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Propagation Phenomena for Man–Environment Epidemic Model with Nonlocal Dispersals

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Abstract

In this paper, we focus on the propagation phenomena of a bistable man–environment epidemic model with nonlocal dispersals, where there exists a positive feedback interaction between the concentration of infectious agent and infectious human population. The monostable and bistable traveling wave solutions and three-wave entire solutions are studied. First, by applying and developing the known results for monostable case, we give a summary of the existence and asymptotic behavior of all monostable traveling wave solutions in two different monostable intervals and further find some relationship between them. The existence of bistable traveling wave solutions is obtained by introducing the results about monotone semiflows with weak compactness. Second, we give twelve types of three-wave entire solutions, which contain all possibilities of three-wave entire solutions originating from three traveling wave solutions with different nonzero wave speeds, by constructing new auxiliary functions and super- and sub-solutions for every type. We also show that these entire solutions are globally Lipschitz continuous with respect to spatial variable. In addition, the nonexistence result of entire solutions originating from more than four traveling wave solutions is obtained.

Keywords Nonlocal dispersal · Epidemic model · Traveling wave solutions · Entire solutions

Mathematics Subject Classification 35K57 · 35R20 · 92D25

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1 Introduction

Dispersal, as a common and important phenomenon in ecology and epidemiology, describes the movement mechanism by which species move into a new area. Mathematically, dispersal can be modeled by the following nonlocal dispersal operator:

$$u \mapsto J * u - u = \int_{\mathbb{R}^N} J(x-y)u(y)\mathrm{d}y - u,$$

which describes the movements of organisms between not only adjacent but also nonadjacent spatial locations. Compared with its local counterpart represented by Laplacian operator (i.e., $u \mapsto \Delta u$), nonlocal dispersal is more effective to study the long-range dispersal phenomena, such as the rapid spread of infectious disease across countries or continents by the travel of infected human (see Cannas et al. 2006; Hallatschek and Fisher 2014). Here, the kernel J is a nonnegative function satisfying $\int_{\mathbb{R}^N} J(x)\mathrm{d}x = 1$, and $J(x-y)$ stands for the probability distribution of the movement jumping from location y to x. Over the past two decades, the following nonlocal dispersal equation with a reaction term

$$u_t = J * u - u + f(u), \ x \in \mathbb{R}^N, \ t \in \mathbb{R}$$

has attracted extensive attention, and we refer to Bates et al. (1997), Carr and Chmaj (2004), Coville et al. (2008), and Schumacher (1980) for the works on traveling wave solutions, and Dong et al. (2021), Li et al. (2010), and Sun et al. (2011) for the works on entire solutions. We also refer to Andreu-Vaillo et al. (2010), Bates (2006), Fife (2003), Kao et al. (2010), and Murray (2003) for the works on other problems about nonlocal dispersals.

In this paper, we focus on an epidemic model where both the dispersals of infectious agent and infectious human population are nonlocal, and there is a positive feedback interaction between the concentration of infectious agent and infectious human population, that is, once the human hosts infected, they will promote the growth rate of bacteria, and in turn, an increase in the concentration of infectious agent can lead to an increase in the infection rate of human population. This model is written as

$$\begin{cases} u_t(x,t) = J_1 * u(x,t) - u(x,t) - \alpha u(x,t) + h(v(x,t)), & x \in \mathbb{R}, t \in \mathbb{R}, \\ v_t(x,t) = J_2 * v(x,t) - v(x,t) - \beta v(x,t) + g(u(x,t)), & x \in \mathbb{R}, t \in \mathbb{R}. \end{cases} \tag{1.1}$$

Here, $u(x,t)$ and $v(x,t)$ biologically stand for the spatial densities of infectious agent and infectious human population at location x and time t, respectively; α denotes the natural death rate of infectious agent; $1/\beta$ is the infectious period of infectious human population; $h(v)$ means the growth rate of infectious agent due to the infectious human population; $g(u)$ stands for the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of epidemic. In the system (1.1), the dispersals of infectious agent and infectious human population are described by

$$J_1 * u(x,t) - u(x,t) = \int_{\mathbb{R}} J_1(x-y)u(y,t)\mathrm{d}y - u(x,t),$$
$$J_2 * v(x,t) - v(x,t) = \int_{\mathbb{R}} J_2(x-y)v(y,t)\mathrm{d}y - v(x,t),$$

respectively. The nonlocal dispersals describe the long-distance movements of infectious agent and infectious humans across cites or countries by air traffic and other long-distance transportation. The system (1.1) can model the spread of the epidemic by fecal–oral transmission such as typhoid fever, cholera, hepatitis A, and poliomyelitis. Fecal–oral transmission occurs when infectious agents in the fecal material from an infected individual contaminate food or water, which is ingested by a second individual. Hence, fecal–oral route transmission is classically associated with contamination of water by human or animal waste. In (1.1), the nonlocal dispersal of infectious agent can be regarded as the long-distance transportation of infected food and water.

If infectious human do not move during the infectious period (for example, they are in sickbeds or quarantined probably), (1.1) reduces to the following degenerate model:

$$\begin{cases} u_t(x,t) = J_1 * u(x,t) - u(x,t) - \alpha u(x,t) + h(v(x,t)), & x \in \mathbb{R}, t \in \mathbb{R}, \\ v_t(x,t) = -\beta v(x,t) + g(u(x,t)), & x \in \mathbb{R}, t \in \mathbb{R}. \end{cases} \tag{1.2}$$

In addition, when the movements of infectious agent and infectious human population happen only between adjacent spatial locations, the local dispersal is more realistic and the local dispersal model is written as

$$\begin{cases} u_t(x,t) = \Delta u(x,t) - \alpha u(x,t) + h(v(x,t)), & x \in \mathbb{R}, t \in \mathbb{R}, \\ v_t(x,t) = \Delta v(x,t) - \beta v(x,t) + g(u(x,t)), & x \in \mathbb{R}, t \in \mathbb{R}. \end{cases} \tag{1.3}$$

The model (1.3) has been previously considered by Capasso and Maddalena (1981, 1982) to study the spread of cholera in the European Mediterranean region in 1973. With some suitable modifications, (1.3) was also used by Capasso and Wilson (1997) to model other epidemics with fecal–oral transmission.

In these epidemic models, the study of the wave propagation phenomena such as traveling wave solutions and entire solutions has attracted a number of researchers. A traveling wave solution is a special solution that keeps its shape while moving at a constant speed, and it can describe the spread of the infectious disease from an outbreak to an endemic disease. An entire solution means a classical solution defined in the whole space and at all time $t \in \mathbb{R}$. Mathematically, the study of entire solutions can help us to understand the structures of the global attractors, and biologically, different entire solutions represent different propagation patterns of disease. Hence, it is important and interesting to find more entire solutions. Since the pioneering works on reaction–diffusion (local dispersal) equations by Hamel and Nadirashvili (1999,

2001) for monostable case and Yagisita Yagisita (2003) for bistable case, there have been large amounts of research on entire solutions for various models, see, e.g., Chen and Guo (2005), Chen et al. (2006), Fukao et al. (2004), Li et al. (2008), and Morita and Ninomiya (2006). Particularly, for the local dispersal model (1.3), we refer to Hsu and Yang (2013), Volpert et al. (1994), Xu and Zhao (2005), and Zhao and Wang (2004) for the works on traveling wave solutions, and Wu and Hsu (2016) for the works on entire solutions. For the nonlocal dispersal model (1.1), we refer to Li et al. (2017) and Meng et al. (2019) for the works of monostable traveling wave solutions and entire solutions. We also refer to Hu et al. (2015) for the works of monostable traveling wave solutions in m-component cooperative systems which contain (1.1) as a special example.

Note that the aforementioned works for (1.1) have essentially considered only monostable traveling wave solutions and two-wave entire solutions. To the best of our knowledge, there is no result about bistable traveling wave solutions or three-wave (or more) entire solutions of (1.1). Here, we say (u, v) is a N-wave entire solution if it originates from the N traveling wave solutions $(c_j, \boldsymbol{\phi}_j)_{j=1,\dots,N}$, in the sense that,

$$\lim_{t\to-\infty}\sum_{j=1}^{N}\left(\sup_{\rho_j(t)\le x\le\rho_{j+1}(t)}\left\|(u,v)(x,t)-\boldsymbol{\phi}_j(x+c_jt+\vartheta_j)\right\|\right)=0, \tag{1.4}$$

where $\{\vartheta_j\}_{j=1,\dots,N}$ denotes some constant and $\{\rho_j(t)\}_{j=1,\dots,N+1}$ is some function satisfying $\rho_j(t) < \rho_{j+1}(t)$, $\rho_1(t) = -\infty$, and $\rho_{N+1}(t) = +\infty$. In fact, even for (1.2), there are almost no results on bistable traveling wave solutions and N-wave entire solutions except the works by Zhang et al. (2016) for two-wave entire solutions and Wu et al. (2018) for three-wave entire solutions. However, the study of (1.1) is much more difficult than (1.2), since (1.1) is essentially a two-component nonlocal dispersal (integral) system but (1.2) can be transformed into a scalar nonlocal dispersal equation by substituting $v(x,t) = e^{-\beta t}v(x,0) + \int_0^t e^{\beta(s-t)}g(u(x,s))\mathrm{d}s$ (from the second equation) into the first equation of (1.2). We also refer to Chen et al. (2018) and Dong et al. (2021) for the three-wave entire solutions of scalar bistable equations, and Guo and Wu (2019) for Lotka–Volterra diffusion–competition models.

The aim of this paper is to study monostable and bistable traveling wave solutions and three-wave entire solutions in the system (1.1) with two stable equilibria $E_0 = (0, 0)$ and $E_2 \in \mathrm{Int}(\mathbb{R}^2_+)$ and an unstable equilibrium $E_1 \in (E_0, E_2)$. Throughout this paper, we always use the standard ordering in $\mathbb{R}^2$. That is, for any $u = (u_1, u_2)$, $v = (v_1, v_2)$, we write $u \ge v$ if $u_i \ge v_i$, $i = 1, 2$; $u \gg v$ if $u_i > v_i$, $i = 1, 2$; and $u > v$ if $u_i \ge v_i$ with $i = 1, 2$ but $u \ne v$. For any $r_1, r_2 \in \mathbb{R}^2$ with $r_1 < r_2$, we define $[r_1, r_2] = \{u \in \mathbb{R}^2 : r_1 \le u \le r_2\}$ and $(r_1, r_2) = \{u \in \mathbb{R}^2 : r_1 < u < r_2\}$. For any $v \in \mathbb{R}^2$, $\|v\|$ means the Euclidean norm.

