Journal of Nonlinear Science (2023) 33:42
https://doi.org/10.1007/s00332-023-09892-3

Journal of Nonlinear Science

On Two Coupled Degenerate Parabolic Equations Motivated by Thermodynamics

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Received: 15 May 2022 / Accepted: 23 January 2023 / Published online: 8 March 2023

Abstract

We discuss a system of two coupled parabolic equations that have degenerate diffusion constants depending on the energy-like variable. The dissipation of the velocity-like variable is fed as a source term into the energy equation leading to conservation of the total energy. The motivation of studying this system comes from Prandtl's and Kolmogorov's one- and two-equation models for turbulence, where the energy-like variable is the mean turbulent kinetic energy. Because of the degeneracies, there are solutions with time-dependent support like in the porous medium equation, which is contained in our system as a special case. The motion of the free boundary may be driven by either self-diffusion of the energy-like variable or by dissipation of the velocity-like variable. The crossover of these two phenomena is exemplified for the associated planar traveling fronts. We provide existence of suitably defined weak and very weak solutions. After providing a thermodynamically motivated gradient structure, we also establish convergence into steady state for bounded domains and provide a conjecture on the asymptotically self-similar behavior of the solutions in $\mathbb{R}^d$ for large times.

Keywords Degenerate parabolic system · Free-boundary problem · Convergence into thermal equilibrium · Gradient structure · Mass and momentum conservation · Energy-dissipation estimates

Communicated by Dejan Slepcev.

The research was partially supported by Deutsche Forschungsgemeinschaft (DFG) via the Collaborative Research Center SFB 910 "Control of self-organizing nonlinear systems" (Project No. 163436311), subproject A5 "Pattern formation in coupled parabolic systems".

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Mathematics Subject Classification 35K65 · 35K40 · 80M30 · 49S05

1 Introduction

On a smooth domain $\Omega \subset \mathbb{R}^d$, we consider the degenerate parabolic system

$$\dot{v} = \operatorname{div}\big(\eta(w)\nabla v\big), \qquad \text{for } (t,x) \in]0,\infty[\times\Omega, \tag{1.1a}$$

$$\dot{w} = \operatorname{div}\big(\kappa(w)\nabla w\big) + \eta(w)|\nabla v|^2 \qquad \text{for } (t,x) \in]0,\infty[\times\Omega, \tag{1.1b}$$

$$0 = \eta(w)\nabla u \cdot \mathrm{n}, \quad 0 = \kappa(w)\nabla w \cdot \mathrm{n} \qquad \text{for } (t,x) \in]0,\infty[\times\partial\Omega, \tag{1.1c}$$

where $v(t,x) \in \mathbb{R}$ can be considered as a shear velocity and $w(t,x) \geq 0$ is an internal energy. Here, the functions $w \mapsto \eta(w)$ and $w \mapsto \kappa(w)$ describe the viscosity law for v and the energy-transport coefficient for w. Throughout this work, we will mainly restrict to the choice

$$\eta(w) = \eta_0 w^\alpha \quad \text{and} \quad \kappa(w) = \kappa_0 w^\beta, \tag{1.2}$$

where $\alpha, \beta, \eta_0, \kappa_0 > 0$ are given parameters.

The main feature of the model is that the shearing dissipation $\eta(w)|\nabla v|^2$ is feeding into the energy equation such that in addition to the total momentum $\mathcal{V}(v,w)$ also the total energy $\mathcal{E}(v,w)$ are conserved along solutions:

$$\mathcal{V}(v,w) := \int_\Omega v(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{E}(v,w) := \int_\Omega \Big(\frac{1}{2}v^2 + w\Big)\,\mathrm{d}x$$

One difficulty of the coupled system is that the viscosity coefficient $\eta(w)$ and the energy-transport coefficient $\kappa(w)$ can be unbounded, but this problem will play a minor role in our work. The main emphasis is on the degeneracies arising from the fact that $\eta(0) = \kappa(0) = 0$ and that the solutions of our interest have a nontrivial support. Thus, we are deriving a theory for solutions that have $(v(t,x), w(t,x) = (0,0))$ in regions of the $Q_T = [0,T]\times\Omega$ of full measure. In particular, we are interested in the free boundary arising at the boundary of the time-dependent support of $w(t,\cdot)$. There is already some existence theory for related models motivated by turbulence in fluids, see Gallouët et al. (2003), Lederer and Lewandowski (2007) and Druet and Naumann (2009) for stationary models and Naumann (2013), Mielke and Naumann (2015), Bulíček and Málek (2019) and Mielke and Naumann (2022) for time-dependent models. However, there it is either assumed that $w^0(x) \geq \underline{w} > 0$ for all $x \in \Omega$ or that $w^0(x) = 0$ for all $x \in \partial\Omega$ and $w^0(x) > 0$ a.e. in Ω such that $\int_\Omega \log w^0(x)\,\mathrm{d}x > -\infty$, see Remark 5.6.

We provide a preliminary existence theory for our coupled system in Sect. 6 which allows for solution with nontrivial support, i.e., $\Omega \setminus \operatorname{sppt}(w(t))$ has nonempty interior. However, we do this just for completeness and we rather focus on the growth behavior of the support, i.e., the moving free boundary. Moreover, we emphasize that some of the statements in this paper are still conjectures, and only a few results are formulated rigorously as propositions or theorems. Nevertheless, we believe that the degenerate coupled system is relevant in applications and opens up new avenues for developing the tools in applied analysis, in particular in the field of free-boundary problems. The system is specific enough to analyze it in more detail, it is close enough to the porous medium equation (PME) to lend some of the tools from there, but it displays a richer structure of nontrivial effects stemming from the coupling between the two scalar equations.

To start with, we remark that (1.1) contains the PME, when restricting to the case $v \equiv 0$:

$$\dot{w} = \operatorname{div}\big(\kappa(w)\nabla w\big) \quad \text{in }]0,\infty[\times\Omega, \qquad \kappa(w)\nabla w\cdot \mathrm{n} = 0 \quad \text{on }]0,\infty[\times\partial\Omega, \tag{1.3}$$

which is known for its solutions with time-dependent support. For $\Omega = \mathbb{R}^d$ and $\kappa(w) = (\beta{+}1)w^\beta$ we have the celebrated self-similar Barenblatt solutions (see Vázquez 2007, Eqn. (1.8))

$$w(t,x) = \frac{1}{(t{+}t_*)^{d\delta}}\Big(\max\Big\{C - \frac{k|x|^2}{(t{+}t_*)^{2\delta}}\,,\,0\Big\}\Big)^{1/\beta} \text{ with } \delta = \frac{1}{2{+}d\beta},\ k = \frac{\delta\beta}{2(\beta{+}1)}. \tag{1.4}$$

Here, $C > 0$ determines the conserved total mass $\int_{\mathbb{R}^d} w(t,x)\,\mathrm{d}x = E_0$, see Sect. 7.1.

For $v \neq 0$, there is a true coupling between the two scalar equations, and its structure is discussed in Sect. 2. In addition to the conservation laws for momentum and energy and symmetries, we show that all entropies of the form $\mathcal{S}(v,w) = \int_\Omega \sigma(w(x))\,\mathrm{d}x$ with nondecreasing and concave σ are growing along solutions (v,w) of (1.1). Moreover, we establish a gradient structure the coupled system: For given σ with $\sigma'(w) > 0$ and $\sigma''(w) < 0$, there exists a state-dependent Onsager operator $\mathbb{K} = \mathbb{K}^* \geq 0$ describing the dissipation mechanisms due to viscosity $\eta(w)$ and to energy transport via $\kappa(w)$:

$$\begin{aligned} \begin{pmatrix}\dot v\\ \dot w\end{pmatrix} &= \mathbb{K}(v,w)\mathrm{D}\mathcal{S}(v,w) = \mathbb{K}(v,w)\begin{pmatrix}0\\ \sigma'(w)\end{pmatrix} \quad \text{with}\\ \mathbb{K}(v,w)\begin{pmatrix}\zeta\\ \xi\end{pmatrix} &= \begin{pmatrix} -\operatorname{div}\big(p_1(w)(\nabla\zeta - \xi\nabla v)\big)\\ -p_1(w)\nabla v\cdot\nabla\zeta + p_1(w)|\nabla v|^2\xi - \operatorname{div}\big(p_2(w)\nabla\xi\big)\end{pmatrix}, \end{aligned} \tag{1.5}$$

$p_1(w){=}\eta(w)/\sigma'(w)$, and $p_2(w){=}\kappa(w)/(-\sigma''(w))$, see Sect. 2.4.

Sections 2.2 and 2.3 are devoted to scaling invariances and self-similar solutions of the coupled system (1.1). We argue that even for general η and κ there are solutions of the form $(v(t,x), w(t,x)) = \big(V(x/(t{+}1)^{1/2}), W((x/(t{+}1)^{1/2})\big)$ where (V, W) may attain nontrivial limits for $y \to \pm\infty$. For $\eta(w) = \kappa(w) = w$, an explicit family of solutions with nontrivial support of W is provided in Example 2.2.

In Sect. 2.4, we consider a bounded domain $\Omega \subset \mathbb{R}^d$ and the case $\alpha = \beta \in\,]0,1[$. By exploiting the gradient structure we show that most solutions converge exponentially to constant states $(v(t), w(t)) \to (\widehat{v}_{V_0,E_0}\mathbf{1}_\Omega, \widehat{w}_{V_0,E_0}\mathbf{1}_\Omega)$, where $\widehat{v}_{V_0,E_0}$ and $\widehat{w}_{V_0,E_0}$ are given explicitly in terms of $V_0 = \mathcal{V}(v^0)$ and $E_0 = \mathcal{E}(v^0, w^0)$. The exponential decay rate is quite explicit. However, we also show that the decay does not hold for all solutions: for instance, because of non-uniqueness we may have $v(t,x) = v^0(x)$ while $w \equiv 0$, which is certainly not decaying to the thermodynamic equilibrium.

We also compare our model to the plasma model discussed (Rosenau and Hyman 1985, 1986; Hyman and Rosenau 1986) for the mass density $\rho \geq 0$ and the temperature $\theta \geq 0$:

$$\rho_t = \operatorname{div}\big(\rho^\gamma \phi_1(\rho,\theta)\nabla\rho\big) \quad \text{and} \quad \big(\rho\theta\big)_t = \operatorname{div}\big(\rho^\delta \phi_2(\rho,\theta)\nabla\theta + \theta\rho^\gamma \phi_1(\rho,\theta)\nabla\rho\big), \tag{1.6}$$

see Sect. 2.5 for more details.

Section 3 is devoted to steady states and traveling fronts. Because of the degeneracy, it is obvious that all functions of the form $(v, w) = (v^0, 0)$ are steady states, which we call trivial steady states. Nontrivial steady states are necessarily spatially constant, i.e., $(v, w) = (v_*, w_*) = \text{const.}$, which provides, for bounded domains, a unique steady state $(\widehat{v}_{V_0,E_0}, \widehat{w}_{V_0,E_0})$ as introduced above.

In Sect. 3.2, we study planar traveling fronts of the form

$$\big(v(t,x), w(t,x)\big) = \big(V(z), W(z)\big) \quad \text{with } z = x_1 + c_\mathrm{F} t,$$

where $c_\mathrm{F} \in \mathbb{R}$ is the front speed. It is well known that the planar fronts play an important role in the theory of the PME (cf. Vázquez 2007, Sec. 4.3), and we expect a similar role for our coupled system (1.1), in particular, for the understanding of the propagation of the boundary of the support. Inserting this ansatz into (1.1) and assuming $V(z) = W(z) = 0$ for $z \leq 0$, which simulates a support propagating with front speed c_F, we obtain after integrating each equations once (see Sect. 3 for details) the two ODEs

$$c_\mathrm{F}\, V = \eta(W)V', \quad c_\mathrm{F}\Big(W - \frac{1}{2}V^2\Big) = \kappa(W)W'.$$

We analyze all solutions of this system, for the different cases occurring for the choices in (1.2). To highlight one of the results, we consider the case $\eta(w) = w$ and $\kappa(w) = \kappa_0 w$, i.e., $\alpha = \beta = 1$. For $\kappa_0 \geq 1/2$ all traveling fronts have the form $(V(z), W(z)) = (0, \frac{1}{\kappa_0} c_\mathrm{F} z)$ for $z \geq 0$, which corresponds to the case of the pure PME with $v \equiv 0$. These solutions still exists for $\kappa_0 \in [0, 1/2[$, but now additional, truly coupled solutions exists:

$$(V(z), W(z)) = \Big(2\sqrt{(1-2\kappa_0)\, c_\mathrm{F} z}\ ,\ 2c_\mathrm{F} z\Big) \quad \text{for } z \geq 0.$$

For these solutions, the propagation of the support of w is not only driven by self-diffusion as for the PME, but it is driven also by the generation of w via the source term $\eta(w)|\nabla u|^2$. This is best seen in the limit $\kappa_0 \to 0$, where self-diffusion disappears but propagation is still possible. In particular we obtain $c_\mathrm{F} = \max\{\kappa_0, 1/2\}W'(0)$, which again shows that for $\kappa_0 < 1/2$ the propagation speed is no longer dominated by self-diffusion alone.

In Sect. 3.3, we conjecture that the typical behavior of w near the boundary of its support is given by $w(t,x) = w_0(z_+)^\gamma$ with $\gamma = \max\{1/\alpha, 1/\beta\}$, which clearly shows that the front is driven by the v-diffusion in case of $\beta > \alpha$. In the critical case $\alpha = \beta$ the switch between the two regimes occurs for $\eta_0 = 2\kappa_0$.

Our definitions of weak and very weak solutions are given in Sect. 4 and are based on a reformulation of the coupled system (1.1) in terms of (1.1a) and the conservation law (2.1) for the energy density $e = \frac{1}{2}v^2 + w$, thus following the ideas in Feireisl and Málek (2006) and Bulíček et al. (2009). This allows us to avoid defect measures. The notion of very weak solutions is based on the *weak weighted gradient* $G_{w^\alpha}v$, where $w^\alpha \nabla v$ is defined in the sense of distributions via $\nabla(w^\alpha v) - v\nabla(w^\alpha)$, thus avoiding any derivatives of v but using ∇w instead, see Definition 4.1, where $G_a v \in \mathrm{L}^1(\Omega)$ is defined for $a \in \mathrm{W}^{1,q}(\Omega)$ and $v \in \mathrm{L}^{q^*}(\Omega)$. This idea seems to be known in applied analysis, but it is difficult to track down a specific and clear reference. For example, in Gallouët et al. (2003, Sec. 3) or Bulíček and Málek (2019, p. 116) closely related arguments are used. Section 4.3 provides an explicit example for non-uniqueness of very weak solutions.

Before showing existence of solution, we provide a series of natural a priori bounds for strong solutions satisfying $w > 0$ in $\overline{Q}_T = [0, T]\times\overline{\Omega}$. Section 5.1 establishes L^p bounds for v and, in the case $\alpha = \beta$, also for w. Section 5.2 provides comparison. Here, we also show that the case $\eta = \kappa$ is very special, as for this case an estimate $|v^0(x)| \leq M_* w^0(x)$ for all $x \in \Omega$ propagates for positive time, i.e., we have $|v(t, x)| \leq M_x w(t, x)$ in all of Q_T. The crucial dissipation estimates are discussed in Sect. 5.3. In particular, for $\alpha \in\,]0, 1[$ and bounded Ω we obtain $\int_{Q_T} |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C(\Omega, \alpha, v^0, w^0)$.

In Sect. 6, we develop our (rather preliminary) existence theory for bounded Ω. For this, we approximate the initial data be smooth functions $(v^0_\varepsilon, w^0_\varepsilon)$ which additionally satisfy $w^0_\varepsilon(x) \geq \varepsilon$. Using the comparison principles derived earlier, we find classical solutions $(v_\varepsilon, w_\varepsilon) : Q_T \to \mathbb{R}^2$ still satisfying $w_\varepsilon(t, x) \geq \varepsilon$. For passing to the limit $\varepsilon \to 0^+$, we use the appropriate a priori bounds proving spatial and temporal compactness such that a suitable version of the Aubin–Lions–Simon lemma provides strong convergence. So far, we are only able to treat the case $\alpha \in\,]0, 1[$, where one can exploit the L^2 a priori bound for ∇v_ε, which was also used in Naumann (2013). This approach allows us to construct weak solutions. For $\alpha = 1$ we only able to handle the case $\eta \equiv \kappa$ and we are only able to establish very weak solutions, where the weak weighted gradient $G_{w^\alpha}v$ is well defined but ∇v may be not. From the expected behavior of the solutions near the boundary of the support, it is clear that gradients may have a blow-up, so that the difference between weak and very weak solutions may be essential.

In the recent paper (Fanelli and Granero-Belinchón 2021), local existence of smooth solutions and blow-up of solutions are investigated for the regime $\alpha, \beta \geq 1$ on the one-dimensional periodic domain $\Omega = \mathbb{T} = \mathbb{S}^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$. Moreover, our existence theory is different from the one developed in Bertsch and Kamin (1990) and Dal Passo and Giacomelli (1999) for the plasma model (1.6), because we enforce global Sobolev regularity, while the latter asks for local regularity of θ on the support of ρ only.

Section 7 provides a few conjectures concerning the longtime behavior in the case $\Omega = \mathbb{R}^d$ and $\eta(w) = \eta_0 w^\alpha$ and $\kappa(w) = \kappa_0 w^\beta$. The whole system does not have any self-similar solution; however, we expect that in many cases v and w behave self-similar in the limit $t \to \infty$. In these cases, we expect that $v(t)$ converges to 0 in $\mathrm{L}^2(\Omega)$ while the momentum is conserved $\mathcal{V}(v(t)) = \int_\Omega v(t, x)\,\mathrm{d}x = \mathcal{V}(v^0)$. Then

$\int_\Omega w(t,x) \to \mathcal{E}(v^0, w^0)$ and we expect that w behaves like the solution of the PME obtained from (1.1) for $v \equiv 0$, but now the total energy is fixed to $\mathcal{E}(v^0, w^0)$.

Finally, in Sect. 8 we show how our coupled model (1.1) is motivated by models from turbulence modeling, where w plays the role of the mean turbulent kinetic energy, such that $e = \frac{1}{2}v^2 + w$ denotes the total kinetic energy. Our model is obtained when the solutions $\mathbf{u}$ of the Navier–Stokes equation are assumed to be parallel flows, namely $\mathbf{u}(t,x) = \big(0, ..., 0, v(t, x_1, ..., x_d)\big)^\top \in \mathbb{R}^{d+1}$ with $d \in \{1, 2\}$. Prandtl's model for turbulence (cf. Prandtl 1946; Bulíček et al. 2011; Naumann 2013) is discussed in Sect. 8.1 relating to our case $\alpha = \beta = 1/2$, while Kolmogorov's two-equation model (Kolmogorov 1942; Spalding 1991) is discussed in Sect. 8.2 relating to our case $\alpha = \beta = 1$. For the rich theory of these models we refer to Lewandowski (1997), Bulíček et al. (2011), Naumann (2013), Chacón Rebello (2014), Mielke and Naumann (2015); Bulíček and Málek (2019) and Mielke and Naumann (2022) and the references therein.

2 The Model and Its Thermodynamical Formulation

Here, we discuss the basic properties of system (1.1), namely the conservation laws of total linear momentum and the total energy, the symmetries and scalings, as well as exact similarity solutions. Section 2.4 provides gradient structures which allow us to show convergence into steady state for the case $\alpha = \beta \in]0, 1[$ and Ω bounded, see Theorem 2.3.

We emphasize that the thermodynamical consideration for our system (1.1) are not directly related to the fluid-dynamical models used as motivation, see Sect. 8. The common structure is the Galilean invariance of both systems and the transfer of the dissipation in the equation for v into the equation for w, leading to the conservation of the total energy. Moreover, both systems have an entropy functional $\mathcal{S}$ that increases along solutions. However, our system (1.1) is indeed a gradient system, whereas the fluid-dynamical models involving the full Navier–Stokes system include the term $\mathbf{u} \cdot \nabla \mathbf{u}$ which cannot be generated by a gradient structure. In particular, this work does not contribute directly to the much more complicated thermodynamics of turbulence modeling.

2.1 Conservation Laws

We first observe that the divergence structure of the equation for v and the no-flux boundary condition provide the conservation of the integral over v, namely

$$\mathcal{V}(v, w) := \int_\Omega v(x)\,\mathrm{d}x.$$

We call this conserved quantity the a momentum because in the thermodynamical interpretation below v should be considered as a velocity, and it should not be mistaken for a concentration of a diffusing species.

In fact, w should be considered as an internal energy such that the energy density

$$e = \frac{1}{2}v^2 + w$$

plays an important role. It satisfies a conservation law without source term, namely

$$\dot{e} = \mathrm{div}\big(\kappa(w)\nabla w + \eta(w)v\nabla v\big) = \mathrm{div}\big(\eta(w)\nabla e + (\kappa(w){-}\eta(w))\nabla w\big). \tag{2.1}$$

Integration over Ω and exploiting the no-flux boundary conditions gives conservation of the total energy

$$\mathcal{E}(v, w) := \int_\Omega \Big(\frac{1}{2}v(x)^2 + w(x)\Big)\,\mathrm{d}x = \text{const.}$$

2.2 Symmetries and Scaling Properties

The full set symmetries of system (1.1) are given for $\Omega = \mathbb{R}^d$. For subsets $\Omega \neq \mathbb{R}^d$ only those symmetries survive that are valid for Ω. These symmetries hold for general functions η and κ.
Euclidean symmetry: For all $x_* \in \mathbb{R}^d$ and $Q \in \mathrm{O}(d) := \big\{\, R \in \mathbb{R}^{d\times d} \;\big|\; R^\top R = I \,\big\}$ and solutions (v, w) of (1.1), the rigidly moved pair

$$\big(\widehat{v}^{Q,x_*}, \widehat{w}^{Q,x_*}\big)(t, x) := \big(v(t, Qx{+}x_*), w(t, Qx{+}x_*)\big)$$

is a solution again.
Time and Galilean invariance: For all $t_* \geq 0$ and $V_* \in \mathbb{R}$ the time and velocity shifted pair

$$\big(\widetilde{v}^{t_*,V_*}, \widetilde{w}^{t_*,V_*}\big)(t, x) := \big(v(t{+}t_*, x) + V_*,\ w(t{+}t_*, x)\big)$$

is a solution again.

Now we discuss scaling properties. Observing that all terms involve either one time derivative or two spatial derivatives we have the following invariance of (1.1):
Scaling S1 (parabolic scaling): Assume $\Omega = \mathbb{R}^d$. If the pair (v, w) is a solution of (1.1) and $\lambda > 0$, then the pair (v_λ, w_λ) is a solution as well, where

$$(v_\lambda, w_\lambda)(t, x) = (v, w)(\lambda^2 t, \lambda x).$$

A more complex symmetry occurs if the viscosity η and the diffusion constant κ are of the same power-law type.
Scaling S2 (nonlinear scaling): Assume that $\eta(w) = \eta_0 w^\alpha$ and $\kappa(w) = \kappa_0 w^\alpha$ for some $\alpha > 0$ and $\eta_0, \kappa_0 \geq 0$. If the pair (v, w) is a solution of (1.1) and $\mu, \lambda > 0$ and $\lambda = 1$ in the case $\Omega \neq \mathbb{R}^d$, then the pair $\big(v^{\mu,\lambda}, w^{\mu,\lambda}\big)$ is a solution as well, where

$$\big(v^{\mu,\lambda}, w^{\mu,\lambda}\big)(t, x) = \big(\mu\, v(\mu^{2\alpha}\lambda^2 t, \lambda x), \mu^2 w(\mu^{2\alpha}\lambda^2 t, \lambda x)\big). \tag{2.2}$$

Note that the energy density $e = \frac{1}{2}v^2 + w$ scales similarly to w and the total conserved quantities satisfy

$$\mathcal{V}(v^{\mu,\lambda}, w^{\mu,\lambda}) = \frac{\mu}{\lambda^d}\mathcal{V}(v, w) \quad \text{and} \quad \mathcal{E}(v^{\mu,\lambda}, w^{\mu,\lambda}) = \frac{\mu^2}{\lambda^d}\mathcal{E}(v, w).$$

The main observation is that the two conserved functionals scale differently. Hence, it is not possible to have exact similarity solutions with both, $\mathcal{V}(v, w)$ and $\mathcal{E}(v, w)$ being finite and different from 0. As nontrivial solutions satisfy $\mathcal{E}(v, w) > 0$ the only choice for similarity solutions is $\mathcal{V}(v, w) = 0$, see also Sect. 7.

2.3 Exact Similarity Solutions

In general, the scaling symmetries can be used to transform into so-called scaling variables via

$$\begin{aligned} &\tau = \log(t{+}t_*), \quad y = (t{+}t_*)^{-\delta}x \quad \text{and} \\ &\big(v(t,x), w(t,x)\big) = \big((t{+}t_*)^{-\gamma/2}\,\widetilde{v}(\tau, y)\,,\ (t{+}t_*)^{-\gamma}\,\widetilde{w}(\tau, y)\big). \end{aligned} \tag{2.3}$$

Parabolic Scaling S1 According to scaling S1 have to choose $\gamma = 0$ and $\delta = 1/2$ and arrive at the transformed parabolic equation

$$\partial_\tau \widetilde{v} - \frac{1}{2}\,y{\cdot}\nabla\widetilde{v} = \mathrm{div}\big(\eta(\widetilde{w})\nabla\widetilde{v}\big), \quad \partial_\tau \widetilde{w} - \frac{1}{2}\,y{\cdot}\nabla\widetilde{w} = \mathrm{div}\big(\kappa(\widetilde{w})\nabla\widetilde{w}\big) + \eta(\widetilde{w})\big|\nabla\widetilde{v}\big|^2. \tag{2.4}$$

Exact similarity solutions are steady states of this coupled system; however, the existence of nontrivial steady states (i.e., with $\widetilde{v} \not\equiv 0$ and $\widetilde{w} \not\equiv 0$) is largely open.

Note that (2.4) cannot have steady states with compact support (or more generally finite energy), because $\widetilde{e} = \widetilde{w} + \frac{1}{2}\widetilde{v}^2$ satisfies $\partial_\tau\widetilde{e} - \frac{1}{2}\,y{\cdot}\nabla\widetilde{e} = \mathrm{div}(\cdots)$ and integrating over $\mathbb{R}^d$ and setting $\widetilde{\mathcal{E}}(\tau) := \int_{\mathbb{R}^d}\widetilde{e}(\tau, y)\,\mathrm{d}y$ gives $\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\mathcal{E}}(\tau) = -\frac{1}{2}\int_{\mathbb{R}^d} y{\cdot}\nabla\widetilde{e}\,\mathrm{d}x = -\frac{d}{2}\widetilde{\mathcal{E}}(\tau)$.

