu+\frac{\chi_2(p-1)}{d_2}\int_\Omega w^pv-\Bigg(b_2p\\&\quad+\frac{\chi_2(p-1)}{d_2}\Bigg)\int_\Omega w^{p+1}+C_{13}\\&\le-\sigma_1\int_\Omega u^{p+1}+\frac{\chi_1(p-1)}{d_4}\int_\Omega u^pz-\sigma_2\int_\Omega w^{p+1}+\frac{\chi_2(p-1)}{d_2}\int_\Omega w^pv+C_{13}\\&\le-\frac{\sigma_1}{2}\int_\Omega u^{p+1}+\left(\frac{2p}{\sigma_1}\right)^p\left(\frac{\chi_1(p-1)}{d_4(p+1)}\right)^{p+1}\int_\Omega z^{p+1}\\&\quad-\frac{\sigma_2}{2}\int_\Omega w^{p+1}+\left(\frac{2p}{\sigma_2}\right)^p\left(\frac{\chi_2(p-1)}{d_2(p+1)}\right)^{p+1}\int_\Omega v^{p+1}+C_{13}\end{aligned}\tag{3.17}$$

for all $t\in(0,T_{\max})$ and some $C_{13}>0$.

By using the Agmon–Douglis–Nirenberg L^p estimates (see Agmon et al. (1959, 1964)) on linear elliptic equations with the homogeneous Neumann boundary conditions, there exists a positive constant C_{14} such that

$$\|v\|_{W^{2,p}(\Omega)}\le C_{14}\|w\|_{L^p(\Omega)}\tag{3.18}$$

for all $t\in(0,T_{\max})$. Then, it follows from Lemma 2.2 and Lemma 2.3 as well as (3.18) that

$$\left(\frac{2p}{\sigma_2}\right)^p\left(\frac{\chi_2(p-1)}{d_2(p+1)}\right)^{p+1}\int_\Omega v^{p+1} = \left(\frac{2p}{\sigma_2}\right)^p\left(\frac{\chi_2(p-1)}{d_2(p+1)}\right)^{p+1}\|v\|_{L^{p+1}(\Omega)}^{p+1}$$
$$\le C_{15}\|D^2v\|_{L^p(\Omega)}^{(p+1)\alpha_2}\|v\|_{L^1(\Omega)}^{(p+1)(1-\alpha_2)} + C_{15}\|v\|_{L^1(\Omega)}^{p+1}$$
$$\le C_{16}\|w\|_{L^p(\Omega)}^{(p+1)\alpha_2} + C_{15}m_0^{p+1} \tag{3.19}$$

for all $t \in (0, T_{\max})$ with $C_{15} = \left(\frac{2p}{\sigma_2}\right)^p\left(\frac{2\chi_2(p-1)C_{GN}}{d_2(p+1)}\right)^{p+1}$ and $C_{16} = C_{15}C_{14}^{(p+1)\alpha_2}m_0^{(p+1)(1-\alpha_2)}$, where $\alpha_2 = \dfrac{n-\frac{n}{p+1}}{n+2-\frac{n}{p}} \in (0,1)$ because of $p > \frac{n}{2}$. It is obvious to compute $(p+1)\alpha_2 < p$. By using Young's inequality with (3.19), we get

$$\begin{aligned}
&\left(\frac{2p}{\sigma_2}\right)^p\left(\frac{\chi_2(p-1)}{d_2(p+1)}\right)^{p+1}\int_\Omega v^{p+1}\\
&\le C_{16}\|w\|_{L^p(\Omega)}^{(p+1)\alpha_2} + C_{15}m_0^{p+1}\\
&\le \frac{\sigma_2}{2}\int_\Omega w^p + \frac{p-(p+1)\alpha_2}{p}C_{16}^{\frac{p}{p-(p+1)\alpha_2}}\left(\frac{b_2 p}{(p+1)\alpha_2}\right)^{-\frac{(p+1)\alpha_2}{p-(p+1)\alpha_2}} + C_{15}m_0^{p+1}\\
&\le \frac{\sigma_2}{2}\int_\Omega w^{p+1} + C_{17}
\end{aligned} \tag{3.20}$$

for all $t \in (0, T_{\max})$ with

$$C_{17} = \frac{\sigma_2 p^p|\Omega|}{2(p+1)^{p+1}} + \frac{p-(p+1)\alpha_2}{p}C_{16}^{\frac{p}{p-(p+1)\alpha_2}}\left(\frac{\sigma_2 p}{2(p+1)\alpha_2}\right)^{-\frac{(p+1)\alpha_2}{p-(p+1)\alpha_2}}$$
$$+C_{15}m_0^{p+1} > 0.$$

Likewise, we have

$$\left(\frac{2p}{\sigma_1}\right)^p\left(\frac{\chi_1(p-1)}{d_4(p+1)}\right)^{p+1}\int_\Omega z^{p+1} \le \frac{\sigma_1}{2}\int_\Omega u^{p+1} + C_{18} \tag{3.21}$$

for all $t \in (0, T_{\max})$ with $C_{18} > 0$.

By combining (3.17)–(3.21), one can find a positive constant C_{19} such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p + \int_\Omega w^p\right) + \left(\int_\Omega u^p + \int_\Omega w^p\right) \le C_{19} \tag{3.22}$$

for all $t \in (0, T_{\max})$. It follows from Lemma 2.4 that

$$\int_\Omega u^p + \int_\Omega w^p \le \max\left\{\int_\Omega u_0^p + \int_\Omega w_0^p, C_{19}\right\}$$

for all $t \in (0, T_{\max})$ and some $p > p_0$, which implies (3.12). The proof of Lemma 3.2 is complete. □

Lemma 3.3 *Let $\tau = 0$, $d_i > 0 (i = 1, 2, 3, 4)$, $\xi_j, \chi_j > 0 (j = 1, 2)$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain. Suppose that (1.10) and (1.11) hold. Then, there exists $C_{20} > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{20} \tag{3.23}$$

for all $t \in (0, T_{\max})$.

Proof When $n = 1$, by Lemma 3.1 and the method in (Zheng and Mu 2017, Lemma 3.6), we can obtain that (3.23) holds. When $n \geq 2$, it follows from Lemma 3.2 and the Moser–Alikakos iteration in Alikakos (1979) (or Lemma A.1 of Tao and Winkler (2012)) that (3.23) holds. Here we omit the details. The proof of Lemma 3.3 is complete. □

Proof of Theorem 1.1. It follows from Lemma 3.3 that $\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} < \infty$ for all $t \in (0, T_{\max})$. According to the well-known elliptic maximum principle and (3.18), we derive $||z(\cdot, t)||_{L^\infty(\Omega)} \leq ||u(\cdot, t)||_{L^\infty(\Omega)} < \infty$ and $||v(\cdot, t)||_{L^\infty(\Omega)} \leq ||w(\cdot, t)||_{L^\infty(\Omega)} < \infty$ for all $t \in (0, T_{\max})$. Hence, it follows from Lemma 2.1 that $T_{\max} = \infty$. The proof of Theorem 1.1 is complete. □

4 Boundedness for $\tau = 1$

The aim of this section is to show the global boundedness of solution to (1.1) with $\tau = 1$ and prove Theorem 1.2.

Lemma 4.1 *Let $\tau = 1$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain. Assume that (1.10) holds. Then, there exists a constant $C_1 > 0$ such that*

$$\int_t^{t+\tau_0} \int_\Omega u^2(\cdot, s) ds + \int_t^{t+\tau_0} \int_\Omega w^2(\cdot, s) ds \leq C_1 \tag{4.1}$$

for all $t \in (0, T_{\max} - \tau_0)$, where $\tau_0 := \min\{1, \frac{T_{\max}}{2}\}$.

Proof From (2.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (u + w) + \min\{l_1, l_2\} \int_\Omega (u^2 + w^2) \leq \max\{a_0, b_0\} \int_\Omega (u + w) \tag{4.2}$$

for all $t \in (0, T_{\max})$. Then, by integrating over $(t, t + \tau_0)$ and using Lemma 2.2, we obtain

$$
\begin{aligned}
&\int_\Omega \Big(u(\cdot, t+\tau_0) + w(\cdot, t+\tau_0)\Big) + \min\{l_1, l_2\} \int_t^{t+\tau_0} \int_\Omega \Big(u^2(\cdot, s) + w^2(\cdot, s)\Big) ds \\
\le &\max\{a_0, b_0\} \int_t^{t+\tau_0} \int_\Omega \Big(u(\cdot, s) + w(\cdot, s)\Big) ds + \int_\Omega \Big(u(\cdot, t) + w(\cdot, t)\Big) \qquad (4.3)\\
\le &\Big(\max\{a_0, b_0\}\tau_0 + 1\Big) M_0
\end{aligned}
$$

for all $t \in (0, T_{\max} - \tau_0)$, which implies that (4.1) holds. The proof of Lemma 4.1 is complete. □

Lemma 4.2 *Let $\tau = 1, d_i > 0 (i = 1, 2, 3, 4), \xi_j, \chi_j > 0 (j = 1, 2)$ and $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a smoothly bounded domain. Assume that (1.10) holds. Then, there exists a constant $C_2 > 0$ such that*

$$
\int_\Omega |\nabla v(\cdot, t)|^2 \le C_2 \tag{4.4}
$$

and

$$
\int_\Omega |\nabla z(\cdot, t)|^2 \le C_2 \tag{4.5}
$$

for all $t \in (0, T_{\max})$, as well as

$$
\int_t^{t+\tau_0} \int_\Omega |\Delta v(\cdot, s)|^2 ds \le C_2 \tag{4.6}
$$

and

$$
\int_t^{t+\tau_0} \int_\Omega |\Delta z(\cdot, s)|^2 ds \le C_2 \tag{4.7}
$$

for all $t \in (0, T_{\max} - \tau_0)$, where $\tau_0 := \min\{1, \frac{T_{\max}}{2}\}$.

Proof Multiplying the second equation in (1.1) by $-2\Delta v$ and using Young's inequality, we derive

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v|^2 + 2\int_\Omega |\nabla v|^2 + 2d_2 \int_\Omega |\Delta v|^2 &= -2\int_\Omega w \cdot \Delta v \\
&\le \frac{1}{d_2} \int_\Omega w^2 + d_2 \int_\Omega |\Delta v|^2,
\end{aligned}
$$

then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla v|^2 + d_2\int_\Omega |\Delta v|^2 \le \frac{1}{d_2}\int_\Omega w^2 \tag{4.8}$$

for all $t \in (0, T_{\max})$.

In view of (Stinner et al. 2014a, Lemma 3.4) and Lemma 4.1, we obtain

$$\int_\Omega |\nabla v(\cdot,t)|^2 \le \max\left\{\int_\Omega |\nabla v_0|^2 + C_1, 3C_1\right\} \tag{4.9}$$

for all $t \in (0, T_{\max})$.

By integrating for (4.8) over $(t, t+\tau_0)$, we get

$$\begin{aligned}&\int_\Omega |\nabla v(\cdot,t+\tau_0)|^2 + \int_t^{t+\tau_0}\int_\Omega |\nabla v(\cdot,s)|^2 ds + d_2\int_t^{t+\tau_0}\int_\Omega |\Delta v(\cdot,s)|^2 ds\\ \le&\frac{1}{d_2}\int_t^{t+\tau_0}\int_\Omega w(\cdot,s)^2 ds + \int_\Omega |\nabla v(\cdot,t)|^2\end{aligned} \tag{4.10}$$

for all $t \in (0, T_{\max} - \tau_0)$, where $\tau_0 := \min\{1, \frac{T_{\max}}{2}\}$. By using Lemma 4.1 and (4.9), we get (4.6). Similarly, we get (4.5) and (4.7). The proof of Lemma 4.2 is complete. □

When $n \le 2$, we shall establish L^2-boundedness of u and w, which is essential to obtain $L^\infty-$boundedness of u and w.

Lemma 4.3 *Let $\tau = 1, d_i > 0 (i = 1, 2, 3, 4), \xi_j, \chi_j > 0 (j = 1, 2)$ and $\Omega \subset \mathbb{R}^n (n \le 2)$ be a smoothly bounded domain. Suppose that (1.12) holds. Then, for all nonnegative initial data (u_0, v_0, w_0, z_0) satisfying (1.3), there exists a constant $C_3 > 0$ such that*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C_3 \tag{4.11}$$

and

$$\|w(\cdot,t)\|_{L^2(\Omega)} \le C_3 \tag{4.12}$$

for all $t \in (0, T_{\max})$.

Proof Multiplying the first equation in (1.1) by $2u$ and integrating by parts over Ω, we derive from Lemma 2.2 and Young's inequality that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 =& -2d_1\int_\Omega |\nabla u|^2 - \xi_1\int_\Omega u^2\cdot\Delta v + \chi_1\int_\Omega u^2\cdot\Delta z\\ &+2\int_\Omega u^2\Big(a_0 - a_1 u - a_2 w - a_3\int_\Omega u - a_4\int_\Omega w\Big)\\ \le& -2d_1\int_\Omega |\nabla u|^2 - \xi_1\int_\Omega u^2\cdot\Delta v + \chi_1\int_\Omega u^2\cdot\Delta z\\ &+2(a_0+2a_5M_0)\int_\Omega u^2 - 2a_1\int_\Omega u^3 + 2(a_2)_-\int_\Omega u^2 w\end{aligned} \tag{4.13}$$

for all $t \in (0, T_{\max})$, where $a_5 := \max\left\{(a_3)_-, (a_4)_-\right\}$.

Similarly, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^2 \leq & -2d_3\int_\Omega |\nabla w|^2 - \xi_2\int_\Omega w^2\cdot\Delta z + \chi_2\int_\Omega w^2\cdot\Delta v \\ & + 2(b_0+2b_5M_0)\int_\Omega w^2 - 2b_2\int_\Omega w^3 + 2(b_1)_-\int_\Omega w^2 u \end{aligned} \tag{4.14}$$

for all $t \in (0, T_{\max})$, where $b_5 := \max\left\{(b_3)_-, (b_4)_-\right\}$. Then, combining (4.13) with (4.14) and using Young's inequality, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(u^2+w^2\right) \leq & -2d_1\int_\Omega |\nabla u|^2 - \xi_1\int_\Omega u^2\cdot\Delta v + \chi_1\int_\Omega u^2\cdot\Delta z \\ & -2d_3\int_\Omega |\nabla w|^2 - \xi_2\int_\Omega w^2\cdot\Delta z + \chi_2\int_\Omega w^2\cdot\Delta v \\ & + 2(a_0+2a_5M_0)\int_\Omega u^2 - 2a_1\int_\Omega u^3 + 2(a_2)_-\int_\Omega u^2 w \\ & + 2(b_0+2b_5M_0)\int_\Omega w^2 - 2b_2\int_\Omega w^3 + 2(b_1)_-\int_\Omega w^2 u \end{aligned} \tag{4.15}$$

for all $t \in (0, T_{\max})$.

We next estimate the boundedness of $\|u\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}$ when $n = 1$ and $n = 2$, respectively.

Case 1: L^2**−boundedness for** $n = 1$.