First, we study both monostable and bistable traveling wave solutions. Note that although (1.1) is bistable in the interval $[E_0, E_2]$, it also has two monostable intervals $[E_0, E_1]$ and $[E_1, E_2]$. In the intervals $[E_0, E_1]$ and $[E_1, E_2]$, we give a summary of the existence and asymptotic behavior of all monostable traveling wave solutions by applying and developing the known results for $[E_0, E_1]$ (see Li et al. 2017; Meng et al. 2019) into $[E_1, E_2]$. Furthermore, we find some relationship of monostable traveling wave solutions between these two monostable intervals, that is, the rightward (or leftward) traveling wave solution in $[E_0, E_1]$ has the same minimal wave speed as that in $[E_1, E_2]$, and when their wave speeds are equal, they also have the same exponential decay rate. In the interval $[E_0, E_2]$, we give the existence of bistable traveling wave solutions by introducing the result given by Fang and Zhao (2015) for monotone semiflows with weak compactness. The asymptotic behavior of these bistable traveling wave solutions is also obtained by studying the associated eigenvalue problems.

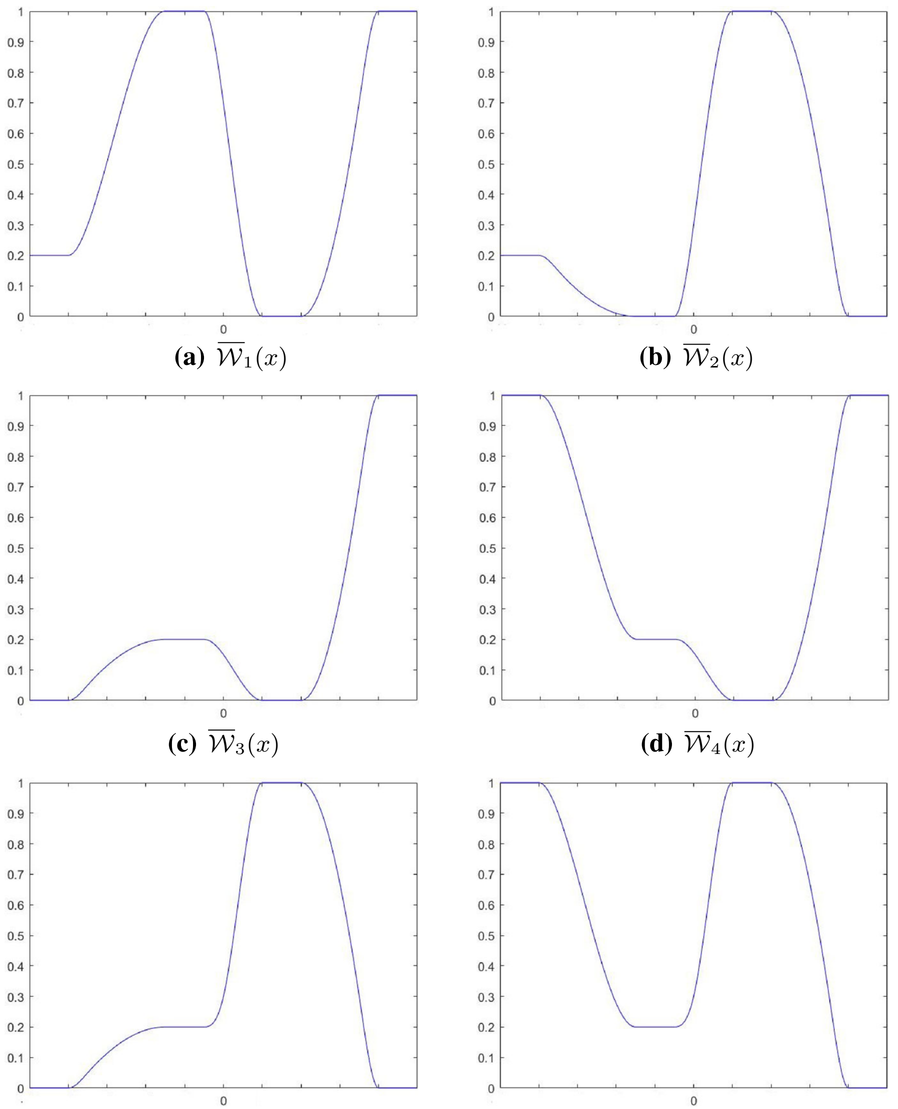
Second, we construct twelve types of three-wave entire solutions. In the construction, we only choose traveling wave solutions with nonzero wave speeds. Under this consideration, these twelve types contain all possibilities of three-wave entire solutions which originate from three traveling wave solutions $(c_j, \boldsymbol{\phi}_j)_{j=1,2,3}$ with different nonzero wave speeds. The sketchy profiles as $t \to -\infty$ of these entire solutions are depicted by Figs. 1 and 2. In fact, the construction of a N-wave entire solution originating from $(c_j, \boldsymbol{\phi}_j)_{j=1,\dots,N}$ is essentially a procedure linking the profiles of $\boldsymbol{\phi}_j$ with $\boldsymbol{\phi}_{j+1}$ for any $j = 1, \dots, N-1$. The tools are a pair of suitable super- and sub-solutions constructed by some appropriate auxiliary functions. Although the idea of linking two traveling wave solutions by super- and sub-solutions comes from Chen et al. (2018) and Morita and Ninomiya (2006), the difficulty is how to construct appropriate auxiliary functions and corresponding super- and sub-solutions for different types of entire solutions and different models. We construct new auxiliary functions and super- and sub-solutions only for the first six types of three-wave entire solutions (see Fig. 1), and the last six types (see Fig. 2) can be easily obtained from the first six types by a symmetric transformation on space $\mathbb{R}$. By applying these auxiliary functions and super- and sub-solutions, the existence of these twelve types of entire solutions is given. Under some appropriate conditions, we have that these twelve types of entire solutions are globally Lipschitz continuous with respect to x. In addition, the nonexistence result of entire solutions originating from more than four traveling wave solutions is obtained.

The rest of this paper is organized as follows. In Sect. 2, we give the existence and asymptotic behavior of all monostable and bistable traveling wave solutions of (1.1). In Sect. 3, we focus only on the construction of the three-wave entire solutions. The nonexistence result of entire solutions originating from more than four traveling wave solutions is also obtained. The proofs of three important lemmas in Sect. 3 are given in the Appendix.

2 Traveling Wave Solutions

In this section, we study monostable and bistable traveling wave solutions of (1.1). We first give the main assumptions. Let $g(\cdot)$ and $h(\cdot)$ be two functions satisfying

(J1) $g \in C^{1+\gamma_1}(\mathbb{R})$, $h \in C^{1+\gamma_2}(\mathbb{R})$ for some $\gamma_1, \gamma_2 \in (0, 1)$, $g(0) = h(0) = 0$, $g'(u) > 0$ for $u \geq 0$ and $h'(v) > 0$ for $v \geq 0$, and the function $f(u) \triangleq h(\frac{g(u)}{\beta}) - \alpha u$ is bistable, in the sense that there exists $u_1^* \in \mathbb{R}^+$ and $u_2^* \in \mathbb{R}^+$ such that $f(u_1^*) = f(u_2^*) = 0$, $f'(0) = g'(0)h'(0)/\beta - \alpha < 0$, $f'(u_1^*) > 0$, and $f'(u_2^*) < 0$;



(a) $\overline{\mathcal{W}}_1(x)$ **(b)** $\overline{\mathcal{W}}_2(x)$

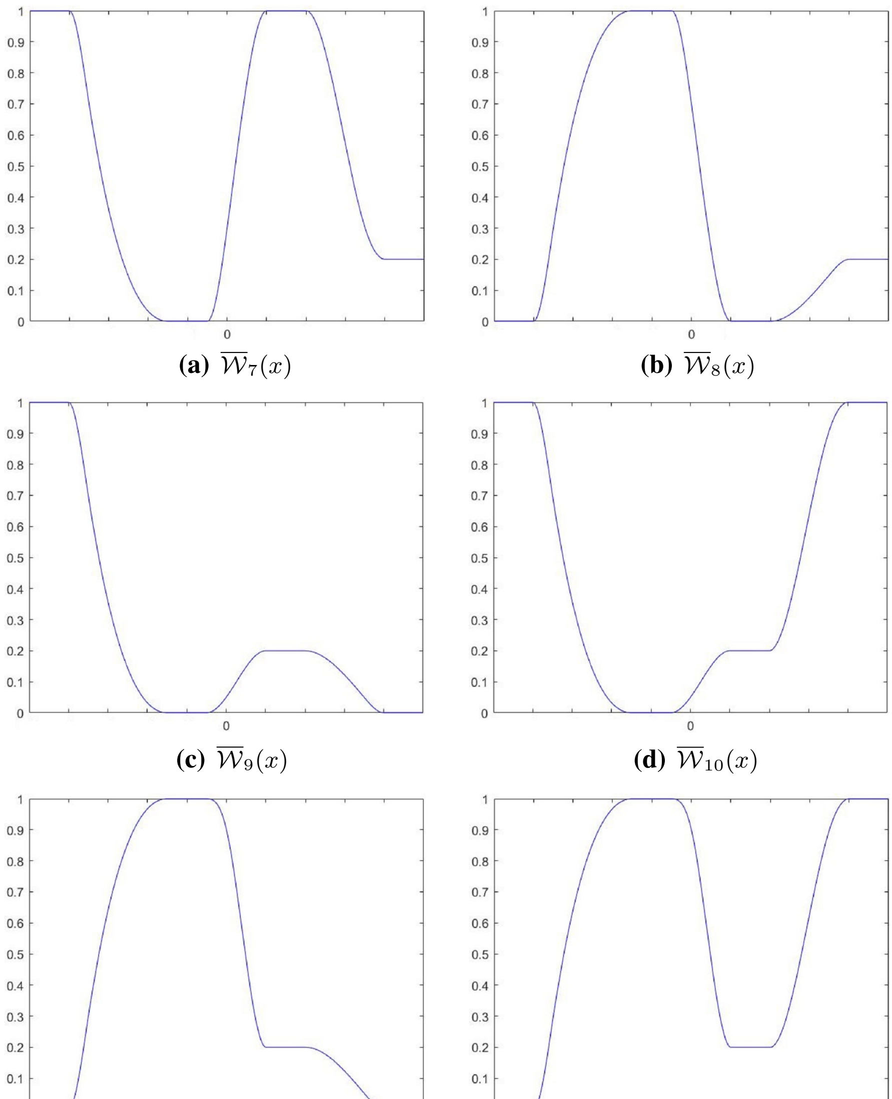
(c) $\overline{\mathcal{W}}_3(x)$ **(d)** $\overline{\mathcal{W}}_4(x)$

(e) $\overline{\mathcal{W}}_5(x)$ **(f)** $\overline{\mathcal{W}}_6(x)$

Fig. 1 The sketchy profiles of the first six entire solutions as $t \to -\infty$

(J2) $g'(u_1^*) \geq (g(u)-g(u_1^*))/(u-u_1^*)$ for $u \in [0, u_2^*]$ and $h'(v_1^*) \geq (h(v)-h(v_1^*))/(v-v_1^*)$ for $v \in [0, v_2^*]$, where $v_1^* \triangleq g(u_1^*)/\beta$ and $v_2^* \triangleq g(u_2^*)/\beta$.