For the case $d = 1$, one can obtain some results for steady states $(\widetilde{v}(\tau, y), \widetilde{w}(\tau, y)) = (V(y), W(y)$ for $y \in \mathbb{R}$, because the problem reduces to the ODE system

$$-\frac{y}{2}V' = \big(\eta(W)V'\big)', \quad -\frac{y}{2}W' = \big(\kappa(W)W'\big)' + \eta(W)(V')^2 \ \text{ for } y \in \mathbb{R}. \tag{2.5}$$

We first introduce $c(y) := \eta(W(y))V'(y)$ such that the first equation reduces to $c'(y) = -\big(y/\eta(W(y))\big)c(y)$. Thus, c cannot change sign, which implies that V' doesn't change sign because $\eta(W) \geq 0$. Hence, for any solution (V, W) the function $y \mapsto V(y)$ must be monotone. This implies that for any nontrivial solution V cannot lie in $\mathrm{L}^p(\mathbb{R})$ for any $p < \infty$. Nevertheless, the function W may be integrable (or even have compact support); then we can integrate the second equation in (2.5) over $\mathbb{R}$ and find

$$\frac{1}{2}\int_{\mathbb{R}} W(y)\,\mathrm{d}y = \int_{\mathbb{R}} \eta(W(y))\big(V'(y)\big)^2\,\mathrm{d}y.$$

Typical solutions of (2.5) will be such that W and V' are even functions. By Galilean invariance we may assume $V(0) = 0$ such that V is odd and $E = W + \frac{1}{2}V^2$ is even again. A special case is given by $\eta \equiv \kappa$, because by (2.1) we then have $-\frac{y}{2}E' = \big(\eta(W)E'\big)'$ which has the unique even solution $E \equiv E(0)$.

Theorem 2.1 *(Similarity solutions) Consider the case* $\eta \equiv \kappa$. *Then, for each pair* $(V_-, V_+) \in \mathbb{R}^2$ *and each* $E_0 > \frac{1}{2}\max\{V_-^2, V_+^2\}$ *there exists a unique solution* (V, W) *of* (2.5) *satisfying*

$$V(y) \to V_\pm \textit{ for } x \to \pm\infty \quad \textit{and} \quad W(y) + \frac{1}{2}V(y)^2 = E_0.$$

In particular, if $V_- \neq V_+$, *then* $y \mapsto V(y)$ *is strictly monotone and* $W(y) > E_0 - \frac{1}{2}\max\{V_-^2, V_+^2\} > 0$.

Proof It suffices to solve the scalar ODE for V namely

$$-\frac{y}{2}V' = \Big(\eta\big(E_0 - \frac{1}{2}V^2\big)V'\Big)' \text{ for } y \in \mathbb{R}, \qquad V(y) \to V_\pm \text{ for } y \to \pm\infty.$$

Then, setting $W(y) = E_0 - \frac{1}{2}V^2$, the pair (V, W) solves (2.5).

The existence and uniqueness of V follows by applying (Gallay and Mielke 1998, Thm. 3.1) or (Mielke and Schindler 2021). For this may assume $V_- \leq V_+$ and define $A \in \mathrm{C}^{1,\mathrm{Lip}}_{\mathrm{loc}}(\mathbb{R})$ via

$$A(V) = \int_{V_-}^{V} \eta\Big(E_0 - \frac{1}{2}v^2\Big)\,\mathrm{d}v \text{ for } V \in [V_-, V_+] \text{ and}$$
$$A'(V) = \eta\Big(E_0 - \frac{1}{2}(V_\pm)^2\Big) + (V - V_\pm) \text{ for } \pm V \geq \pm V_\pm.$$

Thus, A is uniformly convex with upper and lower quadratic bound, and the above-mentioned results are applicable. Since V is monotone, its range stays inside $[V_-, V_+]$; hence, the choice of A outside of $[V_-, V_+]$ is irrelevant. □

For an illustration of these solutions, we refer to the left plot in Fig. 1.

Unfortunately, the previous result does not apply to the degenerate case. For this, we would need $E_0 = \frac{1}{2}\max\{V_-^2, V_+^2\}$. which implies $W(y) \to 0$ for $y \to \infty$ or $y \to -\infty$. We expect that in the case $V_- = -V_+$ and $E_0 = \frac{1}{2}V_-^2 = \frac{1}{2}V_+^2$ we still have a solution (V, W) and that W has compact support. For the special case $\eta(w) = \kappa(w) = w$ this can be confirmed by an explicit solution.

Example 2.2 (The case $\eta(w) = \kappa(w) = w$) In this case, we exploit that $E(y) = B^2$ for all y, where B is arbitrary. Indeed, (2.5) has a one-parameter family of explicit solutions:

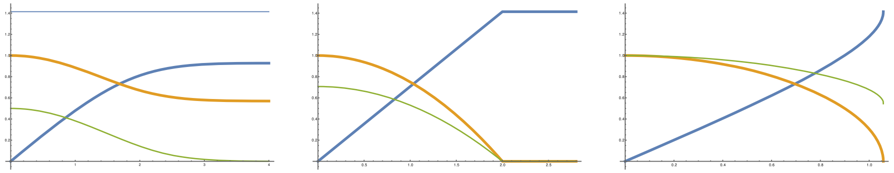

Fig. 1 Solutions $y \mapsto (V(y), W(y))$ (blue,orange) of (2.5) for $V(0) = W'(0) = 0$, $W(0) = 1$, and $V'(0) \in \{0.5, 1/\sqrt{2}, 1.0\}$ (left, middle, right). The green curve displays $W(y)V'(y)$, which for $V'(0) = 1.0$ has a positive limit at y_* where $W(y_*) = 0$ (Color figure online)

$$\big(V(y), W(y)\big) = \begin{cases} \big(y/\sqrt{2}\ ,\ B^2 - y^2/4\big) & \text{for } |y| \leq 2B, \\ \big(\pm\sqrt{2}\,B\ ,\ 0\big) & \text{for } \pm y \geq 2B, \end{cases} \tag{2.6}$$

However, this solution is untypical even for our special case $\eta(w) = \kappa(w) = w$. To see this, we solve (2.5) as an initial-value problem for $y \in [0, \infty[$ with $(V(0), W(0), W'(0))) = (0, B^2, 0)$ and $V'(0) > 0$. As $V'(y) > 0$ and $B^2 = W(y) + \frac{1}{2}V(y)^2$ the solutions stay bounded with $W(y) \in [0, B^2]$ and $V(y) \in [0, \sqrt{2}B]$ as long as they exist.

Starting with $V'(0) \in \,]0, 1/\sqrt{2}[$ we find smooth solutions with $W(\xi) \geq W(+\infty) > 0$. These are the solutions given by Theorem 2.1. When starting with $V'(0) > 1/\sqrt{2}$ the solution (V, W) reaches the point $(\sqrt{2}\,B, 0)$ at a point $y_* \in \,]0, 2B[$ with a square-root type behavior, see Fig. 1 for some plots. In particular, $W(y)V'(y)$ remains bounded from below by a positive constant, which means that the solution cannot be extended by $(V(y), W(y)) = (\sqrt{2}\,B, 0)$ for $y \geq y_*$.

Returning to the case of general η and κ, it remains an open question to discuss whether for all pairs $(V(\infty), W(\infty)) \in \,]0, \infty[$ there exists a unique solution (V, W) (V odd and W even) of (2.5) that attain these limits for $\xi \to \infty$. Moreover, one may prescribe the limits $(V(\infty), 0)$ and the integral $\int_0^\infty W(y)\,\mathrm{d}y \in \,]0, \infty[$. Clearly none of these solutions will have finite energy $\mathcal{E}(V, W) = \int_{\mathbb{R}}(W + \frac{1}{2}V^2)\,\mathrm{d}y$.

Parabolic Scaling S2 In the case $\eta(w) = \eta_0 w^\alpha$ and $\kappa(w) = \kappa_0 w^\alpha$, it is natural to search for similarity solutions induced by the *nonlinear scaling S2*. We again can use the transformation (2.3), where we are no longer forced to use $\gamma = 0$ because we can exploit the scaling properties of η and κ. It suffices to chose $2\delta + \alpha\gamma = 1$ to obtain an equation that is autonomous with respect to τ:

$$\begin{aligned} &\partial_\tau \widetilde{v} - \frac{\gamma}{2}\widetilde{v} - \delta\, y \cdot \nabla \widetilde{v} = \mathrm{div}\big(\eta_0 \widetilde{w}^\alpha\, \nabla \widetilde{v}\big), \quad \text{where } 2\delta + \alpha\gamma = 1, \\ &\partial_\tau \widetilde{w} - \gamma\, \widetilde{w} - \delta\, y \cdot \nabla \widetilde{w} = \mathrm{div}\big(\kappa_0 \widetilde{w}^\alpha \nabla \widetilde{w}\big) + \eta_0 \widetilde{w}^\alpha \big|\nabla \widetilde{v}\big|^2. \end{aligned} \tag{2.7}$$

Again, the existence for nontrivial steady state solutions is totally open.

However, we can say something for finite-energy solutions. If we look for solutions respecting energy conservation, i.e., $\mathcal{E}(\widetilde{v}(\tau), \widetilde{w}(\tau)) = \mathcal{E}(v(t), w(t)) = \mathcal{E}(v(0), w(0)) \in \,]0, \infty[$, then we additionally have to impose $\gamma = d\delta$. Together with $2\delta + \alpha\gamma = 1$ we obtain

$$\delta = \frac{1}{2+d\alpha} \quad \text{and} \quad \gamma = \frac{d}{2+d\alpha}.$$

Again we can show that there are no nontrivial steady states $(\widetilde{v}, \widetilde{w})$ with compact support. To see this, we test the steady-state equation for $\widetilde{v}$ in (2.7) by $|\widetilde{v}|^{-\theta}\widetilde{v}$ for $\theta \in \,]0, 1[$. Integrating by parts the convective part on the left-hand side and the divergence term on the right-hand side leads to the relation

$$\Big(-\frac{\gamma}{2} + \frac{d\delta}{2-\theta}\Big) \int_{\mathbb{R}^d} |\widetilde{v}|^{2-\theta}\,\mathrm{d}y = -\int_{\mathbb{R}^d} \eta_0 \widetilde{w}^{\alpha}(1-\theta)|\widetilde{v}|^{-\theta}|\nabla\widetilde{v}|^2\,\mathrm{d}y.$$

The prefactor on the left-hand side equals $\theta d/(2(2-\theta)(2+d\alpha)) > 0$, whereas the right-hand side is non-positive. Thus, we conclude $\widetilde{v} = 0$ for all compactly supported steady states. For $\widetilde{v} \equiv 0$ the system reduces to the scaled PME and it is well known that all similarity solutions are given by (1.4).

In Sect. 7, we provide some evidence for our conjecture that all finite energy solutions $(\widetilde{v}, \widetilde{w})$ of (2.7) with $\gamma = d\delta$ convergence to the corresponding steady state $(0, W_{E_0})$.

2.4 Gradient Structures and Convergence into Steady State

We now show that the coupled system can be generated by a gradient system $(\mathbf{Q}, \mathcal{S}, \mathbb{K})$, where $\mathbf{Q} = \mathrm{L}^2(\Omega)\times\mathrm{L}^1(\Omega)$ is the state space, $\mathcal{S}$ is an entropy functional, and $\mathbb{K}$ is the Onsager operator satisfying $\mathbb{K} = \mathbb{K}^* \geq 0$. The latter defines the dual entropy-production potential $\mathcal{P}^*(v, w; \zeta_v, \zeta_w) := \frac{1}{2}\langle\boldsymbol{\zeta}, \mathbb{K}\boldsymbol{\zeta}\rangle$. The aim is to show that the coupled system (1.1) can be written in the form

$$\begin{pmatrix}\dot{v}\\ \dot{w}\end{pmatrix} = \mathbb{K}(v, w)\mathrm{D}\mathcal{S}(v, w)$$

for a suitable choice of $\mathcal{S}$ and $\mathbb{K}$, see Peletier (2014) and Mielke (2016) for the general theory on gradient systems. For this, we consider entropies in the form $\mathcal{S}(w) = \int_\Omega \sigma(w)\,\mathrm{d}x$ and obtain, along solutions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(w(t)) = \int_\Omega \Big\{-\sigma''(w)\kappa(w)|\nabla w|^2 + \sigma'(w)\eta(w)|\nabla v|^2\Big\}\,\mathrm{d}x =: \mathfrak{P}(v, w). \quad (2.8)$$

Thus, we have entropy production whenever σ is nondecreasing and concave.

For finding suitable Onsager operators $\mathbb{K}$, we consider dual entropy-production potentials in the form

$$\mathcal{P}^*(v, w; \zeta_v, \zeta_w) := \frac{1}{2}\int_\Omega \Big\{p_1(w)\big|\nabla\zeta_v - \zeta_w\nabla v\big|^2 + p_2(w)|\nabla\zeta_w|^2\Big\}\,\mathrm{d}x$$

with suitable mobilities p_1 and p_2. Here, ζ_v and ζ_w are the variables dual to v and w, respectively. The conservation laws for $\mathcal{V}$ and $\mathcal{E}$ are reflected in the properties

$$\mathcal{P}^*(v, w; \mathrm{D}\mathcal{V}(v, w)) \equiv 0 \quad \text{and} \quad \mathcal{P}^*(v, w; \mathrm{D}\mathcal{E}(v, w)) \equiv 0,$$

where we use $\mathrm{D}\mathcal{V}(v, w) = (1, 0)^\top$ and $\mathrm{D}\mathcal{E}(v, w) = (v, 1)^\top$.

From the formula of $\mathcal{P}^*$, we calculate $\mathbb{K}$ via $\mathbb{K}(u, k) = \mathrm{D}^2_{\boldsymbol{\zeta}}\mathcal{P}^*(v, w; \boldsymbol{\zeta})$, which results in

$$\begin{aligned}\mathbb{K}(v, w)\begin{pmatrix}\zeta_v\\ \zeta_w\end{pmatrix} &= \begin{pmatrix}-\operatorname{div}\big(p_1(w)(\nabla\zeta_v - \zeta_w\nabla v)\big)\\ -p_1(w)\nabla v\cdot\nabla\zeta_v + p_1(w)|\nabla v|^2\varkappa - \operatorname{div}(p_2(w)\nabla\zeta_w)\end{pmatrix}\\ &= \begin{pmatrix}-\operatorname{div}\big(p_1(w)\nabla\square\big) & \operatorname{div}\big(\square p_1(w)\nabla v\big)\\ -p_1(w)\nabla v\cdot\nabla\square & \square p_1(w)|\nabla v|^2 - \operatorname{div}\big(p_2(w)\nabla\square\big)\end{pmatrix}\begin{pmatrix}\zeta_v\\ \zeta_w\end{pmatrix},\end{aligned}$$

where $\square$ indicates the position into which the corresponding component of $\boldsymbol{\zeta} = (\zeta_v, \zeta_w)^\top$ has to be inserted.

Now, calculating $\mathbb{K}(v, w)\mathrm{D}\mathcal{S}(v, w)$ with $\mathrm{D}\mathcal{S}(v, w) = (0, \sigma'(w))^\top$ we see that we obtain our coupled problem (1.1) if the relations

$$\eta(w) = \sigma'(w)p_1(w) \quad \text{and} \quad \kappa(w) = -\sigma''(w)p_2(w) \quad \text{for all } w > 0$$

hold. Moreover, we see that the above entropy-production relation (2.8) takes the general form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(v(t), w(t)) &= \big\langle \mathrm{D}\mathcal{S}(v, w), \mathbb{K}(v, w)\mathrm{D}\mathcal{S}(v, w)\big\rangle = 2\mathcal{P}^*\big(v, w; (0, \sigma'(w))^\top\big)\\ &= \int_\Omega \Big(p_1(w)\big|0 - \sigma'(w)\nabla v\big|^2 + p_2(w)\big|\nabla\sigma'(w)\big|^2\Big)\,\mathrm{d}x\\ &= \int_\Omega \Big(\eta(w)\sigma'(w)\big|\nabla v\big|^2 + \kappa(w)|\sigma''(w)|\big|\nabla w\big|^2\Big)\,\mathrm{d}x =: \mathfrak{P}(v, w)\end{aligned} \tag{2.9}$$

The gradient structure for general choices of σ can be used to obtain a priori estimates, see Sect. 5.3. Moreover, it can be used to prove convergence into steady state on bounded domains $\Omega \subset \mathbb{R}^d$. For this, we observe that taking an increasing and strictly convex σ given the momentum $V_0 := \mathcal{V}(v(0), w(0))$ and the initial energy $E_0 := \mathcal{E}(u(0), w(0))$ there is a unique maximizer of the entropy $\mathcal{S}(v, w)$ on all states in $\mathrm{L}^2(\Omega)\times\mathrm{L}^1_{\geq 0}(\Omega)$ satisfying the constraints $\mathcal{V}(v, w) = V_0$ and $\mathcal{E}(v, w) = E_0$, namely the spatially solutions

$$\widehat{v}_{V_0,E_0} = \frac{1}{|\Omega|}V_0 \quad \text{and} \quad \widehat{w}_{V_0,E_0} = \frac{1}{|\Omega|}W_0 \text{ with } W_0 := \Big(E_0 - \frac{1}{2|\Omega|}V_0^2\Big) \geq 0.$$

By integrating the equation for w over Ω and using $\eta(w)|\nabla v|^2 \geq 0$ and the no-flux boundary conditions, we easily obtain

$$0 \leq \int_\Omega w(0,x)\,\mathrm{d}x \leq \int_\Omega w(t_1,x)\,\mathrm{d}x \leq \int_\Omega w(t_2,x)\,\mathrm{d}x \leq W_0 \quad \text{for } 0 < t_1 < t_2 < \infty, \tag{2.10}$$

where we used energy conservation for the last estimate.

Moreover, under natural conditions all solutions of $\mathfrak{P}(v,w) = 0$ are given by the constant function pair $(v,w) \equiv (v^0, w^0)$ or by pairs of the form $(\widetilde{v}(\cdot), 0)$ for arbitrary $\widetilde{v} \in \mathrm{L}^2(\Omega)$. The latter solutions will be excluded by assuming $\int_\Omega w(0,x)\,\mathrm{d}x > 0$. Hence, in good situations one can hope for convergence of all solutions into the unique thermal equilibrium state $(\widehat{v}_{V_0,E_0}, \widehat{w}_{V_0,E_0})$ depending on the constraint given by the initial conditions.

The following results provide first results and explains the main idea in the simplest case, but we expect that this method generalizes to more general situations, see Mielke and Mittnenzweig (2018) and Haskovec et al. (2018) for related convergence results for systems of diffusion equations or Březina et al. (2021) and Dostalík et al. (2021) and the references therein for viscous fluids. While the latter approaches are quite general, they only provide convergence to the equilibrium without information on the convergence speed. For our special system, we are able to prove exponential decay with an explicit decay rate Λ that is even close to optimal, because we can compare it to the first nontrivial eigenvalue of the linearization. It is surprising that this exponential decay holds globally for our degenerate parabolic system, where the trivial solution $(v(t), w(t)) = (v^0, 0)$ exist and has no decay toward the equilibrium.

We define the entropy density functions $U_s : [0,\infty[\to [0,\infty[$ defined via $U_s''(r) = r^{s-2} \geq 0$ and $U_s(1) = U_s'(1) = 0$ giving the explicit form

$$U_s(r) = \frac{1}{s(s-1)}\Big(r^s - sr - 1 + s\Big).$$

To treat the case $\alpha = \beta \in]0,1[$, we consider the relative entropy functionals

$$\mathcal{H}_\beta(v,w) = \int_\Omega \left(\frac{1}{2}(v-\widehat{v})^2 + \widehat{w}\,\beta U_{1-\beta}\Big(\frac{w}{\widehat{w}}\Big)\right)\mathrm{d}x, \tag{2.11}$$

where we set $(\widehat{v},\widehat{w}) = (\widehat{v}_{V_0,E_0}, \widehat{w}_{V_0,E_0})$ to simplify notation. Observe that $\mathcal{H}_\beta$ is a linear combination of the conserved functionals $\mathcal{V}$ and $\mathcal{E}$ and the entropy $\mathcal{S}(w) = \int_\Omega w^{1-\beta}\,\mathrm{d}x$. In particular, along positive solutions we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\beta(v,w) = -\widehat{w}^\beta\,\mathfrak{P}_\beta(v,w) \\
&\quad\text{with } \mathfrak{P}_\beta(v,w) := \int_\Omega \Big(\frac{\eta(w)}{w^\beta}\big|\nabla v\big|^2 + \frac{\beta\kappa(w)}{w^{\beta+1}}\big|\nabla w\big|^2\Big)\,\mathrm{d}x
\end{aligned} \tag{2.12}$$

Moreover, we have $\mathcal{H}_\beta(v,w)\geq 0$ with equality if and only if $(v,w)=(\widehat{v},\widehat{w})$. Thus, exponential convergence follows by estimating $\mathfrak{P}_\beta$ from below by $\mathcal{H}_\beta$.

Theorem 2.3 *(Convergence to steady state) Consider a bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$ and $\beta\in\,]0,1[$, and assume that*

$$\exists\, c_\eta, c_\kappa>0\;\forall\, w\geq 0: \quad \eta(w)\geq c_\eta w^\beta \text{ and } \kappa(w)\geq c_\kappa w^\beta. \tag{2.13}$$

Then all positive solutions converge exponentially to a spatially constant equilibrium. More precisely, for solutions with $\mathcal{V}(v(0))=V_0$, $\mathcal{E}(v(0),w(0))=E_0$, and $w(0,x)\geq\delta>0$ a.e. in Ω define the relative entropy $\mathcal{H}_\beta$ as in (2.11) with $(\widehat{v},\widehat{w})=(\widehat{v}_{V_0,E_0},\widehat{w}_{V_0,E_0})$ and set $\Lambda=\widehat{w}^\beta\min\{c_\eta,2\beta c_\kappa\}\min\{1,(1-\beta)/\beta\}\,\lambda_{\mathrm{N}}$, where $\lambda_{\mathrm{N}}:=\lambda_{\mathrm{N}}(\Omega)>0$ is the first nontrivial eigenvalue of the Neumann Laplacian on Ω, then for all $t\geq 0$ we have the estimates

$$\mathcal{H}_\beta(v(t),w(t))\leq \mathrm{e}^{-\Lambda t}\,\mathcal{H}_\beta(v(0),w(0)) \quad \textit{and} \tag{2.14a}$$

$$\begin{aligned}&\frac12\big\|v(t)-\widehat{v}\big\|^2_{\mathrm{L}^2}+\big\|\sqrt{w(t)}-\sqrt{\widehat{w}}\,\big\|^2_{\mathrm{L}^2}\\ &\leq \max\Big\{\frac{\beta}{1-\beta},\frac{1-\beta}{\beta}\Big\}\,\mathrm{e}^{-\Lambda t}\Big(\frac12\big\|v(0)-\widehat{v}\big\|^2_{\mathrm{L}^2}+\big\|\sqrt{w(0)}-\sqrt{\widehat{w}}\,\big\|^2_{\mathrm{L}^2}\Big).\end{aligned} \tag{2.14b}$$

Proof To derive (2.14a), we first observe that (2.12) holds along the whole solution because $w(0,x)\geq\delta>0$ implies by Sect. 5.2 (C2) that $w(t,x)\geq\delta>0$ for all $(t,x)\in[0,\infty[\times\overline{\Omega}$. Thus, it suffices to estimate $\mathfrak{P}_\beta$ from below by $\mathcal{H}_\beta$. For this, we define $z(x)=\sqrt{w(x)}$ and use assumption (2.13) to obtain

$$\begin{aligned}\mathfrak{P}_\beta(v,w)&\geq\int_\Omega\Big(c_\eta|\nabla v|^2+\frac{\beta c_\kappa}{w}|\nabla w|^2\Big)\,\mathrm{d}x=\int_\Omega\Big(c_\eta|\nabla v|^2+4\beta c_\kappa|\nabla z|^2\Big)\,\mathrm{d}x\\ &\geq\lambda_{\mathrm{N}}\int_\Omega\Big(c_\eta(v-\widehat{v})^2+4\beta c_\kappa(z-\overline{z})^2\Big)\,\mathrm{d}x \quad\text{with } \overline{z}=\frac{1}{|\Omega|}\int_\Omega z\,\mathrm{d}x.\end{aligned} \tag{2.15}$$

For Poincaré's inequality, we used the first conservation law giving $\int_\Omega v\,\mathrm{d}x=\mathcal{V}(v)=\mathcal{V}(v(0))=|\Omega|\widehat{v}=V_0$. The same does not hold for z because $z=\sqrt{w}$ and $\mathcal{E}(v,w)$ mixes v and w, namely $E_0=\mathcal{E}(v,w)=\int_\Omega\big(\frac12 v^2+z^2\big)\,\mathrm{d}x$. This quadratic structure allows us to exploit the relative entropy $\mathcal{H}_{1/2}$, where the exponent $1/2$ is used instead of $1-\beta$:

$$\begin{aligned}&\mathcal{H}_{1/2}(v,w)=\int_\Omega\Big(\frac12\big(v-\widehat{v}\big)^2+\big(\sqrt{w}-\sqrt{\widehat{w}}\big)^2\Big)\,\mathrm{d}x=|\Omega|\big(\mathcal{A}(v,z)+(\sqrt{\widehat{w}}-\overline{z})^2\big)\\ &\text{with } \mathcal{A}(v,z)=\frac{1}{|\Omega|}\int_\Omega\Big(\frac12\big(v-\widehat{v}\big)^2+\big(z-\overline{z}\big)^2\Big)\,\mathrm{d}x.\end{aligned}$$

To estimate the last term, we use energy conservation as follows:

$$\frac{1}{|\Omega|}E_0=\widehat{w}+\frac12\widehat{v}^2=\frac{1}{|\Omega|}\mathcal{E}(v,z^2)=\mathcal{A}(v,z)+\frac12\widehat{v}^2+\overline{z}^2.$$

From $\mathcal{A}(v,z) \geq 0$, we conclude $\overline{z} \leq \sqrt{\widehat{w}}$ and obtain $\big(\sqrt{\widehat{w}} - \overline{z}\big)^2 \leq \widehat{w} - \overline{z}^2 \leq \mathcal{A}(v,z)$. Thus, $\mathcal{H}_{1/2}(v,w) \leq 2|\Omega|\mathcal{A}(v,z)$ and comparing with (2.15) gives the lower estimate

$$\mathfrak{P}_\beta(v,w) \geq \lambda_N \widehat{c}\,\mathcal{H}_{1/2}(v,w) \quad \text{with } \widehat{c} = \min\{c_\eta, 2\beta c_\kappa\}.$$

It remains to correct the error from replacing $\mathcal{H}_\beta$ by $\mathcal{H}_{1/2}$. For this, we use the estimate

$$\min\{s, 1-s\}\, U_s(z) \leq \frac{1}{2} U_{1/2}(z) \leq \max\{s, 1-s\}\, U_s(z) \quad \text{for all } z \geq 0 \text{ and } s \in \,]0,1[.$$

which immediately implies the estimate

$$\min\Big\{1, \frac{\beta}{1-\beta}\Big\}\mathcal{H}_{1/2}(v,w) \leq \mathcal{H}_\beta(v,w) \leq \max\Big\{1, \frac{\beta}{1-\beta}\Big\}\mathcal{H}_{1/2}(v,w). \tag{2.16}$$

Combining these results, we have obtained $\mathfrak{P}_\beta(v,w) \geq \lambda_{\mathrm{N}} \widehat{c}\, \min\{1, (1-\beta)/\beta\}$ $\mathcal{H}_\beta(v,w)$, and the exponential decay estimate (2.14a) follows from (2.12) by Grönwall's estimate.