By Lemma 2.2, Lemma 2.3 ($n = 1$), Hölder's and Young's inequalities, we deduce

$$\begin{aligned} -\xi_1\int_\Omega u^2\cdot\Delta v \leq & \xi_1\|u\|^2_{L^4(\Omega)}\cdot\|\Delta v\|_{L^2(\Omega)} \\ \leq & C_4\left(\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^1(\Omega)} + \|u\|^2_{L^1(\Omega)}\right)\|\Delta v\|_{L^2(\Omega)} \\ \leq & C_5\left(\|\nabla u\|_{L^2(\Omega)}+1\right)\|\Delta v\|_{L^2(\Omega)} \\ \leq & \frac{2d_1}{3}\int_\Omega |\nabla u|^2 + C_6\int_\Omega |\Delta v|^2 + C_6 \end{aligned} \tag{4.16}$$

with some $C_4, C_5, C_6 > 0$. Similarly, we have

$$\chi_1\int_\Omega u^2\cdot\Delta z \leq \frac{2d_1}{3}\int_\Omega |\nabla u|^2 + C_7\int_\Omega |\Delta z|^2 + C_7 \tag{4.17}$$

and

$$-\xi_2\int_\Omega w^2\cdot\Delta z \leq \frac{2d_3}{3}\int_\Omega |\nabla w|^2 + C_8\int_\Omega |\Delta z|^2 + C_8 \tag{4.18}$$

as well as

$$\chi_2 \int_\Omega w^2 \cdot \Delta v \leq \frac{2d_3}{3} \int_\Omega |\nabla w|^2 + C_9 \int_\Omega |\Delta v|^2 + C_9, \tag{4.19}$$

where $C_7, C_8, C_9 > 0$.

By Young's inequality, we have

$$2(a_2)_- \int_\Omega u^2 w + 2(b_1)_- \int_\Omega w^2 u \leq 2\Big((a_2)_- + (b_1)_-\Big)\Big(\textstyle\int_\Omega u^3 + \int_\Omega u^3\Big). \tag{4.20}$$

Then, using Lemma 2.2, Lemma 2.3 ($n = 1$) and Young's inequality, we derive

$$\begin{aligned} 2\Big((a_2)_- + (b_1)_-\Big) \int_\Omega u^3 \leq & C_{10}\Big(\int_\Omega |\nabla u|^2\Big)^{\frac{2}{3}} \Big(\int_\Omega u\Big)^{\frac{5}{3}} + C_{10}\Big(\int_\Omega u\Big)^3 \\ \leq & \frac{2d_1}{3} \int_\Omega |\nabla u|^2 + C_{11} \end{aligned} \tag{4.21}$$

with some $C_{10}, C_{11} > 0$. Similarly, we have

$$2\Big((a_2)_- + (b_1)_-\Big) \int_\Omega w^3 \leq \frac{2d_3}{3} \int_\Omega |\nabla w|^2 + C_{12} \tag{4.22}$$

with some $C_{12} > 0$.

In view of (4.15)–(4.22), we deduce from Young's inequality that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \big(u^2 + w^2\big) + 2\int_\Omega \big(u^2 + w^2\big) \\ \leq & \big(C_6 + C_9\big) \int_\Omega |\Delta v|^2 + \big(C_7 + C_8\big) \int_\Omega |\Delta z|^2 \\ & + 2(a_0 + 2a_5 M_0 + 1) \int_\Omega u^2 - 2a_1 \int_\Omega u^3 \\ & + 2(b_0 + 2b_5 M_0 + 1) \int_\Omega w^2 - 2b_2 \int_\Omega w^3 \\ & + C_6 + C_7 + C_8 + C_9 + C_{11} + C_{12} \\ \leq & \big(C_6 + C_9\big) \int_\Omega |\Delta v|^2 + \big(C_7 + C_8\big) \int_\Omega |\Delta z|^2 + C_{13} \end{aligned} \tag{4.23}$$

for all $t \in (0, T_{\max})$ with some $C_{13} > 0$.

Accordingly, in view of (Stinner et al. 2014a, Lemma 3.4) and Lemma 4.2, we have

$$\int_\Omega u^2(\cdot, t) + w^2(\cdot, t) < C_{14}$$

for all $t \in (0, T_{\max})$.

Case 2: L^2-boundedness for $n=2$.

By utilizing the Young's inequality, we find that

$$\begin{aligned}&2(a_0+2a_5M_0)\int_\Omega u^2-2a_1\int_\Omega u^3+2(a_2)_-\int_\Omega u^2w\\&+2(b_0+2b_5M_0)\int_\Omega w^2-2b_2\int_\Omega u^3+2(b_1)_-\int_\Omega w^2u\\&\le 2(a_0+2a_5M_0)\int_\Omega u^2-2\left(a_1-\frac{2(a_2)_-}{3}-\frac{(b_1)_-}{3}\right)\int_\Omega u^3\\&+2(b_0+2b_5M_0)\int_\Omega w^2-2\left(b_2-\frac{2(b_1)_-}{3}-\frac{(a_2)_-}{3}\right)\int_\Omega w^3\\&\le C_{15},\end{aligned}\tag{4.24}$$

where $C_{15}=\dfrac{8(a_0+2a_5M_0)^3}{27(a_1-\frac{2(a_2)_-}{3}-\frac{(b_1)_-}{3})^2}|\Omega|+\dfrac{8(b_0+2b_5M_0)^3}{27(b_2-\frac{2(b_1)_-}{3}-\frac{(a_2)_-}{3})^2}|\Omega|>0$ and we have used the facts that

$$a_1-\frac{2(a_2)_-}{3}-\frac{(b_1)_-}{3}>0$$

and

$$b_2-\frac{2(b_1)_-}{3}-\frac{(a_2)_-}{3}>0$$

because of (1.12). Then, combining (4.15) with (4.24) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(u^2+w^2\right)\le&-2d_1\int_\Omega|\nabla u|^2-\xi_1\int_\Omega u^2\cdot\Delta v+\chi_1\int_\Omega u^2\cdot\Delta z\\&-2d_3\int_\Omega|\nabla w|^2-\xi_2\int_\Omega w^2\cdot\Delta z+\chi_2\int_\Omega w^2\cdot\Delta v+C_{15}\end{aligned}\tag{4.25}$$

for all $t\in(0,T_{\max})$.

It follows from Lemma 2.3 ($n=2$), Hölder's and Young's inequalities that

$$\begin{aligned}&-\xi_1\int_\Omega u^2\cdot\Delta v+\chi_1\int_\Omega u^2\cdot\Delta z-\xi_2\int_\Omega w^2\cdot\Delta z+\chi_2\int_\Omega w^2\cdot\Delta v\\\le&\xi_1\left(\int_\Omega u^4\right)^{\frac12}\cdot\left(\int_\Omega|\Delta v|^2\right)^{\frac12}+\chi_1\left(\int_\Omega u^4\right)^{\frac12}\cdot\left(\int_\Omega|\Delta z|^2\right)^{\frac12}\\&+\xi_2\left(\int_\Omega w^4\right)^{\frac12}\cdot\left(\int_\Omega|\Delta z|^2\right)^{\frac12}+\chi_2\left(\int_\Omega w^4\right)^{\frac12}\cdot\left(\int_\Omega|\Delta v|^2\right)^{\frac12}\\\le&\lambda\left(\|u\|^2_{L^4(\Omega)}+\|w\|^2_{L^4(\Omega)}\right)\left(\|\Delta v\|_{L^2(\Omega)}+\|\Delta z\|_{L^2(\Omega)}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq C_{16}\Big(\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}\Big)\Big(\|\Delta v\|_{L^2(\Omega)}+\|\Delta z\|_{L^2(\Omega)}\Big)\\
&\quad + C_{16}\Big(\|\nabla w\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}+\|w\|^2_{L^2(\Omega)}\Big)\Big(\|\Delta v\|_{L^2(\Omega)}+\|\Delta z\|_{L^2(\Omega)}\Big)\\
&\leq 2d_1\|\nabla u\|^2_{L^2(\Omega)}+2d_3\|\nabla w\|^2_{L^2(\Omega)}\\
&\quad + C_{17}\Big(\|u\|^2_{L^2(\Omega)}+\|w\|^2_{L^2(\Omega)}\Big)\Big(\|\Delta v\|^2_{L^2(\Omega)}+\|\Delta z\|^2_{L^2(\Omega)}+1\Big)
\end{aligned}\tag{4.26}
$$

for all $t \in (0, T_{\max})$, where $\lambda := \max\{\xi_1, \chi_1, \xi_2, \chi_2\}$, $C_{16} = \lambda C_{GN}^2 > 0$ and $C_{17} > 0$. By (4.25) and (4.26), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega u^2+\int_\Omega w^2\Big) \leq\, & C_{17}\Big(\int_\Omega u^2+\int_\Omega w^2\Big)\Big(\|\Delta v\|^2_{L^2(\Omega)}+\|\Delta z\|^2_{L^2(\Omega)}+1\Big)\\
&+C_{15}.
\end{aligned}\tag{4.27}
$$

By Lemma 4.1 and Lemma 4.2, there exists some $t_0 \in [t-\tau_0, t] \subset (0, T_{\max})$ such that

$$
\int_\Omega u^2(\cdot,t_0)+\int_\Omega w^2(\cdot,t_0) \leq \max\Big\{\int_\Omega u_0^2+w_0^2, 2C_1\Big\} := C_{18}, \tag{4.28}
$$

where $\tau_0 := \min\{1, \frac{T_{\max}}{2}\}$. By integrating for (4.27) over (t_0, t), we deduce

$$
\begin{aligned}
&\int_\Omega u^2(\cdot,t)+\int_\Omega w^2(\cdot,t)\\
&\leq \int_\Omega \Big(u^2(\cdot,t_0)+w^2(\cdot,t_0)\Big)e^{C_{17}\int_{t_0}^t\left(\|\Delta v(\cdot,s)\|^2_{L^2(\Omega)}+\|\Delta z(\cdot,s)\|^2_{L^2(\Omega)}+1\right)ds}\\
&\quad + C_{15}\int_{t_0}^t e^{C_{17}\int_s^t\left(\|\Delta v(\cdot,s)\|^2_{L^2(\Omega)}+\|\Delta z(\cdot,s)\|^2_{L^2(\Omega)}+1\right)ds}ds.
\end{aligned}\tag{4.29}
$$

With the aid of (4.28), Lemma 4.2 and the fact that $t - t_0 < \tau_0 \leq 1$, we obtain the boundedness of $\int_\Omega u^2(\cdot,t)+w^2(\cdot,t)$ for all $t \in (0, T_{\max})$. The proof of Lemma 4.3 is complete. □

Lemma 4.4 *Let $\tau = 1$, $d_i > 0 (i = 1, 2, 3, 4)$, $\xi_j, \chi_j > 0 (j = 1, 2)$ and $\Omega \subset \mathbb{R}^n (n \leq 2)$ be a smoothly bounded domain. Suppose that (1.12) holds. Then, for $q > 1$ there exists a constant $C_{19} > 0$ such that*

$$
\|u(\cdot,t)\|_{L^q(\Omega)}+\|w(\cdot,t)\|_{L^q(\Omega)} \leq C_{19} \tag{4.30}
$$

for all $t \in (0, T_{\max})$.

Proof Multiplying the first equation in (1.1) by $qu^{q-1}(q > 1)$ and using Young's inequality as well as Lemma 2.2, we infer from a simple calculation that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q + d_1 q(q-1)\int_\Omega u^{q-2}\left|\nabla u\right|^2 \\
&= \xi_1 q(q-1)\int_\Omega u^{q-1}\nabla u\cdot\nabla v - \chi_1 q(q-1)\int_\Omega u^{q-1}\nabla u\cdot\nabla z \\
&+qa_0\int_\Omega u^p - q\int_\Omega u^q\left(a_1 u + a_2 w + a_3\int_\Omega u + a_4\int_\Omega w\right) \\
&\le \xi_1 q(q-1)\int_\Omega u^{q-1}\nabla u\cdot\nabla v - \chi_1 q(q-1)\int_\Omega u^{q-1}\nabla u\cdot\nabla z \\
&+q(a_0+2a_5M_0)\int_\Omega u^q - qa_1\int_\Omega u^{q+1} + q(a_2)_-\left(\int_\Omega u^{q+1} + \int_\Omega w^{q+1}\right)
\end{aligned} \tag{4.31}$$

for all $t \in (0, T_{\max})$, where $a_5 := \max\left\{(a_3)_-, (a_4)_-\right\}$.

In view of (Liu and Tao 2016, Lemma 2.3) ($n \le 2$) and Lemma 4.3, we have

$$\|v\|_{W^{1,4}(\Omega)} \le C_{20} \tag{4.32}$$

and

$$\|z\|_{W^{1,4}(\Omega)} \le C_{20} \tag{4.33}$$

for all $t \in (0, T_{\max})$ with some $C_{20} > 0$.

It follows from (4.32), Young's and Hölder's inequalities that

$$\begin{aligned}
&\xi_1 q(q-1)\int_\Omega u^{q-1}\nabla u\cdot\nabla v \\
&\le \frac{d_1 q(q-1)}{5}\int_\Omega u^{q-2}\left|\nabla u\right|^2 + q(q-1)\xi_1^2\int_\Omega u^q\left|\nabla v\right|^2 \\
&\le \frac{d_1 q(q-1)}{5}\int_\Omega u^{q-2}\left|\nabla u\right|^2 + q(q-1)\xi_1^2\left(\int_\Omega u^{2q}\right)^{\frac{1}{2}}\left(\int_\Omega\left|\nabla v\right|^4\right)^{\frac{1}{2}} \\
&\le \frac{d_1 q(q-1)}{5}\int_\Omega u^{q-2}\left|\nabla u\right|^2 + q(q-1)\xi_1^2 C_{20}^2\left(\int_\Omega u^{2q}\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.34}$$

By Lemma 2.3 and Lemma 4.3, there exist some $C_{21}, C_{22} > 0$ such that

$$\begin{aligned}
q(q-1)\xi_1^2 C_{20}^2\left(\int_\Omega u^{2q}\right)^{\frac{1}{2}} &= q(q-1)\xi_1^2 C_{20}^2\left\|u^{\frac{q}{2}}\right\|_{L^4(\Omega)}^2 \\
&\le C_{21}\left\|\nabla u^{\frac{q}{2}}\right\|_{L^2(\Omega)}^{2\alpha_3}\left\|u^{\frac{q}{2}}\right\|_{L^{\frac{4}{q}}(\Omega)}^{2(1-\alpha_3)} + C_{21}\left\|u^{\frac{q}{2}}\right\|_{L^{\frac{4}{q}}(\Omega)}^2 \\
&\le C_{22}\left\|\nabla u^{\frac{q}{2}}\right\|_{L^2(\Omega)}^{2\alpha_3} + C_{22},
\end{aligned} \tag{4.35}$$

where

$$\alpha_3 := \frac{\frac{nq}{4} - \frac{n}{4}}{1 - \frac{n}{2} + \frac{nq}{4}} \in (0, 1)$$

due to $q > 1$ and $n \leq 2$. Making use of the Young's inequality, we have

$$\begin{aligned} q(q-1)\xi_1^2 C_{20}^2 \left(\int_\Omega u^{2q} \right)^{\frac{1}{2}} \leq & C_{22} \left(\int_\Omega |\nabla u^{\frac{q}{2}}|^2 \right)^{\alpha_3} + C_{22} \\ \leq & \frac{d_1 q(q-1)}{5} \int_\Omega u^{q-2} |\nabla u|^2 + C_{23} \end{aligned} \tag{4.36}$$

with some $C_{23} > 0$. Combining (4.34)–(4.36) yields

$$\xi_1 q(q-1) \int_\Omega u^{q-1} \nabla u \cdot \nabla v \leq \frac{2d_1 q(q-1)}{5} \int_\Omega u^{q-2} |\nabla u|^2 + C_{23}. \tag{4.37}$$

Similarly, we get

$$-\chi_1 q(q-1) \int_\Omega u^{q-1} \nabla u \cdot \nabla z \leq \frac{2d_1 q(q-1)}{5} \int_\Omega u^{q-2} |\nabla u|^2 + C_{24} \tag{4.38}$$

with some $C_{24} > 0$.