By (J1), it is clear that (1.1) admits three nonnegative equilibria: two stable equilibria $E_0 = (0, 0)$ and $E_2 = (u_2^*, v_2^*)$, and an unstable equilibrium $E_1 = (u_1^*, v_1^*)$. Similar to the Fisher–KPP assumption (namely $f(u) \leq f'(0)u$) in a monostable reaction–diffusion equation, the purpose of (J2) is to ensure that the minimal wave speed is



(a) $\overline{\mathcal{W}}_7(x)$ **(b)** $\overline{\mathcal{W}}_8(x)$

(c) $\overline{\mathcal{W}}_9(x)$ **(d)** $\overline{\mathcal{W}}_{10}(x)$

(e) $\overline{\mathcal{W}}_{11}(x)$ **(f)** $\overline{\mathcal{W}}_{12}(x)$

Fig. 2 The sketchy profiles of the last six entire solutions as $t \to -\infty$

linearly selected. For example, according to Corollary 3.2 in Xu et al. (2020), we can obtain that for the monostable traveling wave solution connecting E_1 and E_2, the minimal wave speed $c_{\min}$ must satisfy $c_{\min} \geq c_L$, where c_L is defined in (2.2) later. If $c_{\min} = c_L$, then we say that the minimal wave speed is linearly selected; otherwise, i.e., $c_{\min} > c_L$, we say that the minimal wave speed is nonlinearly selected. Next, we give an example of $g(u)$ and $h(v)$ as follows:

$$g(u) = \beta\left[sin\left(\frac{\pi}{2}u - \frac{\pi}{2}\right) + 1\right], \qquad h(v) = \alpha v.$$

Then, the model (1.1) has three equilibria (0, 0), (1, 1), and (2, 2). Some calculations show that $g'(0)h'(0) = g'(2)h'(2) = 0 < \alpha\beta$, which implies that (0, 0) and (2, 2) are stable, and $g'(1)h'(1) = \frac{\pi}{2}\alpha\beta > \alpha\beta$, which means that (1, 1) is unstable.

Suppose that $J_1(\cdot)$ and $J_2(\cdot)$ are two continuous and nonnegative functions satisfying

(J3) For $i \in \{1, 2\}$, $J_i \in C^1(\mathbb{R})$, $J_i' \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} J_i(x)dx = 1$ and $J_i(x) \geq 0$ for $x \in \mathbb{R}$. Moreover, there exist $x_i^- \in \mathbb{R}^-$ and $x_i^+ \in \mathbb{R}^+$ such that $J_i(x_i^{\pm}) > 0$;

(J4) J_i has compact support and denote $m := \sup\{|y| : y \in \text{supp}(J_i), i = 1, 2\} > 0$.

Note that J_1 and J_2 in this paper may be asymmetric on $\mathbb{R}$.

Let $(u(x,t), v(x,t)) = (\varphi(\xi), \psi(\xi))$ with $\xi = x + ct$ be a traveling wave solution of (1.1) connecting E_i and E_j, where $(\varphi, \psi) \in C^1(\mathbb{R}, \mathbb{R}^2)$. By substituting it into (1.1), we get that

$$\begin{cases} c\varphi'(\xi) = J_1 * \varphi(\xi) - \varphi(\xi) - \alpha\varphi(\xi) + h(\psi(\xi)), \\ c\psi'(\xi) = J_2 * \psi(\xi) - \psi(\xi) - \beta\psi(\xi) + g(\varphi(\xi)), \end{cases} \tag{2.1}$$

and the asymptotic boundary conditions are

$$\lim_{\xi \to -\infty} (\varphi(\xi), \psi(\xi)) = E_i, \qquad \lim_{\xi \to +\infty} (\varphi(\xi), \psi(\xi)) = E_j.$$

The main results of this section consist of two parts. First, in Sect. 2.1, we summarize the results about the existence and asymptotic behavior of all monostable traveling wave solutions in intervals $[E_0, E_1]$ and $[E_1, E_2]$, and further find some relationship between the traveling wave solutions in these two monostable intervals. Second, in Sect. 2.2, we introduce the theory by Fang and Zhao (2015) about the monotone semiflows with weak compactness and obtain the existence of bistable traveling wave solutions. The asymptotic behavior of bistable traveling wave solutions is also given by studying the corresponding eigenvalue problem.

2.1 Monostable Traveling Wave Solutions

Although the system (1.1) satisfies the bistable assumption (J1), it also admits some monostable traveling wave solutions, which play a very important role in the construction of entire solutions in Sect. 3. Indeed, if we restrict (u, v) in the interval $[E_0, E_1]$ or $[E_1, E_2]$, then (1.1) can be regarded as a monostable system. Define

$$c(\lambda) = \frac{1}{2\lambda}\left[F_u(0,\lambda) + F_v(0,\lambda) + \sqrt{(F_u(0,\lambda) - F_v(0,\lambda))^2 + 4g'(u_1^*)h'(v_1^*)}\right] \text{ for } \lambda \neq 0,$$

where

$$F_u(c,\lambda) = \int_{\mathbb{R}} J_1(z)e^{-\lambda z}dz - 1 - c\lambda - \alpha,$$
$$F_v(c,\lambda) = \int_{\mathbb{R}} J_2(z)e^{-\lambda z}dz - 1 - c\lambda - \beta.$$

By Theorem 2.1 in Xu et al. (2020), we can define

$$c_L \triangleq \inf_{\lambda\in\mathbb{R}^+} c(\lambda) \text{ and } c_R \triangleq \sup_{\lambda\in\mathbb{R}^-} c(\lambda), \tag{2.2}$$

and it holds that

$$c_R < c_L. \tag{2.3}$$

The existence of all monostable traveling wave solutions in (1.1) is given by the following two theorems, which are straightforward consequences of Hu et al. (2015, Theorem 3.6), Li et al. (2017, Theorem 2.1), and Xu et al. (2020, Corollary 3.2).

Theorem 2.1 *Assume (J1)–(J4) hold. Then, for any $c_1 \le c_R$, (1.1) admits a nondecreasing monostable traveling wave solution $\Phi_1(x + c_1t)$ satisfying*

$$\Phi_1(-\infty) = E_0, \quad \Phi_1(+\infty) = E_1. \tag{2.4}$$

Moreover, (1.1) has no traveling wave solution $\Phi_1(x + c_1t)$ with $c_1 > c_R$ satisfying (2.4). Similarly, for any $\hat{c}_1 \ge c_L$, (1.1) admits a nonincreasing monostable traveling wave solution $\hat{\Phi}_1(x + \hat{c}_1t)$ satisfying

$$\hat{\Phi}_1(-\infty) = E_1, \quad \hat{\Phi}_1(+\infty) = E_0. \tag{2.5}$$

Moreover, (1.1) has no traveling wave solution $\hat{\Phi}_1(x + \hat{c}_1t)$ with $\hat{c}_1 < c_L$ satisfying (2.5).

Theorem 2.2 *Assume (J1)–(J4) hold. Then, for any $c_2 \ge c_L$, (1.1) admits a nondecreasing monostable traveling wave solution $\Phi_2(x + c_2t)$ satisfying*

$$\Phi_2(-\infty) = E_1, \quad \Phi_2(+\infty) = E_2. \tag{2.6}$$

Moreover, (1.1) has no traveling wave solution $\Phi_2(x + c_2t)$ with $c_2 < c_L$ satisfying (2.6). Similarly, for any $\hat{c}_2 \le c_R$, (1.1) admits a nonincreasing monostable traveling wave solution $\hat{\Phi}_2(x + \hat{c}_2t)$ satisfying

$$\hat{\Phi}_2(-\infty) = E_2, \quad \hat{\Phi}_2(+\infty) = E_1. \tag{2.7}$$

Moreover, (1.1) has no traveling wave solution $\hat{\Phi}_2(x + \hat{c}_2t)$ with $\hat{c}_2 > c_R$ satisfying (2.7).

Remark 2.3 The traveling wave solution with nonzero wave speed in Theorems 2.1 and 2.2 is unique up to translation. When the wave speed is equal to zero, the continuous traveling wave solution is unique up to translation. We refer to Coville et al. (2008) for more details about the uniqueness of traveling wave solutions in a nonlocal dispersal equation.

Now, we consider the asymptotic behavior of these traveling wave solutions. For c and $\lambda \in \mathbb{R}$, define the functions

$$\begin{aligned} \Gamma_0(c,\lambda) &= F_u(c,\lambda)\cdot F_v(c,\lambda) - h'(0)g'(0),\\ \Gamma_1(c,\lambda) &= F_u(c,\lambda)\cdot F_v(c,\lambda) - h'(v_1^*)g'(u_1^*),\\ \Gamma_2(c,\lambda) &= F_u(c,\lambda)\cdot F_v(c,\lambda) - h'(v_2^*)g'(u_2^*). \end{aligned}$$

For any fixed $c \in \mathbb{R}$, we consider the equation $\Gamma_i(c,\lambda) = 0$ with $i \in \{1, 2, 3\}$. The following two lemmas study the number of its roots.

Lemma 2.4 *Assume (J1), (J3), and (J4) hold. Then, for any fixed $c \in \mathbb{R}$, $\Gamma_i(c,\lambda) = 0$ with $i \in \{0, 2\}$ has exactly two negative roots and two positive roots.*

Proof We only consider the equation $\Gamma_0(c,\lambda) = 0$, since the other case is similar. Some simple calculations imply that

$$F_u(c,0) < 0,\ \lim_{\lambda\to\pm\infty} F_u(c,\lambda) = +\infty,\ \text{ and } \frac{\partial^2 F_u(c,\lambda)}{\partial\lambda^2} > 0 \text{ for any } \lambda \in \mathbb{R}.$$

Then, the equation $F_u(c,\lambda) = 0$ has two real roots $\lambda_u^-(c) \in (-\infty, 0)$ and $\lambda_u^+(c) \in (0, +\infty)$ by the continuity of $F_u(c,\lambda)$ with respect to λ. Similarly, the equation $F_v(c,\lambda) = 0$ has two real roots $\lambda_v^-(c) \in (-\infty, 0)$ and $\lambda_v^+(c) \in (0, +\infty)$. It is easy to check that

$$\begin{aligned} &\Gamma_0(c,\pm\infty) = +\infty,\ \Gamma_0(c,0) = \alpha\beta - h'(0)g'(0) > 0,\\ &\Gamma_0(c,\lambda) < 0 \text{ for } \lambda \in [\min\{\lambda_u^+(c), \lambda_v^+(c)\}, \max\{\lambda_u^+(c), \lambda_v^+(c)\}]. \end{aligned}$$

Thus, the equation $\Gamma_0(c,\lambda) = 0$ has a unique root in $(\max\{\lambda_u^+(c), \lambda_v^+(c)\}, +\infty)$ and at least one root in $(0, \min\{\lambda_u^+(c), \lambda_v^+(c)\})$. Applying a similar method to the proof of Li et al. (2017, Lemma 2.6), we can obtain that $\Gamma_0(c,\lambda) = 0$ has at most one root in $(0, \min\{\lambda_u^+(c), \lambda_v^+(c)\})$. Then, the equation $\Gamma_0(c,\lambda) = 0$ has two positive roots. Similarly, the equation $\Gamma_0(c,\lambda) = 0$ has two negative roots. It completes the proof. □