To obtain the L^2 decay estimate (2.14b) we observe that this is an estimate for $\mathcal{H}_{1/2}$, which follows by combining (2.14a) and (2.16). □

We emphasize that the global decay rate Λ obtained in Theorem 2.3 is optimal up to a possible factor of 2, because the linearization of the coupled system at the steady state $(\widehat{v}_{V_0,E_0}, \widehat{w}_{V_0,E_0})$ reads

$$\dot{\widetilde{c}} = \eta(\widehat{w}_{V_0,E_0})\Delta_{\mathrm{N}}\widetilde{c}, \quad \dot{\widetilde{w}} = \kappa(\widehat{w}_{V_0,E_0})\Delta_{\mathrm{N}}\widetilde{w}, \quad \int_\Omega \widetilde{c}\,\mathrm{d}x = 0 = \int_\Omega \widetilde{w}\,\mathrm{d}x.$$

Thus, the decay rate Λ cannot be larger than $\Lambda_{\mathrm{lin}} := 2\min\{\eta(\widehat{w}_{V_0,E_0}), \kappa(\widehat{w}_{V_0,E_0})\}$ $\lambda_{\mathrm{N}}(\Omega)$, because (2.14a) measures quadratic distances from equilibrium. Indeed, assuming $\eta(w) = \eta_0 w^\beta$ and $\kappa(w) = \kappa_0 w^\beta$, which gives $c_\eta = \eta_0$ and $c_\kappa = \kappa_0$, we obtain the estimate

$$\min\Big\{\beta, \frac{1-\beta}{2\beta}\Big\}\Lambda_{\mathrm{lin}} \leq \Lambda \leq \min\Big\{\frac{1}{2}, 1-\beta\Big\}\Lambda_{\mathrm{lin}} \leq \frac{1}{2}\Lambda_{\mathrm{lin}} \quad \text{for all } \beta \in \,]0,1[.$$

This shows that for $\beta \approx 1/2$ the result is optimal up to a factor of 2.

Example 2.4 (Worse decay if supports are disjoint) We want to emphasize that some assumption on the positivity of $w(t,x)$ is necessary in Theorem 2.3, since otherwise the stated decay may not hold. As an example consider the case $\Omega = \,]-\ell, \ell[$ for fixed $\ell \gg 1$, $\eta(w) = \kappa(w) = \frac{3}{2} w^{1/2}$ such that $c_\kappa = c_\eta = 3/2$, and initial data $(v^0, w^0) \in \mathrm{H}^1(\Omega)^2$:

$$v^0(x) = \mathrm{sign}(x)\big(\max\{0, 2|x| - \ell\}\big) \quad \text{and} \quad w^0(x) = \Big(\frac{1}{15}\max\big\{1-x^2, 0\big\}\Big)^2 \text{ for } x \in \Omega.$$

We set $T_\ell = (\ell/2)^{5/2} - 1$ and observe that for $t \in [0, T_\ell]$ we have the explicit solution

$$v(t,x) = v^0(x) \ \text{ and } \ w(t,x) = \frac{1}{(t+1)^{2/5}} \Big(\frac{1}{15} \max\Big\{1 - \frac{x^2}{(t+1)^{4/5}}, 0\Big\}\Big)^2.$$

We observe that we have $\text{sppt}(v(t,\cdot)) = \overline{\Omega} \backslash]-\ell/2, \ell/2[$ and $\text{sppt}(w(t,\cdot)) = [-(t+1)^{2/5}, (t+1)^{2/5}]$, such that the supports are disjoint for $t < T_\ell$. Hence, it is easy to see that both equations are satisfied because of $\eta(w) = 0$ on the support of $v(t,\cdot)$: in the equation v the dynamics is trivial with $\dot{v} = 0$, and in the equation for w we simply have the similarity solution for the PME with $\beta = 1/2$, see (1.4).

To see that this solution contradicts the exponential decay estimate, it is sufficient to calculate the involved terms only in their main term in ℓ, which we indicate by $\sim \ell^\alpha$:

$$\lambda_\text{N} = \frac{\pi^2}{4\ell^2} \sim \ell^{-2}, \quad V_0 = 0, \quad E_0 \sim \ell^2, \quad \widehat{v}_{V_0,E_0} = 0, \quad \widehat{w}_{V_0,E_0} \sim \ell^2.$$

With this, we find the exponential decay rate $\Lambda \sim \ell^{-1}$. Because in (2.14a) the integrals over Ω on both sides are of order ℓ^3 for all $t \in [0, T_\ell]$ and $T_\ell \sim \ell^{5/2}$, we easily obtain a contradiction because $\Lambda T_\ell \sim \ell^{3/2}$ such that $\text{e}^{-\Lambda T_\ell} \ell^3$ is smaller than 1, whereas it is still of order ℓ^3 for the given solution.

Nevertheless, we conjecture that the above decay estimate can be extended to solutions with $\text{sppt}(w^0) \subsetneqq \overline{\Omega}$. The point is that solutions are not unique because of the nonlinearity $w^{1/2}|\nabla v|^2$. Constructing solutions by approximating the initial conditions from above as sketched in Sect. 6, one may obtain solutions satisfying $\text{sppt}(w(t,\cdot)) \supset \text{sppt}(\nabla v(t,\cdot))$ for all $t > 0$, which then satisfy the exponential decay estimate (2.14a).

2.5 A Related Plasma Model

In a series of papers starting with Rosenau and Hyman (1985, 1986) and Hyman and Rosenau (1986) a model of the diffusion of the mass density $\rho \geq 0$ and the heat transport for the temperature in a plasma is developed:

$$\rho_t = \text{div}\big(\rho^\gamma \phi_1(\rho,\theta)\nabla\rho\big) \quad \text{and} \quad \big(\rho\theta\big)_t = \text{div}\big(\rho^\delta \phi_2(\rho,\theta)\nabla\theta + \theta\rho^\gamma\phi_1(\rho,\theta)\nabla\rho\big), \tag{2.17}$$

that conserves mass and energy, namely $\int_\Omega \rho(t,x) \, \text{d}x = M_0$ and $\int_\Omega \rho(t,x)\theta(t,x)\,\text{d}x = E_0$.

Note that the model is such that it allows one to consider a constant temperature $\theta(t,x) = \theta_*$, and it is then sufficient to study the remaining PME for ρ:

$$\dot{\rho} = \text{div}\big(\rho^\gamma \phi_1(\rho,\theta_*)\nabla\rho\big). \tag{2.18}$$

Thus, asymptotic self-similar behavior follows if $\phi_0(,\theta_*) \neq 0$. Moreover, if $\phi(\rho,\theta_*)$ is constant, we even have Barenblatt solutions as in (1.4), see Rosenau and Hyman (1986, Eqn. (5)–(7)).

To have better comparison with our model, we can introduce the energy density $e = \rho\theta$ and obtain the system,

$$\begin{aligned} &\dot{\rho} = \operatorname{div}\big(\rho^\gamma \varphi_1(\rho,e)\nabla\rho\big) \quad \text{and} \\ &\dot{e} = \operatorname{div}\Big(\rho^{\delta-1}\varphi_2(\rho,e)\nabla e + e\big(\rho^{\gamma-1}\varphi_1(\rho,e) - \rho^{\delta-2}\varphi_2(\rho,e)\big)\nabla\rho\Big), \end{aligned} \tag{2.19}$$

where $\varphi_j(\rho,e) = \phi_j(\rho,e/\rho)$. Thus, our case $\eta(w) = \kappa(w)$ corresponds here to the case $\rho^\gamma\varphi_1 = \rho^{\delta-1}\varphi_2$ leading to the special system

$$\dot{\rho} = \operatorname{div}\big(\rho^\gamma\varphi_1(\rho,e)\nabla\rho\big) \quad \text{and} \quad \dot{e} = \operatorname{div}\big(\rho^\gamma\varphi_1(\rho,e)\nabla e\big)$$

which obviously has solutions with $e(t,x) = \rho(t,x)\theta_*$ if ρ satisfies (2.18).

For an existence theory for the model (2.17), we refer to Bertsch and Kamin (1990) and Dal Passo and Giacomelli (1999). However, to the best of the knowledge of the author a systematic study of the motion of the moving front does not exist.

3 Steady States and Traveling Fronts

Here, we provide a few special solutions that will highlight the coupling between the two degenerate equations. To obtain a first feeling about the nontrivial interaction between the two equations we study some simple explicit solutions, namely steady states and traveling fronts.

3.1 Steady States

Steady states are all the solutions of the coupled degenerate elliptic system

$$\begin{aligned} &0 = \operatorname{div}\big(\eta(w)\nabla v\big), \qquad &&0 = \operatorname{div}\big(\kappa(w)\nabla w\big) + \eta(w)|\nabla v|^2 \qquad &&\text{in } \Omega, \\ &0 = \eta(w)\nabla v\cdot \mathrm{n} &&0 = \kappa(w)\nabla w\cdot \mathrm{n} &&\text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Because of $\eta(0) = 0$, it is easy to construct steady states in the form

$$(v,w) = (V_{\mathrm{st}},0) \quad \text{where } V_{\mathrm{st}} \text{ is arbitrary.} \tag{3.2}$$

Thus, there is an infinite-dimensional family of *trivial* steady states, but this family is exceptional. If we assume $w^0 \not\equiv 0$. i.e., $\int_\Omega w^0\,\mathrm{d}x > 0$, we find $\mathcal{E}(v(t),w(t)) \geq \mathcal{E}(v^0,w^0) \geq \int_\Omega w^0\,\mathrm{d}x > 0$ and conclude that these solutions are not relevant any more.

For bounded domains we have the following uniqueness result for nontrivial steady states.

Proposition 3.1 *(Steady states for Ω bounded) Assume that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Then, all steady states are given either by the trivial ones in (3.2) or by the spatially constant ones, namely*

$$(v, w) \equiv \left(\frac{1}{|\Omega|}V_0\,, \frac{1}{|\Omega|}W_0\right), \tag{3.3}$$

where V_0 and W_0 are uniquely given by the conserved quantities $\mathcal{V}(v, w) = V_0$ and $\mathcal{E}(v, w) = E_0 = W_0 + \frac{1}{2|\Omega|}V_0^2$, see Sect. 2.4.

Proof We use the pressure function $\widehat{\pi}(x) = \Pi(w(x))$ with $\Pi(w) = \int_0^w \eta(u)\,\mathrm{d}u$ and observe that for a steady state (v, w) the function $\widehat{\pi}$ satisfies the linear Neumann problem

$$-\Delta\widehat{\pi} = \eta(w)|\nabla v|^2 \ \text{ in } \Omega, \qquad \nabla\widehat{\pi}\cdot \mathrm{n} = 0 \ \text{ on } \partial\Omega.$$

The classical solvability condition for the Neumann problem requires $\int_\Omega \eta(w)|\nabla v|^2\,\mathrm{d}x = 0$. Since the integrand is nonnegative we conclude $\eta(w)|\nabla v|^2 = 0$ a.e. in Ω and $\widehat{\pi} = \pi_*$ =const. Because of the strict monotonicity of Π, we obtain $w = w_*$ =const. and deduce either $w_* = 0$ given the trivial steady states (3.2) or $w_* > 0$ and $v = v_* =$ const. □

In the case of unbounded Ω, we have to allow for solutions with infinite energy and the result is less complete.

Proposition 3.2 *(Steady states for $\Omega = \mathbb{R}^d$) Assume $\Omega = \mathbb{R}^d$ with $d \in \{1, 2\}$. Then, all steady states (i.e., solutions of (3.1)) are given either by the trivial ones in (3.2) or by the spatially constant ones, namely $(v, w) \equiv (v_*, w_*) = const.$*

Proof The pressure $\widehat{\pi}$ introduced in the previous proof still satisfies $-\Delta\widehat{\pi} = \eta(w)|\nabla v|^2 \geq 0$. Moreover, we know $\widehat{\pi} \geq 0$. Hence, $-\widehat{\pi}$ is a subharmonic function that is bounded from above. For $d = 1$, the function $-\widehat{\pi} : \mathbb{R} \to \,]-\infty, 0]$ is convex and hence can only be bounded if it is constant. For $d = 2$, we invoke (Ransford 1995, Cor. 2.3.4) which shows that bounded subharmonic functions on $\mathbb{R}^2$ are constant. In both cases we conclude $\eta(w)|\nabla v|^2 = 0$ a.e. in $\mathbb{R}^d$ and the result concerning v follows. □

It is unclear whether the last result is still true in $\mathbb{R}^d$ with $d \geq 3$.

3.2 Traveling Fronts

The importance of traveling fronts in the PME arises from the fact that they can be used as comparison functions and that they serve as models for the local behavior near the boundary of the support of w. By isotropy of our system (1.1), it is sufficient to study the one-dimensional case $x \in \mathbb{R}^1$. We start from the traveling-wave ansatz

$$v(t, x) = V(x{+}c_\mathrm{F}t) \quad \text{and} \quad w(t, x) = W(x{+}c_\mathrm{F}t),$$

where $c_{\rm F}$ is the front speed. We obtain a coupled system of ODEs for $z = x + c_{\rm F}t$ with the unknowns

$$c_{\rm F}V' = \big(\eta(W)\,V'\big)' \quad \text{and} \quad c_{\rm F}W' = \big(\kappa(W)\,W'\big)' + \eta(W)(V')^2. \tag{3.4}$$

Clearly, the speed $c_{\rm F}$ needs to be determined together with the nontrivial solution (V, W) of (3.4). However, we observe that both right-hand sides in (3.4) contain two derivatives, while both left-hand sides contain only one derivative and one factor $c_{\rm F}$. Hence, we can rescale solutions in such a way that for a solution $(c_{\rm F}, V, W)$ also $(\lambda c_{\rm F}, V(\lambda\,\cdot), W(\lambda\,\cdot))$ is a solution for all $\lambda \in \mathbb{R}$. Since the case $c_{\rm F} = 0$ leads to steady states that were investigated already in Sect. 3.1, it suffices to consider the case $c_{\rm F} = 1$, only.

To analyze the solution set of (3.4), we integrate the first equation obtaining the integration constant v_* and substitute the result for $\eta(W)V'$ in the second equation. Then, the second equation can also be integrated with an integration constant w_*:

$$V - v_* = \eta(W)V', \qquad W - w_* = \kappa(W)W' + \frac{1}{2}(V{-}v_*)^2. \tag{3.5}$$

The integration constants were chosen such that $(V, W) \equiv (v_*, w_*)$ is the only constant solution. The system can be analyzed in the (V, W) phase plane. To see whether (3.5) has other solutions that are defined for all $z \in \mathbb{R}$, we treat for the three cases $\alpha = \beta$, $\alpha < \beta$, and $\alpha > \beta$ separately.

Case $\alpha = \beta$: For $\eta(w) = w^\alpha$ and $\kappa(w) = \kappa_0 w^\alpha$, there is the following helpful observation: the parabola

$$\mathbb{P}_{\kappa_0} : V \mapsto \widehat{w}(V) := w_* + \frac{1}{2(1{-}2\kappa_0)}\,(V{-}v_*)^2$$

is invariant by the flow of (3.5). In the case $\kappa_0 < 1/2$ all points starting above $\mathbb{P}_{\kappa_0}$ stay above, which means they cannot reach $W = 0$ in finite time and hence exist for all $z \in \mathbb{R}$.

For $w_* > 0$, the solutions lying above $\mathbb{P}_{\kappa_0}$ behave as follows (where $\rho_* = w_*^{-\alpha}$)

$$(V(z), W(z)) = (v_*, w_*) + \left(c_1 \mathrm{e}^{\rho_* z}, \frac{c_1^2\,\mathrm{e}^{2\rho_* z}}{2(1{-}2\kappa_0)} + c_2 \mathrm{e}^{\rho_* z/\kappa_0}\right) + \text{h.o.t. for } z \to -\infty,$$
$$(V(z), W(z)) = \big(c_3 z^{\kappa_0/\alpha}, (\tfrac{\alpha}{\kappa_0}\,z)^{1/\alpha}\big) + \text{l.o.t. for } z \to \infty,$$

where $c_1, c_3 \in \mathbb{R}$ and $c_2 > 0$. The two solutions lying exactly on $\mathbb{P}_{\kappa_0} : W = \widehat{w}(V)$ have a slightly different asymptotics. Still in the case $\kappa_0 < 1/2$ one can show that all solutions starting below of $\mathbb{P}_{\eta_0}$ reach $W = 0$ in finite time and cannot be extended for all $z \in \mathbb{R}$, see Fig. 2(right picture).

For the case $\kappa_0 \geq 1/2$, it can be shown that only one solution satisfies $w(z) \geq 0$ for all z, namely the one with $V \equiv v_*$ and $W(z) > w_*$, see Fig. 2 (left picture).

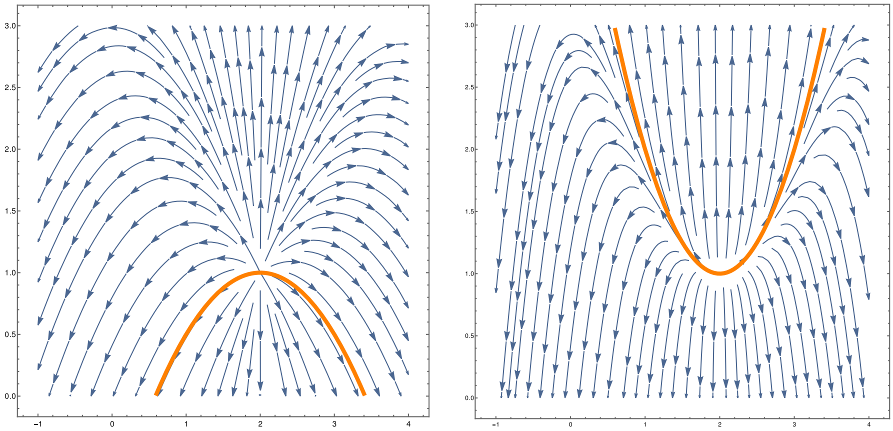

Fig. 2 Phase portraits for (3.5) with $\kappa(w) = \kappa_0\eta(w)$ including the parabola $\mathbb{P}_{\kappa_0}$. Left: $\kappa_0 = 1$. Right: $\kappa_0 = 1/4$

The situation $w_* = 0$ is special, as now we can construct solutions with $(v(z), w(z)) = (v_*, 0)$ for $z \leq 0$ (by Galileian invariance we can set $v_* = 0$ subsequently). These solutions are in particular interesting, because they provide solutions with time-dependent support. Of course, for all κ_0 we have the pure PME traveling wave $(V(z), W(z)) = \big(0, (\frac{\alpha}{\kappa_0} z_+)^{1/\alpha}\big)$ where $z_+ = \max\{0, z\}$.

For $\kappa_0 < 1/2$, the two solutions lying on $\mathbb{P}_{\kappa_0}$ have the explicit form

$$(V(z), W(z)) = \begin{cases} \big(\pm\sqrt{2-4\kappa_0}\,(2\alpha z)^{1/(2\alpha)}, (2\alpha z)^{1/\alpha}\big) & \text{for } z \geq 0, \\ \big(0\;\;0\big) & \text{for } z \leq 0. \end{cases}$$

These solutions will serve as the prototype of solutions with time-dependent support.

As above, there are more traveling waves from the solutions lying above the parabola $\mathbb{P}_{\kappa_0}$, which is now touching the axis $w = 0$ in the origin $(V, W) = (0, 0)$. All these other solutions have the asymptotics

$$(V(z), W(z)) = \big(c(z_+)^{\kappa_0/\alpha}, (\tfrac{\alpha}{\kappa_0} z_+)^{1/\alpha} + \tfrac{c^2}{2}(z_+)^{2\kappa_0/\alpha}\big) + \text{h.o.t.} \quad \text{for } z \to 0^+,$$

where $c \in \mathbb{R}$ is a parameter for choosing the individual solutions above $\mathbb{P}_{\kappa_0}$.

<u>*Case* $\alpha < \beta$:</u> In the case $\eta(w) = w^\alpha$ and $\kappa(w) = w^\beta$ the ODE reads

$$V' = V\,W^{-\alpha}, \quad W' = \Big(W - \frac{1}{2}V^2\Big)W^{-\beta}, \qquad (V(0)), W(0)) = (0, 0). \tag{3.6}$$

In the cases with $\alpha \neq \beta$, we may assume $\eta_0 = \kappa_0$ without loss of generality.

As usual, there is the trivial solution with $V \equiv 0$, but all nontrivial solutions of (3.6) only exist on a subinterval of $\mathbb{R}$, see the right phase plane in Fig. 3. Nontrivial solutions first have to lie above the parabola $W = \frac{1}{2}V^2$ to allow for $W' > 0$, but after finite time they return to the parabola and then W' remains negative until $W(z_*) = 0$ is reached in finite time, where the solution ceases to exist.

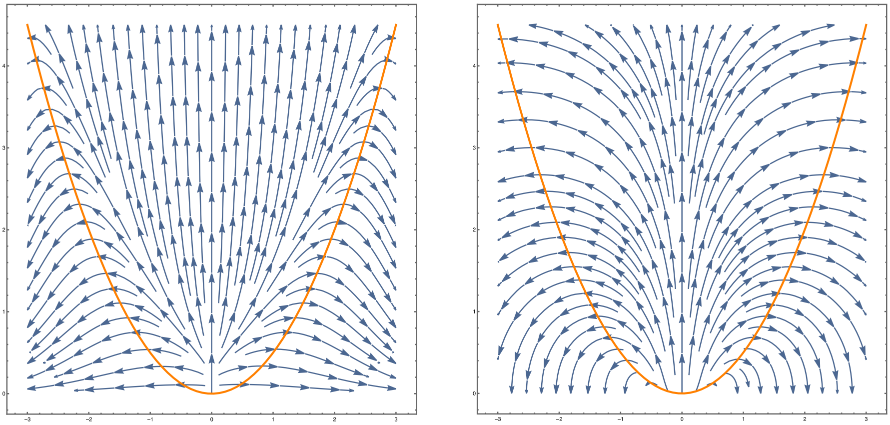

Fig. 3 Phase planes for (3.6) with $(\alpha, \beta) = (2, 1)$ (left) and $(\alpha, \beta) = (1, 2)$ (right)

The solutions can be constructed in the form $W(z) = \widetilde{w}(V(z))$ where $\widetilde{w}$ satisfies the ODE $\widetilde{w}'(V) = \widetilde{w}(V) - \frac{1}{2}V^2/(V\widetilde{w}(V)^{\beta-\alpha})$. From this, one sees that all solutions with $\widetilde{w}(0) = 0$ satisfy the expansion

$$\widetilde{w}(V) = \frac{1}{2}V^2 + 2^{\alpha-\beta}V^{2(1+\beta-\alpha)} + \text{h.o.t.}$$

Inserting this into $V' = V\widetilde{w}(V)^{-\alpha}$, we obtain the expansion

$$\big(V(z), W(z)\big) = \Big((\alpha 2^{1+\alpha})^{1/(2\alpha)}\, z_+^{1/(2\alpha)},\ (2\alpha)^{1/\alpha}\, z_+^{1/\alpha}\Big) + \text{h.o.t.} \tag{3.7}$$

We observe that the local front behavior only depends on the smaller of the two values α and β.

Case $\alpha > \beta$: In this case, we again obtain a one-parameter family of traveling fronts for (3.6) with half-line support.

Proposition 3.3 *(Fronts for $\alpha > \beta$) Assume $\eta(w) = w^\alpha$ and $\kappa(w) = w^\beta$ with $\alpha > \beta$. Then, for all v_∞ there exists a unique traveling front (V, W) solving (3.6), $(V(z), W(z)) = (0, 0)$ for $z \le 0$, and $V(z) \to v_\infty$ for $z \to \infty$. Moreover, for $v_\infty \neq 0$, the functions $\text{sign}(v_\infty)V(\cdot)$ and $W(\cdot)$ are strictly increasing on $[0, \infty[$ and we have the expansion*

$$W(z) \approx (\beta z)^{1/\beta} \text{ for } z \ll 1 \text{ and for } z \gg 1,$$
$$V(z) \approx c \exp\Big(-\frac{\beta}{\alpha-\beta}\, z^{-(\alpha-\beta)/\beta}\Big) \text{ for } z \ll 1.$$

Moreover, there are two traveling fronts $(\pm V_\text{B}, W_\text{B})$ with the expansion

$$(V_\text{B}(z), W_\text{B}(z)) \approx \Big((2\alpha\, z)^{1/(2\alpha)}, \frac{1}{2}(2\alpha\, z)^{1/\alpha}\Big) \textit{ for } z \gg 1.$$

All other solutions are not defined for all $z \in \mathbb{R}$, *see Fig. 3 (left).*

Again we see that the smaller of the two exponents α and β dominates the local behavior of the traveling fronts.

3.3 Conjectured Behavior Near Boundary of Growing Supports

Here, we collect conjectured consequences of the above-established behavior of traveling fronts. Throughout we assume that we are considering sufficiently smooth solutions (v, w) that have the property that $S(t) := \text{sppt}(v(t)) = \text{sppt}(w(t)) \Subset \Omega$.

We conjecture that $t \mapsto S(t)$ has similarly good properties as the support of solutions of the PME, see, e.g., Vázquez (2007). In particular, $t \mapsto S(t)$ is nondecreasing and the boundary $\partial S(t)$ becomes smooth after a suitable waiting time. However, in our coupled system the growth of the support can be steered by different mechanisms depending on the relative size of $\eta(w)$ and $\kappa(w)$ for $w \ll 1$.

To explain the conjectured behavior in more detail, we consider a point $x_* \in \partial S(t_*)$ and assume that $\partial S(t_*)$ is smooth. Without loss of generality we may assume $x_* = 0$ and that the outer normal vector to $S(t_*)$ at x_* is given by $\mathrm{n}_* = -e_1$.

In the PME (1.3) with $\kappa(w) = \kappa_0 w^\beta$ the typical behavior (after waiting time) is that $w(t_*, x) = w_0 (x_1)_+^{1/\beta} + \text{h.o.t.}$ The support $S(t)$ is then growing with propagation speed $c_\text{F} = \kappa_0 w_0^\beta / \beta$.

The behavior for the coupled system depends strongly on the exponents α and β in $\eta(w) = \eta_0 w^\alpha$ and $\kappa(w) = \kappa_0 w^\beta$. In all cases we will address the question of local integrability of ∇v^γ and ∇w^γ near the boundary of $S(t)$. These integrability properties will nicely fit together with the a priori estimates to be derived below, see (5.17) in Proposition 5.5.

$\underline{\alpha > \beta : \textit{support is driven by w as in PME}}$. In the case $\alpha > \beta$ and $w \ll 1$ the energy-transport coefficient $\kappa(w)$ is much bigger than the viscosity coefficient $\eta(w)$. Hence, w will diffuse fast and v will try to keep up by following the growing support. As in the PME the conjectured behavior (after waiting times) is

$$(v(t_*, x), w(t, x)) = \big(0, w_0\, (x_1)_+^{1/\beta}\big) + \text{h.o.t.},$$

where v vanishes faster than $O(|x|^m)$ for any $m \in \mathbb{N}$, see Proposition 3.3. Again the front speed is solely controlled by w alone, namely $c_* = \kappa_0 w_0^\beta / \beta$.

The expansion does not give any information about the integrability of ∇v^γ; however, we see that $\nabla w^\gamma \in \mathrm{L}^p(B_r((t_*, 0)))$ for $\gamma > \beta\,(1-1/p)$.