As a consequence of (4.31), (4.37) and (4.38), we obtain

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^q + \frac{d_1 q(q-1)}{5} \int_\Omega u^{q-2} |\nabla u|^2 \\ \leq & q(a_0 + 2a_5 M_0) \int_\Omega u^q - q a_1 \int_\Omega u^{q+1} + q(a_2)_- \left(\int_\Omega u^{q+1} + \int_\Omega w^{q+1} \right) \\ & + C_{23} + C_{24}. \end{aligned} \tag{4.39}$$

Making use of the similar method on w-equation, we have

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^q + \frac{d_3 q(q-1)}{5} \int_\Omega w^{q-2} |\nabla u|^2 \\ & \leq q(b_0 + 2b_5 M_0) \int_\Omega w^q - q b_2 \int_\Omega w^{q+1} + q(b_1)_- \left(\int_\Omega u^{q+1} + \int_\Omega w^{q+1} \right) + C_{25} \end{aligned} \tag{4.40}$$

with some $C_{25} > 0$. Then, we deduce from (4.39) and (4.40) that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega u^q + \int_\Omega w^q \right) + q \left(\int_\Omega u^q + \int_\Omega w^q \right) \\ & + \frac{d_1 q(q-1)}{5} \int_\Omega u^{q-2} |\nabla u|^2 + \frac{d_3 q(q-1)}{5} \int_\Omega w^{q-2} |\nabla u|^2 \end{aligned}$$

$$\leq q(a_0+2a_5M_0+1)\int_\Omega u^q - qa_1\int_\Omega u^{q+1} + q\Big((a_2)_- + (b_1)_-\Big)\int_\Omega u^{q+1}$$
$$+ q(b_0+2b_5M_0+1)\int_\Omega w^q - qb_2\int_\Omega w^{q+1} + q\Big((a_2)_- + (b_1)_-\Big)\int_\Omega w^{q+1} + C_{26} \tag{4.41}$$

with some $C_{26} > 0$.

By applying Lemma 2.3, Lemma 4.3 and Young's inequality again, there exist some $C_{27}, C_{28}, C_{29} > 0$ such that

$$\begin{aligned}
q\Big((a_2)_- + (b_1)_-\Big)\int_\Omega u^{q+1} &= q\Big((a_2)_- + (b_1)_-\Big)\big\|u^{\frac{q}{2}}\big\|^{\frac{2(q+1)}{q}}_{L^{\frac{2(q+1)}{q}}(\Omega)}\\
&\leq C_{27}\big\|\nabla u^{\frac{q}{2}}\big\|^{\frac{2(q+1)}{q}\alpha_4}_{L^2(\Omega)}\big\|u^{\frac{q}{2}}\big\|^{\frac{2(q+1)}{q}(1-\alpha_4)}_{L^{\frac{4}{q}}(\Omega)} + C_{27}\big\|u^{\frac{q}{2}}\big\|^{\frac{2(q+1)}{q}}_{L^{\frac{4}{q}}(\Omega)}\\
&\leq C_{28}\big\|\nabla u^{\frac{q}{2}}\big\|^{\frac{2(q+1)}{q}\alpha_4}_{L^2(\Omega)} + C_{28}\\
&\leq \frac{d_1q(q-1)}{5}\int_\Omega u^{q-2}\big|\nabla u\big|^2 + C_{29}
\end{aligned} \tag{4.42}$$

where

$$\alpha_4 := \frac{\frac{nq}{4} - \frac{nq}{2(q+1)}}{1-\frac{n}{2}+\frac{nq}{4}} \in (0,1)$$

because of $q > 1$ and $n \leq 2$. Similarly, there exists a positive constant C_{30} such that

$$q\Big((a_2)_- + (b_1)_-\Big)\int_\Omega w^{q+1} \leq \frac{d_3q(q-1)}{5}\int_\Omega w^{q-2}\big|\nabla w\big|^2 + C_{30}. \tag{4.43}$$

By (4.41)–(4.43), we derive from Young's inequality and Lemma 2.4 that (4.30) holds for all $t \in (0, T_{\max})$. The proof of Lemma 4.4 is complete. □

Lemma 4.5 *Let $\tau = 1$, $d_i > 0 (i = 1, 2, 3, 4)$, $\xi_j, \chi_j > 0 (j = 1, 2)$ and $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a smoothly bounded domain. Suppose that (1.13) holds. Then, for $p > 1$ there exists a constant $C_{31} > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} + \|w(\cdot,t)\|_{L^p(\Omega)} \leq C_{31} \tag{4.44}$$

for all $t \in (0, T_{\max})$.

Proof For all $p > 1$, multiplying the first equation in (1.1) by pu^{p-1} and integrating by parts over Ω, we derive from Young's inequality and Lemma 2.2 that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p+1)\int_\Omega u^p \\
=&-\frac{4d_1(p-1)}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 - \xi_1(p-1)\int_\Omega u^p\cdot\Delta v + \chi_1(p-1)\int_\Omega u^p\cdot\Delta z \\
&+(pa_0+p+1)\int_\Omega u^p - p\int_\Omega u^p\left(a_1u + a_2w + a_3\int_\Omega u + a_4\int_\Omega w\right) \\
\le&-\frac{4d_1(p-1)}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + \Big((a_0+2a_5M_0)p + p + 1\Big)\int_\Omega u^p \\
&+\xi_1 p\int_\Omega |\Delta v|^{p+1} + \chi_1 p\int_\Omega |\Delta z|^{p+1} \\
&-\Big(a_1 - (a_2)_- - \xi_1 - \chi_1\Big)p\int_\Omega u^{p+1} + (a_2)_- p\int_\Omega w^{p+1}
\end{aligned}
\tag{4.45}
$$

for all $t \in (0, T_{\max})$, where $a_5 := \max\left\{(a_3)_-, (a_4)_-\right\}$.

It follows from (3.14) and (4.45) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p+1)\int_\Omega u^p \le&\Big(\xi_1 + \chi_1 + (a_2)_- - a_1\Big)p\int_\Omega u^{p+1} + (a_2)_- p\int_\Omega w^{p+1} \\
&+\xi_1 p\int_\Omega |\Delta v|^{p+1} + \chi_1 p\int_\Omega |\Delta z|^{p+1} + C_{32}
\end{aligned}
\tag{4.46}
$$

for all $t \in (0, T_{\max})$ with $C_{32} > 0$.

Let $t_0 \in (0, T_{\max})$ such that $t_0 \le 1$, there exists a positive constant C_{33} such that

$$
\begin{aligned}
\int_\Omega u^p \le&\Big(\xi_1 + \chi_1 + (a_2)_- - a_1\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1}(s)ds \\
&+(a_2)_- p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega w^{p+1}(s)ds \\
&+\xi_1 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds + \chi_1 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds \\
&+e^{-(p+1)(t-t_0)}\int_\Omega u^p(t_0) + C_{32}\int_{t_0}^t e^{-(p+1)(t-s)}ds \\
\le&\xi_1 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds + \chi_1 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds \\
&+\Big(\xi_1 + \chi_1 + (a_2)_- - a_1\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1}(s)ds \\
&+(a_2)_- p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega w^{p+1}(s)ds + C_{33}
\end{aligned}
\tag{4.47}
$$

for all $t \in (t_0, T_{\max})$, where we have used the variation of constants formula to (4.46).

By applying the similar way to w-equation, we get

$$\begin{aligned}\int_\Omega w^p \le& \xi_2 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds + \chi_2 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds\\ &+\Big(\xi_2+\chi_2+(b_1)_- - b_2\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega w^{p+1}(s)ds\\ &+(b_1)_- p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1}(s)ds + C_{34}\end{aligned} \tag{4.48}$$

for all $t \in (t_0, T_{\max})$, where $C_{34} > 0$.

By (4.47) and (4.48), we derive

$$\begin{aligned}\int_\Omega u^p + \int_\Omega w^p \le& \Big(\xi_1+\chi_1+(a_2)_- + (b_1)_- - a_1\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1}(s)ds\\ &+(\chi_1+\xi_2)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds\\ &+\Big(\xi_2+\chi_2+(a_2)_- + (b_1)_- - b_2\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega w^{p+1}(s)ds\\ &+(\xi_1+\chi_2)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds + C_{35}\end{aligned} \tag{4.49}$$

for all $t \in (t_0, T_{\max})$, where $C_{35} = C_{33} + C_{34}$.

By applying Lemma 2.5 with the second equation in (1.1), there exists a positive constant $C_{36} = C_{S_2}(d_2)$ such that

$$\begin{aligned}&\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds\\ \le& C_{36}\int_{t_0}^t e^{(p+1)s}\int_\Omega w^{p+1}(s)ds + C_{36}e^{pt_0}\Big(\|v(\cdot,t_0)\|^{p+1}_{L^{p+1}(\Omega)} + \|\Delta v(\cdot,t_0)\|^{p+1}_{L^{p+1}(\Omega)}\Big).\end{aligned} \tag{4.50}$$

Similarly, we can find $C_{37} = C_{S_1}(d_4) > 0$ such that

$$\begin{aligned}&\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds\\ \le& C_{37}\int_{t_0}^t e^{(p+1)s}\int_\Omega u^{p+1}(s)ds + C_{37}e^{pt_0}\Big(\|z(\cdot,t_0)\|^{p+1}_{L^{p+1}(\Omega)} + \|\Delta z(\cdot,t_0)\|^{p+1}_{L^{p+1}(\Omega)}\Big)\end{aligned} \tag{4.51}$$

for all $t \in (t_0, T_{\max})$.

By collecting (4.49)–(4.51), we get

$$\int_\Omega u^p + \int_\Omega w^p \le -\Big(a_1 - \xi_1 - \chi_1 - (a_2)_- - (b_1)_- - C_{37}(\chi_1+\xi_2)\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1}(s)ds$$

$$-\left(b_2-\xi_2-\chi_2-(a_2)_- - (b_1)_- - C_{36}(\xi_1+\chi_2)\right)p\int_{t_0}^{t}e^{-(p+1)(t-s)}\int_\Omega w^{p+1}(s)ds + C_{38} \tag{4.52}$$

for all $t\in(t_0,T_{\max})$, where

$$\begin{aligned}C_{38}=&C_{36}(\chi_1+\xi_2)pe^{pt_0}\left(\|v(\cdot,t_0)\|^{p+1}_{L^{p+1}(\Omega)}+\|\Delta v(\cdot,t_0)\|^{p+1}_{L^{p+1}(\Omega)}\right)\\&+C_{37}(\xi_1+\chi_2)pe^{pt_0}\left(\|z(\cdot,t_0)\|^{p+1}_{L^{p+1}(\Omega)}+\|\Delta z(\cdot,t_0)\|^{p+1}_{L^{p+1}(\Omega)}\right)+C_{35}.\end{aligned}$$

It follows from (1.13) that the constants $a_1-\xi_1-\chi_1-(a_2)_- - (b_1)_- - C_{37}(\chi_1+\xi_2)$ and $b_2-\xi_2-\chi_2-(a_2)_- - (b_1)_- - C_{36}(\xi_1+\chi_2)$ are positive. Hence, we derive from (4.52) that

$$\int_\Omega u^p+\int_\Omega w^p\le C_{38} \tag{4.53}$$

for all $t\in(t_0,T_{\max})$. By Lemma 2.1, we obtain that $\int_\Omega u^p+\int_\Omega w^p$ is uniformly bounded on $(0,t_0)$. The proof of Lemma 4.5 is complete. □

Proof of Theorem 1.2. Thanks to (Hu and Zheng 2022a, Lemma 3.2) and the boundedness of $\|u\|_{L^k(\Omega)}+\|w\|_{L^k(\Omega)}$ for sufficiently large k, we obtain the boundedness of $\|v\|_{W^{1,\infty}(\Omega)}$, $\|z\|_{W^{1,\infty}(\Omega)}$ for all $t\in(0,T_{\max})$. For the case $n\le 2$, by Lemma 4.4 and the Moser–Alikakos iteration in Alikakos (1979) (or Lemma A.1 of Tao and Winkler (2012)), one can obtain the global boundedness of solution to (1.1). For the case $n\ge 3$, the global boundedness of solution to (1.1) is derived by Lemma 4.5 and the Moser–Alikakos iteration in Alikakos (1979) (or Lemma A.1 of Tao and Winkler (2012)). Hence, it follows from Lemma 2.1 that $T_{\max}=\infty$. The proof of Theorem 1.2 is complete. □

5 Asymptotic Behavior

In this section, under the assumption that locally intraspecific competition and globally interspecific cooperation cases (i.e., $a_0,a_1,b_0,b_2>0$, $a_2,a_4,b_1,b_3<0$), we shall prove Theorems 1.3 and 1.4 by constructing some energy functionals, separately. In what follows, assume that system (1.1) has a unique global classical solution (u,v,w,z) with the property (1.17). To achieve our goals and apart from constructing the energy functionals, we first give the following key lemma.

Lemma 5.1 (see (Bai and Winkler 2016, Lemma 3.1)) *Let $f(t):(1,\infty)\to\mathbb{R}$ be a nonnegative and uniformly continuous function that satisfies $\int_1^\infty f(t)dt<\infty$. Then, $f(t)\to 0$ as $t\to\infty$.*

5.1 Proof of Theorem 1.3

Lemma 5.2 *Let $\tau = 0$ and (u, v, w, z) be a global bounded classical solution to (1.1). Suppose that the conditions of Theorem 1.3 hold. Then, there exists $\delta > 0$ such that*

$$\frac{d}{dt}E_1(t) \leq -\delta \int_\Omega \left((u-u_*)^2 + (v-v_*)^2 + (w-w_*)^2 + (z-z_*)^2\right) \tag{5.1}$$

for all $t > 0$, where

$$E_1(t) := \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right)$$

and (u_*, v_*, w_*, z_*) satisfies (1.16). Moreover,

$$\int_0^\infty \int_\Omega (u-u_*)^2 + \int_0^\infty \int_\Omega (v-v_*)^2 + \int_0^\infty \int_\Omega (w-w_*)^2 + \int_0^\infty \int_\Omega (z-z_*)^2 < \infty.$$

Proof Setting

$$\begin{aligned} A(t) &= \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right), \\ B(t) &= \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right), \end{aligned} \tag{5.2}$$

then $E_1(t)$ can be rewritten as

$$E_1(t) = A(t) + B(t) \tag{5.3}$$

for all $t > 0$.

Firstly, we show the nonnegativity of $E_1(t)$. Let $y(s) := s - u_* \ln s$ for $s > 0$. By applying Taylor's formula, there exists $\sigma \in (0, 1)$ such that

$$\begin{aligned} y(u) - y(u_*) =& y'(u_*) \cdot (u-u_*) + \frac{1}{2} y''[\sigma u + (1-\sigma)u_*] \cdot (u-u_*)^2 \\ =& \frac{u_*}{2[\sigma u + (1-\sigma)u_*]^2}(u-u_*)^2 \geq 0 \end{aligned} \tag{5.4}$$

for $x \in \Omega$ and $t > 0$, which implies that $A(t) = \int_\Omega \left(y(u) - y(u_*)\right) \geq 0$. Similarly, we can obtain $B(t) \geq 0$ for all $t \geq 0$. Thus, $E_1(t)$ is nonnegative.