Lemma 2.5 *Assume (J1)–(J4) hold. For $c > c_L$, $\Gamma_1(c,\lambda) = 0$ has and only has three different positive roots. When $c = c_L$, $\Gamma_1(c,\lambda) = 0$ has two different positive roots. Similarly, for $c < c_R$, $\Gamma_1(c,\lambda) = 0$ has and only has three different negative roots. When $c = c_R$, $\Gamma_1(c,\lambda) = 0$ has two different negative roots.*

Proof For the case $c \geq c_L$, the proof can be obtained by a similar method to Li et al. (2017, Lemma 2.6). For the case $c < c_R$, consider

$$\tilde{\Gamma}(c,\lambda) \triangleq \Gamma_1(-c,-\lambda) = F_u(-c,-\lambda)\cdot F_v(-c,-\lambda) - h'(v_1^*)g'(u_1^*)$$

with

$$F_u(-c,-\lambda) = \int_{\mathbb{R}} J_1(z)e^{\lambda z}\mathrm{d}z - 1 - c\lambda - \alpha,$$
$$F_v(-c,-\lambda) = \int_{\mathbb{R}} J_2(z)e^{\lambda z}\mathrm{d}z - 1 - c\lambda - \beta.$$

By substituting $J_i(z)$ for $J_i(-z)$ with $i \in \{1, 2\}$, we see that the equation $\tilde{\Gamma}(c,\lambda) = 0$ has the same property of roots as $\Gamma_1(c,\lambda) = 0$. Namely, $\tilde{\Gamma}(c,\lambda) = 0$ has and only has three different positive roots for $c > \tilde{c}_L$ and two different positive roots for $c = \tilde{c}_L$, where

$$\tilde{c}_L = \inf_{\lambda\in\mathbb{R}^+}\left\{\frac{1}{2\lambda}\left[F_u(0,-\lambda) + F_v(0,-\lambda) + \sqrt{(F_u(0,-\lambda) - F_v(0,-\lambda))^2 + 4g'(u_1^*)h'(v_1^*)}\right]\right\}.$$

A simple calculation shows that $\tilde{c}_L = -c_R$, which completes the proof. □

For $c \in \mathbb{R}$, we define

$$\begin{aligned} \lambda_0^+(c) &= \text{the smallest positive root of } \Gamma_0(c,\lambda) = 0,\\ \lambda_0^-(c) &= \text{the largest negative root of } \Gamma_0(c,\lambda) = 0,\\ \lambda_2^+(c) &= \text{the smallest positive root of } \Gamma_2(c,\lambda) = 0,\\ \lambda_2^-(c) &= \text{the largest negative root of } \Gamma_2(c,\lambda) = 0, \end{aligned} \tag{2.8}$$

and

$$\lambda_1^+(c) = \text{the smallest positive root of } \Gamma_1(c,\lambda) = 0 \text{ for } c \geq c_L > 0,$$
$$\lambda_1^-(c) = \text{the largest negative root of } \Gamma_1(c,\lambda) = 0 \text{ for } c \leq c_R < 0.$$

The asymptotic behavior of all monostable traveling wave solutions is given by the following two theorems. Since the calculations for different cases in the proof are analogous and similar to Li et al. (2017, Theorem 2.2), we omit them.

Theorem 2.6 *Assume (J1)–(J4) hold. Let $\Phi_1(x + c_1 t)$ and $\hat{\Phi}_1(x + \hat{c}_1 t)$ be the nondecreasing and nonincreasing traveling wave solutions with $c_1 \neq 0$ and $\hat{c}_1 \neq 0$ in (1.1) as given in Theorem 2.1. Then, all the following limits exist in $\mathbb{R}^+ \times \mathbb{R}^+$ and*

(i) *for* $c_1 < c_R$, $\lim_{\xi\to+\infty} (E_1 - \Phi_1(\xi))/e^{\lambda_1^-(c_1)\xi} = -\lim_{\xi\to+\infty} \Phi_1'(\xi)/(\lambda_1^-(c_1)e^{\lambda_1^-(c_1)\xi})$,

for $c_1 = c_R$, $\lim_{\xi\to+\infty} (E_1 - \Phi_1(\xi))/(|\xi|e^{\lambda_1^-(c_1)\xi}) = -\lim_{\xi\to+\infty} \Phi_1'(\xi)/(\lambda_1^-(c_1)|\xi|e^{\lambda_1^-(c_1)\xi})$,

for $c_1 \leq c_R$, $\lim_{\xi\to-\infty} \Phi_1(\xi)/e^{\lambda_0^+(c_1)\xi} = \lim_{\xi\to-\infty} \Phi_1'(\xi)/(\lambda_0^+(c_1)e^{\lambda_0^+(c_1)\xi})$,

(ii) *for* $\hat{c}_1 > c_L$, $\lim_{\xi\to-\infty}(E_1-\hat{\Phi}_1(\xi))/e^{\lambda_1^+(\hat{c}_1)\xi} = -\lim_{\xi\to-\infty}\hat{\Phi}_1'(\xi)/(\lambda_1^+(\hat{c}_1)e^{\lambda_1^+(\hat{c}_1)\xi})$,
for $\hat{c}_1 = c_L$, $\lim_{\xi\to-\infty}(E_1-\hat{\Phi}_1(\xi))/(|\xi|e^{\lambda_1^+(\hat{c}_1)\xi}) = -\lim_{\xi\to-\infty}\hat{\Phi}_1'(\xi)/(\lambda_1^+(\hat{c}_1)|\xi|e^{\lambda_1^+(\hat{c}_1)\xi})$,
for $\hat{c}_1 \geq c_L$, $\lim_{\xi\to+\infty}\hat{\Phi}_1(\xi)/e^{\lambda_0^-(\hat{c}_1)\xi} = \lim_{\xi\to+\infty}\hat{\Phi}_1'(\xi)/(\lambda_0^-(\hat{c}_1)e^{\lambda_0^-(\hat{c}_1)\xi})$.

Theorem 2.7 *Assume (J1)–(J4) hold. Let $\Phi_2(x+c_2t)$ and $\hat{\Phi}_2(x+\hat{c}_2t)$ be the nondecreasing and nonincreasing traveling wave solutions with $c_2 \neq 0$ and $\hat{c}_2 \neq 0$ in (1.1) as given in Theorem 2.2. Then, all the following limits exist in $\mathbb{R}^+\times\mathbb{R}^+$ and*

(i) *for* $c_2 > c_L$, $\lim_{\xi\to-\infty}(\Phi_2(\xi)-E_1)/e^{\lambda_1^+(c_2)\xi} = \lim_{\xi\to-\infty}\Phi_2'(\xi)/(\lambda_1^+(c_2)e^{\lambda_1^+(c_2)\xi})$,
for $c_2=c_L$, $\lim_{\xi\to-\infty}(\Phi_2(\xi)-E_1)/(|\xi|e^{\lambda_1^+(c_2)\xi}) = \lim_{\xi\to-\infty}\Phi_2'(\xi)/(\lambda_1^+(c_2)|\xi|e^{\lambda_1^+(c_2)\xi})$,
for $c_2 \geq c_L$, $\lim_{\xi\to+\infty}(E_2-\Phi_2(\xi))/e^{\lambda_2^-(c_2)\xi} = -\lim_{\xi\to+\infty}\Phi_2'(\xi)/(\lambda_2^-(c_2)e^{\lambda_2^-(c_2)\xi})$,

(ii) *for* $\hat{c}_2 < c_R$, $\lim_{\xi\to+\infty}(\hat{\Phi}_2(\xi)-E_1)/e^{\lambda_1^-(\hat{c}_2)\xi} = \lim_{\xi\to+\infty}\hat{\Phi}_2'(\xi)/(\lambda_1^-(\hat{c}_2)e^{\lambda_1^-(\hat{c}_2)\xi})$,
for $\hat{c}_2 = c_R$, $\lim_{\xi\to+\infty}(\hat{\Phi}_2(\xi)-E_1)/(|\xi|e^{\lambda_1^-(\hat{c}_2)\xi}) = \lim_{\xi\to+\infty}\hat{\Phi}_2'(\xi)/(\lambda_1^-(\hat{c}_2)|\xi|e^{\lambda_1^-(\hat{c}_2)\xi})$,
for $\hat{c}_2 \leq c_R$, $\lim_{\xi\to-\infty}(E_2-\hat{\Phi}_2(\xi))/e^{\lambda_2^+(\hat{c}_2)\xi} = -\lim_{\xi\to-\infty}\hat{\Phi}_2'(\xi)/(\lambda_2^+(\hat{c}_2)e^{\lambda_2^+(\hat{c}_2)\xi})$.

From Theorems 2.1, 2.2, 2.6, and 2.7, we can get some relationship between the traveling wave solutions in the intervals $[E_0, E_1]$ and $[E_1, E_2]$. We see that the two rightward traveling wave solutions Φ_1 and $\hat{\Phi}_2$ have the same minimal wave speed. And when their wave speeds are equal, they also have the same exponential decay rate as $\xi \to +\infty$. The difference is that Φ_1 is in the interval $[E_0, E_1]$, while $\hat{\Phi}_2$ is in $[E_1, E_2]$, and hence the convergence $\Phi_1 \to E_1$ is from below, while the convergence $\Phi_2 \to E_1$ is from above. Similarly, for the two leftward traveling wave solutions $\hat{\Phi}_1$ and Φ_2, although the convergence $\hat{\Phi}_1 \to E_1$ is from below, while the convergence $\Phi_2 \to E_1$ is from above as $\xi \to -\infty$, they have the same minimal wave speed, and when their wave speeds are equal, they also have the same exponential decay rate.

2.2 Bistable Traveling Wave Solutions

In this subsection, we introduce the theory developed by Fang and Zhao (2015) about the monotone semiflows with weak compactness, and study the existence and asymptotic behavior of bistable traveling wave solutions in (1.1).