$\underline{\alpha < \beta : \textit{support is driven by v}}$. Now $\eta(w) \gg \kappa(w)$ for $w \ll 1$; hence, v can easily diffuse to the boundary of the support and pile up there. We conjecture that the typical behavior (after waiting times) is given by (3.7), namely

$$(v(t_*, x), w(t, x)) = \Big(v_0\, (x_1)_+^{1/(2\alpha)}, \frac{1}{2} v_0^2\, (x_1)_+^{1/\alpha}\Big) + \text{h.o.t.}.$$

The corresponding propagation speed is then given by $c_* = \eta_0 v_0^{2\alpha-2} / (\alpha 2^{\alpha+1})$.

The reason of this behavior is that in the equation for w the energy transport via κ can be neglected and the growth of the support is controlled by the source term $\eta(w)|\nabla v|^2$. This leads to a equipartition of energy near the boundary of the support giving $w \approx \frac{1}{2}v^2$, or using $e = \frac{1}{2}v^2 + w$ we have $\frac{1}{2}v^2 \approx \frac{1}{2}e \approx w$.

In this case, we see that $\nabla v^\gamma \in \mathrm{L}^p(B_r((t_*, 0)))$ for $\gamma > 2\alpha\,(1-1/p)$ and $\nabla w^\gamma \in \mathrm{L}^p(B_r((t_*, 0)))$ for $\gamma > \alpha\,(1-1/p)$.

Critical case $\alpha = \beta$. We now consider $\eta(w) = w^\alpha$ and $\kappa(w) = \kappa_0 w^\alpha$ and will see that the both previous cases appear, because $\kappa(w)/\eta(w) = \kappa_0$ may be large or small depending on κ_0.

For $\kappa_0 > 1/2$, the support is driven by w as in the PME; however, now v can follow fast enough. The conjectured behavior is

$$(v(t_*, x), w(t_*, x)) = (v_0(x_1)_+^{\kappa_0/\alpha}, w_0(x_1)_+^{1/\alpha}) + \text{h.o.t.}$$

Here, $v_0 \in \mathbb{R}$ is arbitrary, and the propagation speed $c_* = \kappa_0 w_0^\alpha/\alpha$ depends only on w_0 as for the PME.

For $\kappa_0 < 1/2$, the front is driven by a combination of v and w. The conjectured expansion takes the form

$$\big(v(t_*, x), w(t_*, x)\big) = \Big(v_0\ (x_1)_+^{1/(2\alpha)}\ ,\ \frac{v_0^2}{2(1-2\kappa_0)}\ (x_1)_+^{1/\alpha}\Big) + \text{h.o.t.}$$

We observe that the leading terms of v and w are coupled together by the relation $w \approx v^2/(2-4\kappa_0)$. Moreover, the limit $\kappa_0 \to 0^+$ is consistent with the equipartition in the case $\alpha < \beta$. The propagation speed is given by $c_* = w_0^\alpha/(2\alpha)$ which is different from $c_* = \kappa_0 w_0^\alpha/(2\alpha)$ in the case $\kappa_0 > 1/2$. Thus, the interaction with the v component prevents the deterioration of the wave speed for the limit $\kappa_0 \to 0^+$.

In both subcases, we see that $\nabla v^\gamma \in \mathrm{L}^p(B_r((t_*, 0)))$ for $\gamma > 2\alpha\,(1-1/p)$ and $\nabla w^\gamma \in \mathrm{L}^p(B_r((t_*, 0)))$ for $\gamma > \alpha\,(1-1/p)$.

In summary, we find that the behavior of w near the boundary of the support is given by $w(t, x) = w_0(x_1)_+^\gamma$ with $\gamma = \max\{1/\alpha, 1/\beta\}$, which clearly shows that the front is driven by the v-diffusion in case of $\beta > \alpha$. In the critical case $alpha = \beta$ the switch between the two regimes occurs for $\eta_0 = 2\kappa_0$.

4 Weak and Very Weak Solutions

In general, we cannot expect to have strong solutions for our degenerate coupled parabolic system. Hence, we define a suitable notions of weak and very solutions. The problem is that the degeneracies of the viscosity η and the energy-transport coefficient κ, do not allows us to use parabolic regularity, which is most easily seen for the trivial solutions $(v(t, x), w(t, x)) = (v^0(x), 0)$ that do not regularize at all. Hence, we provide a proper definition of weak and very weak solutions in Sect. 4.1, then discuss a compactly supported explicit solution in Sect. 4.2, and finally show non-uniqueness of very weak solutions in Sect. 4.3.

4.1 Definition of Weak and Very Weak Solutions

Moreover, there is an intrinsic problem in passing to the limit in the "L^1" source term $\eta(w)|\nabla v|^2$, which typically generates a nonnegative defect measure. This is particularly difficult because of the degeneracies $\eta(0) = \kappa(0)$ in the viscosity η and energy-transport coefficient κ. To avoid this problem, we use the strategy introduced by Feireisl and Málek (2006) and Bulíček et al. (2009). This means we replace the "partial energy equation" (1.1b) by the equation for the total energy $e = \frac{1}{2}v^2 + w$ as given in (2.1). Thus, we are studying suitably defined weak solutions of the coupled system

$$\dot{v} = \operatorname{div}\big(\eta(w)\nabla v\big), \quad \partial_t\Big(\frac{1}{2}v^2 + w\Big) = \operatorname{div}\big(\kappa(w)\nabla w + \eta(w)v\nabla v\big) \tag{4.1}$$

completed by no-flux conditions for v and w at the boundary $\partial\Omega$.

Below, we will give two different solution concepts, the first being a classical weak solution. However, since ∇v only occurs together with $\eta(w)$ it is difficult to obtain good a priori estimates guaranteeing that a limit v^0 obtained from $v_\varepsilon \in L^2(0, T; H^1(\Omega))$ remains in that space. Hence, we define a second concept called very weak solutions, which can be defined for $v \in L^p(Q_T)$. For the latter we use the notion of a *weak weighted gradient* generalizing terms of the form $a\nabla v$, where a takes the role of $\eta(w)$ and may degenerate. We will generalize to a mapping $G_a v$ that is valid under weak assumptions on v if a is sufficiently well behaved.

Definition 4.1 (Weak weighted gradient) Let $q \in \,]1, \infty[$, $a \in W^{1,q}(\Omega)$ and $v \in L^{q^*}(\Omega)$ with $1/q + 1/q^* = 1$. We say that $g \in L^p(\Omega)$ with $p \in [1, \infty]$ is the a-weighted weak gradient of v and write $g = G_a v$ if

$$\forall\, \Psi \in C^1_N(\overline{\Omega}; \mathbb{R}^d): \quad \int_\Omega g \cdot \Psi \, dx = -\int_\Omega v\, \big(\Psi \cdot \nabla a + a \operatorname{div} \Psi\big)\, dx, \tag{4.2}$$

where $C^1_N(\overline{\Omega}; \mathbb{R}^d) := \big\{\, \Psi \in C^1(\overline{\Omega}; \mathbb{R}^d) \;\big|\; \Psi \cdot n = 0 \text{ on } \partial\Omega \,\big\}$.

The idea behind the notion of weak weighted gradients seems to used in analysis for many years, see, e.g., the occurrences of the identity $a\nabla v = \nabla(av) - v\nabla a$ in Gallouët (2003, Sec. 3) and Bulíček and Málek (2019, p. 116).

As in the classical definition of weak derivatives, we see that g is uniquely defined by the pair $(a, v) \in W^{1,q}(\Omega) \times L^{q^*}(\Omega)$. Moreover, for $v \in W^{1,q^*}(\Omega)$ we obviously have $G_a v = a\nabla v$ by applying Gauß' divergence theorem and using $\Psi \cdot n = 0$. However, function for which ∇v does not exist may have a weighted gradient if a is canceling the singularity. For instance, on $\Omega = \,]-1, 1[$ we may choose $v : x \mapsto \operatorname{sign}(x)$ and $a(x) = |x|^\alpha$ for $\alpha > 0$, then $G_a v$ exists and equals 0.

The next result shows that the notion of weighted gradients is stable under limit passages, which will be crucial for constructing weak and very weak solutions.

Lemma 4.2 *(A closedness result for weak weighted gradients) Let* $p, q \in\,]1,\infty[$ *and consider* $a_\varepsilon \in \mathrm{W}^{1,q}(\Omega)$ *and* $v_\varepsilon \in \mathrm{L}^{q^*}(\Omega)$ *such that*

$$a_\varepsilon \rightharpoonup a_0 \;\; in\; \mathrm{W}^{1,q}(\Omega), \quad v_\varepsilon \to v_0 \;\; in\; \mathrm{L}^{q^*}(\Omega), \quad and\; g_\varepsilon = G_{a_\varepsilon} v_\varepsilon \rightharpoonup g_0 \in \mathrm{L}^p(\Omega).$$

Then, we have $g_0 = G_{a_0} v_0$.

Proof We simply consider the defining identity (4.2) for $\varepsilon > 0$ and observe that we can pass to the limit $\varepsilon \to 0^+$ in all three terms. On the right-hand side, it is crucial to have strong convergence of v_ε. □

We are now ready to give our two notions of solutions, where in the second definition we have written out the definition of weak weighted gradients explicitly to emphasize that the definition does not involve derivatives of v and that the test function must be smoother. Moreover, as common in the PME we use the pressure function

$$\Pi(w) := \int_0^w \kappa(s)\,\mathrm{d}s \quad \text{for } s \geq 0.$$

Definition 4.3 (Weak and very weak solutions) Given $T \in\,]0,\infty]$ and initial conditions $(v^0, w^0) \in \mathrm{L}^2(\Omega)\times\mathrm{L}^1_\geq(\Omega)$ we call a pair $(v, w) \in \mathrm{L}^\infty([0,T];\mathrm{L}^2(\Omega))\times\mathrm{L}^\infty([0,T];\mathrm{L}^1_\geq(\Omega))$ a *weak solution* of system (1.1) if the following holds:

$$\nabla v,\; \eta(w)\nabla v,\; \eta(w)v\nabla v,\; \nabla\Pi(w) \in \mathrm{L}^1(Q_T;\mathbb{R}^d), \tag{4.3a}$$

$$-\int_\Omega v^0\varphi(0,\cdot)\,\mathrm{d}x - \iint_{Q_T} v\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t = -\iint_{Q_T} \eta(w)\nabla v\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \varphi \in \mathrm{C}^1_\mathrm{c}([0,T[\times\overline{\Omega}), \tag{4.3b}$$

$$-\int_\Omega \big(\tfrac{1}{2}(v^0)^2 + w^0\big)\xi(0,\cdot)\,\mathrm{d}x - \iint_{Q_T} \big(\tfrac{1}{2}v^2 + w\big)\partial_t\xi\,\mathrm{d}x\,\mathrm{d}t = -\iint_{Q_T}\Big(\nabla\big(\Pi(w)\big)\cdot\nabla\xi + \eta(w)v\nabla v\cdot\nabla\xi\Big)\,\mathrm{d}x\,\mathrm{d}t \; \text{ for all } \xi \in \mathrm{C}^1_\mathrm{c}([0,T[\times\overline{\Omega}). \tag{4.3c}$$

A pair $(v, w) \in \mathrm{L}^\infty([0,T];\mathrm{L}^2(\Omega))\times\mathrm{L}^\infty([0,T];\mathrm{L}^1_\geq(\Omega))$ is called *very weak solution* of system (1.1) if the following holds:

$$(v+v^2)\eta(w) \in \mathrm{L}^1(Q_T), \;\; (v+v^2)\nabla(\eta(w)),\; \nabla\Pi(w) \in \mathrm{L}^1(Q_T;\mathbb{R}^d), \tag{4.4a}$$

$$-\int_\Omega v^0\varphi(0,\cdot)\,\mathrm{d}x - \iint_{Q_T} v\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} v\,\big(\eta(w)\Delta\varphi + \nabla(\eta(w))\cdot\nabla\varphi\big)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \varphi \in \mathrm{C}^2_\mathrm{c}([0,T[\times\overline{\Omega}) \text{ with } \nabla\varphi\cdot\mathrm{n} = 0, \tag{4.4b}$$

$$-\int_\Omega \big(\tfrac{1}{2}(v^0)^2 + w^0\big)\xi(0,\cdot)\,\mathrm{d}x - \iint_{Q_T} \big(\tfrac{1}{2}v^2 + w\big)\partial_t\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$= \iint_{Q_T} \Big(-\nabla(\Pi(w)) \cdot \nabla\xi + \frac{v^2}{2} \big(\nabla(\eta(w)) \cdot \nabla\xi + \eta(w)\Delta\xi \big) \Big) \,\mathrm{d}x\,\mathrm{d}t$$
$$\text{for all } \xi \in \mathrm{C}^2_{\mathrm{c}}([0,T[\times\overline{\Omega}) \text{ with } \nabla\xi\cdot\mathrm{n} = 0. \tag{4.4c}$$

We first observe that functions of the form $(v(t,x), w(t,x)) = (v^0(x), 0)$ with $v^0 \in \mathrm{L}^2(\Omega)$ are very weak solutions. Because of $w \equiv 0$, it is trivial to see that (4.4) is satisfied.

We remark that weak solutions are not necessarily very weak solutions, because the degeneracies do not allow us to transfer the necessary integrabilities easily.

For both notions of solutions, we have conservation of momentum and energy. To see this, we simply consider spatially constant test functions $(t,x) = \varphi(t)$ and $(t,x) = \xi(t)$. As the spatial gradients of the test functions vanish, we obtain

$$\mathcal{V}(v^0)\varphi(0) = \int_0^t \mathcal{V}(v(t))\dot{\varphi}(t)\,\mathrm{d}t \quad \text{and} \quad \mathcal{E}(v^0, w^0)\xi(0) = \int_0^t \mathcal{E}(v(t), w(t))\dot{\xi}(t)\,\mathrm{d}t.$$

By the lemma of Du Bois–Reymond, we conclude $\mathcal{V}(v(t)) = \mathcal{V}(v^0)$ and $\mathcal{E}(v(t), w(t)) = \mathcal{E}(v^0, w^0)$ for a.a. $t \in [0,T]$.

Of course, weak or very weak solutions that are sufficiently regular are even strong solutions.

4.2 An Explicit Compactly Supported Solution

We consider the case $\Omega = \mathbb{R}$ and $\eta(w) = \kappa(w) = w$ and provide a nontrivial solution with compact support that grows in time. The solution is obtained by combining two self-similar solutions as discussed in Example 2.2 in a suitable way.

We choose real positive parameters B, x_*, and t_* such that $t_* < x_*^2/(4B^2)$ and set $T := x_*^2/(4B^2) - t_* > 0$. For $(t,x) \in [0,T]\times\mathbb{R}$ we define the functions (v^*, w^*) via

$$v^*(t,x) = \begin{cases} \frac{x+x_*+2B\sqrt{t+t_*}}{\sqrt{2}\,\sqrt{t+t_*}} & \text{for } |x+x_*| \le 2B\sqrt{t+t_*}, \\ 2\sqrt{2}\,B & \text{for } |x| \le x_* - 2B\sqrt{t+t_*}, \\ \frac{x_*+2B\sqrt{t+t_*}-x}{\sqrt{2}\,\sqrt{t+t_*}} & \text{for } |x-x_*| \le 2B\sqrt{t+t_*}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$
$$w^*(t,x) = \begin{cases} B^2 - \frac{(x+x_*)^2}{4(t+t_*)} & \text{for } |x+x_*| \le 2B\sqrt{t+t_*}, \\ B^2 - \frac{(x-x_*)^2}{4(t+t_*)} & \text{for } |x-x_*| \le 2B\sqrt{t+t_*}, \\ 0 & \text{otherwise.} \end{cases}$$

For a plot of the functions $v^*(t,\cdot)$ and $w^*(t,\cdot)$ we refer to Fig. 4.

A direct calculation shows that (v^*, w^*) is a strong as well as a weak solution. Moreover, we have $e^* := \frac{1}{2}(v^*)^2 + w^* = \sqrt{2}B\,v^*$, which is consistent with the fact that $\eta \equiv \kappa$ implies that e^* and v^* satisfy the same equation, namely $\dot{e}^* = (w^* e^*_x)_x$ and $\dot{v}^* = (w^* v^*_x)_x$.

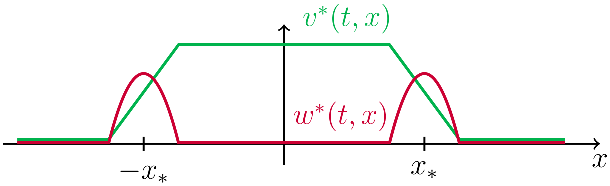

Fig. 4 Graph of the explicit solution (v^*, w^*) from (4.5) having growing support $[-\xi(t), \xi(t)]$ where $\xi(t) = x_* + 2B\sqrt{t+t_*}$

A simple calculation using the piecewise linear structure of $v^*(t,\cdot)$ and the piecewise parabolic structure of $w^*(t,\cdot)$ gives the relations

$$\int_{\mathbb{R}} v^*(t,x)\,\mathrm{d}x = \sqrt{32}\,Bx_*, \quad \int_{\mathbb{R}} \frac{(v^*)^2}{2}\,\mathrm{d}x = 8B^2x_* - \tfrac{16}{3}B^3\sqrt{t+t_*},$$
$$\int_{\mathbb{R}} w^*(t,x)\,\mathrm{d}x = \tfrac{16}{3}B^3\sqrt{t+t_*}\,.$$

This confirms the conservation of the total momentum and the total energy.

We also note that the source term $\eta(w^*)(v^*_x)^2$ reduces here to the simple expression $w^*/(2t+2t_*)$, which vanishes at the boundary of $\mathrm{sppt}(w^*(t,\cdot))$. Hence, in this case the source term does not contribute to the growth of the support.

4.3 Non-Uniqueness of Very Weak Solutions

We now consider the pair (v^0, w^0) that is obtained from (v^*, w^*) by keeping B and x_* fixed but taking the limit $t_* \to 0^+$. Then, (v^0, w^0) has the initial values $(v^0_0, w^0_0) = (2\sqrt{2}\,B\,\mathbf{1}_{[-x_*,x_*]}, 0) \in \mathrm{L}^2(\Omega)\times\mathrm{L}^1_{\geq}(\Omega)$. Next we observe that $\nabla v = v_x$ is piecewise constant with values $\pm 1/\sqrt{2t}$ in the intervals $|x\pm x_*| \leq 2B\sqrt{t}$ and 0 otherwise. Hence, we have $\nabla v \in \mathrm{L}^p(Q_T)$ for $p \in [1, 3[$. With $\Pi(w) = w^2/2$ we find $\nabla\Pi(w) = ww_x \in \mathrm{L}^q(Q_T)$ for all $q \in [1, 2[$. Using $(v^0, w^0) \in \mathrm{L}^\infty(Q_t)$ we have the conditions (4.3a) and (4.4a). Moreover, inserting (v^0, w^0) into the weak form (4.3b)+(4.3c) of the very weak form (4.4b)+(4.4c) we can use the explicit formula for (v^0, w^0) to undo the integrations by parts and see that (v^0, w^0) is indeed a weak solution as well as a very weak solution.

However, there is the trivial second very weak solution, namely $(v(t,\cdot), w(t,\cdot)) = (v^0_0, 0) = (2\sqrt{2}\,B\,\mathbf{1}_{[-x_*,x_*]}, 0)$. Thus, we definitely have non-uniqueness in the class of very weak solutions.

Indeed, we have a two-parameter family of very weak solutions for the initial conditions $(v^0_0, 0) = (2\sqrt{2}\,B\,\mathbf{1}_{[-x_*,x_*]}, 0)$. The point is that we may keep the solution constant in time for an arbitrary $t_+ > 0$ for $x \geq 0$ and then start with a delayed version of (v^0, w^0). Moreover, we may choose $t_- > 0$ for starting a delayed version of (v^0, w^0) for $x \leq 0$. More precisely, we choose $t_+, t_- \in [0, T]$ and set

$$(\widetilde{v}(t,x), \widetilde{w}(t,x)) = \begin{cases} (v_0^0(x), 0) & \text{for } x \geq 0 \text{ and } t \in [0, t_+], \\ (v^0(t-t_+, x), w^0(t-t_+, x)) & \text{for } x \geq 0 \text{ and } t \in [t_+, T], \\ (v_0^0(x), 0) & \text{for } x \leq 0 \text{ and } t \in [0, t_-], \\ (v^0(t-t_-, x), w^0(t-t_-, x)) & \text{for } x \leq 0 \text{ and } t \in [t_-, T]. \end{cases}$$

We emphasize that the different delays do not produce any nonsmoothness, because we have $(v^0(t,0), w^0(t,0)) = (2\sqrt{2}B, 0)$ for all $t \in [0, T]$. A direct calculation shows that $(\widetilde{v}, \widetilde{w})$ is a very weak solution for all the choices of $t_+, t_- \in [0, T]$.

4.4 Case $\eta \equiv \kappa$: Families of Solutions with Growing Support

In the case $\kappa \equiv \eta$, we have the additional simple equation for $e = \frac{1}{2}v^2 + w$, namely

$$\dot{e} = \operatorname{div}\big(\kappa(w)\nabla e\big), \quad \kappa(w)\nabla e \cdot \mathrm{n} = 0.$$

Thus, we obtain exactly the same equation as for v and may restrict to a solution class defined via the relation $e = Bv$ for some fixed constant $B > 0$. This leads to the relation $w = e - \frac{1}{2}v^2 = Bv - \frac{1}{2}v^2$, where we now have the restrictions $v \in [0, 2B]$ and $w \in [0, B^2/2]$. With this, the pair $(v, w) = (v, Bv - \frac{1}{2}v^2)$ solves the coupled system (1.1) if and only if v solves the scalar equation

$$\dot{v} = \operatorname{div}\Big(\widetilde{\kappa}_B(v)\nabla v\Big), \quad \widetilde{\kappa}_B(v)\nabla v \cdot \mathrm{n} = 0, \quad \text{where } \widetilde{\kappa}_B(v) := \kappa\Big(Bv - \tfrac{1}{2}v^2\Big). \tag{4.6}$$

For this scalar equation, the general existence theory for the PME (cf. Vázquez 2007) can be applied and a huge set of solutions with compact and growing support for v are known to exist.

For example, we may choose $\eta \equiv \kappa$ in the form

$$\kappa(w) = \eta(w) = \begin{cases} B - \sqrt{B^2 - 2w} & \text{for } w \in [0, 3B^2/8], \\ 2w/B - B/4 & \text{for } w \geq 3B^3/8. \end{cases} \tag{4.7}$$

Then, for $v \in [0, B/2]$ we have $\widetilde{\kappa}_B(v) = \kappa\big(Bv - v^2/2\big) = v$ and arrive at the classical PME $\dot{v} = \operatorname{div}(v\nabla v) = \Delta(v^2/2)$, which has explicit similarity solutions satisfying $v \in [0, B/2]$ for sufficiently large t, see (1.4).

5 General A Priori Estimates

To provide a first existence theory for the coupled system (1.1), we restrict now to the case that $\Omega \subset \mathbb{R}^d$ is a smooth and bounded domain. This will simplify certain compactness arguments. Moreover, throughout this section we will assume that the solutions (v, w) are classical solutions.

To derive general a priori estimate, we consider a smooth function $(v, w) \mapsto \varphi(v, w)$ and find for solutions (v, w) of (1.1) the relation

$$\partial_t \varphi(v, w) = \mathrm{div}\big(\eta(w)\varphi_v(v, w)\nabla v + \kappa(w)\varphi_w(v, w)\nabla w)\big) + R_\varphi(v, w, \nabla v, \nabla w), \tag{5.1a}$$

where the remainder R is given explicitly via

$$\begin{aligned} R_\varphi(v, w, \nabla v, \nabla w) = {}& \eta(w)\,(\varphi_w(v, w) - \varphi_{vv}(v, w))\big|\nabla v\big|^2 \\ & - (\eta(w)+\kappa(w))\,\varphi_{vw}(v, w)\,\nabla v\cdot\nabla w - \kappa(w)\,\varphi_{ww}(v, w)\big|\nabla w\big|^2. \end{aligned} \tag{5.1b}$$

Integrating (5.1a) and using the boundary condition we find along solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi(v, w)\,\mathrm{d}x = \int_\Omega R_\varphi(v, w, \nabla v, \nabla w)\,\mathrm{d}x. \tag{5.2}$$

5.1 Estimates for the L^p Norms

Clearly, choosing $\varphi(v, w) = v$ or $\varphi(v, w) = \frac{1}{2}v^2 + w$ gives $R_\varphi \equiv 0$, which is the conservation of momentum $\mathcal{V}$ and energy $\mathcal{E}$ as discussed in Sect. 2.1.

Moreover, choosing $\varphi(v, w) = \phi(v)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \phi(v(t, x))\,\mathrm{d}x = -\int_\Omega \phi''(v)\eta(w)\big|\nabla v\big|^2\,\mathrm{d}x$$

Hence, for all convex functions ϕ we obtain that $\int_\Omega \phi(v(t, x))\,\mathrm{d}x$ is nondecreasing in t. This implies the decay of all L^p norms, namely

$$\forall\, p \in [1, \infty]\ \forall\, t \geq 0 : \quad \|v(t, \cdot)\|_{\mathrm{L}^p(\cdot)} \leq \|v(0, \cdot)\|_{\mathrm{L}^p(\cdot)}. \tag{5.3}$$

For the $w \geq 0$, we obviously have an a priori bound in $\mathrm{L}^1(\Omega)$ via

$$\int_\Omega w(t)\,\mathrm{d}x \leq \mathcal{E}(v(t), w(t)) = \mathcal{E}(v^0, w^0).$$

However, because of the L^1-right-hand side $\eta(w)|\nabla v|^2$ it is difficult to derive a priori bounds for high L^p norms.

The class $\varphi(v, w) = \Phi(w)$ is also important and leads to the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi(w(t))\,\mathrm{d}x = \int_\Omega \Big\{\eta(w)\Phi'(w)|\nabla v|^2 - \kappa(w)\Phi''(w)|\nabla w|^2\Big\}\,\mathrm{d}x. \tag{5.4}$$

Thus, we have growth of the integral functional if $\Phi' \geq 0$ and $\Phi'' \leq 0$, e.g., for $\Phi(k) = -k^\alpha$ with $\alpha \in [0, 1]$. Such functionals include the physically relevant entropies discussed in Sect. 2.4.

Remark 5.1 (L^p norms for $\eta \equiv \kappa$) For the case $\eta \equiv \kappa$, we are in a special situation, where we can use $\varphi(u,k) = \phi\big(\frac{1}{2}u^2{+}k\big)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \phi\big(\frac{1}{2}u^2{+}k\big)\,\mathrm{d}x = -\int_\Omega \eta(k)\,\phi''\big(\frac{1}{2}u^2{+}k\big)\,\big|u\nabla u{+}\nabla k\big|^2\,\mathrm{d}x. \tag{5.5}$$

Thus, we have decay for all convex ϕ, whereas concave ϕ leads to growth. This estimate is also easily derived from the simple equation $\dot e = \operatorname{div}\big(\eta(w)\nabla e\big)$, which holds for $\eta \equiv \kappa$. Together with (5.3) we find

$$\begin{aligned}\forall\, p\in[1,\infty]\ \forall\, t\geq 0:\quad &\|w(t)\|_{\mathrm{L}^p(\Omega)} \leq \|e(t)\|_{\mathrm{L}^p(\Omega)}\\ &\leq \|e_0\|_{\mathrm{L}^p(\Omega)} \leq \|w^0\|_{\mathrm{L}^p(\Omega)} + \frac{1}{2}\|v^0\|^2_{\mathrm{L}^{2p}(\Omega)}.\end{aligned} \tag{5.6}$$

To see that L^p bounds for w can also be derived for cases without $\eta \equiv \kappa$ we consider now the situation $\kappa(w) = \kappa_0\eta(w)$. For this case, we can write R_φ as a multiple of $\eta(w)$ in the form

$$R_\varphi(v,w,\nabla v,\nabla w) = -\,\eta(w)\begin{pmatrix}\nabla v\\ \nabla w\end{pmatrix}\cdot \mathbb{A}_\varphi^{\kappa_0}\begin{pmatrix}\nabla v\\ \nabla w\end{pmatrix}\ \text{ with}$$
$$\mathbb{A}_\varphi^{\kappa_0} = \begin{pmatrix}\varphi_{vv}-\varphi_w & \frac{1+\kappa_0}{2}\varphi_{vw}\\ \frac{1+\kappa_0}{2}\varphi_{vw} & \kappa_0\varphi_{ww}\end{pmatrix}.$$

Thus, it suffices to show that $\mathbb{A}_\varphi^{\kappa_0}(v,w)$ is positive semidefinite for all arguments to obtain decay estimates for $\int_\Omega \varphi(v,w)\,\mathrm{d}x$.