Next, we will prove (5.1). By a simple calculation with (1.1), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}A(t) =& \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(u-u_*-u_*\ln\frac{u}{u_*}\right)\\
=& \int_\Omega\left(1-\frac{u_*}{u}\right)\left(d_1\Delta u-\xi_1\nabla\cdot(u\nabla v)+\chi_1\nabla\cdot(u\nabla z)\right)\\
&+\int_\Omega\left(u-u_*\right)\left(a_0-a_1u-a_2w-a_3\int_\Omega u-a_4\int_\Omega w\right)\\
=& -d_1u_*\int_\Omega\left|\frac{\nabla u}{u}\right|^2+\xi_1u_*\int_\Omega\frac{\nabla u}{u}\cdot\nabla v-\chi_1u_*\int_\Omega\frac{\nabla u}{u}\cdot\nabla z\\
&+\int_\Omega\left(u-u_*\right)\left(a_0-a_1u-a_2w-a_3\int_\Omega u-a_4\int_\Omega w\right).
\end{aligned}
\tag{5.5}
$$

It follows from $a_0=\big(a_1+a_3|\Omega|\big)u_*+\big(a_2+a_4|\Omega|\big)w_*$, Hölder's and Young's inequalities that

$$
\begin{aligned}
&\int_\Omega\left(u-u_*\right)\left(a_0-a_1u-a_2w-a_3\int_\Omega u-a_4\int_\Omega w\right)\\
=&\int_\Omega\left(u-u_*\right)\left(a_1(u_*-u)+a_2(w_*-w)+a_3\int_\Omega(u_*-u)+a_4\int_\Omega(w_*-w)\right)\\
=&-a_1\int_\Omega(u-u_*)^2-a_2\int_\Omega(u-u_*)(w-w_*)-a_3\left(\int_\Omega(u-u_*)\right)^2\\
&-a_4\int_\Omega(u-u_*)\cdot\int_\Omega(w-w_*)\\
\le&-\left(a_1-(a_3)_-|\Omega|-\frac{(a_4)_-|\Omega|}{2}\right)\int_\Omega(u-u_*)^2-a_2\int_\Omega(u-u_*)(w-w_*)\\
&+\frac{(a_4)_-|\Omega|}{2}\int_\Omega(w-w_*)^2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}A(t)\le& -d_1u_*\int_\Omega\left|\frac{\nabla u}{u}\right|^2+\xi_1u_*\int_\Omega\frac{\nabla u}{u}\cdot\nabla v-\chi_1u_*\int_\Omega\frac{\nabla u}{u}\cdot\nabla z\\
&-\left(a_1-(a_3)_-|\Omega|-\frac{(a_4)_-|\Omega|}{2}\right)\int_\Omega(u-u_*)^2\\
&-a_2\int_\Omega(u-u_*)(w-w_*)+\frac{(a_4)_-|\Omega|}{2}\int_\Omega(w-w_*)^2.
\end{aligned}
\tag{5.6}
$$

By using a similar method for $B(t)$, we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}B(t)\le -d_3w_*\int_\Omega\left|\frac{\nabla w}{w}\right|^2+\xi_2w_*\int_\Omega\frac{\nabla w}{w}\cdot\nabla z-\chi_2w_*\int_\Omega\frac{\nabla w}{w}\cdot\nabla v
$$

$$
\begin{aligned}
&-\left(b_2-(b_4)_-|\Omega|-\frac{(b_3)_-|\Omega|}{2}\right)\int_\Omega (w-w_*)^2\\
&-b_1\int_\Omega (u-u_*)(w-w_*)+\frac{(b_3)_-|\Omega|}{2}\int_\Omega (u-u_*)^2
\end{aligned}
\tag{5.7}
$$

due to the fact that $b_0=\big(b_1+b_3|\Omega|\big)u_*+\big(b_2+b_4|\Omega|\big)w_*$.

By combining (5.3), (5.6) and (5.7), we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)\le &-d_1u_*\int_\Omega\left|\frac{\nabla u}{u}\right|^2+\xi_1u_*\int_\Omega\frac{\nabla u}{u}\cdot\nabla v-\chi_1u_*\int_\Omega\frac{\nabla u}{u}\cdot\nabla z\\
&-d_3w_*\int_\Omega\left|\frac{\nabla w}{w}\right|^2+\xi_2w_*\int_\Omega\frac{\nabla w}{w}\cdot\nabla z-\chi_2w_*\int_\Omega\frac{\nabla w}{w}\cdot\nabla v\\
&-\left(a_1-(a_3)_-|\Omega|-\frac{(a_4)_-|\Omega|}{2}-\frac{(b_3)_-|\Omega|}{2}\right)\int_\Omega(u-u_*)^2\\
&-(a_2+b_1)\int_\Omega(u-u_*)(w-w_*)\\
&-\left(b_2-(b_4)_-|\Omega|-\frac{(b_3)_-|\Omega|}{2}-\frac{(a_4)_-|\Omega|}{2}\right)\int_\Omega(w-w_*)^2.
\end{aligned}
\tag{5.8}
$$

By applying Young's inequality, we have

$$
\begin{aligned}
-\chi_1\int_\Omega\frac{\nabla u}{u}\cdot\nabla z&\le\frac{d_1}{2}\int_\Omega\left|\frac{\nabla u}{u}\right|^2+\frac{\chi_1^2}{2d_1}\int_\Omega|\nabla z|^2,\\
\xi_1\int_\Omega\frac{\nabla u}{u}\cdot\nabla v&\le\frac{d_1}{2}\int_\Omega\left|\frac{\nabla u}{u}\right|^2+\frac{\xi_1^2}{2d_1}\int_\Omega|\nabla v|^2
\end{aligned}
$$

and

$$
-\chi_2\int_\Omega\frac{\nabla w}{w}\cdot\nabla v\le\frac{d_3}{2}\int_\Omega\left|\frac{\nabla w}{w}\right|^2+\frac{\chi_2^2}{2d_3}\int_\Omega|\nabla v|^2
$$

as well as

$$
\xi_2\int_\Omega\frac{\nabla w}{w}\cdot\nabla z\le\frac{d_3}{2}\int_\Omega\left|\frac{\nabla w}{w}\right|^2+\frac{\xi_2^2}{2d_3}\int_\Omega|\nabla z|^2.
$$

Then, we deduce

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)\le &-\left(a_1-(a_3)_-|\Omega|-\frac{(a_4)_-|\Omega|}{2}-\frac{(b_3)_-|\Omega|}{2}\right)\int_\Omega(u-u_*)^2\\
&-(a_2+b_1)\int_\Omega(u-u_*)(w-w_*)\\
&-\left(b_2-(b_4)_-|\Omega|-\frac{(b_3)_-|\Omega|}{2}-\frac{(a_4)_-|\Omega|}{2}\right)\int_\Omega(w-w_*)^2
\end{aligned}
$$

$$+\frac{1}{2}\Big(\frac{\xi_1^2 u_*}{d_1}+\frac{\chi_2^2 w_*}{d_3}\Big)\int_\Omega |\nabla v|^2+\frac{1}{2}\Big(\frac{\chi_1^2 u_*}{d_1}+\frac{\xi_2^2 w_*}{d_3}\Big)\int_\Omega |\nabla z|^2. \tag{5.9}$$

It follows from $0=d_2\Delta v-v+w$ and $v_*=w_*$ that

$$d_2\int_\Omega |\nabla v|^2=-\int_\Omega (v-v_*)^2+\int_\Omega (v-v_*)(w-w_*). \tag{5.10}$$

Similarly, we have

$$d_4\int_\Omega |\nabla z|^2=-\int_\Omega (z-z_*)^2+\int_\Omega (z-z_*)(u-u_*). \tag{5.11}$$

Together with (5.9)–(5.11), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)\le&-\left(a_1-(a_3)_-|\Omega|-\frac{(a_4)_-|\Omega|}{2}-\frac{(b_3)_-|\Omega|}{2}\right)\int_\Omega (u-u_*)^2\\&-(a_2+b_1)\int_\Omega (u-u_*)(w-w_*)\\&-\left(b_2-(b_4)_-|\Omega|-\frac{(b_3)_-|\Omega|}{2}-\frac{(a_4)_-|\Omega|}{2}\right)\int_\Omega (w-w_*)^2\\&-\frac{1}{2d_2}\Big(\frac{\xi_1^2u_*}{d_1}+\frac{\chi_2^2w_*}{d_3}\Big)\int_\Omega (v-v_*)^2+\frac{1}{2d_2}\Big(\frac{\xi_1^2u_*}{d_1}+\frac{\chi_2^2w_*}{d_3}\Big)\int_\Omega (v-v_*)(w-w_*)\\&-\frac{1}{2d_4}\Big(\frac{\chi_1^2u_*}{d_1}+\frac{\xi_2^2w_*}{d_3}\Big)\int_\Omega (z-z_*)^2+\frac{1}{2d_4}\Big(\frac{\chi_1^2u_*}{d_1}+\frac{\xi_2^2w_*}{d_3}\Big)\int_\Omega (z-z_*)(u-u_*).\end{aligned} \tag{5.12}$$

Since the conditions (1.18) and (1.19) hold, $\gamma_1:=a_1-(a_3)_-|\Omega|-\frac{(a_4)_-|\Omega|}{2}-\frac{(b_3)_-|\Omega|}{2}$ and $\gamma_2:=b_2-(b_4)_-|\Omega|-\frac{(b_3)_-|\Omega|}{2}-\frac{(a_4)_-|\Omega|}{2}$ are positive, and there exist $\theta_1,\theta_2\in(0,1)$ such that

$$4\theta_1\theta_2\gamma_1\gamma_2>(a_2+b_1)^2, \tag{5.13}$$

$$\gamma_1>\max\left\{\frac{u_*\chi_1^2}{4d_1d_4(1-\theta_1)},\frac{w_*\xi_2^2}{4d_3d_4(1-\theta_1)}\right\} \tag{5.14}$$

and

$$\gamma_2>\max\left\{\frac{u_*\xi_1^2}{4d_1d_2(1-\theta_2)},\frac{w_*\chi_2^2}{4d_2d_3(1-\theta_2)}\right\}. \tag{5.15}$$

According to (5.13)–(5.15) and $u_*,w_*>0$, one can find some $\delta>0$ satisfying

$$\begin{aligned}\delta\le\min\Bigg\{&\frac{4\theta_1\theta_2\gamma_1\gamma_2-(a_2+b_1)^2}{\theta_1\theta_2(\gamma_1+\gamma_2)},\frac{4(1-\theta_1)\gamma_1d_1d_4u_*\chi_1^2-u_*^2\chi_1^4}{4(1-\theta_1)d_1d_4(d_1d_4\gamma_1+u_*\chi_1^2)},\frac{4(1-\theta_1)\gamma_1d_3d_4w_*\xi_2^2-w_*^2\xi_2^4}{4(1-\theta_1)d_3d_4(d_3d_4\gamma_1+w_*\xi_2^2)},\\&\frac{4(1-\theta_2)\gamma_2d_1d_2u_*\xi_1^2-u_*^2\xi_1^4}{4(1-\theta_2)d_1d_2(d_1d_2\gamma_2+u_*\xi_1^2)},\frac{4(1-\theta_2)\gamma_2d_2d_3w_*\chi_2^2-w_*^2\chi_2^4}{4(1-\theta_2)d_2d_3(d_2d_3\gamma_2+w_*\chi_2^2)},\gamma_1,\gamma_2\Bigg\}.\end{aligned} \tag{5.16}$$

Therefore, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) \leq &-\delta\int_\Omega\Big((u-u_*)^2+(v-v_*)^2+(w-w_*)^2+(z-z_*)^2\Big)\\ &+\int_\Omega\big(h_1+h_2+h_3+h_4+h_5\big),\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
h_1 &= -\theta_1(\gamma_1-\delta)(u-u_*)^2-(a_2+b_1)(u-u_*)(w-w_*)-\theta_2(\gamma_2-\delta)(w-w_*)^2,\\
h_2 &= -\frac{1-\theta_1}{2}(\gamma_1-\delta)(u-u_*)^2+\frac{u_*\chi_1^2}{2d_1d_4}(u-u_*)(z-z_*)-\frac{1}{2}\Big(\frac{u_*\chi_1^2}{d_1d_4}-\delta\Big)(z-z_*)^2,\\
h_3 &= -\frac{1-\theta_1}{2}(\gamma_1-\delta)(u-u_*)^2+\frac{w_*\xi_2^2}{2d_3d_4}(u-u_*)(z-z_*)-\frac{1}{2}\Big(\frac{w_*\xi_2^2}{d_3d_4}-\delta\Big)(z-z_*)^2,\\
h_4 &= -\frac{1-\theta_2}{2}(\gamma_2-\delta)(w-w_*)^2+\frac{u_*\xi_1^2}{2d_1d_2}(w-w_*)(v-v_*)-\frac{1}{2}\Big(\frac{u_*\xi_1^2}{d_1d_2}-\delta\Big)(v-v_*)^2,\\
h_5 &= -\frac{1-\theta_2}{2}(\gamma_2-\delta)(w-w_*)^2+\frac{w_*\chi_2^2}{2d_2d_3}(w-w_*)(v-v_*)-\frac{1}{2}\Big(\frac{w_*\chi_2^2}{d_2d_3}-\delta\Big)(v-v_*)^2.
\end{aligned}$$

For each the discriminant of $h_i(i=1,2,3,4,5)$ and by (5.16), we have

$$\begin{aligned}
\triangle_1 &= (a_2+b_1)^2(w-w_*)^2-4\theta_1\theta_2(\gamma_1-\delta)(\gamma_2-\delta)(w-w_*)^2\leq 0,\\
\triangle_2 &= \frac{u_*^2\chi_1^4}{4d_1^2d_4^2}(z-z_*)^2-4\times\frac{1-\theta_1}{2}(\gamma_1-\delta)\times\frac{1}{2}\Big(\frac{u_*\chi_1^2}{d_1d_4}-\delta\Big)(z-z_*)^2\leq 0,\\
\triangle_3 &= \frac{w_*^2\xi_2^4}{4d_3^2d_4^2}(z-z_*)^2-4\times\frac{1-\theta_1}{2}(\gamma_1-\delta)\times\frac{1}{2}\Big(\frac{w_*\xi_2^2}{d_3d_4}-\delta\Big)(z-z_*)^2\leq 0,\\
\triangle_4 &= \frac{u_*^2\xi_1^4}{4d_1^2d_2^2}(v-v_*)^2-4\times\frac{1-\theta_2}{2}(\gamma_2-\delta)\times\frac{1}{2}\Big(\frac{u_*\xi_1^2}{d_1d_2}-\delta\Big)(v-v_*)^2\leq 0,\\
\triangle_5 &= \frac{w_*^2\chi_2^4}{4d_2^2d_3^2}(v-v_*)^2-4\times\frac{1-\theta_2}{2}(\gamma_2-\delta)\times\frac{1}{2}\Big(\frac{w_*\chi_2^2}{d_2d_3}-\delta\Big)(v-v_*)^2\leq 0,
\end{aligned} \tag{5.18}$$

which concludes

$$h_i\leq 0, i=1,2,3,4,5. \tag{5.19}$$

By (5.17) and (5.19), we directly obtain (5.1). Finally, integrating (5.1) over $(0,\infty)$, we get

$$\int_0^\infty\int_\Omega(u-u_*)^2+\int_0^\infty\int_\Omega(v-v_*)^2+\int_0^\infty\int_\Omega(w-w_*)^2+\int_0^\infty\int_\Omega(z-z_*)^2<\infty.$$

The proof of Lemma 5.2 is complete. □

Lemma 5.3 *Suppose that the conditions of Theorem 1.3 hold. Then, the solution of (1.1) satisfies*

$$\begin{aligned}&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}+\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}+\|w(\cdot,t)-w_*\|_{L^\infty(\Omega)}\\&\quad+\|z(\cdot,t)-z_*\|_{L^\infty(\Omega)}\to 0\end{aligned} \tag{5.20}$$

as $t\to\infty$.

Proof A combination of Lemma 5.1 and Lemma 5.2 implies this lemma. The proof of Lemma 5.3 is complete. □

Lemma 5.4 *Suppose that the conditions of Theorem 1.3 hold. Then, there exist* $C_1>0$ *and* $\kappa>0$ *such that the solution* (u,v,w,z) *of (1.1) satisfies*

$$\begin{aligned}&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}+\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}\\&\quad+\|w(\cdot,t)-w_*\|_{L^\infty(\Omega)}+\|z(\cdot,t)-z_*\|_{L^\infty(\Omega)}\le C_1e^{-\kappa t}\end{aligned} \tag{5.21}$$

for all $t>t_1$*, where* $t_1>0$ *is some fixed time.*

Proof This idea of proof is similar to (Bai and Winkler 2016, Lemma 3.7). For reader's convenience, we give the sketch of the proof.