First, we give some notations. Let $\mathcal{X}$ be an ordered Banach space with the norm $\|\cdot\|_{\mathcal{X}}$ and the cone $\mathcal{X}^+$. Assume that $\mathrm{Int}(\mathcal{X}^+)$ is not empty. Let $\mathcal{C}$ denote the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathcal{X}$ and let $\mathcal{M}$ denote the set of all nondecreasing functions from $\mathbb{R}$ to $\mathcal{X}$. We equip $\mathcal{C}$ and $\mathcal{M}$ with the compact open topology and the standard cone consisting of all nonnegative functions. For any $u, v \in \mathcal{H}$ with $\mathcal{H} \in \{\mathcal{X}, \mathcal{C}, \mathcal{M}\}$, we write $u \geq v$ if $u - v \in \mathcal{H}^+$; $u > v$ if $u \geq v$ but $u \neq v$; and $u \gg v$ if $u - v \in \mathrm{Int}(\mathcal{H}^+)$. For any $r_1, r_2 \in \mathcal{X}$ with $r_1 < r_2$, we define $\mathcal{H}_{[r_1,r_2]} = \{u \in \mathcal{H} : r_1 \leq u \leq r_2\}$ and write $\mathcal{H}_{[0,r]}$ as $\mathcal{H}_r$ for short. We say

$\{Q_t\}_{t\geq 0}$ is a semiflow on $\mathcal{M}_{\mathbf{b}}$ when it satisfies that $Q_0 = I$ where I is a identity map, $Q_t \circ Q_s = Q_{t+s}$ for any $t, s > 0$, and $Q_{t_n}[\varphi_n](y) \to Q_t[\varphi](y)$ in $\mathcal{X}_{\mathbf{b}}$ for almost all $y \in \mathbb{R}$ whenever $t_n \to t$ and $\varphi_n \to \varphi$ in $\mathcal{M}_{\mathbf{b}}$.

Let Q be a map from $\mathcal{M}_{\mathbf{b}}$ to $\mathcal{M}_{\mathbf{b}}$ with two fixed points $\mathbf{0}$ and $\mathbf{b}$. We denote E by the set consisting of all fixed points of Q restricted on $\mathcal{X}_{\mathbf{b}}$. The following definition states the stability of fixed points.

Definition 2.8 For $Q : \mathcal{M}_{\mathbf{b}} \to \mathcal{M}_{\mathbf{b}}$, a fixed point $\mathbf{a} \in E$ is said to be strongly stable from below if there is a constant $\delta > 0$ and a unit vector $\mathbf{e} \in \mathrm{Int}(\mathcal{X}^+)$ such that for any $\eta \in (0, \delta]$,

$$Q[\mathbf{a} - \eta\mathbf{e}] \gg \mathbf{a} - \eta\mathbf{e}. \tag{2.9}$$

Strong instability from below is defined by reversing the inequality (2.9). Similarly, we can define strong stability and strong instability from above.

Now, we give some fundamental assumptions on Q.

(B1) (*Translation Invariance*) $T_y \circ Q[\phi] = Q \circ T_y[\phi]$ for all $\phi \in \mathcal{M}_{\mathbf{b}}$ and $y \in \mathbb{R}$, where $T_y[\phi](x) := \phi(x - y)$.

(B2) (*Continuity*) $Q : \mathcal{M}_{\mathbf{b}} \to \mathcal{M}_{\mathbf{b}}$ is continuous in the sense that if $\phi_n \to \phi$ in $\mathcal{M}_{\mathbf{b}}$, then $Q[\phi_n](x) \to Q[\phi](x)$ in $\mathcal{X}_{\mathbf{b}}$ for almost all $x \in \mathbb{R}$.

(B3) (*Monotonicity*) Q is order preserving in the sense that $Q[\phi] \geq Q[\varphi]$ whenever $\phi \geq \varphi$.

(B4) (*Weak Compactness*) For any fixed $y \in \mathbb{R}$, the set $Q[\mathcal{M}_{\mathbf{b}}](y)$ is precompact in $\mathcal{X}_{\mathbf{b}}$.

(B5) (*Bistability*) Fixed points $\mathbf{0}$ and $\mathbf{b}$ are strongly stable from above and below, respectively, for the map $Q : \mathcal{X}_{\mathbf{b}} \to \mathcal{X}_{\mathbf{b}}$. The set $E \setminus \{\mathbf{0}, \mathbf{b}\} \subset \mathcal{X}_{\mathbf{b}}$ is totally unordered.

For each $\mathbf{a} \in E \setminus \{\mathbf{0}, \mathbf{b}\}$, it follows from the assumption (B5) that Q has no other fixed point in both $\mathcal{M}_{[\mathbf{a},\mathbf{b}]} \setminus \{\mathbf{a}, \mathbf{b}\}$ and $\mathcal{M}_{[\mathbf{0},\mathbf{a}]} \setminus \{\mathbf{0}, \mathbf{a}\}$. In view of the assumption (B5), there are constants $\delta_1, \delta_2 > 0$ and unit vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{Int}(\mathcal{X}^+)$ such that $Q[\eta\mathbf{e}_1] \ll \eta\mathbf{e}_1$ for $\eta \in (0, \delta_1]$ and $Q[\mathbf{b} - \eta\mathbf{e}_2] \gg \mathbf{b} - \eta\mathbf{e}_2$ for $\eta \in (0, \delta_2]$. Denote $\theta^- = \sup\{\theta \in [0, 1] : \theta\mathbf{a} + (1 - \theta)\mathbf{b} \in \mathcal{X}_{[\mathbf{b}-\delta_2\mathbf{e}_2,\mathbf{b}]}\}$ and $\theta^+ = \sup\{\theta \in [0, 1] : \theta\mathbf{a} \in \mathcal{X}_{[0,\delta_1\mathbf{e}_1]}\}$. Let $\nu^- = \theta^-\mathbf{a} + (1 - \theta^-)\mathbf{b}$ and $\nu^+ = \theta^+\mathbf{a}$. Define

$$c_-^*(\mathbf{a}, \mathbf{b}) = \sup\left\{c \in \mathbb{R} : \lim_{n\to\infty, x\geq -cn} Q^n[\phi_{\mathbf{a}}^-](x) = \mathbf{b}\right\}, \tag{2.10}$$

$$c_+^*(\mathbf{0}, \mathbf{a}) = \sup\left\{c \in \mathbb{R} : \lim_{n\to\infty, x\leq cn} Q^n[\phi_{\mathbf{a}}^+](x) = \mathbf{0}\right\}, \tag{2.11}$$

where the continuous initial functions $\phi_{\mathbf{a}}^{\pm} \in \mathcal{M}_{\mathbf{b}}$ satisfy that

$$\phi_{\mathbf{a}}^-(x) = \mathbf{a} \text{ for } x \leq -1,\ \phi_{\mathbf{a}}^-(x) = \nu^- \text{ for } x \geq 0,$$
$$\phi_{\mathbf{a}}^+(x) = \mathbf{a} \text{ for } x \geq 1,\ \phi_{\mathbf{a}}^+(x) = \nu^+ \text{ for } x \leq 0.$$

We further assume that

(B6) (*Counter-propagation*) For each $\mathbf{a} \in E \setminus \{\mathbf{0}, \mathbf{b}\}$, $c_-^*(\mathbf{a}, \mathbf{b}) + c_+^*(\mathbf{0}, \mathbf{a}) > 0$.

As stated in Fang and Zhao (2015), assumption (B6) can ensure that the spatial propagation directions of the two monostable systems restricted on $\mathcal{M}_{[\mathbf{a},\mathbf{b}]}$ and $\mathcal{M}_{[\mathbf{0},\mathbf{a}]}$ with front-like initial data are opposite.

Let $C(M, \mathbb{R}^d)$ denote the set of all continuous functions from the compact metric space M to the d-dimensional Euclidean space $\mathbb{R}^d$ with the maximum norm and the standard cone consisting of all nonnegative functions. The following theorem shows the existence of nondecreasing bistable traveling wave solutions in monotone semiflows with weak compactness. It is a key tool to study bistable traveling wave solutions in (1.1).

Theorem 2.9 (see Fang and Zhao 2015, Theorem 5.3) *Let $\mathcal{X} = C(M, \mathbb{R}^d)$. Assume that $\{Q_t\}_{t\geq 0}$ is a semiflow on $\mathcal{M}_{\boldsymbol{b}}$ and for any $t > 0$, the map Q_t satisfies (B1) and (B3)–(B6). Then, there is a constant $c \in \mathbb{R}$ and a nondecreasing function $\phi \in \mathcal{M}_{\boldsymbol{b}}$ connecting $\boldsymbol{0}$ to $\boldsymbol{b}$ such that $Q_t[\phi](x) = \phi(x + ct)$ for $x \in \mathbb{R}$.*

In what follows, we apply Theorem 2.9 and prove the existence of bistable traveling solutions connecting E_0 and E_2 in (1.1). Note that we usually choose $\mathcal{X} = C(M, \mathbb{R}^d)$ to study some more complex systems, such as the time-periodic reaction–diffusion systems and the nonlocal dispersal equations with time delay (see, e.g., Fang and Zhao 2014, 2015). However, in (1.1), we only need to choose $\mathcal{X} = \mathbb{R}^2$ with the Euclidean norm $\|\cdot\|$, which means that M is a singleton. Let $P(t)$ be the solution semigroup of the following nonlocal dispersal equation:

$$\begin{cases} u_t = J * u - u, \ \ x \in \mathbb{R}, t > 0, \\ u(x, 0) = \varphi(x), \ \ x \in \mathbb{R}. \end{cases}$$

As stated in Weng and Zhao (2006), $P(t)$ can be written as

$$P(t)[\varphi](x) = e^{-t} \sum_{m=0}^{\infty} \frac{t^m}{m!} a_m(\varphi)(x), \tag{2.12}$$

where $a_0(\varphi) = \varphi$ and $a_m(\varphi) = J * a_{m-1}(\varphi)$ for $m \geq 1$. Now, we consider the Cauchy problem of the system (1.1) as follows:

$$\begin{cases} u_t(x,t) = J_1 * u(x,t) - u(x,t) - \alpha u(x,t) + h(v(x,t)), & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ v_t(x,t) = J_2 * v(x,t) - v(x,t) - \beta v(x,t) + g(u(x,t)), & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ (u(x,0), v(x,0)) = \phi(x), & x \in \mathbb{R}. \end{cases} \tag{2.13}$$

Let

$$\boldsymbol{\omega} = (u, v)^T, \ \ \mathbf{F}(\boldsymbol{\omega}) = \begin{pmatrix} -\alpha u + h(v) \\ -\beta v + g(u) \end{pmatrix} \ \ \text{and} \ \ \mathbf{P}(t) = \operatorname{diag}\left(P_1(t), P_2(t)\right),$$

where $P_i(t)$ with $i \in \{1, 2\}$ is defined by (2.12) with substituting J_i for J. Then, (2.13) can be written as

$$\boldsymbol{\omega}(x,t) = \mathbf{P}(t)[\phi](x) + \int_0^t \mathbf{P}(t-s)[\mathbf{F}(\boldsymbol{\omega}(\cdot,s))](x)\mathrm{d}s, \quad x \in \mathbb{R}, \ t \ge 0.$$

By applying the standard contracting mapping theorem or the theory of abstract functional–differential equations in Martin and Smith (1990), the existence and uniqueness of the solution of (2.13) can be obtained as follows.