Lemma 5.2 *(Higher norms in Spec. II) For all integers $m\in\mathbb{N}$ and all $\kappa_0>0$, there exist coefficients $a_j\geq 0$ for $j=1,\ldots,m$ such that $\varphi_m(v,w) = w^m + \sum_{j=1}^m a_j w^{m-j}v^{2j}$ satisfies $\mathbb{A}_{\varphi_m}^{\kappa_0}(v,w)\geq 0$ for all $(v,w)\in\mathbb{R}{\times}[0,\infty[$.*

Proof The case $m=1$ is solved by $\varphi_1(v,w) = w+\frac{1}{2}v^2$ independently of κ_0. For case $m=2$ a direct calculation for $\varphi(v,w) = w^2 + a_1wv^2 + a_2v^4$ yields

$$\mathbb{A}_{\varphi_2}^{\kappa_0} = \begin{pmatrix}(12a_2-a_1)v^2+(2a_1-2)w & (1+\kappa_0)a_1v\\ (1+\kappa_0)a_1v & 2\kappa_0\end{pmatrix},$$

which is positive semidefinite for all (v,w) if and only if $a_1\geq 1$ and $2\kappa_0(12a_2{-}a_1)\geq (1{+}\kappa_0)^2a_1^2$. Clearly, it suffices to choose $a_1=1$ and then $a_2\geq 1/12{+}(1{+}\kappa_0)^2/(24\kappa_0)$ to fulfill all requirements in the case $m=2$.

For general $m\geq 3$ we set $e = w+\frac{1}{2}v^2$ and define $\phi_j^m(v,w) := e^{m-j}v^{2j}$,

$$\varphi_m(v,w) = \sum_{j=0}^m b_j\phi_j^m(v,w),\quad \mathbb{V}^{\kappa_0} := \begin{pmatrix}v^2 & \frac{1+\kappa_0}{2}v\\ \frac{1+\kappa_0}{2}v & \kappa_0\end{pmatrix},\quad \mathbb{E} := \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

With $\kappa_* = (1-\kappa_0)^2/(4\kappa_0)$ we obtain $\mathbb{V}^{\kappa_0} \geq -\kappa_* v^2\mathbb{E}$ in the sense of positive definite matrices. Since $\mathbb{A}^{\kappa_0}_{\phi}$ depends linearly, we first calculate the individual terms for ϕ^m_j:

$$\begin{aligned}\mathbb{A}^{\kappa_0}_{\phi^m_j} &= (m-j)(m-j-1)e^{m-j-2}v^{2j}\mathbb{V}^{\kappa_0} + \big((m-j)4je^{m-j-1}v^{2j} + 2j(2j-1)e^{m-j}v^{2j-2}\big)\mathbb{E}\\ &\geq -\kappa_*(m-j)(m-j-1)e^{m-j-2}v^{2j+2}\mathbb{E} + j(4m-2j-1)e^{m-j-1}v^{2j}\mathbb{E},\end{aligned}$$

where we used $e \geq \frac{1}{2}v^2$ for the last term in the upper line. Summing the estimates over j and using $b_j \geq 0$, we obtain the estimate $\mathbb{A}^{\kappa_0}_{\varphi_m} \geq b_*(v,e)\mathbb{E}$ with

$$\begin{aligned}b_*(v,e) &= \sum_{j=0}^{m} b_j\Big(j(4m-2j-1)e^{m-j-1}v^{2j} - -\kappa_*(m-j)(m-j-1)e^{m-j-2}v^{2j+2}\Big)\\ &= \sum_{i=1}^{m}\big(i(4m-2i-1)b_i - \kappa_*(m-i+1)(m-i)b_{i-1}\big)e^{m-i-1}v^{2i}.\end{aligned}$$

Clearly, $\mathbb{A}^{\kappa_0}_{\varphi_m}(v,w) \geq 0$ for all (v,w), if $b_*(v,e) \geq 0$ for all (v,e), and this is easily reach by starting with $b_0 = 1$ and then choosing iteratively b_i via $i(4m-2i-1)b_i = \kappa_*(m-i+1)(m-i)b_{i-1}$ for $i = 1,\dots,m$. □

The following result follows easily by applying the previous lemma with (5.2) and the fact that φ_m can be estimated from above and below by $v^{2m} + w^m$.

Proposition 5.3 *(L^p bounds for (v,w)) For the case $\kappa \equiv \kappa_0\eta$ and $p = m \in \mathbb{N}$, there exists a constant $C(\kappa_0,p)$ such that all smooth solutions (v,w) of (1.1) satisfy*

$$\|v(t)\|^2_{\mathrm{L}^{2p}(\Omega)} + \|w(t)\|_{\mathrm{L}^p(\Omega)} \leq C(\kappa_0,p)\big(\|v^0\|^2_{\mathrm{L}^{2p}(\Omega)} + \|w^0\|_{\mathrm{L}^p(\Omega)}\big)\ \textit{for all}\ t \in [0,T].$$

5.2 Estimates Based on Comparison Principles

As our system is given in terms of two scalar diffusion equations, we can apply comparison principles when taking care of the interaction between the two equations.
(C1) We first observe that $w^0(x) \geq 0$ immediately implies $w(t,x) \geq 0$ for all $t \geq 0$ and $x \in \Omega$. Of course this similarly holds for v, but we do not need a sign condition for u.
(C2) Moreover, if (v,w) is a smooth solution of our system and $\underline{w}$ solves the scalar PME

$$\dot{\underline{w}} = \mathrm{div}\,\big(\kappa(\underline{w})\nabla\underline{w}\big), \quad \underline{w}(0,\cdot) = w^0,$$

then we have $w(t,x) \geq \underline{w}(t,x)$ for all $t \geq 0$ and $x \in \Omega$, see Vázquez (2007) for a proof. In particular, if $w^0 \geq c > 0$ then $w(t,x) \geq c$ for all (t,x).
(C3) If $v_* \leq v^0(x) \leq v^*$ for all $x \in \Omega$, then $v(t,x) \in [v_*, v^*]$ for all $t \geq 0$ and $x \in \Omega$.

The next result is more advanced and truly uses the interaction of the two equations. However, it is restricted to the case $\kappa \equiv \eta$. Under this assumption, Eq. (5.1) gives

$$\partial_t \varphi = \operatorname{div}(\eta \nabla \varphi) - \eta \widehat{R} \quad \text{with } \widehat{R} = (\varphi_{vv} - \varphi_w)|\nabla v|^2 + 2\varphi_{vw} \nabla v \cdot \nabla w + \varphi_{ww} |\nabla w|^2.$$

We are interested in the case $\widehat{R} \geq 0$, for which initial conditions with $\varphi(v^0, w^0) \leq 0$ immediately implies $\varphi(v, w) \leq 0$ for all $t \in [0, T]$ and $x \in \Omega$.

Proposition 5.4 *(Comparison for $\eta \equiv \kappa$) Consider $a, b, c \in \mathbb{R}$ with $c \leq 2b$. If (v, w) is a classical solution of our coupled system* (1.1), *then we have*

$$a v^0 + b (v^0)^2 + c w^0 \leq 0 \text{ on } \Omega \quad \Longrightarrow \quad a v + b v^2 + c w \leq 0 \text{ on } [0, T] \times \Omega.$$

In particular, $|v^0| \leq M_* w^0$ on Ω implies $|v(t, x)| \leq M_* w(t, x)$ for $(t, x) \in [0, T] \times \Omega$.

Proof We simply observe that the function $\varphi(v, w) = av + bv^2 + cw$ gives $\widehat{R}(v, w) = (2b - c)|\nabla v|^2$ which is nonnegative because of $c \leq 2b$. Then, the maximum principle applied to $\zeta(t, x) = \varphi(v(t, x), w(t, x))$ shows that $\zeta_0 \leq 0$ implies $\zeta(t, x) \leq 0$, which is the desired result.

The last assertion follows by choosing $(a, b, c) = (1, 0, -M_*)$ for finding $v \leq M_* w$ and by choosing $(a, b, c) = (-1, 0, -M_*)$ for finding $-v \leq M_* w$. □

5.3 Dissipation Estimates

Here, we provide space-time estimates for various quantities, which will allow us to derive suitable compactness for approximating sequences.

We start with the estimate obtained in Sect. 5.1 for the choice $\varphi(v, w) = \frac{1}{p}|v|^p$ with $p > 1$. Integration in time gives the estimate

$$\iint_{Q_T} \eta(w)(p-1)|v|^{p-2} \big|\nabla v\big|^2 \,\mathrm{d}x \leq \int_\Omega \frac{1}{p} |v^0|^p \,\mathrm{d}x. \tag{5.7}$$

The special case $p = 2$ shows that it is sufficient to use the initial energy, namely

$$\iint_{Q_T} \eta(w) \big|\nabla v\big|^2 \,\mathrm{d}x \leq \int_\Omega \frac{1}{2} |v^0|^p \,\mathrm{d}x \leq \mathcal{E}(v^0, w^0). \tag{5.8}$$

This estimate shows that the right-hand side in the w-equation in (1.1) is always in $\mathrm{L}^1(Q_T)$, but it will be difficult to obtain higher integrability.

Choosing $\varphi(v, w) = \Phi(w)$ leads to the dissipation relation

$$\iint_{Q_T} \big(\kappa(w)\Phi''(w)|\nabla w|^2 - \eta(w)\Phi'(w)|\nabla v|^2\big) \,\mathrm{d}x \,\mathrm{d}t = \int_\Omega \big\{\Phi(w^0) - \Phi(w(T))\big\} \,\mathrm{d}x, \tag{5.9}$$

which is obtained by integrating (5.4). Here, we additionally have to impose $w(t, x) > 0$ in the case that $\Phi''(w) \to \infty$ for $w \to 0^+$.

The last relation can be used in two different ways to obtain a bound (i) on ∇w and (ii) on ∇v.

(i) Estimates for ∇w: we observe that we may use (5.8) if Φ satisfies

$$0 = \Phi(0) \le \Phi(w) \le C_\Phi w, \quad 0 \le \Phi'(w) \le C_\Phi, \quad \Phi''(k) \ge 0 \tag{5.10}$$

for some $C_\Phi > 0$. Then, we find

$$\iint_{Q_T} \kappa(w)\Phi''(w)|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \le C_\Phi \iint_{Q_T} \eta(w)|\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t + C_\Phi \int_\Omega w^0 \,\mathrm{d}x \le 2C_\Phi \mathcal{E}(v^0, w^0).$$

In particular, one may want to make $\Phi''(w)$ big for $w \approx 0$ such that $\kappa(w)\Phi''(w)$ is large there. Of course, we need integrability of Φ'' on $]0, \infty[$ to allow for $0 \le \Phi'(w) \le C_*$. Hence, generalizing (Boccardo and Gallouët 1989; Boccardo et al. 1997; Mielke and Naumann 2022) we will use the family

$$\Phi(w) = \frac{1}{w^\delta(1+w)} \quad \text{for } w > 0, \quad \text{where } \delta \in]0, 1[. \tag{5.11}$$

The case $0 < \delta \ll 1$ provides good results for large w, while $0 < 1-\delta \ll 1$ produces good results for small w: Thus, for all $\delta \in]0, 1[$ there exists $C(\delta) > 0$ such that for all classical solutions (v, w) with $w(t, x) > 0$ on $[0, T]\times\overline{\Omega}$ we obtain the estimate

$$\forall\, \delta \in]0, 1[\; \exists\, C_\delta > 0 : \quad \iint_{Q_T} \frac{\kappa(w)}{w^\delta(1+w)} \big|\nabla w\big|^2 \,\mathrm{d}x\,\mathrm{d}t \le C_\delta\, \mathcal{E}(v^0, w^0). \tag{5.12}$$

Alternatively, we may also consider a function $\varphi(v, w) = \Psi(w)$ such that

$$0 \ge \Psi(w) \ge -C_\Psi(1+w), \quad \Psi'(w) \le 0, \quad \Psi''(w) > 0 \quad \text{for all } w > 0, \tag{5.13}$$

where $\Psi(w) = -w^\gamma$ with $\gamma \in]0, 1[$ is a typical candidate. We can then drop the term $-\eta(w)\Psi'(w)|\nabla v|^2 \ge 0$ and take advantage of the fact that $\Psi''(w)$ can be much bigger than $\Phi'(w)$ for Φ satisfying (5.10). The estimates for Ψ and the identity (5.9) show that positive classical solutions satisfy

$$\begin{aligned} \iint_{Q_T} \Big(\Psi''(w)\kappa(w)\big|\nabla w\big|^2 - \Psi'(w)\eta(w)\big|\nabla v\big|^2\Big) \,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega \big\{\Psi(w^0) - \Psi(w(T))\big\} \,\mathrm{d}x \\ \le \int_\Omega C_\Psi(1 + w(T)) \,\mathrm{d}x \le C_\Psi\big(|\Omega| + \mathcal{E}(v^0, w^0)\big), \end{aligned} \tag{5.14}$$

where we used that $\Omega \subset \mathbb{R}^d$ has a finite Lebesgue volume.

(ii) Estimates for ∇v: The best estimate for ∇v can be obtained if the function $w \mapsto 1/\eta(w)$ is integrable near $w = 0$, which is certainly the case for $\eta(w) = w^\alpha$ with

$\alpha \in]0,1[$. In this case, we define the negative function Ψ_η via

$$\Psi_\eta(w) := -\int_0^w \frac{1}{\eta(\varkappa)}\,\mathrm{d}\varkappa.$$

The choice leads to the relation $\Psi_\eta'(w) = -1/\eta(w)$ and $\Psi_\eta''(w) = \eta'(w)/\eta(w)^2 \geq 0$, such that the conditions (5.13) hold and (5.14) leads to the special case

$$\iint_{Q_T} \Big(\big|\nabla v\big|^2 + \frac{\eta'(w)\kappa(w)}{\eta(w)^2}\big|\nabla w\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \leq C_{\Psi_\eta}\big(|\Omega| + \mathcal{E}(v^0, w^0)\big), \tag{5.15}$$

see Naumann (2013, Eqn. (4.6)) for an earlier occurrence for the case $\alpha = 1/2$.

Another way of estimating ∇u can be derived from (5.8) if we additionally have a pointwise estimate of the form $|v(t,x)| \leq M_* w(t,x)$, which was derived in Proposition 5.4 for the special case $\kappa \equiv \eta$. Using the monotonicity $\eta'(w) \geq 0$ we have $\eta(|v|/M_*) \leq \eta(w)$ and conclude

$$\iint_{Q_T} \eta\Big(\frac{1}{M_*}|v|\Big)|\nabla v|^2\,\mathrm{d}x\,\mathrm{d}t \leq \mathcal{E}(v^0, w^0). \tag{5.16}$$

We summarize the dissipation estimate by restricting to the homogeneous case $\eta(w) = \eta_0 w^\alpha$ and $\kappa(w) = \kappa_0 w^\beta$. More general cases can easily be deduced in the same fashion. We explicitly show the dependence on $|\Omega|$ such that unbounded domains like $\Omega = \mathbb{R}^d$ can be treated as well in cases where no dependence on $|\Omega|$ is indicated.

Proposition 5.5 (Dissipation estimates) *Consider the case $\eta(w) = \eta_0 w^\alpha$ and $\kappa(w) = \kappa_0 w^\beta$ with $\alpha, \beta, \eta_0, \kappa_0 > 0$ and bounded $\Omega \subset \mathbb{R}^d$. Then for all $\delta \in]0,1[$ there exists a constant $C_* > 0$ such that smooth solutions (v, w) of (1.1) with $w(t,x) \geq \underline{w} > 0$ satisfy*

$$\beta > 0: \quad \iint_{Q_T} \frac{1}{1{+}w}\Big|\nabla\big(w^{1+(\beta-\delta)/2}\big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t \leq C_*\mathcal{E}(v^0, w^0), \tag{5.17a}$$

$$\beta > 0: \quad \iint_{Q_T} \Big|\nabla\big(w^{(\beta+\delta)/2}\big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t \leq C_*\big(|\Omega| + \mathcal{E}(v^0, w^0)\big), \tag{5.17b}$$

$$\alpha \in]0,1[: \quad \iint_{Q_T} \big|\nabla v\big|^2\,\mathrm{d}x\,\mathrm{d}t \leq C_*\big(|\Omega| + \mathcal{E}(v^0, w^0)\big). \tag{5.17c}$$

If we additionally have $\kappa \equiv \eta$ (i.e., $\alpha = \beta$ and $\kappa_0 = \eta_0$) and $|v^0(x)| \leq M_* w^0(x)$, then for C_* now depending on δ and M_* only, we further have

$$\alpha = \beta: \quad \iint_{Q_T} \Big|\nabla\big(v^{1+\alpha/2}\big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t \leq C_*\mathcal{E}(v^0, w^0), \tag{5.17d}$$

$$\alpha = \beta \geq 1{-}\delta: \quad \iint_{Q_T} \Big|\nabla\big(v^{(1+\alpha+\delta)/2}\big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t \leq C_*\big(|\Omega| + \mathcal{E}(v^0, w^0)\big). \tag{5.17e}$$

Throughout, the constant C_* is independent of the lower bound $\overline{w}$.

Proof The results are obtained by applying the above estimate for suitable functions Φ and Ψ. Throughout use $w^\gamma|\nabla w|^2 = c_\gamma\big|\nabla(w^{1+\gamma/2})\big|^2$ for the smooth solutions.

Estimate (5.17a) follows directly from (5.12).

For (5.17b), we exploit (5.14) with $\Psi(w) = -w^\delta$.

Estimate (5.17c) is a consequence of (5.15), because $1/w^\alpha$ is integrable for $\alpha < 1$.

For (5.17d), we simply use (5.16).

Finally, (5.17e) follows by combining (5.14) with $\Psi(w) = -w^\delta$, the estimate $|v| \leq M_* w$ from Proposition 5.4, and the monotonicity of $w \mapsto |\Psi'(w)|\eta(w) = cw^{\alpha-1+\delta}$. □

Based on the conjectures concerning the typical front behavior, which were discussed in Sect. 3.3, we see that $\nabla v \in \mathrm{L}^2(Q_T)$ can be expected only in the case $\alpha < 1$, i.e., the restriction $\alpha \in\,]0, 1[$ in (5.17c) seems to be sharp. Similarly, Sect. 3.3 show that $\nabla w^\gamma \in \mathrm{L}^2(Q_T)$ implies $\gamma > \beta/2$, which corresponds to the restriction $\delta > 0$ in (5.17b).

Remark 5.6 ($\int_\Omega \log w^0\,\mathrm{d}x$ finite) In their treatment of Kolmogorov's two-equation model in Bulíček and Málek (2019), the authors consider the situation corresponding to $\eta(w) = \eta_0 w$ and $\kappa(w) = \kappa_0 w$, i.e., $\alpha = \beta = 1$. Moreover, they assume that w^0 is positive almost everywhere such that $L_0 := -\int_\Omega \log w^0\,\mathrm{d}x < \infty$. Choosing the convex functional $\Psi(w) = -\log w$ the dissipation relation (5.9) leads to

$$\iint_{Q_T}\Big(\frac{\kappa_0}{w}|\nabla w|^2 + \eta_0|\nabla v|^2\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega\big(\log w(T) - \log w^0\big)\,\mathrm{d}x$$
$$\leq \int_\Omega w(T)\,\mathrm{d}x + L_0 \leq \mathcal{E}(v^0, w^0) + L_0.$$

Of course, the assumption $L_0 < \infty$ does not allow to study solutions with nontrivial support, i.e., it implies $\mathrm{sppt}(w(t)) = \overline{\Omega}$ for all $t \geq 0$.

6 Existence of Solutions

Through this section, we use the following standard assumptions:

$$\Omega \subset \mathbb{R}^d \text{ is bounded with } \mathrm{C}^2 \text{ boundary}, \tag{6.1a}$$
$$\exists\, \alpha, \beta, \eta_0, \kappa_0 > 0\ \forall\, w \geq 0: \quad \eta(w) = \eta_0 w^\alpha \text{ and } \kappa(w) = \kappa_0 w^\beta. \tag{6.1b}$$

Below, we will derive three different results on the global existence of weak or very weak solutions. A local existence theory for smooth solutions is established in Fanelli and Granero-Belinchón (2021) in the one-dimensional periodic domain $\Omega = \mathbb{T} = \mathbb{S}^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$. Moreover, there it is also shown that smooth solutions may blow up in finite time. All these results are valid for the exponents α, β satisfying $\alpha, \beta \geq 1$ and maybe further conditions, whereas our results are restricted to the case $\alpha \in\,]0, 1[$ or $\alpha = \beta > 0$. Of course, global existence of weak solutions and blow-up of smooth solutions may occur in one equation.

6.1 Three Prototypical Existence Results

For given initial conditions $(v^0, w^0) \in \mathrm{L}^2(\Omega)\times\mathrm{L}^1_{\geq}(\Omega)$, we choose a sequence of smooth initial data $(v^0_\varepsilon, w^0_\varepsilon) \in \mathrm{C}^\infty(\overline{\Omega})^2$ such that

$$w^0_\varepsilon(x) \geq \varepsilon > 0 \text{ on } \Omega, \tag{6.2a}$$

$$(v^0_\varepsilon, w^0_\varepsilon) \to (v^0, w^0) \in \mathrm{L}^2(\Omega)\times\mathrm{L}^1(\Omega), \tag{6.2b}$$

$$v^0 \in \mathrm{L}^{p_0}(\Omega) \quad \text{and} \quad v^0_\varepsilon \to v^0 \text{ in } \mathrm{L}^{p_0}(\Omega). \tag{6.2c}$$

Condition (6.2b) implies the convergence of the conservation laws $\mathcal{V}(v^0_\varepsilon) \to \mathcal{V}(v^0)$ and $\mathcal{E}(v^0_\varepsilon, w^0_\varepsilon) \to \mathcal{E}(v^0, w^0)$. Condition (6.2c) with $p_0 \in [1, 2]$ follows from (6.2b), hence, it will only be an extra condition for $p_0 > 2$.

With this, we have a classical parabolic system with positive viscosity such that local existence of solutions $(v_\varepsilon, w_\varepsilon)$ is classical, see, e.g., Amann (1989) and Wiegner (1992) or Lunardi (1995, Cha. 8). Of course all these solutions satisfy our a priori bounds, in particular, we have $w_\varepsilon(t, x) \geq \varepsilon$ as long as the solution exists, cf. Sect. 5.2 (C2). This implies that the solution stays as smooth. Moreover, the structure of the equation for v implies the global L^∞ bound $\|v_\varepsilon(t)\|_{\mathrm{L}^\infty(\cdot)} \leq \|v^0_\varepsilon\|_{\mathrm{L}^\infty(\cdot)}$, see (5.3). Finally, the energy conservation $\mathcal{E}(v_\varepsilon(t), w_\varepsilon(t)) = \mathcal{E}(v^0_\varepsilon, w^0_\varepsilon)$ which also implies $\iint_{Q_T} \eta(w_\varepsilon)|\nabla v_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \mathcal{E}(v^0_\varepsilon, w^0_\varepsilon)$. Thus, blow-up is not possible and the classical solutions exist for all time.

The aim is now to show that the solutions $(v_\varepsilon, w_\varepsilon)$, or better a suitable subsequence thereof, converge to a limit (v, w) that is a weak or very weak solution of our coupled system. The problem here is that the limit w may have a nontrivial support strictly contained in Ω. As a consequence, an integral bound

$$\iint_{Q_T} \big(w^\gamma_\varepsilon |\nabla w^\alpha_\varepsilon|^2 + w^\delta_\varepsilon |\nabla v_\varepsilon|^2\big) \,\mathrm{d}x\,\mathrm{d}t \leq C_* \tag{6.3}$$

does not necessarily imply spatial compactness for v_ε. Thus, we will provide two different existence results, in the case $\eta(w) = \eta_0 w^\alpha$ with $\alpha < 1$ one may obtain the exponent $\delta = 0$ and a bound for ∇v_ε in $\mathrm{L}^2(Q_T)$ follows. In the case, $\alpha \geq 1$, we can only treat the case $\eta \equiv \kappa$ under the additional assumption $|v^0(x)| \leq M_* w^0(x)$, which allows us to obtain a bound on $\nabla(v_\varepsilon)^{1+\delta/2}$ in $\mathrm{L}^2(Q_T)$.

Another way of obtaining weak solutions with a positive lower bound $\varepsilon > 0$ would be to adapt the methods in Naumann (2013), Bulíček and Málek (2019) and Mielke and Naumann (2022) to the present case. Our approach is different is similar to the one taken in Bertsch and Kamin (1990) and Dal Passo and Giacomelli (1999); however, our goal is more restrictive. There, for the solutions (ρ, θ) of the plasma model (1.6) one only asks for $\rho^{\delta/2}\nabla\theta \in \mathrm{L}^2(\mathcal{P})$ where $\mathcal{P} := \big\{ (t, x) \in [0, T]\times\Omega \,\big|\, \rho(t, x) > 0 \big\}$.

Throughout the rest of this section, we use the following short-hand notations:

(a) $u_\varepsilon \overset{\text{bdd}}{\in} X \iff \exists\, C > 0\, \forall\, \varepsilon \in\,]0, 1[\, : \quad u_\varepsilon \in X \text{ and } \|u_\varepsilon\|_X \leq C,$

(b) $\mathrm{L}^p\mathrm{L}^q := \mathrm{L}^p\big([0, T]; \mathrm{L}^q(\Omega)\big)$ and similarly $\mathrm{L}^q\mathrm{H}^1$ or $\mathrm{L}^p\mathrm{W}^{s,q}$.

We will use the following standard interpolation in $\mathrm{L}^s\mathrm{L}^p$ (cf. Mielke and Naumann 2022, Lem. 4.2):

$$\begin{aligned}&\forall\,\theta\in[0,1],\ s,s_1,s_2,p,p_1,p_2\in[1,\infty] \text{ with } \tfrac{1}{s}=\tfrac{1-\theta}{s_1}+\tfrac{\theta}{s_2} \text{ and } \tfrac{1}{p}=\tfrac{1-\theta}{p_1}+\tfrac{\theta}{p_2}\\ &\exists\, C>0\ \forall\, u\in \mathrm{L}^{s_1}\mathrm{L}^{p_1}\cap \mathrm{L}^{s_2}\mathrm{L}^{p_2}: \quad \|u\|_{\mathrm{L}^s\mathrm{L}^p}\le C\|u\|^{1-\theta}_{\mathrm{L}^{s_1}\mathrm{L}^{p_1}}\|u\|^{\theta}_{\mathrm{L}^{s_2}\mathrm{L}^{p_2}}. \end{aligned} \tag{6.4}$$

Moreover, we define 2^*_d as the optimal exponent in the embedding $\mathrm{H}^1(\Omega)\subset \mathrm{L}^{2^*_d}(\Omega)$, i.e., $2^*_d=2d/(d-2)$ for $d\ge 3$, $2^*_1=\infty$, and $2^*_2<\infty$.