According to the function $y(s)=s-u_*\ln s$ for $s>0$ that is given in Lemma 5.2 and L'Hôpital's rule, we have

$$\lim_{s\to u_*}\frac{y(s)-y(u_*)}{(s-u_*)^2}=\frac{1}{2u_*}. \tag{5.22}$$

Similarly, we have

$$\lim_{s\to w_*}\frac{s-w_*-w_*\ln\frac{s}{w_*}}{(s-w_*)^2}=\frac{1}{2w_*}. \tag{5.23}$$

By Lemma 5.3, there exist some $t_1>0$ and $C_2, C_3>0$ such that

$$C_2\int_\Omega(u-u_*)^2\le\int_\Omega\left(u-u_*-u_*\ln\frac{u}{u_*}\right)\le C_3\int_\Omega(u-u_*)^2 \tag{5.24}$$

and

$$C_2\int_\Omega(w-w_*)^2\le\int_\Omega\left(w-w_*-w_*\ln\frac{w}{w_*}\right)\le C_3\int_\Omega(w-w_*)^2 \tag{5.25}$$

for all $t>t_1$.

By means of the definition of $E_1(t)$, it follows from the second inequalities in (5.24) and (5.25) that there exists $C_4>0$ such that

$$C_4E_1(t)\le\int_\Omega(u-u_*)^2+\int_\Omega(v-v_*)^2+\int_\Omega(w-w_*)^2+\int_\Omega(z-z_*)^2 \tag{5.26}$$

for all $t > t_1$. With the aid of Lemma 5.2, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) \leq -\delta\big(\int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2 + \int_\Omega (z-z_*)^2\big) \\ \leq -C_4\delta E_1(t), \tag{5.27}$$

which implies

$$E_1(t) \leq E_1(t_1)e^{-C_4\delta(t-t_1)} \tag{5.28}$$

for all $t > t_1$. Then, by combining (5.28) with the first inequalities in (5.24) and (5.25), we derive a positive constant $C_5 > 0$ such that

$$\int_\Omega (u-u_*)^2 + \int_\Omega (w-w_*)^2 \leq C_5E_1(t) \leq C_5E_1(t_1)e^{-C_4\delta(t-t_1)} \tag{5.29}$$

for all $t > t_1$.

By applying the Gagliardo–Nirenberg inequality and (1.17), there exist some positive constants C_6, C_7 and C_8 such that

$$\begin{aligned} &\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)} + \|w(\cdot,t)-w_*\|_{L^\infty(\Omega)} \\ \leq &C_6\|u(\cdot,t)-u_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|u(\cdot,t)-u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\ &+ C_6\|w(\cdot,t)-w_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|w(\cdot,t)-w_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\ \leq &C_7\|u(\cdot,t)-u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} + C_7\|w(\cdot,t)-w_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\ \leq &C_8\bigg(\|u(\cdot,t)-u_*\|_{L^2(\Omega)}^2 + \|w(\cdot,t)-w_*\|_{L^2(\Omega)}^2\bigg)^{\frac{1}{n+2}} \\ \leq &C_8\bigg(C_5E_1(t_1)\bigg)^{\frac{1}{n+2}} e^{-\frac{C_4\delta(t-t_1)}{n+2}} \end{aligned} \tag{5.30}$$

for all $t > t_1$.

According to $\tau = 0$ in (1.1) and $v_* = w_*$, $z_* = u_*$, we get

$$\begin{aligned} &\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)} + \|z(\cdot,t)-z_*\|_{L^\infty(\Omega)} \\ \leq &\|w(\cdot,t)-w_*\|_{L^\infty(\Omega)} + \|u(\cdot,t)-u_*\|_{L^\infty(\Omega)} \\ \leq &C_8\bigg(C_5E_1(t_1)\bigg)^{\frac{1}{n+2}} e^{-\frac{C_4\delta(t-t_1)}{n+2}} \end{aligned} \tag{5.31}$$

by the application of the elliptic maximum principle and (5.30). The proof of Lemma 5.4 is complete. □

Proof of Theorem 1.3. Lemma 5.4 directly shows the results of Theorem 1.3. □

5.2 Proof of Theorem 1.4

When $\tau = 1$, we firstly introduce the following functional

$$E_2(t) := \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right) + \frac{\rho_1}{2} \int_\Omega \left(v - v_*\right)^2 + \frac{\rho_2}{2} \int_\Omega \left(z - z_*\right)^2,$$

where $\rho_1, \rho_2 > 0$ shall be determined and (u_*, v_*, w_*, z_*) satisfies (1.16).

Lemma 5.5 *Let $\tau = 1$ and (u, v, w, z) be a global bounded classical solution of (1.1). Assume that the conditions of Theorem 1.4 hold. Then, there exists $\beta > 0$ such that*

$$\frac{d}{dt} E_2(t) \leq -\beta \int_\Omega \left((u - u_*)^2 + (v - v_*)^2 + (w - w_*)^2 + (z - z_*)^2\right) \tag{5.32}$$

for all $t > 0$. Moreover,

$$\int_0^\infty \int_\Omega (u - u_*)^2 + \int_0^\infty \int_\Omega (v - v_*)^2 + \int_0^\infty \int_\Omega (w - w_*)^2 + \int_0^\infty \int_\Omega (z - z_*)^2 < \infty.$$

Proof Firstly, $E_2(t)$ can be rewritten as

$$E_2(t) = A(t) + B(t) + C(t) + D(t), \tag{5.33}$$

where $A(t)$, $B(t)$ are given by (5.2) and $C(t) := \frac{\rho_1}{2} \int_\Omega \left(v - v_*\right)^2$, $D(t) := \frac{\rho_2}{2} \int_\Omega \left(z - z_*\right)^2$. We obtain the nonnegativity of $E_2(t)$ by (5.4).

Next, we will prove (5.32). It follows from a simple calculation that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} C(t) =& \rho_1 \int_\Omega (v - v_*)(d_2 \Delta v - v + w) \\ =& -\rho_1 d_2 \int_\Omega |\nabla v|^2 - \rho_1 \int_\Omega (v - v_*)^2 + \rho_1 \int_\Omega (v - v_*)(w - w_*) \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} D(t) =& \rho_2 \int_\Omega (z - z_*)(d_4 \Delta z - z + u) \\ =& -\rho_2 d_4 \int_\Omega |\nabla z|^2 - \rho_2 \int_\Omega (z - z_*)^2 + \rho_2 \int_\Omega (z - z_*)(u - u_*), \end{aligned} \tag{5.35}$$

where we have used the facts that $v_* = w_*$ and $z_* = u_*$.

By combining (5.9), (5.34) and (5.35), we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) \le &-\left(a_1-(a_3)_-|\Omega|-\frac{(a_4)_-|\Omega|}{2}-\frac{(b_3)_-|\Omega|}{2}\right)\int_\Omega (u-u_*)^2\\
&-(a_2+b_1)\int_\Omega (u-u_*)(w-w_*)\\
&-\left(b_2-(b_4)_-|\Omega|-\frac{(b_3)_-|\Omega|}{2}-\frac{(a_4)_-|\Omega|}{2}\right)\int_\Omega (w-w_*)^2\\
&+\left(\frac{1}{2}\Big(\frac{\xi_1^2u_*}{d_1}+\frac{\chi_2^2w_*}{d_3}\Big)-\rho_1d_2\right)\int_\Omega|\nabla v|^2\\
&+\left(\frac{1}{2}\Big(\frac{\chi_1^2u_*}{d_1}+\frac{\xi_2^2w_*}{d_3}\Big)-\rho_2d_4\right)\int_\Omega|\nabla z|^2\\
&-\rho_1\int_\Omega (v-v_*)^2+\rho_1\int_\Omega (v-v_*)(w-w_*)\\
&-\rho_2\int_\Omega (z-z_*)^2+\rho_2\int_\Omega (z-z_*)(u-u_*).
\end{aligned}
\tag{5.36}
$$

Since the conditions (1.21) and (1.22) hold, $\gamma_3 := a_1-(a_3)_-|\Omega|-\frac{(a_4)_-|\Omega|}{2}-\frac{(b_3)_-|\Omega|}{2}$ and $\gamma_4 := b_2-(b_4)_-|\Omega|-\frac{(b_3)_-|\Omega|}{2}-\frac{(a_4)_-|\Omega|}{2}$ are positive, and there exist $\theta_3,\theta_4\in(0,1)$ such that one can find two positive constants ρ_1,ρ_2 satisfying

$$
(a_2+b_1)^2 < 4\theta_3\theta_4\gamma_3\gamma_4, \tag{5.37}
$$

$$
\frac{1}{2d_2}\Big(\frac{\xi_1^2u_*}{d_1}+\frac{\chi_2^2w_*}{d_3}\Big) < \rho_1 < 4\gamma_4(1-\theta_4) \tag{5.38}
$$

and

$$
\frac{1}{2d_4}\Big(\frac{\chi_1^2u_*}{d_1}+\frac{\xi_2^2w_*}{d_3}\Big) < \rho_2 < 4\gamma_3(1-\theta_3). \tag{5.39}
$$

By (5.37)–(5.39) and $u_*, w_* > 0$, one can find some $\beta > 0$ fulfilling

$$
\beta \le \min\left\{\frac{4\theta_3\theta_4\gamma_3\gamma_4-(a_2+b_1)^2}{\theta_3\theta_4(\gamma_3+\gamma_4)}, \frac{4(1-\theta_3)\gamma_3\rho_2-\rho_2^2}{4(1-\theta_3)(\gamma_3+\rho_2)}, \frac{4(1-\theta_4)\gamma_4\rho_1-\rho_1^2}{4(1-\theta_4)(\gamma_4+\rho_1)}, \gamma_3, \gamma_4\right\}. \tag{5.40}
$$

Therefore, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) \le &-\beta\int_\Omega\Big((u-u_*)^2+(v-v_*)^2+(w-w_*)^2+(z-z_*)^2\Big)\\
&+\int_\Omega \big(f_1+f_2+f_3\big),
\end{aligned}
\tag{5.41}
$$

where

$$
\begin{aligned}
f_1 &= -\theta_3(\gamma_3-\beta)(u-u_*)^2-(a_2+b_1)(u-u_*)(w-w_*)-\theta_4(\gamma_4-\beta)(w-w_*)^2,\\
f_2 &= -(1-\theta_3)(\gamma_3-\beta)(u-u_*)^2+\rho_2(u-u_*)(z-z_*)-(\rho_2-\beta)(z-z_*)^2,\\
f_3 &= -(1-\theta_4)(\gamma_4-\beta)(w-w_*)^2+\rho_1(w-w_*)(v-v_*)-(\rho_1-\beta)(v-v_*)^2.
\end{aligned}
$$

For each the discriminant of $f_i(i=1,2,3)$ and by (5.40), we have

$$
\begin{aligned}
\triangle_1 &= (a_2+b_1)^2(w-w_*)^2-4\theta_3\theta_4(\gamma_3-\beta)(\gamma_4-\beta)(w-w_*)^2\le 0,\\
\triangle_2 &= \rho_2^2(z-z_*)^2-4(1-\theta_3)(\gamma_3-\beta)(\rho_2-\beta)(z-z_*)^2\le 0,\\
\triangle_3 &= \rho_1^2(v-v_*)^2-4(1-\theta_4)(\gamma_4-\beta)(\rho_1-\beta)(v-v_*)^2\le 0,
\end{aligned}
\tag{5.42}
$$

which concludes

$$
f_i\le 0,\ i=1,2,3. \tag{5.43}
$$

By (5.41) and (5.43), we directly obtain (5.32). Moreover, we get

$$
\int_0^\infty\int_\Omega (u-u_*)^2+\int_0^\infty\int_\Omega (v-v_*)^2+\int_0^\infty\int_\Omega (w-w_*)^2+\int_0^\infty\int_\Omega (z-z_*)^2<\infty
$$

by integrating (5.32) over $(0,\infty)$. The proof of Lemma 5.5 is complete. □

Lemma 5.6 *Suppose that the conditions of Theorem 1.4 hold. Then, the solution of (1.1) satisfies*

$$
\begin{aligned}
&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}+\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}+\|w(\cdot,t)\\
&\quad-w_*\|_{L^\infty(\Omega)}+\|z(\cdot,t)-z_*\|_{L^\infty(\Omega)}\to 0
\end{aligned}
\tag{5.44}
$$

as $t\to\infty$.

Proof A combination of Lemma 5.1 and Lemma 5.5 implies this lemma. The proof of Lemma 5.6 is complete. □

Lemma 5.7 *Suppose that the conditions of Theorem 1.4 hold. Then, there exist $C_9>0$ and $\lambda>0$ such that the solution of (1.1) satisfies*

$$
\begin{aligned}
&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}+\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}+\|w(\cdot,t)\\
&\quad-w_*\|_{L^\infty(\Omega)}+\|z(\cdot,t)-z_*\|_{L^\infty(\Omega)}\le C_9e^{-\lambda t}
\end{aligned}
\tag{5.45}
$$

for all $t>t_2$, where $t_2>0$ is some fixed time.

Proof This proof is similar to ones of Lemma 5.4, thus we only give the sketch. By (5.26), Lemma 5.5 and Lemma 5.6, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) &\leq -\beta\bigg(\int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2 + \int_\Omega (z-z_*)^2\bigg)\\ &\leq -C_4\beta E_2(t),\end{aligned} \tag{5.46}$$

which implies

$$E_2(t) \leq E_2(t_2)e^{-C_4\beta(t-t_2)} \tag{5.47}$$

for all $t > t_2$. Then, by combining (5.47) with the first inequalities in (5.24) and (5.25), we derive a positive constant $C_{10} > 0$ such that

$$\begin{aligned}&\textstyle\int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2 + \int_\Omega (z-z_*)^2\\ &\qquad \leq C_{10}E_2(t) \leq C_{10}E_2(t_2)e^{-C_4\beta(t-t_2)}\end{aligned} \tag{5.48}$$

for all $t > t_2$.

By applying the Gagliardo–Nirenberg inequality and (1.17), there exist some positive constants C_{11}, C_{12} and C_{13} such that

$$\begin{aligned}&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)} + \|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}\\ &\quad + \|w(\cdot,t)-w_*\|_{L^\infty(\Omega)} + \|z(\cdot,t)-z_*\|_{L^\infty(\Omega)}\\ \leq& C_{11}\|u(\cdot,t)-u_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|u(\cdot,t)-u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}}\\ &+ C_{11}\|v(\cdot,t)-v_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|v(\cdot,t)-v_*\|_{L^2(\Omega)}^{\frac{2}{n+2}}\\ &+ C_{11}\|w(\cdot,t)-w_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|w(\cdot,t)-w_*\|_{L^2(\Omega)}^{\frac{2}{n+2}}\\ &+ C_{11}\|z(\cdot,t)-z_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|z(\cdot,t)-z_*\|_{L^2(\Omega)}^{\frac{2}{n+2}}\\ \leq& C_{12}\|u(\cdot,t)-u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} + C_{12}\|v(\cdot,t)-v_*\|_{L^2(\Omega)}^{\frac{2}{n+2}}\\ &+ C_{12}\|w(\cdot,t)-w_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} + C_{12}\|z(\cdot,t)-z_*\|_{L^2(\Omega)}^{\frac{2}{n+2}}\\ \leq& C_{13}\bigg(\|u(\cdot,t)-u_*\|_{L^2(\Omega)}^2 + \|v(\cdot,t)-v_*\|_{L^2(\Omega)}^2 + \|w(\cdot,t)-w_*\|_{L^2(\Omega)}^2 + \|z(\cdot,t)-z_*\|_{L^2(\Omega)}^2\bigg)^{\frac{1}{n+2}}\\ \leq& C_{13}\bigg(C_{10}E_2(t_2)\bigg)^{\frac{1}{n+2}}e^{-\frac{C_4\beta(t-t_2)}{n+2}}\end{aligned}$$

for all $t > t_2$. The proof of Lemma 5.7 is complete. □

Proof of Theorem 1.4. Lemma 5.7 directly shows the results of Theorem 1.4. □

6 Numerical Experiments

The main purpose of this section is to exhibit the spatiotemporal dynamics of system (1.1). Numerical simulations of model (1.1) are carried out by using the finite difference method (central difference scheme) for the spatial derivatives and the method of lines (MOL) for the time integration. Numerical integration is obtained with the help of the

MATLAB ode15s solver, and all the space steps are set to $\Delta x = 0.1$ in $1D$ domain as well as $\Delta x_1 = \Delta x_2 = 0.1$ in $2D$ domain. Here we only consider the fully parabolic system (i.e., $\tau = 1$) and let the parameters $d_1 = d_2 = d_3 = d_4 = \xi_1 = \chi_1 = \xi_2 = \chi_2 = a_0 = b_0 = 1$, $a_2 = a_3 = a_4 = b_1 = b_3 = b_4 = -1$.