Lemma 2.10 *Assume (J1) and (J3) hold. For any $\phi \in \mathcal{M}_{E_2}$ (or $\mathcal{C}_{E_2}$) with $E_0 \le \phi \le E_2$, the system (1.1) has a unique solution $\boldsymbol{\omega}(x,t;\phi)$ on $\mathbb{R} \times [0,\infty)$ satisfying $\boldsymbol{\omega}(x,0;\phi) = \phi(x)$, and moreover, $E_0 \le \boldsymbol{\omega}(x,t;\phi) \le E_2$.*

For $t > 0$, let $Q_t : \mathcal{M}_{E_2} \to \mathcal{M}_{E_2}$ satisfy $Q_t(\phi) = \boldsymbol{\omega}(\cdot,t;\phi)$. It is easy to check that Q_t satisfies assumptions (B1) and (B4) for any $t > 0$. Next, we show that $\{Q_t\}_{t\ge0}$ is a semiflow on $\mathcal{M}_{E_2}$ and Q_t satisfies assumption (B3) for any $t > 0$.

Lemma 2.11 *Assume (J1) and (J3) hold. Then, $\{Q_t\}_{t\ge0}$ is a monotone semiflow on $\mathcal{M}_{E_2}$.*

Proof For any $t > 0$, we first show the monotonicity of Q_t on $\mathcal{M}_{E_2}$. By (J1), the function $\mathbf{F} : \mathcal{X}_{E_2} \to \mathbb{R}^2$ is Lipschitz continuous and cooperative. Thus, we can choose a sufficiently large constant $a > 0$ such that the function $\mathbf{F}_a(\boldsymbol{\omega}) := \mathbf{F}(\boldsymbol{\omega}) + a\boldsymbol{\omega}$ is nondecreasing on $\mathcal{X}_{E_2}$. Then, (2.13) can be written as an abstract integral equation on $\mathcal{M}_{E_2}$, namely

$$\boldsymbol{\omega}(t) = e^{-at}\mathbf{P}(t)[\phi] + \int_0^t e^{-a(t-s)}\mathbf{P}(t-s)[\mathbf{F}_a(\boldsymbol{\omega}(s))]\mathrm{d}s, \ t \ge 0.$$

By a similar argument to Thieme (1979, Lemma 3.2), we can prove that $Q_t : \mathcal{M}_{E_2} \to \mathcal{M}_{E_2}$ is monotone for any $t > 0$.

It suffices to prove that Q_t is continuous for $(t,\phi) \in \mathbb{R}^+ \times \mathcal{M}_{E_2}$ with respect to the compact open topology. For any given $\phi \in \mathcal{M}_{E_2}$, it follows from (1.1) that $\frac{\partial}{\partial t}\boldsymbol{\omega}(x,t;\phi)$ is bounded in $\mathbb{R}^2$ for any $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Then, there exists a constant $L = L(\phi) > 0$ such that

$$\|\boldsymbol{\omega}(x,t_1;\phi) - \boldsymbol{\omega}(x,t_2;\phi)\| \le L|t_1 - t_2| \ \text{ for any } x \in \mathbb{R} \text{ and } t_1, t_2 \ge 0.$$

Therefore, for any $\phi \in \mathcal{M}_{E_2}$, the function $t \mapsto Q_t(\phi) = \boldsymbol{\omega}(\cdot,t;\phi)$ from $\mathbb{R}^+$ to $\mathcal{M}_{E_2}$ is continuous with respect to the compact open topology.

For any $\phi_1, \phi_2 \in \mathcal{M}_{E_2}$, $\epsilon > 0$, and $t_0 > 0$, we define

$$\begin{aligned}
&\gamma(x,t) = (\gamma_1(x,t), \gamma_2(x,t)) = (u(x,t;\phi_1) - u(x,t;\phi_2), v(x,t;\phi_1) - v(x,t;\phi_2)),\\
&k_1 = \sup_{x\in\mathbb{R}, t\in[0,t_0]} \gamma_1(x,t), \quad k_2 = \sup_{x\in\mathbb{R}, t\in[0,t_0]} \gamma_2(x,t),\\
&N_1 = \max_{[0,u_2^*]} g'(u), \quad N_2 = \max_{[0,v_2^*]} h'(v),
\end{aligned}$$

$$\epsilon' = \frac{\sqrt{2}\epsilon}{2(1+\max\{N_1, N_2\}t_0)e^{N_1N_2t_0^2}}, \quad \varepsilon = \frac{\epsilon'}{2+e^{t_0}+\max\{N_1, N_2\}t_0},$$
$$\Omega_\rho(z) = [-\rho+z, \rho+z] \text{ for } \rho > 0, z \in \mathbb{R}, \quad |\varphi|_{\Omega_\rho(z)} = \sup_{x\in\Omega_\rho(z)} |\varphi(x)|.$$

Then, there exist (x_1^*, t_1^*) and (x_2^*, t_2^*) in $\mathbb{R} \times [0, t_0]$ such that

$$\gamma_1(x,t) \le \gamma_1(x_1^*, t_1^*) + \varepsilon, \ \gamma_2(x,t) \le \gamma_2(x_2^*, t_2^*) + \varepsilon, \ (x,t) \in \mathbb{R} \times [0, t_0].$$

We choose M_i with $i = 1, 2$ sufficiently large such that

$$\int_{|x_i^*-y|\ge M_i} J_i(y)\mathrm{d}y \le \frac{\varepsilon}{\max\{k_1, k_2\}}, \ i = 1, 2.$$

First, we consider the case $\phi_1 \ge \phi_2$. The monotonicity of Q_t implies $\boldsymbol{\omega}(x,t;\phi_1) \ge \boldsymbol{\omega}(x,t;\phi_2)$. It follows that

$$\begin{cases} \gamma_1(x,t) \le P_1(t)[\gamma_1(\cdot,0)](x) + \displaystyle\int_0^t P_1(t-s)[N_2\gamma_2(\cdot,s)](x)\mathrm{d}s, \\ \gamma_2(x,t) \le P_2(t)[\gamma_2(\cdot,0)](x) + \displaystyle\int_0^t P_2(t-s)[N_1\gamma_1(\cdot,s)](x)\mathrm{d}s. \end{cases}$$

According to Weng and Zhao (2006, Lemma 3.1), there is a sufficiently large number M_i' in $(M_i, +\infty)$ and a small number $\delta > 0$ such that $\mathbf{P}(t)[\phi](x_i^*) < (1+e^{t_0})$" for $t \in [0, t_0]$, provided that $\phi \in \mathcal{M}$ satisfies $\|\phi(x)\| < \delta$ for any $x \in \Omega_{M_i'}(x_i^*)$. Choose $M > \max\{M_1, M_2\}$ sufficiently large such that $\Omega_{M_i'}(x_i^*) \subset \Omega_M(0)$. If $\|\phi_1(x) - \phi_2(x)\| < \delta$ for $x \in \Omega_M(0)$, then

$$\begin{aligned} |\gamma_1(\cdot,t)|_{\Omega_M(0)} &\le \varepsilon + \gamma_1(x_1^*, t_1^*) \le \varepsilon + P_1(t_1^*)[\gamma_1(\cdot,0)](x_1^*) \\ &\quad + N_2 \int_0^{t_1^*} P_1(t_1^*-s)[\gamma_2(\cdot,s)](x_1^*)\mathrm{d}s \\ &\le \varepsilon + (1+e^{t_0})\varepsilon + N_2t_1^*\varepsilon + N_2\int_0^{t_1^*} |\gamma_2(\cdot,s)|_{\Omega_{M_1'}(x_1^*)}\mathrm{d}s \\ &\le \epsilon' + N_2\int_0^{t_0} |\gamma_2(\cdot,s)|_{\Omega_M(0)}\mathrm{d}s. \end{aligned}$$

Similarly, we have

$$|\gamma_2(\cdot,t)|_{\Omega_M(0)} \le \epsilon' + N_1\int_0^{t_0} |\gamma_1(\cdot,s)|_{\Omega_M(0)}\mathrm{d}s.$$

For any $t \in (0, t_0)$, it follows that

$$\begin{cases} |\gamma_1(\cdot, t)|_{\Omega_M(0)} \leq (1 + N_2 t_0)\epsilon' + N_1 N_2 t_0 \displaystyle\int_0^{t_0} |\gamma_1(\cdot, s)|_{\Omega_M(0)} \mathrm{d}s, \\ |\gamma_2(\cdot, t)|_{\Omega_M(0)} \leq (1 + N_1 t_0)\epsilon' + N_1 N_2 t_0 \displaystyle\int_0^{t_0} |\gamma_2(\cdot, s)|_{\Omega_M(0)} \mathrm{d}s. \end{cases}$$

By Gronwall's inequality, it is easy to check that

$$\begin{cases} |\gamma_1(\cdot, t)|_{\Omega_M(0)} \leq (1 + N_2 t_0)\epsilon' e^{N_1 N_2 t_0^2}, \\ |\gamma_2(\cdot, t)|_{\Omega_M(0)} \leq (1 + N_1 t_0)\epsilon' e^{N_1 N_2 t_0^2}. \end{cases}$$

We get that

$$\|\boldsymbol{\omega}(0, t; \phi_1) - \boldsymbol{\omega}(0, t; \phi_2)\| \leq \epsilon \text{ for } t \in [0, t_0].$$

Then, the spatial translation invariance in (1.1) implies that

$$\|\boldsymbol{\omega}(z, t; \phi_1) - \boldsymbol{\omega}(z, t; \phi_2)\| \leq \epsilon \text{ for } t \in [0, t_0],$$

provided $\|\phi_1(x) - \phi_2(x)\| < \delta$ for any $x \in \Omega_M(z)$.

Second, we consider the case $\phi_1 \not\geq \phi_2$. Define

$$\Psi_1(x) = \max\{\phi_1(x), \phi_2(x)\}, \quad \Psi_2(x) = \min\{\phi_1(x), \phi_2(x)\}.$$

We have that

$$\begin{aligned} &\Psi_1(x) - \Psi_2(x) = |\phi_1(x) - \phi_2(x)|, \\ &\boldsymbol{\omega}(x, t; \phi_1) \leq \boldsymbol{\omega}(x, t; \Psi_1), \text{ and } \boldsymbol{\omega}(x, t; \Psi_2) \leq \boldsymbol{\omega}(x, t; \phi_2). \end{aligned}$$

It follows that

$$\|\boldsymbol{\omega}(x, t; \phi_1) - \boldsymbol{\omega}(x, t; \phi_2)\| \leq \|\boldsymbol{\omega}(x, t; \Psi_1) - \boldsymbol{\omega}(x, t; \Psi_2)\|.$$

Then, we have that Q_t is continuous in ϕ with respect to the compact open topology, uniformly for t in any bounded interval. It completes the proof. □

Similarly, we have that the map $Q_t : \mathcal{C}_{E_2} \to \mathcal{C}_{E_2}$ is continuous for $(t, \phi) \in \mathbb{R}^+ \times \mathcal{C}_{E_2}$ with respect to the compact open topology. Next, we verify the stability of fixed points in (B5).