We are now ready to state three different existence results. The first result provides weak solutions and relies on the restriction (6.5) concerning the exponents α and β as well as the integrability power p_0 for v^0. We do not expect that condition (6.5) is sharp.

Theorem 6.1 (Existence of weak solutions for $\alpha<1$) *Assume that* (6.1) *holds with* $\alpha\in\,]0,1[$. *Moreover, consider* $p_0\ge 2$ *such that*

$$\frac{\alpha}{\beta+2-2/2^*_d}+\frac{1}{p_0+2-2p_0/2^*_d}<\frac{1}{2}. \tag{6.5}$$

Then, for all initial data $(v^0,w^0)\in \mathrm{L}^{p_0}(\Omega)\times\mathrm{L}^1_{\ge}(\Omega)$ *there exists a weak solution* (v,w) *in the sense of Definition* 4.3. *Moreover, this solution satisfies*

$$\nabla v\in \mathrm{L}^2(Q_T)\quad \textit{and}\quad w\in \mathrm{L}^{\beta+2-2/2^*_d}(Q_T).$$

Condition (6.5) is somehow restrictive and shows that small α and large β are desirable. Even assuming $v^0\in \mathrm{C}(\overline{\Omega})\subset\mathrm{L}^\infty(\Omega)$, i.e., $p_0=\infty$, we still have to satisfy

$$2\alpha<\beta+2-\max\{0,1{-}2/d\}.$$

Because of $\alpha\in\,]0,1[$, this is always satisfied for $d\le 2$, but may provide a nontrivial lower bound for $d\ge 3$ and $\alpha>1/2$. However, the important case $\alpha=\beta\in\,]0,1[$ is always possible, even for finite $p_0\in[2,\infty[$.

For example, assuming $\alpha=\beta=1/2$ (as in Kolmogorov's one-equation or Prandtl's model discussed in Sect. 8.1) we obtain, after some computation, the restriction

$$p_0\ge 2\quad\text{and}\quad p_0>\frac{2d^2}{d{+}4}.$$

For the next result, we stay in the case $\alpha\in\,]0,1[$ and show that very weak solutions can be obtained in a previously inaccessible regime. However, the usage of the weighted gradients as introduced in Definition 4.1 forces us to control $\nabla w^\alpha_\varepsilon$ in $\mathrm{L}^1(Q_T)$, whereas the equation provides control on $\nabla w^{\beta/2}_\varepsilon$ only. Thus, the restriction $\beta<2\alpha$ will be needed.

Theorem 6.2 (Existence of very weak solutions for $\alpha < 1$) *Assume that* (6.1) *holds with* $0 < \beta < 2\alpha < 2$. *Moreover, consider* $p_0 \geq 2$ *such that*

$$\frac{2\alpha + 1 - 2/2_d^*}{2(\beta+2-2/2_d^*)} + \frac{1}{p_0 + 2 - 2p_0/2_d^*} < 1. \tag{6.6}$$

Then, for all initial data $(v^0, w^0) \in \mathrm{L}^{p_0}(\Omega)\times\mathrm{L}^1_{\geq}(\Omega)$ *there exists a very weak solution* (v, w) *in the sense of Definition* 4.3. *Moreover, this solution satisfies*

$$\nabla v \in \mathrm{L}^2(Q_T) \quad \textit{and} \quad w \in \mathrm{L}^{\beta+2-2/2_d^*}(Q_T).$$

We emphasize that condition (6.6) is weaker than (6.5). Indeed, consider the case $p_0 = \infty$ for simplicity, then (6.6) is equivalent to $2\alpha < 2\beta + 3 - 2/2_d^*$, which is automatically satisfied for $0 \leq \beta < 2\alpha < 2$, because of $2\alpha \leq \alpha + 1 < 2\beta + 1 < 2\beta + 3 - 2/2_d^*$.

However, for $p_0 = \infty$ (6.5) reduces to $2\alpha < \beta + 2 - 2/2_d^*$ which is violated for $d \geq 3$ and suitable α and β, e.g., for $(d, \alpha, \beta) = (4, 7/8, 1/8)$ we have

$$0 < \beta = 1/8 < 2\alpha = 7/4 < 2, \quad \text{and} \quad 2\alpha = 7/4 \not< 13/8 = 2+1/8-1/2 = 2+\beta-2/2_d^*.$$

Thus, there are cases where Theorem 6.2 can be applied but Theorem 6.1 cannot.

Finally, we treat the case $\eta \equiv \kappa$ which is special in two ways: first the energy density $e = \frac{1}{2}v^2 + w$ satisfies the simple equation $\dot{e} = \mathrm{div}\big(\eta(w)\nabla e\big)$, and second we can pointwise bound v by w, see Proposition 5.4. The case $\kappa \equiv \eta = w^\alpha$ is very special, but it is presently the only case where $\alpha \geq 1$ can be handled.

Theorem 6.3 (Existence of very weak solutions for $\eta \equiv \kappa$) *Assume that* (6.1) *holds with* $\eta(w) = \kappa(w) = w^\alpha$ *with* $\alpha > 0$. *Moreover, consider* $p_0 \geq 2$ *such that*

$$p_0\Big(1 - \frac{2}{2_d^*}\Big) > \sqrt{8 + 4\alpha + \alpha^2/4} - \frac{\alpha}{2}. \tag{6.7}$$

Then, for all $M_* > 0$ *and all initial data* $(v^0, w^0) \in \mathrm{L}^{p_0}(\Omega)\times\mathrm{L}^{p_0/2}_{\geq}(\Omega)$ *satisfying*

$$|v^0(x)| \leq M_* w^0(x) \ \textit{for a.a.}\ x \in \Omega \tag{6.8}$$

there exists a very weak solution (v, w) *in the sense of Definition* 4.3. *Moreover, this solution satisfies*

$$|v(t, x)| \leq M_* w(t, x) \ \textit{for a.a.}\ (t, x) \in Q_T.$$

The right-hand side in (6.7) is strictly increasing with range $]2, 4[$. Hence, the assumption $p_0 \geq \max\{4, 2d\}$ is sufficient for all $\alpha > 0$.

The proofs of these three results are the contents of the following three subsections.

6.2 Limit Passage to Weak Solutions in the Case $\alpha < 1$

Here, we provide the proof of existence of weak solutions first, and then show what has to be changed for obtaining very weak solutions.

Proof of Theorem 6.1 We proceed in three steps: 1. a priori estimates, 2. compactness, and 3. identification of nonlinear limits.

Step 1: A priori estimates.

Because of $\mathcal{E}(v_\varepsilon(t), w_\varepsilon(t)) = E_0$ and (5.17c), we obviously have $v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{H}^1$.

For w_ε, we use (5.17b) to obtain $\nabla w_\varepsilon^{(\beta+\delta)/2} \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{L}^2$ for all $\delta \in]0, 1[$. Since energy conservation implies $w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^\infty\mathrm{L}^1$, we conclude $w_\varepsilon^{(\beta+\delta)/2} \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{H}^1$. For bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ we have the embedding $\mathrm{H}^1(\Omega) \subset \mathrm{L}^{2^*_d}(\Omega)$ with $2^*_1 = \infty$, $2^*_2 < \infty$, and $2^*_d = 2d/(d-2)$ for $d \geq 3$. We obtain $w_\varepsilon^{(\beta+\delta)/2} \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{L}^{2^*_d}$, or equivalently $w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{\beta+\delta}\mathrm{L}^q$ for $q = (\beta+\delta)2^*_d/2$. Using the interpolation (6.4) with $w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^\infty\mathrm{L}^1$ gives

$$w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^p(Q_T) = \mathrm{L}^p\mathrm{L}^p \text{ with } p = p(\delta) := \beta + \delta + 1 - 2/2^*_d. \tag{6.9}$$

For controlling the gradient of $\Pi(w) = \frac{1}{\beta+1} w^{\beta+1}$ in $\mathrm{L}^{1+\eta}(Q_T) = \mathrm{L}^{1+\eta}\mathrm{L}^{1+\eta}$ for some $\eta > 0$, we use the simple identity

$$\nabla(\Pi(w)) = w^\beta \nabla w = \frac{1}{\gamma_2} w^{\gamma_1} \nabla\big(w^{\gamma_2}\big) \quad \text{for } \gamma_1 + \gamma_2 = \beta + 1.$$

Choosing $\gamma_2 = (\beta+\delta)/2$ and using $\nabla w_\varepsilon^{(\beta+\delta)/2} \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{L}^2$ we find $\nabla\Pi(w_\varepsilon) \overset{\text{bdd}}{\in} \mathrm{L}^{1+\eta}(Q_T)$ if we have $w_\varepsilon^{\gamma_1} \overset{\text{bdd}}{\in} \mathrm{L}^q\mathrm{L}^q$ or $w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{\gamma_1 q}\mathrm{L}^{\gamma_1 q}$ with $1/2+1/q < 1$. Hence, it suffices to show $p(\delta) > 2\gamma_1(\delta) =: \beta + 2 - \delta$, which holds for some $\delta \in]0, 1[$ because of the continuity of $\delta \mapsto (\gamma_2(\delta), p(\delta))$ on $[0, 1]$ and $p(1) = \beta + 2 - 2/2^*_d > 2\gamma_2(1) = \beta + 1$. In summary, we have established

$$\exists\, \eta > 0: \qquad \nabla\big(\Pi(w_\varepsilon)\big) \overset{\text{bdd}}{\in} \mathrm{L}^{1+\eta}(Q_T). \tag{6.10}$$

Clearly, the classical dissipation estimate (5.8) for the v-equation gives

$$U_\varepsilon := \eta(w_\varepsilon)^{1/2}\nabla v_\varepsilon = w_\varepsilon^{\alpha/2}\nabla v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^2(Q_T) \tag{6.11}$$

Similarly, using $w_\varepsilon^{\alpha/2} \overset{\text{bdd}}{\in} \mathrm{L}^\infty\mathrm{L}^{2/\alpha} \subset \mathrm{L}^{2/\alpha}(Q_T)$ and $\eta(w_\varepsilon)\nabla v_\varepsilon = w_\varepsilon^{\alpha/2} w_\varepsilon^{\alpha/2}\nabla v_\varepsilon$ yields

$$\eta(w_\varepsilon)\nabla v_\varepsilon = w_\varepsilon^{\alpha}\nabla v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{2/(1+\alpha)}(Q_T). \tag{6.12}$$

Step 2: Compactness.

As usual we apply an Aubin–Lions–Simon theorem, see Simon (1987, Cor. 4, Lions 1969, Thm. 5.1, Roubíček 2013, Lem. 7.7) and for nonlinear versions see Moussa (2016, Thm. 1).

For this, we first create *spatial compactness*, which is trivial for v_ε because of $v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{H}^1$ with $\mathrm{H}^1 = \mathrm{H}^1(\Omega)$ compactly embedded into $\mathrm{L}^2(\Omega)$.

For w_ε, we have the difficulty that we only control the derivative of power of w in (6.10), namely $w_\varepsilon^{1+\beta} \overset{\text{bdd}}{\in} \mathrm{L}^{1+\eta}\mathrm{W}^{1,1+\eta}$. However, employing the fact that the Nemitskii operator $u \mapsto u^\theta$ maps $\mathrm{W}^{s,p}(\Omega)$ into $\mathrm{W}^{\theta s,p/\theta}(\Omega)$ for all $s \in]0,1[$ and $p \in]1,\infty[$, see Proposition 6.4. This leads to

$$w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{(1+\eta)(1+\beta)}\mathrm{W}^{\sigma,(1+\eta)(1+\beta)} \quad \text{for all } \sigma \in \Big[0, \tfrac{1}{1+\beta}\Big[.$$

Clearly, $\mathrm{W}^{\sigma,(1+\eta)(1+\beta)}(\Omega)$ is still compactly embedded into $\mathrm{L}^{1+\eta}(\Omega)$.

To derive temporal compactness, we use the PDEs for $\dot v_\varepsilon$ and $\dot w_\varepsilon$, see (1.1). From (1.1a) and (6.12) we obtain

$$\dot v_\varepsilon \in \mathrm{L}^{2/(1+\alpha)}\mathrm{W}_0^{-1,2/(1+\alpha)} = \mathrm{L}^{2/(1+\alpha)}\big(\mathrm{W}^{1,2/(1-\alpha)}(\Omega)\big)^*. \tag{6.13}$$

Starting from $\nabla\Pi(w_\varepsilon) \overset{\text{bdd}}{\in} \mathrm{L}^{1+\eta}(Q_T)$ and $\eta(w_\varepsilon)|\nabla v_\varepsilon|^2 \overset{\text{bdd}}{\in} \mathrm{L}^1(Q_T)$ the PDE (1.1b) for w_ε gives

$$\dot w_\varepsilon \in \mathrm{L}^{1+\eta}\mathrm{W}_0^{-1,1+\eta} + \mathrm{L}^1\mathrm{L}^1 \subset \mathrm{L}^1\big(\mathrm{W}^{1,1+1/\eta}(\Omega)\big)^*. \tag{6.14}$$

Now applying Banach's selection principle for weak convergence and Aubin-Lions-Simon theorem strong convergence we obtain a limit pair (v, w) with

$$v \in \mathrm{C_w}([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2\mathrm{H}^1 \quad \text{and} \quad w \in \mathrm{L}^\infty\mathrm{L}^1 \cap \mathrm{L}^{(1+\eta)(1+\beta)}\mathrm{W}^{\sigma,(1+\eta)(1+\beta)}$$

and, along a suitable subsequence, the convergences

$$(v_\varepsilon(x), w_\varepsilon(x)) \to (v(x), w(x)) \qquad \text{a.e. in } \Omega; \tag{6.15a}$$

$$(v_\varepsilon, w_\varepsilon) \to (v, w) \qquad \text{in } \mathrm{L}^2(Q_T)\times\mathrm{L}^{1+\eta}(Q_T); \tag{6.15b}$$

$$v_\varepsilon \rightharpoonup v \qquad \text{in } \mathrm{L}^2\mathrm{H}^1, \tag{6.15c}$$

$$w_\varepsilon \rightharpoonup w \qquad \text{in } \mathrm{L}^{1+\beta}\mathrm{W}^{1/(2+\beta),1+\beta}. \tag{6.15d}$$

Step 3: Identification of nonlinear limits.
It remains to pass to the limit in the nonlinear terms of the weak formulation, namely in

$$\text{(i) } \eta(w_\varepsilon)\nabla v_\varepsilon, \qquad \text{(ii) } \nabla\Pi(w_\varepsilon), \qquad \text{(iii) } \eta(w_\varepsilon)v_\varepsilon\nabla v_\varepsilon.$$

In the trivial nonlinear term $\iint_{Q_T} v_\varepsilon^2 \partial_t \xi \,\mathrm{d}x\,\mathrm{d}t$ in the left-hand side of (4.3c), the limit passage follows easily from the strong convergence of v_ε in $\mathrm{L}^2(Q_T)$.

Concerning the term (i) we recall the decomposition

$$w_\varepsilon^\alpha \nabla v_\varepsilon = w_\varepsilon^{\alpha/2} U_\varepsilon \quad \text{with } U_\varepsilon = w_\varepsilon^{\alpha/2}\nabla v_\varepsilon.$$

By (6.15b) we have $w_\varepsilon^{\alpha/2} \to w^{\alpha/2}$ in $\mathrm{L}^{2/\alpha}$. On the one hand, we have $U_\varepsilon \rightharpoonup U$ in $\mathrm{L}^2(Q_T)$ because of (6.11). On the other hand, the strong convergence of $w_\varepsilon^{\alpha/2}$ and $\nabla v_\varepsilon \rightharpoonup \nabla v$ in $\mathrm{L}^2(Q_T)$ implies $U_\varepsilon \rightharpoonup w^{\alpha/2}\nabla v$ in $\mathrm{L}^{2/(1+\alpha)}(Q_T)$. Thus, $U = w^{\alpha/2}\nabla v$ and, with the same argument we find

$$w_\varepsilon^\alpha \nabla v_\varepsilon = w_\varepsilon^{\alpha/2} U_\varepsilon \rightharpoonup w^{\alpha/2} U = w^\alpha \nabla v \quad \text{in } \mathrm{L}^{2/(1+\alpha)}. \tag{6.16}$$

For term (ii), we use $\Pi(w) = \frac{1}{1+\beta} w^{1+\beta}$ and the a priori estimates from Step 2, such that along a further subsequence (not relabeled) we have, for all $q \in [1, 1+\beta 2_d^*/2[$,

$$w_\varepsilon \rightharpoonup w \text{ in } \mathrm{L}^q(Q_T), \quad w_\varepsilon^{1+\beta} \rightharpoonup g \text{ in } \mathrm{L}^{1+\eta}(Q_T), \quad \nabla w_\varepsilon^{1+\beta} \rightharpoonup G \text{ in } \mathrm{L}^{1+\eta}(Q_T).$$

Together with the strong convergence (6.15b), we obtain $w_\varepsilon \to w$ in $\mathrm{L}^q(\mathbb{Q}_T)$ for the same q such that $w_\varepsilon^{1+\beta} \to w^{1+\beta}$ in $\mathrm{L}^1(Q_T)$, which implies $g = w^{1+\beta}$ as desired, and $G = \nabla g = \nabla \Pi(w)$ follows.

For the term (iii), we may need the extra condition $v^0 \in \mathrm{L}^{p_0}(\Omega)$, which implies $v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^\infty \mathrm{L}^{p_0}$. Interpolation with $v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{H}^1 \subset \mathrm{L}^2\mathrm{L}^{2_d^*}$ via (6.4) we obtain

$$v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{q_0}(Q_T) \text{ with } q_0 := p_0 + 2 - 2p_0/2_d^*, \quad \text{and}$$
$$v_\varepsilon \to v \text{ in } \mathrm{L}^q(Q_T) \text{ for } q \in [1, q_0[,$$

where the second statement uses (6.15b). Similarly, (6.9) and (6.15b) yield

$$w_\varepsilon^\alpha \to w^\alpha \text{ in } \mathrm{L}^{p/\alpha}(Q_T) \text{ for } p \in [1, \beta{+}2{-}2/2_d^*[.$$

Combining these two strong convergences and exploiting the condition (6.5), we conclude $\eta(w_\varepsilon)v_\varepsilon \to \eta(w)v$ in $\mathrm{L}^2(Q_T)$. Together with the weak convergence $\nabla v_\varepsilon \to \nabla v$ (cf. (6.15c)), we obtain the desired weak convergence for term (iii), i.e., $\eta(w_\varepsilon)v_\varepsilon\nabla v_\varepsilon \to \eta(w)v\nabla v$ in $\mathrm{L}^1(Q_T)$. Thus, the limit passage for all terms on the weak formulation (4.3), which shows that (v, w) is indeed a weak solution.

This finishes the proof of Theorem 6.1. □

6.3 Limit Passage to Very Weak Solutions for $\alpha < 1$

Proof of Theorem 6.2 We follow the same three steps as in the proof of Theorem 6.1. Step 1 and Step 2 are in fact identical, and in Step 3 we have to exploit the idea of weighted gradients for $\eta(w)\nabla v$, which means that we have to prove weak convergence in $\mathrm{L}^1(Q_T)$ for the following five terms:

(i)' $v_\varepsilon \nabla \eta(w_\varepsilon)$, (ii) $\nabla \Pi(w_\varepsilon)$, (iii)' $v_\varepsilon^2 \nabla \eta(w_\varepsilon)$, (i)" $v_\varepsilon \eta(w_\varepsilon)$, (iii)" $v_\varepsilon^2 \eta(w_\varepsilon)$.

Clearly, (ii) works as before and (i)' and (iii)' look as (i) and (iii) above, but the gradient is moved from v_ε to $\eta(w_\varepsilon)$, while in (i)" and (iii)" there are no gradients at all.

It is easy to see that the most critical term (iii)'. If the convergences work for this case, then they also work for all the other ones.

The problem is now that we need to control $\nabla w_\varepsilon^\alpha$ which is nontrivial if α is small, because (5.17a) and (5.17b) provide bounds for $\nabla w_\varepsilon^\gamma$ for γ defined in terms of β. We hence assume

$$\beta < 2\alpha < 2.$$

Using $\nabla w^\alpha = \frac{\alpha}{\gamma} w^{\alpha-\gamma}\nabla w^\gamma$ for $\gamma \in]0,\alpha[$ and choosing $\gamma = (\beta{+}\delta)/2$ for some $\delta \in]0,1[$, we obtain from (5.17b) and (6.9) the boundedness

$$\begin{aligned}
&\alpha \in \big]\tfrac{\beta}{2}, \tfrac{\beta+1}{2}\big[: \ \ \nabla w_\varepsilon^\alpha \overset{\text{bdd}}{\in} \mathrm{L}^2(Q_T)\\
&\alpha \in \big[\tfrac{\beta+1}{2}, 1\big[: \ \ \nabla w_\varepsilon^\alpha \overset{\text{bdd}}{\in} \mathrm{L}^r(Q_T) \ \text{ for all } r \in [1, r_0[\text{ with } r_0 = \frac{2(\beta{+}2{-}2/2^*_d)}{2\alpha{+}1{-}2/2^*_d}.
\end{aligned}$$

Because of $r_0 > 1$ and the strong convergence (6.15b), we obtain $\nabla w_\varepsilon^\alpha \rightharpoonup \nabla w^\alpha$ in $\mathrm{L}^r(Q_T)$.

As before, we have the strong convergence $v_\varepsilon^2 \to v^2$ in $\mathrm{L}^{q/2}(Q_T)$ for all $q \in [1, q_0[$ and now condition (6.6) is exactly made to allows us to conclude that $v_\varepsilon^2 \nabla\eta(w_\varepsilon) \rightharpoonup v^2\nabla\eta(w)$, as desired. This finishes the proof of Theorem 6.2. □

6.4 Limit Passage in the Case $\eta = \kappa$

For $a \in \mathbb{R}$ and $\gamma > 0$ we set $\{a\}^\gamma := \mathrm{sign}(a)\,|a|^\gamma$.

Proof of Theorem 6.3 We proceed along the same three steps as in Sect. 6.2 but now use that it is easy to construct smooth initial conditions $(v_\varepsilon^0, w_\varepsilon^0)$ such that (6.2) holds together with $|v_\varepsilon^0(x)| \le M_* w_\varepsilon^0(x)$ for $x \in \Omega$. For this one simply mollifies the nonnegative functions $a^\pm := M_* w^0 \pm v^0$ to obtain smooth and nonnegative $a_\varepsilon^\pm$. Then, we set $v_\varepsilon^0 = (a_\varepsilon^+{-}a_\varepsilon^-)/2$ and $w_\varepsilon^0 = (a_\varepsilon^+{+}a_\varepsilon^-)/(2M_*)$. With Proposition 5.4 we obtain

$$|v_\varepsilon(t,x)| \le M_* w_\varepsilon(t,x) \quad \text{for all } (t,x) \in Q_T. \tag{6.17}$$

Step 1: a priori bounds. We first observe $e^0 := \frac{1}{2}(v^0)^2 + w^0 \in \mathrm{L}^{p_0/2}(\Omega)$, and $\dot e = \mathrm{div}(w^\alpha e)$ implies $\|e(t)\|_{\mathrm{L}^{p_0/2}} \le \|e^0\|_{\mathrm{L}^{p_0/2}}$. Thus, we have $v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^\infty\mathrm{L}^{p_0}$ and $w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^\infty\mathrm{L}^{p_0/2}$, and using (5.7) and (6.17) we obtain $\{v_\varepsilon\}^{(p_0+\alpha)/2} \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{H}^1 \le \mathrm{L}^2\mathrm{L}^{2^*_d}$ and conclude $v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{p_0+\alpha}\mathrm{L}^{(p_0+\alpha)2^*_d/2}$. Using $v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^\infty\mathrm{L}^{p_0}$ and the interpolation (6.4) we arrive at

$$v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{q_\mathrm{v}}(Q_T) \ \text{ with } q_\mathrm{v} := \alpha + 2p_0\big(\tfrac{1}{2} - \tfrac{1}{2^*_d}\big). \tag{6.18}$$

From (5.17b) with $\alpha = \beta$ we find $w^{(\alpha+\delta)/2} \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{H}^1$ for all $\delta \in]0,1[$ and interpolation with $w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^\infty\mathrm{L}^{p_0/2}$ yields

$$w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^q(Q_T) \text{ for all } q \in [1, q_\mathrm{w}[\text{ with } q_\mathrm{w} := 1 + \alpha + p_0\big(\tfrac{1}{2} - \tfrac{1}{2^*_d}\big). \tag{6.19}$$

Step 2: compactness. From $\{v_\varepsilon\}^{(p_0+\alpha)/2} \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{H}^1$ and $w^{\alpha+\delta} \overset{\text{bdd}}{\in} \mathrm{L}^2\mathrm{H}^1$ we again obtain

$$v_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{p_0+\alpha}\mathrm{W}^{2s/(p_0+\alpha),p_0+\alpha} \text{ and } w_\varepsilon \overset{\text{bdd}}{\in} \mathrm{L}^{2(\alpha+\delta)}\mathrm{W}^{s/(\alpha+\delta),2(\alpha+\delta)}$$
$$\text{for } s \in]0,1[\text{ and } \delta \in]1/2,1[,$$

where we used Proposition 6.4. Thus, compactness works as in Step 2 of Sect. 6.2. In particular, we have

$$v_\varepsilon \to v \text{ in } \mathrm{L}^p(Q_T) \text{ for } p \in [1,q_\mathrm{v}[\quad \text{and} \quad w_\varepsilon \to w \text{ in } \mathrm{L}^q(Q_T) \text{ for } q \in [1,q_\mathrm{w}[. \tag{6.20}$$

Using the pointwise convergence a.e. in Q_T, the limits (v,w) still satisfies (6.17).
Step 3: Identification of nonlinear limits. We have to show that the five terms

$$\text{(i)' } v_\varepsilon\nabla\eta(w_\varepsilon), \quad \text{(ii) } \nabla\Pi(w_\varepsilon), \quad \text{(iii)' } v_\varepsilon^2\nabla\eta(w_\varepsilon), \quad \text{(i)'' } v_\varepsilon\eta(w_\varepsilon), \quad \text{(iii)'' } v_\varepsilon^2\eta(w_\varepsilon)$$

converge to their respective limits.

Clearly, $w_\varepsilon^{1+\alpha} \overset{\text{bdd}}{\in} \mathrm{L}^p(Q_T)$ for all $p \in [1, q_\mathrm{w}/(1+\alpha)[$ and by compactness converges strongly to $w^{1+\alpha}$. Moreover, $\nabla w_\varepsilon^{(\alpha+\delta)/2} \overset{\text{bdd}}{\in} \mathrm{L}^2(Q_T)$ and (6.19) imply

$$\nabla w_\varepsilon^{1+\alpha} = w_\varepsilon^{1+(\alpha-\delta)/2}\nabla w_\varepsilon^{(\alpha+\delta)/2} \overset{\text{bdd}}{\in} \mathrm{L}^p(Q_T) \text{ for } 1 \le p < p_\Pi := \frac{2q_\mathrm{w}}{q_\mathrm{w}+1+\alpha} > 1.$$

Here, we need to choose $\delta \approx 1$. With this $\nabla w_\varepsilon^{1+\alpha} \rightharpoonup \nabla w^{1+\alpha}$ in $\mathrm{L}^r(Q_T)$ follows, and the case (ii) is settled.