Firstly, in order to illustrate the conditions of Theorem 1.2 and Theorem 1.4 in one-dimensional space and two-dimensional space, we take the domain $\Omega = (0, R)^n$ for $R > 0$. Hence, the conditions (1.10) and (1.12) of Theorem 1.2 imply that when locally intraspecific competition coefficients satisfy

$$a_1, b_2 > 1 + 2R^n \tag{6.1}$$

with $n = 1, 2$, then the system (1.1) has a unique positive globally bounded solution in $n-$dimensional domain.

Moreover, we take $\theta_3 = \theta_4 = \frac{1}{2}$ in (1.21) and (1.22), then Theorem 1.4 implies that when the locally intraspecific competition coefficients satisfy the condition (6.1) and

$$a_1 > \max\left\{\frac{u_* + w_*}{4} + 2R^n, \frac{8}{b_2 - 2R^n} + 2R^n\right\}, b_2 > \frac{u_* + w_*}{4} + 2R^n \tag{6.2}$$

with $n = 1, 2$, then the system (1.1) converges to a positive constant equilibrium (u_*, v_*, w_*, z_*) as $t \to \infty$, where

$$\begin{aligned} u_* &= \frac{b_2 + 1}{(a_1 - R^n)(b_2 - R^n) - (R^n + 1)^2}, \; v_* = \frac{a_1 + 1}{(a_1 - R^n)(b_2 - R^n) - (R^n + 1)^2}, \\ w_* &= \frac{a_1 + 1}{(a_1 - R^n)(b_2 - R^n) - (R^n + 1)^2}, \; z_* = \frac{b_2 + 1}{(a_1 - R^n)(b_2 - R^n) - (R^n + 1)^2}. \end{aligned} \tag{6.3}$$

Next, our numerical simulations are divided into the following two parts.

Part I: In this part, we mainly study the effect of locally intraspecific competition in system (1.1). Above all, we take $\Omega = (0, 1)^n$ with $n = 1, 2$, then the conditions in (6.1) are changed into

$$a_1, b_2 > 3 \tag{6.4}$$

and the conditions in (6.2) are turned into

$$a_1 > \max\left\{\frac{u_* + w_*}{4} + 2, \frac{8}{b_2 - 2} + 2\right\}, b_2 > \frac{u_* + w_*}{4} + 2. \tag{6.5}$$

Case 1: If we take appropriately large locally intraspecific competition coefficients $a_1 = 4$ and $b_2 = 8$, then the conditions (6.4) and (6.5) hold. Therefore, the system (1.1) is stable and converges to $(u_*, v_*, w_*, z_*) = (\frac{9}{17}, \frac{5}{17}, \frac{5}{17}, \frac{9}{17})$ as $t \to \infty$ with

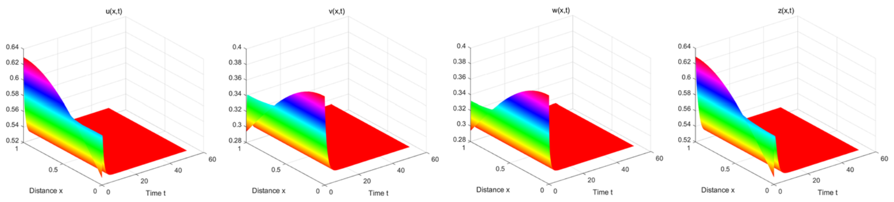

Fig. 1 Stability of (u_*, v_*, w_*, z_*) for 1D simulations with $a_1 = 4, b_2 = 8$

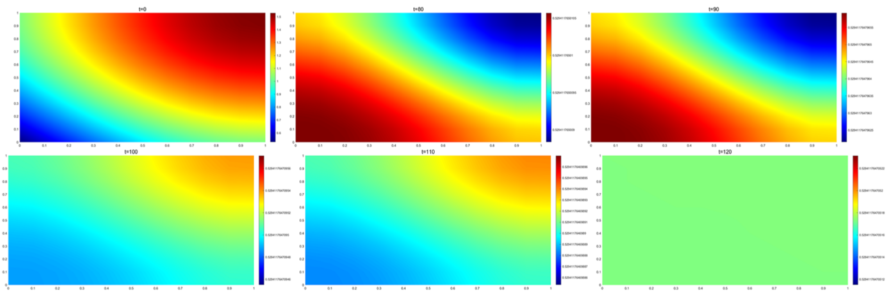

Fig. 2 Stability of u for 2D simulations with $a_1 = 4, b_2 = 8$ at time steps $t = 0, t = 80, t = 90, t = 100, t = 110, t = 120$

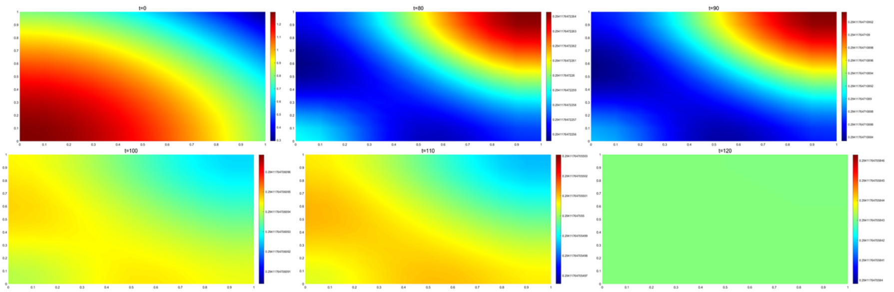

Fig. 3 Stability of w for 2D simulations with $a_1 = 4, b_2 = 8$ at time steps $t = 0, t = 80, t = 90, t = 100, t = 110, t = 120$

the following perturbation of the positive equilibrium

$$\begin{aligned} u_0(x) &= 0.1 \sin\left(\frac{\pi x}{2}\right) + u_*, \quad v_0(x) = 0.1 \cos\left(\frac{\pi x}{2}\right) + v_*, \\ w_0(x) &= 0.1 \cos\left(\frac{\pi x}{2}\right) + w_*, \quad z_0(x) = 0.1 \sin\left(\frac{\pi x}{2}\right) + z_* \end{aligned} \tag{6.6}$$

in one-dimensional space, see Fig. 1.

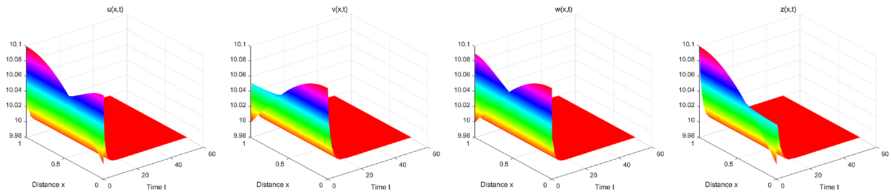

Fig. 4 Stability of (u_*, v_*, w_*, z_*) for 1D simulations with $a_1 = b_2 = 3.1$

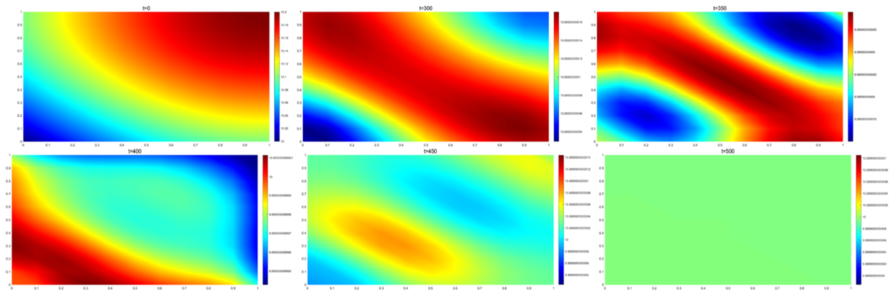

Fig. 5 Stability of u for 2D simulations with $a_1 = b_2 = 3.1$ at time steps $t = 0, t = 300, t = 350, t = 400, t = 450, t = 500$

When $n = 2$, we take the same value of parameters and $\Omega = (0, 1)^2$, then the same effect is observed in 2D simulations with initial data

$$\begin{aligned} u_0(x_1, x_2) &= 0.1 \sin\left(\frac{\pi x_1}{2}\right) + 0.1 \sin\left(\frac{\pi x_2}{2}\right) + u_*, \ v_0(x_1, x_2) \\ &= 0.1 \cos\left(\frac{\pi x_1}{2}\right) + 0.1 \cos\left(\frac{\pi x_2}{2}\right) + v_*, \\ w_0(x_1, x_2) &= 0.1 \cos\left(\frac{\pi x_1}{2}\right) + 0.1 \cos\left(\frac{\pi x_2}{2}\right) + w_*, \ z_0(x_1, x_2) \\ &= 0.1 \sin\left(\frac{\pi x_1}{2}\right) + 0.1 \sin\left(\frac{\pi x_2}{2}\right) + z_*, \end{aligned} \tag{6.7}$$

see Fig. 2 and Fig. 3.

Case 2: Taking the appropriate locally intraspecific competition coefficients $a_1 = b_2 = 3.1$, we can only obtain that the global bounded condition (6.4) holds. However, it is interesting to see that through numerical simulations in one-dimensional space, we find the system (1.1) is still stable and converges to $(u_*, v_*, w_*, z_*) = (10, 10, 10, 10)$ as $t \to \infty$ with the same initial perturbation in (6.6), see Fig. 4. Therefore, we preliminarily judge that Theorem 1.4 only gives a sufficient condition, and it is still an open problem to determine the optimal condition for the stabilization of system (1.1). Moreover, the same effect is also observed in 2D simulations with the same initial perturbation in (6.7), see Fig. 5 and Fig. 6.

Case 3: When $n = 1$, taking $\Omega = (0, 1)$ and the small locally intraspecific competition coefficients $a_1 = 2, b_2 = 3$ such that the condition (6.4) of global boundedness does not hold. Then, it may lead to finite-time blow-up of population density u and population density w with the slight initial perturbation $u_0(x) =$

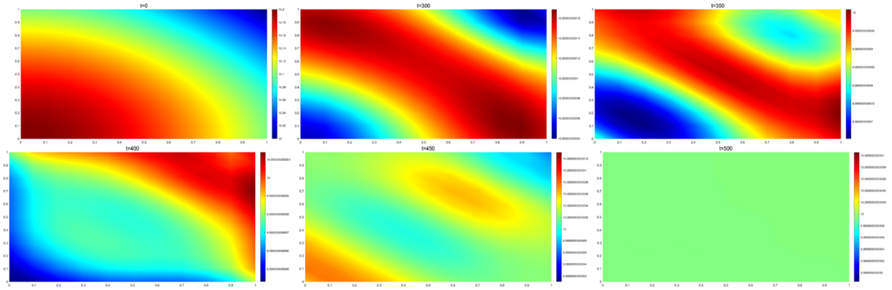

Fig. 6 Stability of w for 2D simulations with $a_1 = b_2 = 3.1$ at time steps $t = 0, t = 300, t = 350, t = 400, t = 450, t = 500$

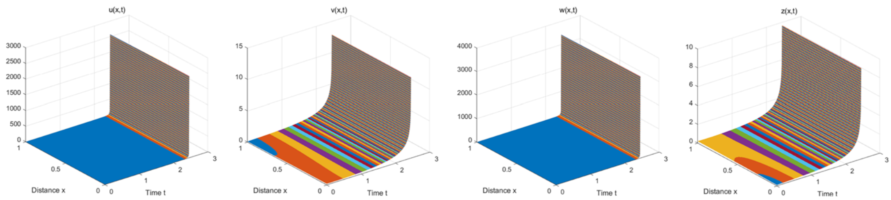

Fig. 7 Finite-time blow-up for u, w and spatiotemporal pattern for v, z with $a_1 = 2, b_2 = 3$ and slight perturbation in $1D$

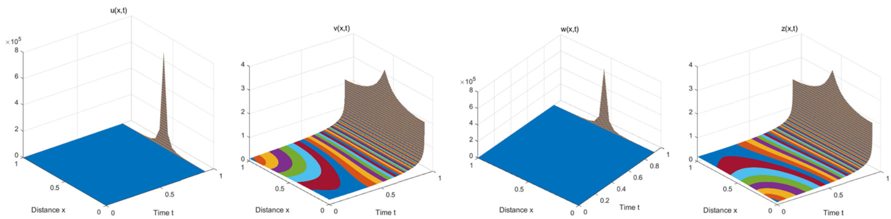

Fig. 8 Finite-time blow-up for u, w and spatiotemporal pattern for v, z with $a_1 = b_2 = -1$ and slight perturbation in $1D$

$0.1\sin(\frac{\pi x}{2}) + 0.1$, $v_0(x) = 0.1\cos(\frac{\pi x}{2}) + 0.1$, $w_0(x) = 0.1\cos(\frac{\pi x}{2}) + 0.1$, $z_0(x) = 0.1\sin(\frac{\pi x}{2}) + 0.1$, see Fig. 7. Further, if we take the smaller $a_1 = b_2 = -1$, which becomes local intraspecific cooperative effect, then the possible blow-up phenomenon can be observed earlier, see Fig. 8. However, it is still open in the mathematical theory whether finite-time blow-up of solutions occurs.

Therefore, from case 1 to case 3, we can conclude that locally intraspecific competition has an inhibitory effect on the blow-up phenomenon of system (1.1).