Lemma 2.12 *Assume (J1) and (J3) hold. For $Q_t : \mathcal{M}_{E_2} \to \mathcal{M}_{E_2}$ with $t > 0$, we have that*

(1) *E_0 and E_2 are strongly stable from above and below, respectively;*
(2) *E_1 is strongly unstable from both above and below.*

Proof We write the restriction of Q_t on $\mathcal{X}_{E_2}$ as $\bar{Q}_t$. Let $\bar{M}_{0,t}$ be the derivation $D\bar{Q}_t[E_0]$ of $\bar{Q}_t$. Then, $\bar{M}_{0,t}$ is the solution map of the following system:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = -\alpha u + h'(0)v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = g'(0)u - \beta v. \end{cases}$$

Consider the following eigenvalue problem:

$$\begin{cases} -\alpha\varphi + h'(0)\psi = \lambda\varphi, \\ g'(0)\varphi - \beta\psi = \lambda\psi. \end{cases} \tag{2.14}$$

It is obvious that (2.14) admits two negative eigenvalues λ_1 and λ_2. Without loss of generality, we assume that $\lambda_2 < \lambda_1 < 0$. Then, there exists an eigenfunction $\mathbf{e}_0 \gg E_0$ associated with the eigenvalue λ_1, namely $\bar{M}_{0,t}[\mathbf{e}_0] = e^{\lambda_1 t}\mathbf{e}_0$. For any $t > 0$, there is $\delta_0(t) > 0$ such that

$$\begin{aligned} \bar{Q}_t[\delta\mathbf{e}_0] &= \bar{Q}_t[E_0] + D\bar{Q}_t[E_0][\delta\mathbf{e}_0] + \mathbf{o}(\delta^2) = \delta\bar{M}_{0,t}[\mathbf{e}_0] + \mathbf{o}(\delta^2) \\ &= \delta e^{\lambda_1 t}\mathbf{e}_0 + \mathbf{o}(\delta^2) = \delta\mathbf{e}_0 + \delta[e^{\lambda_1 t} - 1]\mathbf{e}_0 + \mathbf{o}(\delta^2) \\ &\ll \delta\mathbf{e}_0, \ \ \delta \in (0, \delta_0(t)]. \end{aligned}$$

Similarly, there exist $\delta_1(t)$, $\delta_2(t)$, $\mathbf{e}_1$, and $\mathbf{e}_2$ such that

$$\bar{Q}_t[E_2 - \delta\mathbf{e}_2] \gg E_2 - \delta\mathbf{e}_2, \ \ \delta \in (0, \delta_2(t)]$$

and

$$\bar{Q}_t[E_1 + \delta\mathbf{e}_1] \gg E_1 + \delta\mathbf{e}_1, \ \ \bar{Q}_t[E_1 - \delta\mathbf{e}_1] \ll E_1 - \delta\mathbf{e}_1, \ \ \delta \in (0, \delta_1(t)].$$

It completes the proof. □

Next, we study some propagating properties of the two monostable systems restricted on $\mathcal{M}_{[E_0,E_1]}$ and $\mathcal{M}_{[E_1,E_2]}$, respectively, to verify the assumption (B6).

Lemma 2.13 *Assume (J1)–(J4) hold. Then,* $c^*_-(E_1, E_2) + c^*_+(E_0, E_1) > 0$.

Proof For the rightward propagation dynamics of $\{Q_t\}_{t\geq 0}$, we restrict $\{Q_t\}_{t\geq 0}$ on $\mathcal{M}_{E_1}$. Define $(\tilde{u}(x,t), \tilde{v}(x,t)) = (u_1^*, v_1^*) - (u(x,t), v(x,t))$ and then the rightward propagation dynamics is equivalent to that of the following system restricted on $\mathcal{M}_{E_1}$

$$\begin{cases} \dfrac{\partial\tilde{u}}{\partial t} = J_1 * \tilde{u} - \tilde{u} - \alpha\tilde{u} - h(v_1^* - \tilde{v}) + h(v_1^*), \\ \dfrac{\partial\tilde{v}}{\partial t} = J_2 * \tilde{v} - \tilde{v} - \beta\tilde{v} - g(u_1^* - \tilde{u}) + g(u_1^*). \end{cases} \tag{2.15}$$

Define a family of operators $\{\tilde{Q}_t\}_{t\geq 0}$ on $\mathcal{M}_{E_1}$ by $\tilde{Q}_t(\phi) = \tilde{\boldsymbol{\omega}}(\cdot, t; \phi)$, where $\tilde{\boldsymbol{\omega}}(\cdot, t; \phi)$ is the solution of (2.15) satisfying $\tilde{\boldsymbol{\omega}}(\cdot, 0; \phi) = \phi \in \mathcal{M}_{E_1}$. We have that $\tilde{Q}_t(E_0) = E_0$ and $\tilde{Q}_t(E_1) = E_1$ for all $t \geq 0$. It is easy to check that $\tilde{Q}_t : \mathcal{C}_{E_1} \to \mathcal{C}_{E_1}$ is continuous and satisfies the assumptions (A1), (A3)–(A5) in Fang and Zhao (2014) with $\mathcal{M}_{E_1}$ replaced by $\mathcal{C}_{E_1}$. According to Fang and Zhao (2014, Remark 3.7), we have that $\tilde{Q}_1 : \mathcal{M}_{E_1} \to \mathcal{M}_{E_1}$ admits a rightward spreading speed, which is no more than $c_+^*(E_0, E_1)$. For (1.1), Theorem 3.1 in Xu et al. (2020) shows that the rightward spreading speed is equal to $-c_R$, which is defined by (2.2). Then, we have that

$$c_+^*(E_0, E_1) \geq -c_R.$$

Similarly, for the leftward propagation dynamics of $\{Q_t\}_{t\geq 0}$, we restrict $\{Q_t\}_{t\geq 0}$ on $\mathcal{M}_{[E_1,E_2]}$. Define $(\hat{u}(x,t), \hat{v}(x\ ,t)) = (u(x,t), v(x,t)) - (u_1^*, v_1^*)$ and consider

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} = J_1 * \hat{u} - \hat{u} - \alpha\hat{u} + h(\hat{v} + v_1^*) - h(v_1^*), \\ \dfrac{\partial \hat{v}}{\partial t} = J_2 * \hat{v} - \hat{v} - \beta\hat{v} + g(\hat{u} + u_1^*) - g(u_1^*). \end{cases} \tag{2.16}$$

Let $E_1^* := E_2 - E_1$. We define $\{\hat{Q}_t\}_{t\geq 0}$ on $\mathcal{M}_{E_1^*}$ by $\hat{Q}_t(\phi) = \hat{\boldsymbol{\omega}}(\cdot, t; \phi)$, where $\hat{\boldsymbol{\omega}}(\cdot, t; \phi)$ is the solution of (2.16) satisfying $\hat{\boldsymbol{\omega}}(\cdot, 0; \phi) = \phi \in \mathcal{M}_{E_1^*}$. Then, $\hat{Q}_1 : \mathcal{M}_{E_1^*} \to \mathcal{M}_{E_1^*}$ admits a leftward spreading speed c_L defined by (2.2) and it holds that

$$c_-^*(E_1, E_2) \geq c_L.$$

Finally, the proof is completed by (2.3). □

Now, we are ready to give the existence of bistable traveling wave solutions of (1.1).

Theorem 2.14 *Assume (J1)–(J4) hold. Then, there is a constant $c_0 \in \mathbb{R}$ such that (1.1) admits an increasing traveling wave solution $\Phi(x + c_0 t)$ satisfying*

$$\Phi(-\infty) = E_0 \text{ and } \Phi(+\infty) = E_2.$$

Moreover, $\Phi(\cdot)$ is strictly increasing.

Proof In view of Lemmas 2.11, 2.12, and 2.13 , the existence of traveling wave solutions in Theorem 2.9 implies that there is $c_0 \in \mathbb{R}$ and $\Phi \in \mathcal{M}_{E_2}$ such that $Q_t(\Phi)(x) = \Phi(x + c_0 t)$ for any $x \in \mathbb{R}$. Next, we prove that Φ is a classical solution of (1.1). It is easy to verify that

$$\frac{\partial}{\partial t}\mathbf{P}(t)[\Phi](x) = -\mathbf{P}(t)[\Phi](x) + J * (\mathbf{P}(t)[\Phi])(x). \tag{2.17}$$

Note that $\boldsymbol{\omega}(x,t) := \Phi(x + c_0 t)$ satisfies the following integral equation:

$$\boldsymbol{\omega}(x,t) = \mathbf{P}(t)[\boldsymbol{\omega}(\cdot,0)](x) + \int_0^t \mathbf{P}(t-s)[\mathbf{F}(\boldsymbol{\omega}(\cdot,s))](x)\mathrm{d}s, \quad x \in \mathbb{R},\ t \geq 0. \tag{2.18}$$

Since the right side of (2.18) is differentiable with respect to variable t, it follows from (2.17) that $\boldsymbol{\omega}(x,t)$ is a classical solution of (1.1).

It remains to prove that the nondecreasing function $\Phi(\cdot) = (\varphi, \psi)(\cdot)$ is strictly increasing. By a contradiction argument, without loss of generality, we assume that $\varphi(\cdot)$ is not strictly increasing; namely, there exist $x_1 < x_2$ such that $\varphi(\xi) = \varphi(x_1)$ for any $\xi \in [x_1, x_2]$. It follows that $\varphi'(\xi) = 0$ for $\xi \in [x_1, x_2]$. By (2.1), we have that

$$\begin{aligned} 0 = &\int_{\mathbb{R}} J_1(y)(\varphi(x_1 - y) - \varphi(x_2 - y))\mathrm{d}y - \alpha(\varphi(x_1) - \varphi(x_2)) \\ &+ h(\psi(x_1)) - h(\psi(x_2)). \end{aligned}$$

It follows that

$$\begin{aligned} 0 \geq &\int_{\mathbb{R}} J_1(y)(\varphi(x_1 - y) - \varphi(x_2 - y))\mathrm{d}y = \alpha(\varphi(x_1) - \varphi(x_2)) \\ &- h(\psi(x_1)) + h(\psi(x_2)) \geq 0, \end{aligned}$$

which implies $\int_{\mathbb{R}} J_1(y)(\varphi(x_1 - y) - \varphi(x_2 - y))\mathrm{d}y = 0$. Then, $\varphi(x_1 - y) = \varphi(x_2 - y)$ for any $y \in \mathrm{supp}(J_1)$. Rechoose x_1 and x_2 and repeat the process above, we can obtain that $\varphi(\cdot)$ is a constant function, which contradicts $\varphi(-\infty) = 0$ and $\varphi(+\infty) = u_2^*$. It completes the proof. □

Since J_1 and J_2 are not required to be symmetric, the corresponding nondecreasing and nonincreasing traveling wave solutions can be different in shape. The following theorem gives the existence of nonincreasing traveling wave solution.

Theorem 2.15 *Assume (J1)–(J4) hold. Then, there is a constant $\hat{c}_0 \in \mathbb{R}$ such that (1.1) admits a decreasing traveling wave solution $\hat{\Phi}(x + \hat{c}_0 t)$ satisfying*

$$\hat{\Phi}(-\infty) = E_2 \quad \text{and} \quad \hat{\Phi}(+\infty) = E_0.$$

Moreover, $\hat{\Phi}(\cdot)$ is strictly decreasing.