In exactly the same way with $\eta(w) = w^\alpha$, we obtain

$$\nabla\eta(w_\varepsilon) \rightharpoonup \nabla\eta(w) \text{ in } \mathrm{L}^p(Q_T) \text{ for all } p \in [1,p_\eta[\text{ with } p_\eta := \frac{2q_\mathrm{w}}{q_\mathrm{w}+\alpha-1} > p_\Pi. \tag{6.21}$$

By (6.20), we also have $\eta(w_\varepsilon) \to \eta(w)$ in $\mathrm{L}^p(Q_T)$ for all $p \in [1, q_\mathrm{w}/\alpha[$. Thus, for the four weak convergences of the terms in (i)', (iii)', (i)'', and (iii)'' in $\mathrm{L}^{1+\zeta}(Q_T)$ for some $\zeta > 0$, we need the four relations $1/q_\mathrm{v} + 1/p_\eta < 1$, $2/q_\mathrm{v} + 1/p_\eta < 1$, $1/q_\mathrm{v} + \alpha/q_\mathrm{w} < 1$, and $2/q_\mathrm{v} + \alpha/q_\mathrm{w} < 1$, respectively. Because of $q_\mathrm{w}/\alpha < p_\eta$, all four inequalities follow if the second holds. Inserting the definitions of q_v and p_η into $2/q_\mathrm{v} + 1/p_\eta < 1$ shows that this condition is equivalent to our assumption (6.7).

Thus, the proof of Theorem 6.3 is finished. □

6.5 Fractional Sobolev Spaces

Here, we provide a result that was used in the compactness arguments.

For $\gamma > 0$, we set $\{u\}^\gamma : u \mapsto \mathrm{sign}(u)|u|^\gamma$. Then, we have the relations

$$\forall \gamma \ge 1\ \forall a,b \in \mathbb{R}: \quad \big|\{a\}^{1/\gamma} - \{b\}^{1/\gamma}\big|^\gamma \le 2^{\gamma-1}\,\big|a-b\big|. \tag{6.22}$$

With this, we easily obtain the following statement.

Proposition 6.4 *(Fractional powers) For all* $\gamma > 1$, $s \in]0,1[$, $p \in]1,\infty[$, *and* $u \in \mathrm{W}^{s,p}(\Omega)$, *we have* $\{u\}^{1/\gamma} \in \mathrm{W}^{s/\gamma,p\gamma}(\Omega)$ *with the estimate*

$$\big\|\{u\}^{1/\gamma}\big\|_{\mathrm{W}^{s/,\mathrm{p}}(\Omega)} \leq 2^{(1-1/\gamma)/p}\,\|u\|^{1/\gamma}_{\mathrm{W}^{s,\mathrm{p}}(\Omega)}. \tag{6.23}$$

Proof Clearly, we have $\|\{u\}^{1/\gamma}\|^{p\gamma}_{\mathrm{L^p}} = \|u\|^p_{\mathrm{L^p}}$. For the Sobolev-Slobodeckij semi-norm $[[\,\cdot]]_{s,p}$ we apply (6.22) and obtain

$$[\![\{u\}^{1/\gamma}]\!]^{p\gamma}_{s/\gamma,p\gamma} := \iint_{\Omega\times\Omega} \frac{\big|\{u(x)\}^{1/\gamma}-\{u(y)\}^{1/\gamma}\big|^{p\gamma}}{|x-y|^{d+(s/\gamma)p\gamma}}\,\mathrm{d}x\,\mathrm{d}y \leq 2^{\gamma-1}[\![u]\!]^p_{s,p}.$$

This proves (6.23). □

7 Conjectures on the Self-Similar Behavior

In this section, we speculate about the longtime behavior of solutions on the full space $\Omega = \mathbb{R}^d$. First, we recall the similarity solution w_{PME} for the classical PME in Sect. 7.1. In Sect. 7.2, we discuss the case $\alpha = \beta$, where we have full scaling invariance. We expect $\int_{\mathbb{R}^d} \frac{1}{2}v(t,x)^2\,\mathrm{d}x \to 0$ such that $\int_{\mathbb{R}^d} w(t,x)\,\mathrm{d}x \to \mathcal{E}(v^0,w^0)$ for $t \to \infty$. Moreover, we conjecture that $w(t,\cdot)$ behaves like the w_{PME} for the corresponding mass $\mathcal{E}(v^0,w^0)$, while $v(t)$ behaves like $t^\gamma(w_{\mathrm{PME}})^{\kappa_0/\eta_0}$, which again shows that $\eta_0 \gg \kappa_0$ leads to singular behavior of v at the boundary of the support. Finally, Sect. 7.3 addresses the cases $\alpha > \beta > 0$ and $\beta > \alpha > 0$. In the latter we still expect that $w(t)$ behaves like w_{PME} while $v(t)$ should behave like $ct^{-d\delta}\mathbf{1}_{B_{t^\delta/b}(0)}(\cdot)$. In the former case, a similar result should not be expected.

7.1 Similarity Solution for the Classical PME

To describe the exact similarity solutions, we introduce the shape functions

$$\mathbb{W}_\sigma(y) = \big(1-|y|^2\big)^\sigma_+.$$

We have the obvious relations

$$\mathbb{W}^\gamma_\sigma = \mathbb{W}_{\sigma\gamma}, \quad \mathbb{W}_\sigma\mathbb{W}_\gamma = \mathbb{W}_{\sigma+\gamma}, \quad \Theta(\sigma,d) := \int_{\mathbb{R}^d} \mathbb{W}_\sigma(y)\,\mathrm{d}y = \frac{\pi^{d/2}\Gamma(1+\sigma)}{\Gamma(1+\sigma+d/2)}, \tag{7.1a}$$

$$\nabla\mathbb{W}_\sigma(y) = -2\sigma\mathbb{W}_{\sigma-1}(y)\,y, \tag{7.1b}$$

$$\mathrm{div}\big(\mathbb{W}_\gamma\nabla\mathbb{W}_\sigma\big)(y) = -2d\sigma\mathbb{W}_{\gamma+\sigma-1}(y) - 2\sigma\, y\cdot\nabla\mathbb{W}_{\gamma+\sigma-1}(y) \quad \text{for } |y| \neq 1. \tag{7.1c}$$

With this, we easily obtain the following well-known, explicit form of the similarity solution for the PME.

Lemma 7.1 *(Similarity solution for PME) The function w_{PME} defined via $w_{\mathrm{PME}}(t,x) = c(t+t_*)^{-d\delta}\mathbb{W}_\sigma(b(t+t_*)^{-\delta}x)$ is a solution of the PME$_\beta$ given by*

$$\dot{w} = \mathrm{div}(\kappa_0 w^\beta \nabla w), \qquad \int_{\mathbb{R}^d} w(t,x)\,\mathrm{d}x = E_0$$

if and only if the we choose the parameters such that

$$E_0 = c\,b^{-d}\,\Theta\big(1/\beta, d\big), \quad \sigma = 1/\beta, \quad \delta = 1/(2+d\beta), \quad 2\kappa_0 bc^\beta = \delta\beta. \tag{7.2}$$

Proof The first relation stems from the integral constraint for w.

A direct calculation using $y = bt^{-\delta}x$ gives

$$\dot{w} = -c\delta t^{-d\delta-1}\big(d\mathbb{W}_\sigma(y) + y\cdot\nabla\mathbb{W}_\sigma(y)\big),$$
$$\mathrm{div}(\kappa_0 w^\beta\nabla w) = -2\sigma\kappa_0 b^2 c^{\beta+1} t^{-(\beta+1)d\delta-2\delta}\big(d\mathbb{W}_{\beta\sigma+\sigma-1}(y) + y\cdot\nabla\mathbb{W}_{\beta\sigma+\sigma-1}(y)\big).$$

Thus, we first see that matching the two lines needs $\beta\sigma = 1$ and $\delta(2+d\beta) = 1$. Finally, we compare the prefactor which gives the last relation. □

The convergence of all nonnegative solutions w to the self-similar profile is one of the major achievements in the theory of PME, see Vázquez (2007, Cha. 16), Carrillo and Toscani (2000), or Otto (2001, Sec. 3). We believe that a corresponding result on asymptotic self-similarity for our system, at least if $\alpha = \beta$. However, in the following we only present conjectures together with some supporting observations, including a numerical simulation.

7.2 Conjectured Longtime Behavior for $\beta = \alpha$

To substantiate our conjecture, we use the scaling properties of the coupled system for $\beta = \alpha$. Slightly generalizing (2.3) we consider the two possible transformations, namely

$$\begin{aligned} &\tau = \log(t+1), \quad y = (t+1)^{-\delta}x \quad \text{and} \\ &\big(v(t,x), w(t,x)\big) = \big((t+1)^{-d\theta\delta}\,\widetilde{v}(\tau,y)\,,\ (t+1)^{-d\delta}\,\widetilde{w}(\tau,y)\big), \end{aligned} \tag{7.3}$$

where $\delta = 1/(2+d\beta)$ as above, and the new parameter θ either equals

- $\theta = 1/2$: the "*energy-conserving scaling*" or
- $\theta = 1$: the "*momentum-conserving scaling*."

The reason for these two choices is that the variable v occurs with different powers in the two conserved quantities $\mathcal{V}(v,w) = \int_{\mathbb{R}^d} v(x)\,\mathrm{d}x$ and $\mathcal{E}(v,w) = \int_{\mathbb{R}^d}\big(\frac{1}{2}v^2+w\big)\,\mathrm{d}x$. Thus, a given scaling can conserve at most one of the two functionals.

The transformed coupled PDE system for $(\widetilde{v},\widetilde{w})$ reads

$$\partial_\tau\widetilde{v} = \mathrm{div}\big(\delta\,\widetilde{v}\,y + \eta_0\widetilde{w}^\beta\nabla\widetilde{v}\big) - (1-\theta)\delta d\,v, \tag{7.4a}$$

$$\partial_\tau \widetilde{w} = \operatorname{div}\big(\delta \widetilde{w}\, y + \kappa_0 \widetilde{w}^\beta \nabla \widetilde{w}\big) + \mathrm{e}^{(1-2\theta)\delta d\, \tau} \eta_0 \widetilde{w}^\beta |\nabla \widetilde{v}|^2. \tag{7.4b}$$

The conserved total momentum $\mathcal{V}$ and total energy $\mathcal{E}$ transform as follows:

$$\widetilde{\mathcal{V}}(\tau, \widetilde{v}, \widetilde{w}) := \int_{y\in\mathbb{R}^d} \mathrm{e}^{(1-\theta)\delta d\tau} \widetilde{v}(y)\,\mathrm{d}y = \int_{x\in\mathbb{R}^d} v(x)\,\mathrm{d}x = \mathcal{V}(v, w) = \mathcal{V}(v^0, w^0) = V_0, \tag{7.5a}$$

$$\begin{aligned} \widetilde{\mathcal{E}}\big(\tau, \widetilde{v}(\tau), \widetilde{w}(\tau)\big) &:= \int_{\mathbb{R}^d} \big(\tfrac{1}{2}\, \mathrm{e}^{(1-2\theta)d\delta\, \tau}\, \widetilde{v}(\tau, y)^2 + \widetilde{w}(\tau, y)\big)\,\mathrm{d}y \\ &= \mathcal{E}\big(v(\mathrm{e}^\tau - 1), w(\mathrm{e}^\tau - 1)\big) = \mathcal{E}(v^0, w^0) = E_0. \end{aligned} \tag{7.5b}$$

Thus, the linear momentum $\mathcal{V}$ remains a conserved quantity (i.e., $\widetilde{\mathcal{V}}(\tau, \cdot) = \mathcal{V}$) for $\theta = 1$ only. In contrast, the energy $\mathcal{E}$ remains a conserved quantity (i.e., $\widetilde{\mathcal{E}}(\tau, \cdot) = \mathcal{E}$) for $\theta = 1/2$ only. Moreover, system (7.4) is a autonomous system if and only if $\theta = 1/2$.

7.2.1 Energy-Conserving Scaling with $\theta = 1/2$

For $\theta = 1/2$ the transformed system (7.4) takes the explicit form

$$\partial_\tau \widetilde{v} = \operatorname{div}\big(\delta\, \widetilde{v}\, y + \eta_0 \widetilde{w}^\beta \nabla \widetilde{v}\big) - \frac{1}{2}\delta d\, v, \tag{7.6a}$$

$$\partial_\tau \widetilde{w} = \operatorname{div}\big(\delta \widetilde{w}\, y + \kappa_0 \widetilde{w}^\beta \nabla \widetilde{w}\big) + \eta_0 \widetilde{w}^\beta |\nabla \widetilde{v}|^2, \tag{7.6b}$$

which is an autonomous evolutionary system with the conserved quantity $\mathcal{E}(\widetilde{v}, \widetilde{w})$. To understand the longtime behavior, we can test (7.6a) by $|\widetilde{v}|^{p-2}\widetilde{v}$ for $p \in]1, 2]$ and find

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathbb{R}^d} \frac{1}{p} |\widetilde{v}|^p\,\mathrm{d}y = -(p-1) \int_{\mathbb{R}^d} \eta_0 \widetilde{w}^\beta\, |\widetilde{v}|^{p-2}\, |\nabla \widetilde{v}|^2\,\mathrm{d}y + \delta d \Big(\frac{p-1}{p} - \frac{1}{2}\Big) \int_{\mathbb{R}^d} |\widetilde{v}|^p\,\mathrm{d}y. \tag{7.7}$$

This implies exponential decay of $\|\widetilde{v}(\tau, \cdot)\|_{\mathrm{L}^p}$ for $p \in]1, 2[$. Moreover, for $p = 2$ all steady states (where the left-hand side in the above relation is 0) satisfy $\nabla \widetilde{v} = 0$. As $\mathcal{E}(\widetilde{v}, \widetilde{w}) = E_0 < \infty$ we conclude $\widetilde{v} = 0$ and obtain the following result.

Corollary 7.2 (Unique nonnegative steady state) *Given arbitrary $E_0 \geq 0$ there is exactly one nonnegative steady state for $(7.4)_{\theta=1/2}$, namely $(\widetilde{v}_{\mathrm{st}}^{E_0}, \widetilde{w}_{\mathrm{st}}^{E_0}) = \big(0, c\mathbb{W}_{1/\beta}(b\cdot)\big)$ where b and c depend on E_0 as given in (7.2).*

We conjecture that all solutions of $(7.4)_{\theta=1/2}$ with initial condition satisfying $\mathcal{E}(v^0, w^0) = E_0$ converge to $(\widetilde{v}_{\mathrm{st}}^{E_0}, \widetilde{w}_{\mathrm{st}}^{E_0})$. The reasoning is simple because we know the convergence of $\widetilde{v}$ to 0, so we expect that for large times the dynamics of the PME for $\widetilde{w}$ (now in transformed variables) and $\widetilde{v} \equiv 0$ will determine the longtime behavior. However, to show this one needs to show that *all* kinetic energy is converted to heat, i.e., $\|\widetilde{v}(\tau, \cdot)\|_{\mathrm{L}^2} \to 0$ for $\tau \to \infty$, which would need a more careful analysis than the simple estimate (7.7).

7.2.2 Momentum-Conserving Scaling with $\theta = 1$

For $\theta = 1$ the transformed system (7.4) takes the explicit form

$$\partial_\tau \widehat{v} = \operatorname{div}\big(\delta\, \widehat{v}\, y + \eta_0 \widetilde{w}^\beta \nabla \widehat{v}\big), \tag{7.8a}$$

$$\partial_\tau \widetilde{w} = \operatorname{div}\big(\delta \widetilde{w}\, y + \kappa_0 \widetilde{w}^\beta \nabla \widetilde{w}\big) + \mathrm{e}^{-\delta d\, \tau} \eta_0 \widetilde{w}^\beta |\nabla \widehat{v}|^2, \tag{7.8b}$$

We now use $\widehat{v}$ which is related to $\widetilde{v}$ in Sect. 7.2.1 by $\widehat{v}(\tau) = \mathrm{e}^{\delta d\tau/2}\widetilde{v}(\tau)$. It is important to keep in mind that the component $\widetilde{w}$ remains exactly the same in both systems, (7.6) and (7.8).

The new system is no longer autonomous but the time dependence occurs via an exponentially decaying term, which hopefully does not influence the longtime behavior even if $\widehat{v}$ is not decaying but stays suitably bounded. From Sect. 7.2.1, we still expect that $\widetilde{w}(\tau)$ converges to $w_{\mathrm{st}}^{E_0}$.

However, the new feature of (7.8) is that $\widehat{v}$ again satisfies a PDE in divergence form, namely (7.8a), which implies that $\mathcal{V}(\widehat{w}(\tau) = V_0$ for all $\tau \geq 0$. Moreover, as we already expect $\widetilde{w}(\tau) \to w_{\mathrm{st}}^{E_0}$ we may expect that $\widehat{v}(\tau)$ converges to a steady state of the linear diffusion equation obtained from (7.8a) by replacing $\widetilde{w}$ by its limit $w_{\mathrm{st}}^{E_0}$. The surprising fact is that this equation has a unique steady state which is given by an explicit formula.

Proposition 7.3 *(Similarity solutions $(\widehat{v}_{\mathrm{st}}, \widetilde{w}_{\mathrm{st}})$) For each pair $(V_0, E_0) \in \mathbb{R}\times]0, \infty[$ there is a unique solution $(\widehat{v}_{\mathrm{st}}, \widetilde{w}_{\mathrm{st}})$ of the system (where $\delta = 1/(2+d\beta)$)*

$$0 = \operatorname{div}\big(\delta \widehat{v}\, y + \eta_0 \widetilde{w}^\beta \nabla \widehat{v}\big), \quad 0 = \operatorname{div}\big(\delta \widetilde{w}\, y + \kappa_0 \widetilde{w}^\beta \nabla \widetilde{w}\big),$$
$$\int_{\mathbb{R}^d} \widehat{v}\,\mathrm{d}y = V_0, \quad \int_{\mathbb{R}^d} \widetilde{w}\,\mathrm{d}y = E_0.$$

This solution is given explicitly in the form

$$\big(\widehat{v}_{\mathrm{st}}(y), \widetilde{w}_{\mathrm{st}}(y)\big) = \Big(a\mathbb{W}_\sigma(by)\,,\ c\mathbb{W}_{1/\beta}(by)\Big) \ \ \textit{with}\ \sigma = \frac{\kappa_0}{\eta_0\beta}, \quad b^2 = \frac{\beta\delta}{2c^\beta\kappa_0},$$
$$V_0 = \frac{a}{b^d}\Theta(\kappa_0/(\beta\eta_0), d), \quad E_0 = \frac{c}{b^d}\Theta(1/\beta, d).$$

Proof The result follows easily with (7.1b). Indeed, we even have $\delta\widehat{v}\, y + \eta_0\widetilde{w}^\beta\nabla\widehat{v} \equiv 0$ and $\delta\widetilde{w}\, y + \kappa_0\widetilde{w}^\beta\nabla\widetilde{w} \equiv 0$ on $\mathbb{R}^d$. □

Of course, justifying the convergence to the steady states $(\widehat{v}_{\mathrm{st}}, \widetilde{w}_{\mathrm{st}})$ for a suitable solution class $(\widehat{v}, \widetilde{w})$ of the full system (7.4) is a nontrivial task. This can be seen by inserting the steady state into the exponentially decaying term which gives the perturbation

$$\mathrm{e}^{-d\delta\tau}\eta_0\widetilde{w}_{\mathrm{st}}^\alpha|\nabla\widehat{v}_{\mathrm{st}}|^2 = \widetilde{c}\,\mathrm{e}^{-d\delta\tau}\mathbb{W}_{2\kappa_0/(\beta\eta_0)-1}(by)|y|^2.$$

This term lies in $\mathrm{L}^\infty(0, T; \mathrm{L}^1(\mathbb{R}^d))$ but is singular at $b|y| = 1$ for $\beta > 2\kappa_0/\eta_0$.

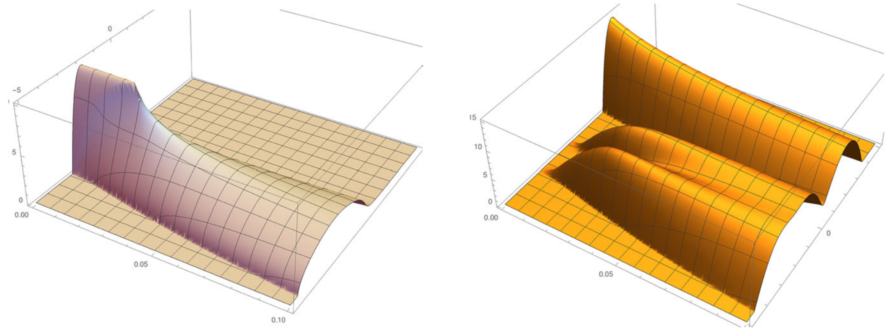

Fig. 5 The solution $v(t, x)$ (left) and $w(t, x)$ (right) for $t \in [0, 0.1]$ and $x \in [-5, 5]$

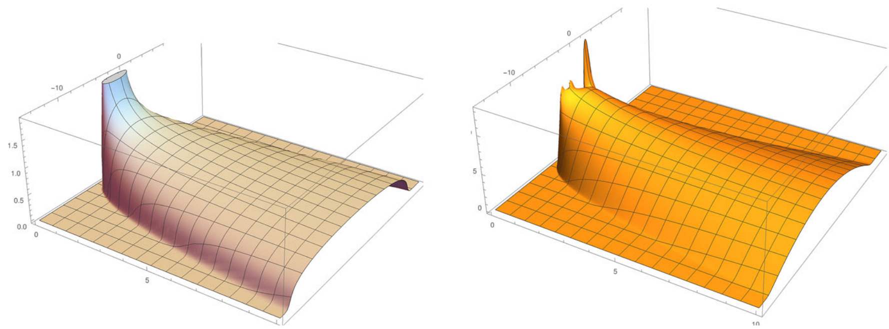

Fig. 6 The solution $v(t, x)$ (left) and $w(t, x)$ (right) for $t \in [0, 10]$ and $x \in [-10, 10]$

7.2.3 A Numerical Simulation Showing Convergence

A simple numerical experiment covers the one-dimensional case $\Omega = \mathbb{R}^1$ with $\alpha = \beta = 1$, $\eta_0 = 2$, and $\kappa_0 = 1/2$. We start with initial conditions $v^0(x) = \max\{0, 10 - 10(x+2)^2\}$ and $w^0(x) = \max\{0, 15 - 15(x-2)^2\}$. In Fig. 5, we show that solution for the short initial time interval $t \in [0, 0.1]$ which shows that kinetic energy is dissipated fast and turned into heat.

In Fig. 6, we show that (unscaled) solutions on the longer time interval $t \in [0, 10]$, where the self-similar behavior becomes visible.

In Fig. 7, we show that the rescaled solutions $(t, y) \mapsto (1+t)^\delta v(t, (1+t)^\delta y)$ and $(t, y) \mapsto (1+t)^\delta w(t, (1+t)^\delta y)$ for $t \in [0, 1.5]$ where convergence into a self-similar profile is already evident. It is clearly seen that w develops the simple Barenblatt profile $c_w \max\{0, b-y^2\}$, whereas v develops the more singular profile $c_v \max\{0, b-y^2\}^{1/4}$ because of $\kappa_0/\eta_0 = 1/4$.

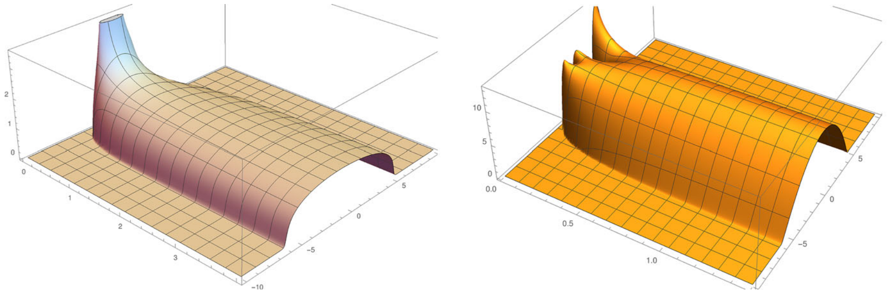

Fig. 7 Scaled solutions $(1+t)^{\delta} v(t, (1+t)^{\delta} y)$ (left) and $(1+t)^{\delta} w(t, (1+t)^{\delta} y)$ (right) for $t \in [0, 1.5]$ and $y \in [-10, 10]$

7.3 Conjectured Longtime Behavior for $\beta \neq \alpha$

With the transformation (7.3) with $\delta = 1/(2+d\beta)$, we now find the transformed system

$$\partial_\tau \widetilde{v} = \operatorname{div}\big(\delta\, \widetilde{v}\, y + \mathrm{e}^{\gamma\tau} \eta_0 \widetilde{w}^{\beta} \nabla \widetilde{v}\big) \qquad \text{with } \gamma = (\beta-\alpha)d\delta, \tag{7.9a}$$

$$\partial_\tau \widetilde{w} = \operatorname{div}\big(\delta \widetilde{w}\, y + \kappa_0 \widetilde{w}^{\beta} \nabla \widetilde{w}\big) + \mathrm{e}^{(\gamma-d\delta)\tau} \eta_0 \widetilde{w}^{\beta} |\nabla \widetilde{v}|^2. \tag{7.9b}$$

We consider the cases $\alpha > \beta > 0$ and $\beta > \alpha > 0$ separately.

<u>*Case* $\gamma = (\beta-\alpha)d\delta > 0$</u> : The chosen scaling leads to the exponentially growing prefactor for the diffusion in (7.9a). Hence, it is dominating the concentration term such that we expect for large τ that $\nabla v(\tau)$ is small. This can also be seen by testing (7.9a) by $\mathrm{e}^{-d\delta\tau} v(\tau)$ which leads to

$$\iint_{Q_T} \mathrm{e}^{(\gamma-d\delta)\tau} \eta_0 \widetilde{w}^{\beta} |\nabla \widetilde{v}|^2 \,\mathrm{d}y\,\mathrm{d}\tau \le \frac{1}{2} \|\widetilde{v}(0)\|^2_{\mathrm{L}^2(\mathrm{R}^{\mathrm{d}})}.$$

Thus, it is reasonable to expect that $\widetilde{v}(t)$ converges to $c\, \mathbf{1}_{B_{1/b}(0)}(\cdot)$ while $\widetilde{w}$ converges to $\widetilde{w}_{\mathrm{st}}$ as for the case $\alpha = \beta$.

A proof of this conjecture will be even more difficult, because $\nabla v(\tau)$ must develop a singularity near the boundary of $B_{1/b}(0) \subset \mathbb{R}^d$ to allow for the convergence $\widetilde{v}(\tau) \to c\, \mathbf{1}_{B_{1/b}(0)}(\cdot)$ in $\mathrm{L}^2(\mathbb{R}^d)$.