Part II: In this part, we study the effect of nonlocal terms in the system (1.1) in terms of spatial dimensions and domain sizes. Firstly, we take $a_1 = b_2 = 7$, then the condition (6.1) is changed into

$$R^n < 3, \tag{6.8}$$

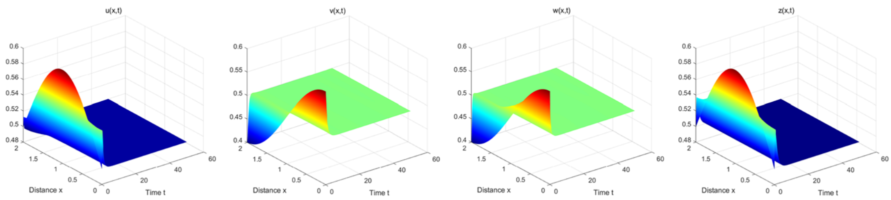

Fig. 9 Stability of (u_*, v_*, w_*, z_*) for 1D simulations with $a_1 = b_2 = 7$, $R = 2$

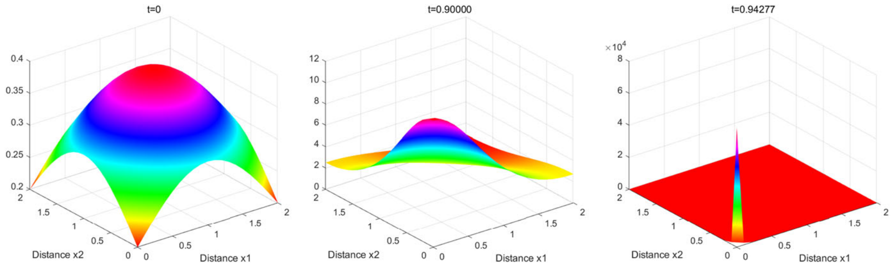

Fig. 10 Finite-time blow-up of u for 2D simulations with $a_1 = b_2 = 7$, $R = 2$

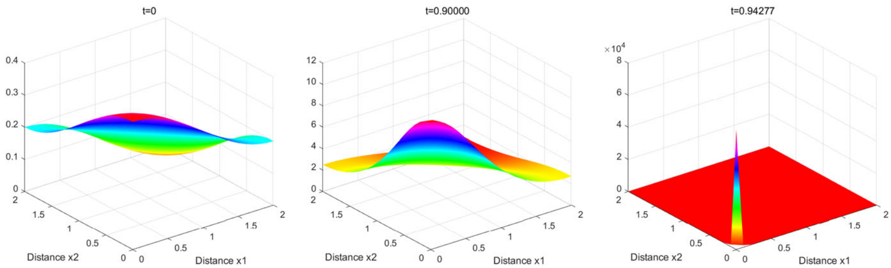

Fig. 11 Finite-time blow-up of w for 2D simulations with $a_1 = b_2 = 7$, $R = 2$

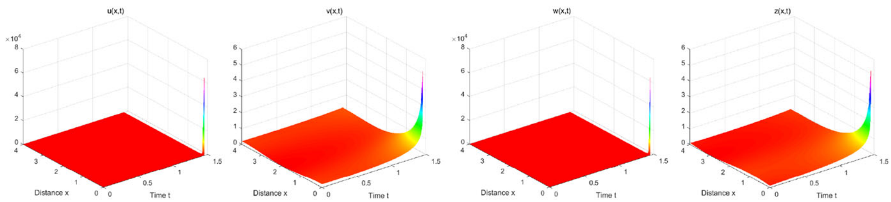

Fig. 12 Finite-time blow-up for u, w and spatiotemporal pattern for v, z in $1D$ with $a_1 = b_2 = 7$, $R = 4$

and the conditions in (6.2) are turned into

$$\max\left\{\frac{u_* + w_*}{4} + 2R^n, \frac{8}{b_2 - 2R^n} + 2R^n\right\} < 7, \frac{u_* + w_*}{4} + 2R^n < 7 \tag{6.9}$$

with $n = 1, 2$.

In one-dimensional space, if we take $R = 2$, then the conditions (6.8) and (6.9) hold. Therefore, the system (1.1) is stable and converges to $(u_*, v_*, w_*, z_*) = (0.5, 0.5, 0.5, 0.5)$ as $t \to \infty$ with the same initial perturbation in (6.6), see Fig. 9. However, in two-dimensional space, the conditions (6.8) and (6.9) are invalid when $R = 2$, then the possible blow-up phenomenon of u, w can be observed by simulations with the slight initial perturbation $u_0(x) = 0.1\sin(\frac{\pi x_1}{2}) + 0.1\sin(\frac{\pi x_2}{2}) + 0.1$, $v_0(x) = 0.1\cos(\frac{\pi x_1}{2}) + 0.1\cos(\frac{\pi x_2}{2}) + 0.1$, $w_0(x) = 0.1\cos(\frac{\pi x_1}{2}) + 0.1\cos(\frac{\pi x_2}{2}) + 0.1$, $z_0(x) = 0.1\sin(\frac{\pi x_1}{2}) + 0.1\sin(\frac{\pi x_2}{2}) + 0.1$, see Fig. 10 and Fig. 11. Moreover, if we take $R = 4$ in one-dimensional space, the conditions (6.8) and (6.9) are invalid, and the possible blow-up phenomenon of u, w can also be observed by simulations with the slight initial perturbation $u_0(x) = 0.1\sin(\frac{\pi x}{2}) + 0.1$, $v_0(x) = 0.1\cos(\frac{\pi x}{2}) + 0.1$, $w_0(x) = 0.1\cos(\frac{\pi x}{2}) + 0.1$, $z_0(x) = 0.1\sin(\frac{\pi x}{2}) + 0.1$, see Fig. 12. In general, we can observe that appropriately large R^n may cause the phenomenon of blow-up. Therefore, when $a_3 = a_4 = b_3 = b_4 = -1$, the nonlocal terms will enhance the possible blow-up occurrence of the populations u, w.

7 Conclusion

It is well known that the interplay of diffusion, chemotaxis and logistic growth has been an interesting topic (see Budrene and Berg (1991) and references therein). The research result in Luca et al. (2003) has implied that a population can produce both chemoattractant and chemorepellent. Based on the mathematical model in Liu and Dai (2021), we not only considered the random diffusion (i.e., $d_1\Delta u$, $d_2\Delta w$) and attraction–repulsion mechanism (i.e., $-\xi_1\nabla\cdot(u\nabla v) + \chi_1\nabla\cdot(u\nabla z)$, $-\xi_2\nabla\cdot(w\nabla z) + \chi_2\nabla\cdot(w\nabla v)$), but also the intrinsic growth, intraspecific and interspecific relations (i.e., $g_1(u, w)$, $g_1(u, w)$) in this paper. In other words, the random diffusion, chemoattractant, chemorepellent and kinetics can influence the movements of population. Therefore, system (1.1) can describe more abundant and interesting biological phenomena between two species, such as the cell sorting process (Painter 2009). We mainly studied the qualitative analysis on the solutions of system (1.1) as follows: (i) Relying on some a priori estimates and the Moser–Alikakos iteration in Alikakos (1979) (or Lemma A.1 of Tao and Winkler (2012)), we obtained the existence and boundedness of global solutions to system (1.1) in any spatial dimension (see Theorems 1.1 and 1.2). (ii) When the locally intraspecific competition and globally interspecific cooperation exist, we established the globally asymptotic stabilization of coexistence by constructing Lyapunov functionals (see Theorems 1.3 and 1.4).

Furthermore, by some numerical simulations, we verified our analytically theoretical results and found some new and interesting phenomena. Our results complemented the existing results in Zheng and Hu (Preprint), where the chemo-repulsive mechanisms or nonlocal terms were not considered. For the parabolic–elliptic–parabolic–elliptic case (i.e., $\tau = 0$), the results in Theorem 1.1 imply that the chemo-repulsive mechanisms effectively prevent the occurrence of cells aggregation under the same conditions in Liu and Dai (2022). And for two cases $\tau = 0$ and $\tau = 1$, the results in Theorems 1.1 and 1.2 also showed that the size of domain Ω influences the mass of population under the globally intraspecific and interspecific cooperation.

In a biological sense, if the strengths of locally intraspecific competition are large enough in two populations, then they can keep the coexistence steady state all the time. On the other hand, when the strength of the chemorepellent is the same as the chemoattractant (i.e., $\xi_i = \chi_i = 1, i = 1, 2$) and the global cooperation is stronger than the locally intraspecific competition, we numerically showed that it may lead to the occurrence of the cells aggregation. Moreover, when there is only the mechanism of cooperation in (1.1) (i.e., $a_0, b_0 > 0, a_i, b_i < 0 (i = 1, 2, 3, 4)$ in (1.2)), the numerical simulations indicated that it may also lead to the occurrence of cells aggregation. However, the rigorous mathematical analysis is still open.

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Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declaration

Conflict of interest The authors declare that this work does not have any conflicts of interests.

Appendix A. Proof of Lemma 2.1

Proof The ideas of proof are similar to (Winkler 2010a, Lemma 1.1) and (Stinner et al. 2014b, Lemma 2.1). For reader's convenience, we give the sketch of the proof.

(i) Existence. Under the assumptions of Lemma 2.1, we claim that for all $L > 0$ there exists $T = T(L) > 0$ such that $\|u_0\|_{L^\infty(\Omega)} \leq L$, $\|w_0\|_{L^\infty(\Omega)} \leq L$, $\|v_0\|_{W^{1,q}(\Omega)} \leq L$ and $\|z_0\|_{W^{1,q}(\Omega)} \leq L$, then system (1.1) is classically solvable in $\Omega \times (0, T)$. As a consequence of a standard extension argument, this will imply the existence of a maximal existence time $T_{\max}$ satisfying (2.1).

Now, we prove the local existence of solutions for system (1.1) when $\tau = 1$ and $\tau = 0$, respectively.

When $\tau = 1$, according to the well-known Neumann heat semigroup $\left(e^{t\Delta}\right)_{t\geq 0}$ in (Winkler 2010b, Lemma 3.1), we can pick $K > 0$ such that $\|e^{t\Delta}v\|_{W^{1,q}(\Omega)} \leq K\|v\|_{W^{1,q}(\Omega)}$ and $\|e^{t\Delta}z\|_{W^{1,q}(\Omega)} \leq K\|z\|_{W^{1,q}(\Omega)}$ for all $v, z \in W^{1,q}(\Omega)$. For small $T \in (0, 1)$ to be fixed below, we introduce the Banach space

$$X := C^0\Big([0,T]; C^0(\overline{\Omega})\Big) \times C^0\Big([0,T]; W^{1,q}(\Omega)\Big) \times C^0\Big([0,T]; C^0(\overline{\Omega})\Big)$$
$$\times C^0\Big([0,T]; W^{1,q}(\Omega)\Big),$$

and the close subset

$$F := \Big\{(u,v,w,z) \in X \mid \|u\|_{L^\infty((0,T);L^\infty(\Omega))} \le L+1,\ \|v\|_{L^\infty((0,T);W^{1,q}(\Omega))} \le KL+1,$$
$$\|w\|_{L^\infty((0,T);L^\infty(\Omega))} \le L+1,\ \|z\|_{L^\infty((0,T);W^{1,q}(\Omega))} \le KL+1\Big\}.$$

For $(u,v,w,z) \in F$ and $t \in (0,T)$, we define the mapping

$$\Psi_1(u,v,w,z)(t) := \begin{pmatrix} \Psi_{11}(u,v,w,z)(t) \\ \Psi_{12}(u,v,w,z)(t) \\ \Psi_{13}(u,v,w,z)(t) \\ \Psi_{14}(u,v,w,z)(t) \end{pmatrix} :=$$

$$\begin{pmatrix} e^{td_1\Delta}u_0 - \xi_1 \int_0^t e^{(t-s)d_1\Delta}\nabla\cdot\big(u(s)\nabla v(s)\big)ds + \chi_1\int_0^t e^{(t-s)d_1\Delta}\nabla\cdot\big(u(s)\nabla z(s)\big)ds + \int_0^t e^{(t-s)d_1\Delta}g_1(u,w)ds \\ e^{t(d_2\Delta-1)}v_0 + \int_0^t e^{(t-s)(d_2\Delta-1)}w(s)ds \\ e^{td_3\Delta}w_0 - \xi_2 \int_0^t e^{(t-s)d_3\Delta}\nabla\cdot\big(w(s)\nabla z(s)\big)ds + \chi_2\int_0^t e^{(t-s)d_3\Delta}\nabla\cdot\big(w(s)\nabla v(s)\big)ds + \int_0^t e^{(t-s)d_3\Delta}g_2(u,w)ds \\ e^{t(d_4\Delta-1)}z_0 + \int_0^t e^{(t-s)(d_4\Delta-1)}u(s)ds \end{pmatrix}.$$

Then, we have

$$\begin{aligned} \|\Psi_{11}(u,v,w,z)(t)\|_{L^\infty(\Omega)} \le & \|e^{td_1\Delta}u_0\|_{L^\infty(\Omega)} + \xi_1\int_0^t \Big\|e^{(t-s)d_1\Delta}\nabla\cdot\Big(u(s)\nabla v(s)\Big)\Big\|_{L^\infty(\Omega)}ds \\ & + \chi_1\int_0^t \Big\|e^{(t-s)d_1\Delta}\nabla\cdot\Big(u(s)\nabla z(s)\Big)\Big\|_{L^\infty(\Omega)}ds \\ & + \int_0^t \Big\|e^{(t-s)d_1\Delta}g_1(u,w)\Big\|_{L^\infty(\Omega)}ds \\ \le & \xi_1\int_0^t \Big\|e^{(t-s)d_1\Delta}\nabla\cdot\Big(u(s)\nabla v(s)\Big)\Big\|_{L^\infty(\Omega)}ds \\ & + \chi_1\int_0^t \Big\|e^{(t-s)d_1\Delta}\nabla\cdot\Big(u(s)\nabla z(s)\Big)\Big\|_{L^\infty(\Omega)}ds \\ & + L + T\cdot\|g_1(u,w)\|_{L^\infty((-R-1,R+1))}, \end{aligned} \tag{7.1}$$

where by the maximum principle

$$\|e^{td_1\Delta}u_0\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \le L \tag{7.2}$$

and

$$\begin{aligned} \int_0^t \|e^{td_1\Delta}g_1(u,w)\|_{L^\infty(\Omega)}ds \le & \int_0^t \|g_1(u,w)\|_{L^\infty(\Omega)}ds \\ \le & T\cdot\|g_1(u,w)\|_{L^\infty((-R-1,R+1))}, \end{aligned} \tag{7.3}$$

for all $t \in (0,T)$. Furthermore, by picking any $p > \frac{nq}{q-n}$ and then $\alpha \in \big(\frac{n}{p}, \frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})\big)$, we obtain $p\alpha > n$ and the fractional power A^α of the sectorial operator

$A := -d_1\Delta+1$ with Neumann data in $L^p(\Omega)$ satisfies $\|\phi\|_{L^\infty(\Omega)} \le C\|A^\alpha\phi\|_{L^p(\Omega)}$ as well as $\|A^\alpha e^{\rho d_1\Delta}\phi\|_{L^p(\Omega)} \le C\rho^{-\alpha}\|\phi\|_{L^p(\Omega)}$ for all $\phi \in C_0^\infty(\Omega)$ (cf. Henry (1981)). Here and below, $C_i\,(i = 1, 2, \cdots, 23)$ denote generic positive constants. Therefore, by $T < 1$, $\alpha \in \left(\frac{n}{p}, \frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})\right)$ and $\|e^{\rho d_1\Delta}\nabla\cdot\psi\|_{L^p(\Omega)} \le C\rho^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|\psi\|_{L^q(\Omega)}$ for $\rho < 1$ and all $\mathbb{R}-$valued $\psi \in C_0^\infty(\Omega)$ (cf. Weinberger (1982)), we have

$$\begin{aligned}
&\xi_1\int_0^t \left\|e^{(t-s)d_1\Delta}\nabla\cdot\Big(u(s)\nabla v(s)\Big)\right\|_{L^\infty(\Omega)} ds\\
&\le C_1\int_0^t \left\|A^\alpha e^{(t-s)d_1\Delta}\nabla\cdot\Big(u(s)\nabla v(s)\Big)\right\|_{L^p(\Omega)} ds\\
&\le C_2\int_0^t (t-s)^{-\alpha}\left\|e^{\frac{t-s}{2}d_1\Delta}\nabla\cdot\Big(u(s)\nabla v(s)\Big)\right\|_{L^p(\Omega)} ds\\
&\le C_3\int_0^t (t-s)^{-\alpha}\cdot(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\left\|u(s)\nabla v(s)\right\|_{L^q(\Omega)} ds\\
&\le C_4 T^{\frac{1}{2}-\alpha-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\cdot(L+1)\cdot(KL+1)
\end{aligned} \tag{7.4}$$

for all $t \in (0, T)$. Similarly, we obtain

$$\begin{aligned}
&\chi_1\int_0^t \left\|e^{(t-s)d_1\Delta}\nabla\cdot\Big(u(s)\nabla z(s)\Big)\right\|_{L^\infty(\Omega)} ds\\
&\le C_5 T^{\frac{1}{2}-\alpha-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\cdot(L+1)\cdot(KL+1)
\end{aligned} \tag{7.5}$$

for all $t \in (0, T)$. For the term $\|\Psi_{13}(u, v, w, z)(t)\|_{L^\infty(\Omega)}$, we can use the similar way.

For $\|\Psi_{12}(u, v, w, z)(t)\|_{L^\infty(\Omega)}$, we have

$$\begin{aligned}
\|\Psi_{12}(u, v, w, z)(t)\|_{W^{1,q}(\Omega)} &\le e^{-t}\|e^{td_1\Delta}v_0\|_{W^{1,q}(\Omega)} + C_6\int_0^t (t-s)^{-\frac{1}{2}}\|w(s)\|_{L^q(\Omega)}ds\\
&\le K\|v_0\|_{W^{1,q}(\Omega)} + C_7\int_0^t (t-s)^{-\frac{1}{2}}\|w(s)\|_{L^\infty(\Omega)}ds\\
&\le KL + C_8 T^{\frac{1}{2}}\cdot(L+1)
\end{aligned} \tag{7.6}$$

for all $t \in (0, T)$. Similarly, we can estimate the term $\|\Psi_{14}(u, v, w, z)(t)\|_{L^\infty(\Omega)}$. Then, it follows from (7.1)–(7.6) that if we fix $T_0 \in (0, 1)$ small enough such that $T \in (0, T_0)$, then Ψ_1 maps F into itself.

Moreover, using the same ideas with (7.4), for $(u, v, w, z) \in F$ and $(\bar{u}, \bar{v}, \bar{w}, \bar{z}) \in F$, we get

$$\begin{aligned}
&\|\Psi_{11}(u, v, w, z)(t) - \Psi_{11}(\bar{u}, \bar{v}, \bar{w}, \bar{z})(t)\|_{L^\infty(\Omega)}\\
&\le C_9\int_0^t \left\|A^\alpha e^{(t-l)d_1\Delta}\nabla\cdot\Big(u(s)\nabla v(s) - \bar{u}(s)\nabla\bar{v}(s)\Big)\right\|_{L^p(\Omega)} ds
\end{aligned}$$

$$
\begin{aligned}
&+C_9 \int_0^t \left\| A^\alpha e^{(t-s)d_1\Delta} \nabla \cdot \Big(u(s)\nabla z(s) - \bar{u}(s)\nabla \bar{z}(s) \Big) \right\|_{L^p(\Omega)} ds \\
&+\int_0^t \left\| e^{(t-s)d_1\Delta} \Big(g_1(u,w) - g_1(\bar{u},\bar{w}) \Big) \right\|_{L^\infty(\Omega)} ds \\
&\le C_{10} \int_0^t (t-s)^{-\alpha-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \| u(s)\nabla v(s) - \bar{u}(s)\nabla \bar{v}(s)\|_{L^q(\Omega)} ds \\
&+C_{10} \int_0^t (t-s)^{-\alpha-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \| u(s)\nabla z(s) - \bar{u}(s)\nabla \bar{z}(s)\|_{L^q(\Omega)} ds \\
&+\int_0^t \| g_1(u,w) - g_1(\bar{u},\bar{w})\|_{L^\infty(\Omega)} ds \\
&\le C_{11} T^{\frac{1}{2}-\alpha-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \Big((L+1) + (KL+1) \Big) \cdot \|(u,v,w,z) - (\bar{u},\bar{v},\bar{w},\bar{z})\|_X \\
&+T \cdot \left\| g_1' \right\|_{L^\infty((-L-1,L+1))} \cdot \|(u,v,w,z) - (\bar{u},\bar{v},\bar{w},\bar{z})\|_X .
\end{aligned} \tag{7.7}
$$

Similarly, we have

$$
\begin{aligned}
&\|\Psi_{13}(u,v,w,z)(t) - \Psi_{13}(\bar{u},\bar{v},\bar{w},\bar{z})(t)\|_{L^\infty(\Omega)} \\
\le & C_{12} T^{\frac{1}{2}-\alpha-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \Big((L+1) + (KL+1) \Big) \cdot \|(u,v,w,z) - (\bar{u},\bar{v},\bar{w},\bar{z})\|_X \\
&+ T \cdot \left\| g_2' \right\|_{L^\infty((-L-1,L+1))} \cdot \|(u,v,w,z) - (\bar{u},\bar{v},\bar{w},\bar{z})\|_X
\end{aligned} \tag{7.8}
$$

and

$$
\begin{aligned}
&\|\Psi_{12}(u,v,w,z)(t) - \Psi_{12}(\bar{u},\bar{v},\bar{w},\bar{z})(t)\|_{L^\infty(\Omega)} \\
&\le C_{13} \int_0^t (t-s)^{-\frac{1}{2}} \|w(s) - \bar{w}(s)\|_{L^q(\Omega)} ds \\
&\le C_{14} T^{\frac{1}{2}} \cdot \|(u,v,w,z) - (\bar{u},\bar{v},\bar{w},\bar{z})\|_X
\end{aligned} \tag{7.9}
$$

as well as

$$
\begin{aligned}
&\|\Psi_{14}(u,v,w,z)(t) - \Psi_{14}(\bar{u},\bar{v},\bar{w},\bar{z})(t)\|_{L^\infty(\Omega)} \\
&\le C_{15} T^{\frac{1}{2}} \cdot \|(u,v,w,z) - (\bar{u},\bar{v},\bar{w},\bar{z})\|_X
\end{aligned} \tag{7.10}
$$

for all $t \in (0, T)$, which shows that Ψ is a contraction mapping if $T \in (0, T_0)$ is small enough. Then, by using the Banach fixed point theorem, we know that the existence of some $(u, v, w, z) \in F$ such that $\Psi_1(u, v, w, z) = (u, v, w, z)$. Once again using standard arguments involving semigroup estimates, it can easily be checked that in fact (u, v, w, z) lies in the asserted regularity class and is a classical solution of (1.1) in $\Omega \times (0, T)$. Since $g_1(0, 0) \ge 0$ and $g_2(0, 0) \ge 0$ hold, the maximum principle moreover ensures that u, w, v, z are nonnegative.

When $\tau = 0$, we introduce the Banach space

$$
\bar{X} := C^0\big([0,T]; C^0(\overline{\Omega})\big) \times C^0\big([0,T]; C^0(\overline{\Omega})\big),
$$

and consider the close subset

$$\bar{F} := \left\{ (u, w) \in X \mid \|u\|_{L^\infty((0,T);L^\infty(\Omega))} \leq L+1, \|w\|_{L^\infty((0,T);L^\infty(\Omega))} \leq L+1 \right\},$$

where $T \in (0, 1)$ is small. Similarly, we define the mapping

$$\Psi_2(u, v, w, z)(t) := \begin{pmatrix} \Psi_{21}(u, v, w, z)(t) \\ \Psi_{22}(u, v, w, z)(t) \end{pmatrix}$$

$$:= \begin{pmatrix} e^{td_1\Delta}u_0 + \int_0^t e^{(t-s)d_1\Delta}\nabla \cdot \Big(\chi_1 u(s)\nabla z(s) - \xi_1 u(s)\nabla v(s)\Big)ds + \int_0^t e^{(t-s)d_1\Delta} g_1(u, w)ds \\ e^{td_3\Delta}w_0 + \int_0^t e^{(t-s)d_3\Delta}\nabla \cdot \Big(\chi_2 w(s)\nabla v(s) - \xi_2 w(l)\nabla z(s)\Big)ds + \int_0^t e^{(t-s)d_3\Delta} g_2(u, w)ds \end{pmatrix}.$$

for $(u, w) \in \bar{F}$ and $t \in (0, T)$, where $\left(e^{td_i\Delta}\right)_{t\geq 0}$ denotes the Neumann heat semigroup. From the second and fourth equation in (1.1), we have $-d_1\Delta v + v = w$ and $-d_3\Delta z + z = u$ under homogeneous Neumann boundary conditions. According to the same methods in case of $\tau = 1$, we get that Ψ_2 is a contraction mapping on $\bar{F}$ if $T \in (0, T_0)$ is sufficiently small. Hence, the Banach fixed point theorem implies the existence of some $(u, w) \in \bar{F}$ such that $\Psi_2(u, w) = (u, w)$. Moreover, by applying the similar arguments and the strong maximum principle, we deduce that (u, w) is nonnegative. And by the strong elliptic maximum principle applied to the second and fourth equation in (1.1), we also obtain the nonnegativity of (v, z).

(ii) Uniqueness. Proceeding as in Gajewski and Zacharias (1998), for given $T > 0$ and two solutions (u, v, w, z), $(\bar{u}, \bar{v}, \bar{w}, \bar{z})$ in $\Omega \times (0, T)$, we fix $T_1 \in (0, T)$ and set $U := u - \bar{u}$, $V := v - \bar{v}$, $W := w - \bar{w}$, $Z := z - \bar{z}$. By applying straightforward testing procedures to (1.1), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2 + 2d_1\int_\Omega |\nabla U|^2 = 2\xi_1\int_\Omega U\nabla v \cdot \nabla U + 2\xi_1\int_\Omega \bar{u}\nabla U \cdot \nabla V \\ - 2\chi_1\int_\Omega U\nabla z \cdot \nabla U - 2\chi_1\int_\Omega \bar{u}\nabla U \cdot \nabla Z \\ + 2\int_\Omega \Big(g_1(u, w) - g_1(\bar{u}, \bar{w})\Big)U \end{aligned} \tag{7.11}$$

and

$$\begin{aligned} \int_\Omega W^2 + 2d_3\int_\Omega |\nabla W|^2 = 2\xi_2\int_\Omega W\nabla z \cdot \nabla W + 2\xi_2\int_\Omega \bar{w}\nabla W \cdot \nabla Z \\ - 2\chi_2\int_\Omega W\nabla v \cdot \nabla W - 2\chi_2\int_\Omega \bar{w}\nabla W \cdot \nabla V \\ + 2\int_\Omega \Big(g_2(u, w) - g_2(\bar{u}, \bar{w})\Big)W \end{aligned} \tag{7.12}$$

for all $t \in (0, T_1)$.

When $\tau = 1$, by the second and fourth equations in (1.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla V|^2 + 2d_2\int_\Omega |\Delta V|^2 + 2\int_\Omega |\nabla V|^2 = -2\int_\Omega W\Delta V \tag{7.13}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla Z|^2 + 2d_4\int_\Omega |\Delta Z|^2 + 2\int_\Omega |\nabla Z|^2 = -2\int_\Omega U\Delta Z \tag{7.14}$$

for all $t \in (0, T_1)$. By the Hölder, Young and Gagliardo–Nirenberg inequalities, we get

$$\begin{aligned} 2\xi_1\int_\Omega U\nabla v\cdot\nabla U &\le 2\xi_1\Big(\int_\Omega |\nabla U|^2\Big)^{\frac{1}{2}}\cdot\Big(\int_\Omega |\nabla v|^q\Big)^{\frac{1}{q}}\cdot\Big(\int_\Omega U^{\frac{2q}{q-2}}\Big)^{\frac{q-2}{2q}} \\ &\le C_{22}\Big(\int_\Omega |\nabla U|^2\Big)^{\frac{1}{2}+\frac{n}{2q}}\cdot\Big(\int_\Omega |\nabla v|^q\Big)^{\frac{1}{q}}\cdot\Big(\int_\Omega U^2\Big)^{\frac{q-n}{2q}} \\ &\le \frac{d_1}{2}\int_\Omega |\nabla U|^2 + C_{16}\int_\Omega U^2, \end{aligned} \tag{7.15}$$

where we have used the fact that $\int_\Omega U = 0$ by a simple integration of (1.1), and $\|\nabla v\|_{L^q(\Omega)} \le C_{17}$ for $t \in (0, T_1)$ as well as $q > n \ge 2$. By using the same method with (7.15), we have

$$-2\chi_1\int_\Omega U\nabla z\cdot\nabla U \le \frac{d_1}{2}\int_\Omega |\nabla U|^2 + C_{18}\int_\Omega U^2. \tag{7.16}$$

Furthermore, we have

$$2\xi_1\int_\Omega \bar{u}\nabla U\cdot\nabla V \le \frac{d_1}{2}\int_\Omega |\nabla U|^2 + C_{17}\int_\Omega |\nabla V|^2 \tag{7.17}$$

and

$$-2\chi_1\int_\Omega \bar{u}\nabla U\cdot\nabla Z \le \frac{d_1}{2}\int_\Omega |\nabla U|^2 + C_{19}\int_\Omega |\nabla Z|^2 \tag{7.18}$$

as well as

$$2\int_\Omega \Big(g_1(u, w) - g_1(\bar{u}, \bar{w})\Big)U \le C_{20}\int_\Omega U^2, \tag{7.19}$$

in view of the boundedness of u and $\bar{u}$ in $\Omega\times(0, T_1)$ and the local Lipschitz continuity of g_1. Then, by substituting (7.15)–(7.19) into (7.11), we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2 \le (C_{16}+C_{18}+C_{20})\int_\Omega U^2 + C_{17}\int_\Omega |\nabla V|^2 + C_{19}\int_\Omega |\nabla Z|^2. \tag{7.20}$$

By using the same method to (7.12), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega W^2 \le C_{21}\int_\Omega W^2 + C_{21}\int_\Omega |\nabla V|^2 + C_{21}\int_\Omega |\nabla Z|^2. \tag{7.21}$$

By Young's inequality, we obtain from (7.13) and (7.14) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla V|^2 + 2\int_\Omega |\nabla V|^2 \le \frac{1}{2d_2}\int_\Omega W^2 \tag{7.22}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla Z|^2 + 2\int_\Omega |\nabla Z|^2 \le \frac{1}{2d_4}\int_\Omega U^2. \tag{7.23}$$

By combining (7.20)–(7.23), one can find a positive constant C_{22} such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega U^2 + \int_\Omega W^2 + \int_\Omega |\nabla V|^2 + \int_\Omega |\nabla Z|^2\right)\\ &\le C_{22}\left(\int_\Omega U^2 + \int_\Omega W^2 + \int_\Omega |\nabla V|^2 + \int_\Omega |\nabla Z|^2\right).\end{aligned} \tag{7.24}$$

When $\tau = 0$, by a straightforward computation, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2 \le C_{23}\left(\int_\Omega U^2 + \int_\Omega W^2 + \int_\Omega |\nabla V|^2 + \int_\Omega |\nabla Z|^2\right) \tag{7.25}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega W^2 \le C_{23}\left(\int_\Omega U^2 + \int_\Omega W^2 + \int_\Omega |\nabla V|^2 + \int_\Omega |\nabla Z|^2\right) \tag{7.26}$$

as well as

$$\int_\Omega |\nabla V|^2 \le \int_\Omega W^2 \tag{7.27}$$

and

$$\int_\Omega |\nabla Z|^2 \le \int_\Omega U^2. \tag{7.28}$$

Then, by combining (7.25)–(7.28), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega U^2 + \int_\Omega W^2\right) \le C_{23}\left(\int_\Omega U^2 + \int_\Omega W^2\right). \tag{7.29}$$

Now with the aid of Grönwall's lemma, we obtain that $U \equiv 0$, $V \equiv 0$, $W \equiv 0$, $Z \equiv 0$ in $\Omega \times (0, T_1)$. Hence, we obtain $(u, v, w, z) \equiv (\bar{u}, \bar{v}, \bar{w}, \bar{z})$ in $\Omega \times (0, T)$, because $T_1 \in (0, T)$ is arbitrary. The proof of Lemma 2.1 is complete. $\square$

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