<u>*Case* $\gamma = (\beta-\alpha)d\delta < 0$</u> : In this case we expect a different behavior because of the decaying diffusion in the equation for $\widetilde{v}$. For large τ the dominating term in (7.9a) is the transport term $\operatorname{div}\big(\delta \widetilde{v} y\big)$ which is simply due to the similarity rescaling. Hence, for $\tau > \tau_0 \gg 1$ one expects $\widetilde{v}(\tau, y) \approx \mathrm{e}^{d\delta(t-\tau_0)} \widetilde{v}\big(\tau_0, \mathrm{e}^{\delta(\tau-\tau_0)} y\big)$, which in the original physical coordinates means the same as $v(t, x) \approx v(t_0, x)$ for $t > t_0 \gg 1$. Thus, it is likely that $\|v(t)\|^2_{\mathrm{L}^2} = \mathrm{e}^{-d\delta\tau} \|\widetilde{v}(\tau)\|^2_{\mathrm{L}^2}$ does not converge to 0 for $\tau = \log(t+1) \to \infty$. Consequently, the nondecreasing function $\widetilde{E}(t) := \int_{\mathbb{R}^d} \widetilde{w}(\tau, y)\, \mathrm{d}y$ is converging to some $\widetilde{E}_\infty \in [E(0), E_0]$ with $E_0 = \mathcal{E}(v^0, w^0)$. There is still a chance that $\widetilde{w}$ converges to some $\widetilde{w}_{\mathrm{st}}$ but now with mass $\widetilde{E}_\infty$ instead of E_0.

8 Parabolic Systems in Turbulence Modeling

The author's motivation for studying the given class of coupled degenerate parabolic equation comes from turbulence modeling. However, for the sake of a concise presentation the model was simplified considerably, but still keeping the main feature of a velocity-type variable with a degenerate viscosity depending on the energy-like variable, which is now the mean turbulent kinetic energy $k \geq 0$ playing the role of w. We refer to Lewandowski (1997), Gallouët et al. (2003), Lederer and Lewandowski (2007), Druet and Naumann (2009), Naumann (2013) and Chacón Rebello (2014) for a mathematical exposition of the ideas behind of such a turbulent modeling and to Wilcox (1993, Ch. 4) for a fluid mechanical approach.

We first introduce Prandtl's model for turbulence and then Kolmogorov's two-equation model. In both cases, we introduce the full model with the macroscopically mean velocity $\mathbf{u} \in \mathbb{R}^{d+1}$ satisfying the incompressibility $\operatorname{div} \mathbf{u} = 0$, the pressure p, and the turbulent kinetic energy $k \geq 0$. Then, we show that restricting to

$$\text{simple shear flows} \qquad \mathbf{u}(t,x) = \big(0,\dots,0,v(t,x_1,\dots,x_d)\big)^\top \in \mathbb{R}^{d+1} \tag{8.1}$$

we obtain a system that has the form of our coupled system (1.1) with some simple extra terms.

In both models below, we consider a fluid viscosity $k \mapsto \eta(k)$ and a diffusivity $k \mapsto \kappa(k)$ with the properties $\eta(0) = 0 = \kappa(0)$. This means that the model is physically valid on very large spatial scales such that the molecular viscosity $\eta(0)$ or diffusivity $\kappa(0)$ can be neglected, i.e., set to 0, in comparison to the viscosity and diffusivity induced by turbulent mixing. In practice it only means that the ratio between a $\eta(0)$ and a typical $\eta(k)$ should be very large. Indeed, the numerical simulations in Figs. 5 and 6 were done with $\eta(0) = \kappa(0) = 10^{-4} > 0$, and one still observes a sharp transition between the region $w(t,x) \approx 0$ and the effective support with $w(t,x) > w_*$ for a fixed small and positive w_*.

8.1 Kolmogorov's One-Equation Model = Prandtl's Model

Prandtl's one-equation model for turbulence was developed from 1925 to 1945, see Prandtl (1925, 1946) and the historical remarks in Bulíček et al. (2011, Sec. 1.2) and Naumann (2013, p. 20).

$$\begin{aligned}
&\dot{\mathbf{u}} + \mathbf{u}\cdot\overline{\nabla}\mathbf{u} + \overline{\nabla}p = \overline{\operatorname{div}}\big(\ell\sqrt{k}\,\overline{\mathrm{D}}(\mathbf{u})\big) + f, \quad \overline{\operatorname{div}}\mathbf{u} = 0,\\
&\dot{k} + \mathbf{u}\cdot\overline{\nabla}k = \widehat{\kappa}\,\overline{\operatorname{div}}\big(\ell\sqrt{k}\,\overline{\nabla}k\big) + \ell\sqrt{k}\,|\overline{\mathrm{D}}(\mathbf{u})|^2 - a\,\frac{k\sqrt{k}}{\ell},
\end{aligned}$$

where $a \geq 0$ is a dimensional parameter and $\ell > 0$ is a constant characteristic length that may depend on position (e.g., via the distance to the wall). Moreover, $\overline{\nabla}$, $\overline{\mathrm{D}}$, and $\overline{\operatorname{div}}$ stand for the differential operators in $\mathbb{R}^{d+1}$ with $d \in \{1,2\}$. We refer to Bulíček et al. (2011) for a general existence theory in bounded domains for dissipation

$\eta(k) \sim (1+k)^\alpha$ (in the Navier–Stokes equation) and $\kappa(k) \sim (1+k)^\beta$) (in the heat equation).

Looking for shear flows as in (8.1), assuming that the external force vanishes, i.e., $f \equiv 0$, and that k is independent of x_{d+1} we first observe that the pressure gradient has the form $\overline{\nabla} p = (0, \dots, 0, g(t, x_1, \dots, x_d))^\top \in \mathbb{R}^{d+1}$ for some function g. As $\mathrm{D}^2 p \in \mathbb{R}^{(d+1)\times(d+1)}$ must be symmetric, we conclude that the downstream pressure gradient $\partial_{x_{d+1}} p = g$ can only depend on t but not on $(x_1, \dots, x_d)$. Such a pressure gradient is important for pipe flows in a pipe with a bounded cross-section $\Sigma \subset \mathbb{R}^d$ to compensate for the friction of the walls when using no-slip boundary conditions. We refer to the derivation of the pipe-flow models in Naumann and Wolf (2013, Eqn. (2.1)) for the stationary case and Naumann (2013, Eqn. (1.12)) for the time-dependent case, where f and g, respectively, are exactly the pressure gradient.

In the case of unbounded cross sections, like $\Sigma = \mathbb{R}^d$ in our case, it is commonly assumed $g \equiv 0$ without loss of generality. Indeed, for $\Sigma = \mathbb{R}^d$ the case of a spatially constant $g(t) \not\equiv 0$ can be transformed by to the case $g \equiv 0$ by Galileian invariance, namely be replacing $v(t, x)$ by $v(t, x) - \int_0^t g(s)\,\mathrm{d}s$. Hence, assuming $g \equiv 0$, we obtain the coupled system

$$\dot{v} = \mathrm{div}\big(\eta_0 k^{1/2} \nabla v\big), \qquad \dot{k} = \mathrm{div}\big(\kappa_0 k^{1/2} \nabla k\big) + \eta_0 k^{1/2} |\nabla v|^2 - \frac{a}{\ell}\, k^{3/2} \tag{8.2}$$

with $\eta_0 = \ell/2$ and $\kappa_0 = \widehat{\kappa}\ell$. Clearly, when neglecting the term involving a, we arrive at our coupled system in the case $\alpha = \beta = 1/2$.

The case $a > 0$ does not pose any additional problem when constructing solutions. In particular, the term reduces the total energy in the form $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v, k) = -\frac{a}{\ell} \int_\Omega k^{3/2}\,\mathrm{d}x \leq 0$. Hence, the a priori estimates of this equation remain the same, such that existence of weak solutions with $k(t, x) \geq \underline{k} > 0$ was shown already in Naumann (2013). Our existence result for weak solutions in Theorem 6.1 works for $p_0 = 2$ because of $\alpha = \beta = 1/2$.

The general scaling (2.3) leading to (2.7) can still be applied, but to have the same scaling behavior for the terms $\dot{k}$ and $k^{3/2}$ we have only one choice, namely $\gamma = 2$ and with $\alpha = 1/2$ and $2\delta + \alpha\gamma = 1$ we find $\delta = 0$. Hence, for $(\widehat{v}, \widehat{w}) = \big(\frac{1}{t} v, \frac{1}{t^2} w\big)$ we find the rescaled equation

$$\begin{aligned} \partial_\tau \widehat{v} - \widehat{v} &= \mathrm{div}\big(\eta_0 \widehat{k}^{1/2}\, \nabla \widehat{v}\big), \\ \partial_\tau \widehat{k} - 2\widehat{k} &= \mathrm{div}\big(\kappa_0 \widehat{k}^{1/2} \nabla \widehat{k}\big) + \eta_0 \widehat{k}^{1/2} \big|\nabla \widehat{v}\big|^2 - \frac{a}{\ell}\, \widehat{k}^{3/2}. \end{aligned} \tag{8.3}$$

We may still consider the conserved quantities and find the relations

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_\Omega \widehat{v}\,\mathrm{d}y = \int_\Omega \widehat{v}\,\mathrm{d}y \;\text{ and }\; \frac{\mathrm{d}}{\mathrm{d}\tau} \int_\Omega \big(\tfrac{1}{2}\widehat{v}^2 + \widehat{k}\big)\,\mathrm{d}y = 2 \int_\Omega \big(\tfrac{1}{2}\widehat{v}^2 + \widehat{k}\big)\,\mathrm{d}y - \int_\Omega \frac{a}{\ell}\, \widehat{k}^{3/2}\,\mathrm{d}y.$$

Thus, we may expect that (8.3) has steady states $(v_{\mathrm{st}}, k_{\mathrm{st}})$ with nontrivial $k_{\mathrm{st}} \geq 0$ (but necessarily with $\int_\Omega v_{\mathrm{st}}\,\mathrm{d}y = 0$) that correspond to solutions of (8.2) that decay, namely $(v(t, x), k(t, x)) = (\frac{1}{t} v_{\mathrm{st}}(x), \frac{1}{t^2} k_{\mathrm{st}}(x))$.

8.2 Kolmogorov's *k*-*ω* Model

Kolmogorov's model Kolmogorov (1942) (see Bulíček and Málek 2019, Sec. 1.2 for historical remarks and Spalding (1991) for a full translation, sometimes also called Wilcox k-ω model because of Wilcox (1993)) consists of the Navier–Stokes equation for the velocity $\mathbf{v} \in \mathbb{R}^{d+1}$ and the pressure p coupled to two scalar equations, namely for the specific dissipation rate $\omega > 0$ (or better dissipation per unit turbulent kinetic energy) and the turbulent energy $k \geq 0$:

$$\dot{\mathbf{u}} + \mathbf{u} \cdot \overline{\nabla}\mathbf{u} + \overline{\nabla} p = \mu_1 \,\overline{\mathrm{div}}\big(\tfrac{k}{\omega}\overline{\mathrm{D}}(\mathbf{u})\big), \quad \overline{\mathrm{div}}\,\mathbf{u} = 0, \tag{8.4a}$$

$$\dot{\omega} + \mathbf{u} \cdot \overline{\nabla}\omega = \mu_2 \,\overline{\mathrm{div}}\big(\tfrac{k}{\omega}\overline{\nabla}\omega\big) - \alpha_1\omega^2, \tag{8.4b}$$

$$\dot{k} + \mathbf{u} \cdot \overline{\nabla}k = \mu_3 \,\overline{\mathrm{div}}\big(\tfrac{k}{\omega}\overline{\nabla}k\big) + \mu_1 \tfrac{k}{\omega}\,\big|\overline{\mathrm{D}}(\mathbf{u})\big|^2 - \alpha_2 k\omega. \tag{8.4c}$$

The main point is that all quantities are convected with the fluid velocity $\mathbf{v}$ and that all quantities diffuse with "viscosities" that are proportional to k/ω by the dimensionless factors $\mu_j > 0$. There are sink terms in the equations for ω and k, namely $-\alpha_1\omega^2$ and $-\alpha_2 k\omega$ with dimensionless nonnegative constants α_j. The nonlinearities in system (8.4) are devised in a specific way to allow for a two-dimensional scaling group, see Bulíček and Málek (2019, p. 115) and Mielke and Naumann (2022, Sec. 2), which exactly corresponds to our *Scaling S2* in (2.2). In the former reference the scaling was mainly used to argue about regularity properties of the solutions on bounded domains, while here and in Mielke and Naumann (2022) the scaling is used to study self-similar behavior on $\Omega = \mathbb{R}^d$.

The existence of weak solutions to the above coupled system is studied in Mielke and Naumann (2015), Bulíček and Málek (2019) and Mielke and Naumann (2022). However, in both of these papers the construction of the solutions for (8.4) strongly relies on lower bounds on $k(t, x)$. In Mielke and Naumann (2015) and Mielke and Naumann (2022), the assumption $k_0(x) = k(0, x) \geq k_* > 0$ a.e. in Ω was used, whereas Bulíček and Málek (2019) shows that the weaker assumption $\int_\Omega \big|\log k_0(x)\big|\,\mathrm{d}x < \infty$ is sufficient.

Here, we are interested in solutions that may have compact support, i.e., $k(t, x) = 0$ is allowed on a set of positive measure. In those regions, the regularizing diffusion terms $\mu_j\,\mathrm{div}(\frac{k}{\omega}\nabla\,\cdot\,)$ disappear. Moreover, we restrict to shear flows (without pressure gradient) as above:

$$\begin{aligned} &\dot{v} = \tfrac{\mu_1}{2}\,\mathrm{div}\big(\tfrac{k}{\omega}\nabla v\big), \qquad \dot{\omega} = \mu_2\,\mathrm{div}\big(\tfrac{k}{\omega}\nabla\omega\big) - \omega^2, \\ &\dot{k} = \mu_3\,\mathrm{div}\big(\tfrac{k}{\omega}\nabla k\big) + \tfrac{\mu_1}{2}\,\tfrac{k}{\omega}|\nabla v|^2 - ak\omega. \end{aligned} \tag{8.5}$$

The special symmetry of the nonlinearities allows for more scalings than in the Prandtl equation (8.2), because (8.5) only contains the dimensionless parameters μ_j and a, whereas (8.2) contains the Prandtl length ℓ.

We can look for solutions in the following form (cf. (2.3)):

$$\tau = \log(t{+}1), \quad y = (t{+}1)^{-\delta}x, \quad \text{and}$$

$$\big(v(t,x),\omega(t,x),k(t,x)\big)=\Big(\frac{\widehat v(\tau,y)}{(t{+}1)^{\gamma/2}},\ \frac{\widehat\omega(\tau,y)}{(t{+}1)^{1}},\ \frac{\widehat k(\tau,y)}{(t{+}1)^{\gamma}}\Big).$$

If the exponents γ and δ are chosen with $\gamma+2\delta=2$, we are led to the rescaled system

$$\partial_\tau\widehat v-\frac{\gamma}{2}\,\widehat v-\delta\,y\cdot\nabla\widehat v=\frac{\mu_1}{2}\,\mathrm{div}\big(\tfrac{\widehat k}{\widehat\omega}\,\nabla\widehat v\big), \tag{8.6a}$$

$$\partial_\tau\widehat\omega-\ \widehat\omega\ -\delta\,y\cdot\nabla\widehat\omega=\mu_2\,\mathrm{div}\big(\tfrac{\widehat k}{\widehat\omega}\,\nabla\widehat v\big)-\widehat\omega^2, \tag{8.6b}$$

$$\partial_\tau\widehat k-\gamma\,\widehat k-\delta\,y\cdot\nabla\widehat k=\mu_3\,\mathrm{div}\big(\tfrac{\widehat k}{\widehat\omega}\,\nabla\widehat k\big)+\frac{\mu_1}{2}\widehat k|\nabla\widehat v|^2-a\,\widehat k. \tag{8.6c}$$

The case $(\gamma,\delta)=(2,0)$ can be applied in bounded domains Ω and it corresponds to the scaling used for the Prandtl equation to obtain (8.3).

In principle, it would be interesting to investigate the system for $(\widehat v,\widehat\omega,\widehat k)$ for steady states, which correspond to self-similar solutions for (8.5). However, the singularity occurring via $\widehat k/\widehat\omega$ will create difficulties; thus from now on we restrict to the case $\widehat\omega\equiv1$ which is always a solution of (8.6b) independently of $\widehat v$ and $\widehat k$. The choice $\widehat\omega\equiv1$ can also be justified by the upper and lower estimates for ω that are derived in Mielke and Naumann (2022, Eqn. (4.3)). They imply uniform convergence of $\widehat\omega(\tau,\cdot)$ to 1 for $\tau\to\infty$ if the initial condition $\widehat\omega(0,\cdot)$ has positive lower and upper bounds.

The reduced system with $\widehat\omega\equiv1$ takes a form similar to our coupled system (7.8):

$$\partial_\tau\widehat v-\frac{\gamma}{2}\,\widehat v-\delta\,y\cdot\nabla\widehat v=\frac{\mu_1}{2}\,\mathrm{div}\big(\widehat k\,\nabla\widehat v\big), \tag{8.7a}$$

$$\partial_\tau\widehat k-\gamma\,\widehat k-\delta\,y\cdot\nabla\widehat k=\mu_3\,\mathrm{div}\,\big(\widehat k\,\nabla\widehat k\big)+\frac{\mu_1}{2}\widehat k|\nabla\widehat v|^2-a\,\widehat k. \tag{8.7b}$$

Clearly, we are in the case $\alpha=\beta=1$ with two additional linear terms. Thus, the existence theory of very weak solutions in Theorem (6.3) only applies in the case $\mu_1=2\mu_3$.

For the case $\delta>0$, we consider $\Omega=\mathbb{R}^d$, such that the rescaled momentum $V(\widehat v,\widehat k)=\int_{\mathbb{R}^d}\widehat v(y)\,\mathrm{d}y$ and the rescaled energy $E(\widehat v,\widehat k)=\int_{\mathbb{R}^d}\big(\frac12\widehat v^2+\widehat k\big)\,\mathrm{d}y$ now satisfy, along solutions $(\widehat v(\tau),\widehat k(\tau))$, the linear relations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V(\widehat v,\widehat k)=\big(\frac{\gamma}{2}-d\delta\big)V(\widehat v,\widehat k)\quad\text{and}\quad\frac{\mathrm{d}}{\mathrm{d}\tau}E(\widehat v,\widehat k)=(\gamma-d\delta)E(\widehat v,\widehat k)-a\int_{\mathbb{R}^d}\widehat k\,\mathrm{d}y. \tag{8.8}$$

Because of $2=\gamma+2\delta$, the case $(\gamma,\delta)=\big(\frac{2d}{1+d},\ \frac{1}{1+d}\big)$ is special as it leads to

$$\binom{\gamma}{\delta}=\frac{1}{1+d}\binom{2d}{1}\quad\Longrightarrow\quad\frac{\mathrm{d}}{\mathrm{d}\tau}V(\widehat v,\widehat k)=0\ \text{ and }\ \frac{\mathrm{d}}{\mathrm{d}\tau}E(\widehat v,\widehat k)=\frac{d}{1+d}E(\widehat v,\widehat k)-a\int_{\mathbb{R}^d}\widehat k\,\mathrm{d}y.$$

For $a < \frac{d}{1+d}$, we obtain $\frac{\mathrm{d}}{\mathrm{d}\tau} E(\widehat{v}, \widehat{k}) > 0$ as long as $E(\widehat{v}, \widehat{k}) > 0$, so no steady state can exist. However, for $a > \frac{d}{1+d}$ one can expect the existence of a family of state states $(v_{\mathrm{st}}, k_{\mathrm{st}})$ such that $V(v_{\mathrm{st}}, k_{\mathrm{st}}) = V_0 \in \mathbb{R}$.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability Statement Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

Amann, H.: Dynamic theory of quasilinear parabolic systems. III: global existence. Math. Z. **202**:2 (1989) 219–250, Erratum Vol. 205, page 331

Bertsch, M., Kamin, S.: A system of degenerate parabolic equations. SIAM J. Math. Anal. **21**(4), 905–916 (1990)

Bulíček, M., Feireisl, E., Málek, J.: A Navier–Stokes–Fourier system for incompressible fluids with temperature dependent material coefficients. Nonlinear Anal. RWA **10**(2), 992–1015 (2009)

Březina, J., Feireisl, E., Novotný, A.: On convergence to equilibria of flows of compressible viscous fluids under in/out-flux boundary conditions. Discrete Contin. Dyn. Syst. **41**(8), 3615–3627 (2021)

Boccardo, L., Gallouët, T., Orsina, L.: Nonlinear eparabolic equations with measure data. J. Funct. Anal. **147**(1), 237–258 (1997)

Bulíček, M., Lewandowski, R., Málek, J.: On evolutionary Navier–Stokes–Fourier type systems in three spatial dimensions. Comment. Math. Univ. Carolin. **52**(1), 89–114 (2011)

Boccardo, L., Gallouët, T.: Nonlinear elliptic and parabolic equations involving measure data. J. Funct. Anal. **87**(1), 149–169 (1989)

Bulíček, M., Málek, J.: Large data analysis for Kolmogorov's two-equation model of turbulence. Nonlinear Anal. Real World Appl. **50**, 104–143 (2019)

Carrillo, J.A., Toscani, G.: Asymptotic L^1-decay of solutions of the prous medium equation to self-similarity. Indiana Univ. Math. J. **49**(1), 113–142 (2000)

Chacón Rebello, T., Lewandowski, R.: Mathematical and Numerical Foundations of Turbulence Models and Applications. Springer, New York (2014)

Dal Passo, R., Giacomelli, L.: Weak solutions of a strongly coupled degenerate parabolic system. Adv. Differ. Equ. **4**(5), 617–638 (1999)

Dostalík, M., Průša, V., Rajagopal, K.R.: Unconditional finite amplitude stability of a fluid in a mechanically isolated vessel with spatially non-uniform wall temperature. Contin. Mech. Thermodyn. **33**(2), 515–543 (2021)

Druet, P.-É., Naumann, J.: On the existence of weak solutions to a stationary one-equation RANS model with unbounded eddy viscosities. Ann. Univ. Ferrara **55**, 67–87 (2009)

Fanelli, F., Granero-Belinchón, R.: Finite time blow-up for some parabolic systems arising in turbulence theory. Z. angew. Math. Physik **73**, 180 (2022) arXiv: 2204.04934

Feireisl, E., Málek, J.: On the Navier–Stokes equations with temperature-dependent transport coefficients. Differ. Equ. Nonl. Mech. Art. ID **90616**, 1–14 (2006)

Gallay, T., Mielke, A.: Diffusive mixing of stable states in the Ginzburg–Landau equation. Commun. Math. Phys. **199**(1), 71–97 (1998)

Gallouët, T., Lederer, J., Lewandowski, R., Murat, F., Tartar, L.: On a turbulent system with unbounded eddy viscosities. Nonlinear Anal. **52**(4), 1051–1068 (2003)

Haskovec, J., Hittmeir, S., Markowich, P.A., Mielke, A.: Decay to equilibrium for energy-reaction–diffusion systems. SIAM J. Math. Anal. **50**(1), 1037–1075 (2018)

Hyman, J.M., Rosenau, P.: Analysis of nonlinear parabolic equations modeling plasma diffusion across a magnetic field. Nonlinear systems of partial differential equations in applied mathematics. Lect. Appl. Math. **23**, 219–245 (1986)

Kolmogorov, A.N.: The equations of turbulent motion of an incompressible fluid. Izv. Akad. Nauk SSSR Ser. Fiz. **6**(1–2), 56–58 (1942)

Lederer, J., Lewandowski, R.: A RANS 3D model with unbounded eddy viscosities. Ann. I.H. Poincaré Ana. Nonlinear **24**, 413–441 (2007)

Lewandowski, R.: The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier–Stokes equation with an eddy viscosity. Nonlinear Anal. TMA **28**, 393–417 (1997)

Lions, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod et Gauthier Villars, Paris (1969)

Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhäuser, Basel (1995)

Mielke, A.: On evolutionary Γ-convergence for gradient systems (Ch. 3), Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity. In: Muntean, A., Rademacher, J., Zagaris, A. (eds.) Proceedings of Summer School in Twente University, June 2012. Lecture Notes in Applied Mathematics Mechanics, vol. 3, pp. 187–249. Springer, Berlin (2016)

Mielke, A., Mittnenzweig, M.: Convergence to equilibrium in energy-reaction–diffusion systems using vector-valued functional inequalities. J. Nonlinear Sci. **28**(2), 765–806 (2018)

Mielke, A., Naumann, J.: Global-in-time existence of weak solutions to Kolmogorov's two-equation model of turbulence. C.R. Acad. Sci. Paris Ser. I **353**, 321–326 (2015)

Mielke, A., Naumann, J.: On the existence of global-in-time weak solutions and scaling laws for Kolmogorov's two-equation model of turbulence. Z. Angew. Math. Mech. (ZAMM) **102**:9 e202000019/1–31 (2022)

Mielke, A., Schindler, S.: Existence of similarity profiles for systems of diffusion equations. Preprint arXiv:2301.10360 (2023)

Moussa, A.: Some variants of the classical Aubin–Lions lemma. J. Evol. Equ. **16**(1), 65–93 (2016)

Naumann, J.: Degenerate parabolic problems in turbulence modelling. Boll. Accad. Gioenia Sci. Nat. Catania **46**(376), 18–43 (2013)

Naumann, J., Wolf, J.: On Prandtl's turbulence model: existence of weak solutions to the equations of stationary turbulent pipe-flow. Discr. Cont. Dynam. Systems Ser. S **6**(5), 1371–1390 (2013)

Otto, F.: The geometry of dissipative evolution equations: the porous medium equation. Commun. Partial Differ. Equ. **26**, 101–174 (2001)

Peletier, M.A.: Variational modelling: energies, gradient flows, and large deviations. arXiv:1402.1990 (2014)

Prandtl, L.: Bericht über Untersuchungen zur ausgebildeten Turbulenz. Z. Angew. Math. Mech. (ZAMM) **5**, 136–139 (1925)

Prandtl, L.: Über ein neues Formelsystem für die ausgebildete Turbulenz. Nachr. Akad. Wiss. Göttingen. Math. Phys. **1**, 6–19 (1946)

Ransford, T.: Potential Theory in the Complex Plane. London Mathematical Society, London (1995)

Rosenau, P., Hyman, J.M.: Analysis of nonlinear mass and energy diffusion. Phys. Rev. A **32**, 2370–2373 (1985)

Rosenau, P., Hyman, J.M.: Plasma diffusion across a magnetic field. Physica D **20**, 444–446 (1986)

Roubíček, T.: Nonlinear partial differential equations with applications (2nd edition), ISNM. International Series of Numerical Mathematics 153, Birkhäuser Verlag, Basel (2013)

Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. (4) **146**, 65–96 (1987)

Spalding, D.B.: Kolmogorov's two-equation model of turbulence. Proc. R. Soc. Lond. Ser. A **434**:1890 (1991) 211–216, Turbulence and stochastic processes: Kolmogorov's ideas 50 years on

Vázquez, J.L.: The Porous Medium Equation. Mathematical Theory. Clarendon Press, Oxford (2007)

Wiegner, M.: Global solutions to a class of strongly coupled parabolic systems. Math. Ann. **292**(4), 711–727 (1992)

Wilcox, D.C.: Turbulence Modeling for CFD. DCW Industries, La Canada (1993)