Proof According to Fang and Zhao (2015, Remark 2.1), when we consider the nonincreasing traveling waves, assumption (B6) should be replaced by

$$c_+^*(E_1, E_2) + c_-^*(E_0, E_1) > 0. \tag{2.19}$$

Here, $c_+^*(E_1, E_2)$ and $c_-^*(E_0, E_1)$ are defined similarly to (2.10) and (2.11), respectively. By the same argument as the proof of Lemma 2.13, we have that $c_+^*(E_1, E_2) \geq -c_R$ and $c_-^*(E_0, E_1) \geq c_L$. Then, (2.19) follows from $c_L - c_R > 0$. The rest of the proof is similar to Theorem 2.14. □

Now, we give the result about the asymptotic behavior of bistable traveling wave solutions. The proof is omitted since we can use the same techniques as in Theorems 2.6 and 2.7.

Theorem 2.16 *Assume (J1)–(J4) hold. Let $\Phi(x + c_0 t)$ and $\hat{\Phi}(x + \hat{c}_0 t)$ be the nondecreasing and nonincreasing traveling wave solutions of* (1.1) *as in Theorems* 2.14 *and* 2.15 *with $c_0 \neq 0$ and $\hat{c}_0 \neq 0$, respectively. Then, all of the following limits exist in $\mathbb{R}^+ \times \mathbb{R}^+$ and*

$$\begin{aligned}
&\lim_{\xi \to -\infty} \Phi(\xi)/e^{\lambda_0^+(c_0)\xi} = \lim_{\xi \to -\infty} \Phi'(\xi)/(\lambda_0^+(c_0) e^{\lambda_0^+(c_0)\xi}), \\
&\lim_{\xi \to +\infty} (E_2 - \Phi(\xi))/e^{\lambda_2^-(c_0)\xi} = - \lim_{\xi \to +\infty} \Phi'(\xi)/(\lambda_2^-(c_0) e^{\lambda_2^-(c_0)\xi}), \\
&\lim_{\xi \to +\infty} \hat{\Phi}(\xi)/e^{\lambda_0^-(\hat{c}_0)\xi} = \lim_{\xi \to +\infty} \hat{\Phi}'(\xi)/(\lambda_0^-(\hat{c}_0) e^{\lambda_0^-(\hat{c}_0)\xi}), \\
&\lim_{\xi \to -\infty} (E_2 - \hat{\Phi}(\xi))/e^{\lambda_2^+(\hat{c}_0)\xi} = - \lim_{\xi \to -\infty} \hat{\Phi}'(\xi)/(\lambda_2^+(\hat{c}_0) e^{\lambda_2^+(\hat{c}_0)\xi}),
\end{aligned}$$

where $\lambda_i^+(c)$ and $\lambda_i^-(c)$ with $i \in \{0, 2\}$ are defined by (2.8).

3 Entire Solutions

In this section, we focus on the construction of entire solutions of (1.1). As stated in Sect. 1, we only choose traveling wave solutions with nonzero wave speeds c_0, $\hat{c}_0$, c_i, and $\hat{c}_i$ $(i = 1, 2)$. By super- and sub-solutions method, we construct six types of entire solutions, and each of them originates from three traveling wave solutions. Figure 1 in Section 1 depicts the sketchy profiles of these entire solutions as $t \to -\infty$. The existence results of these six types of entire solutions are given in Sects. 3.2–3.7, respectively. Since the construction methods for these six types are similar, we give details only for the first type. And for the other five types, the main differences are shown in the corresponding subsections. Moreover, we prove that the entire solutions established in Sects. 3.2–3.7 are globally Lipschitz continuous with respect to x under the following assumption:

(J5) $$\sup_{u \in [0, u_2^*]} g'(u) \cdot \sup_{v \in [0, v_2^*]} h'(v) < (1 + \alpha)(1 + \beta).$$

Finally, when $c_0 \neq \hat{c}_0$, we obtain the nonexistence of entire solutions originating from more than four traveling wave solutions in Sect. 3.8.

3.1 Preliminaries

This subsection gives some fundamental properties of (1.1), including the comparison principle, the Lipschitz continuity of solution, and the study of some ordinary differential equations. First, we give the definition of super- and sub-solutions of (1.1).

Definition 3.1 For $\tau, T \in \mathbb{R}$ with $\tau < T$, a function $\boldsymbol{\omega} = (u, v)$ is called a super-solution (sub-solution) of (1.1) on $\mathbb{R} \times [\tau, T)$, if $\boldsymbol{\omega} \in C^{0,1}(\mathbb{R} \times [\tau, T), \mathbb{R}^2)$ and

$$\begin{cases} u_t(x,t) \geq (\leq) J_1 * u(x,t) - u(x,t) - \alpha u(x,t) + h(v(x,t)), \\ v_t(x,t) \geq (\leq) J_2 * v(x,t) - v(x,t) - \beta v(x,t) + g(u(x,t)) \end{cases}$$

for any $(x, t) \in \mathbb{R} \times [\tau, T)$. Moreover, $\boldsymbol{\omega} = (u, v)$ is called a super-solution (sub-solution) of (1.1) on $\mathbb{R} \times (-\infty, T)$, if for any $\tau < T$, it is a super-solution (sub-solution) on $\mathbb{R} \times [\tau, T)$.

The following lemma states the comparison principle proven in Li et al. (2017, Theorem 3.6).

Lemma 3.2 *Assume (J1) and (J3) hold. Let $\boldsymbol{\omega}^+(x,t)$ and $\boldsymbol{\omega}^-(x,t)$ be the super- and sub-solutions of (1.1) on $\mathbb{R} \times [0, \infty)$, respectively. If $\boldsymbol{\omega}^+(x,0) \geq \boldsymbol{\omega}^-(x,0)$ for $x \in \mathbb{R}$ and $E_0 \leq \boldsymbol{\omega}^-(x,t), \boldsymbol{\omega}^+(x,t) \leq E_2$, then $\boldsymbol{\omega}^-(x,t) \leq \boldsymbol{\omega}^+(x,t)$ for $t \in [0, \infty)$(x is fixed).*

In the study of entire solutions, the Lipschitz continuity of solution as given by the following lemma is a very important property. We refer to Li et al. (2017, Lemma 3.8) for its proof.

Lemma 3.3 *Assume (J1)–(J5) hold. Let $(u(x,t), v(x,t))$ be a solution of the corresponding Cauchy problem of (1.1) with bounded initial data. Then, there is a constant $M > 0$ (independent of initial data) such that*

$$|u_t(x,t)|, |u_{tt}(x,t)|, |v_t(x,t)|, |v_{tt}(x,t)| \leq M \quad \textit{for any } x \in \mathbb{R},\ t > 0.$$

In addition, if the initial data are globally Lipschitz continuous, namely there is $L_0 > 0$ such that

$$|u(x+\tilde{x},0) - u(x,0)| \leq L_0\tilde{x},\ |v(x+\tilde{x},0) - v(x,0)| \leq L_0\tilde{x} \quad \textit{for any } x \in \mathbb{R},\ \tilde{x} > 0,$$

then for any $t > 0$, the solution is also globally Lipschitz continuous and

$$\begin{aligned} &|u(x+\tilde{x},t) - u(x,t)|,\ |v(x+\tilde{x},t) - v(x,t)| \leq M'\tilde{x}, \\ &|\frac{\partial u}{\partial t}(x+\tilde{x},t) - \frac{\partial u}{\partial t}(x,t)|,\ |\frac{\partial v}{\partial t}(x+\tilde{x},t) - \frac{\partial v}{\partial t}(x,t)| \leq M''\tilde{x}, \end{aligned}$$

where M' and M'' are two positive constants independent of x and $\tilde{x}$.

The construction of entire solutions also involves the solutions of some ordinary differential equations. For any real numbers $s_1 < s_2 < s_3$, define

$$\nu_1 := (s_2 - s_1)/2 > 0,\ \nu_2 := [2s_3 - (s_1 + s_2)]/2 > \nu_1. \tag{3.1}$$

Consider the following ordinary differential equations: (c.f. Chen et al. 2018)

$$\begin{cases} p_1' = \nu_1 - Le^{\kappa p_1}, & -\infty < t < 0, \ p_1(0) = p_0, \\ p_2' = \nu_2 + Le^{\kappa p_1}, & -\infty < t < 0, \ p_2(0) = r_0, \\ r_1' = \nu_1 + Le^{\kappa r_1}, & -\infty < t < 0, \ r_1(0) = r_0, \\ r_2' = \nu_2 - Le^{\kappa r_1}, & -\infty < t < 0, \ r_2(0) = p_0, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \tilde{p}_1' = \nu_1 + Le^{\kappa \tilde{p}_1}, & -\infty < t < 0, \ \tilde{p}_1(0) = p_0, \\ \tilde{p}_2' = \nu_2 + Le^{\kappa \tilde{p}_1}, & -\infty < t < 0, \ \tilde{p}_2(0) = p_0, \\ \tilde{r}_1' = \nu_1 - Le^{\kappa \tilde{r}_1}, & -\infty < t < 0, \ \tilde{r}_1(0) = r_0, \\ \tilde{r}_2' = \nu_2 - Le^{\kappa \tilde{r}_1}, & -\infty < t < 0, \ \tilde{r}_2(0) = r_0, \end{cases} \tag{3.3}$$

where L, p_0, and r_0 are some constants determined later and

$$\kappa := \min\left\{\eta_1, \eta_2, \frac{(\nu_2 - \nu_1)\eta_1}{4\nu_1}, \frac{(\nu_2 - \nu_1)\eta_2}{4\nu_1}\right\}. \tag{3.4}$$

We can solve (3.2) and (3.3) explicitly as

$$\begin{aligned} p_1(t) &= \nu_1 t - \frac{1}{\kappa}\ln\left[e^{-\kappa p_0} - \frac{L(1-e^{\kappa\nu_1 t})}{\nu_1}\right], \ p_2(t) = \nu_2 t + \frac{1}{\kappa}\ln\left[e^{-\kappa p_0} - \frac{L(1-e^{\kappa\nu_1 t})}{\nu_1}\right] + p_0 + r_0, \\ r_1(t) &= \nu_1 t - \frac{1}{\kappa}\ln\left[e^{-\kappa r_0} + \frac{L(1-e^{\kappa\nu_1 t})}{\nu_1}\right], \ r_2(t) = \nu_2 t + \frac{1}{\kappa}\ln\left[e^{-\kappa r_0} + \frac{L(1-e^{\kappa\nu_1 t})}{\nu_1}\right] + p_0 + r_0, \\ \tilde{p}_1(t) &= \nu_1 t - \frac{1}{\kappa}\ln\left[e^{-\kappa p_0} + \frac{L(1-e^{\kappa\nu_1 t})}{\nu_1}\right], \ \tilde{p}_2(t) = \nu_2 t - \frac{1}{\kappa}\ln\left[e^{-\kappa p_0} + \frac{L(1-e^{\kappa\nu_1 t})}{\nu_1}\right], \\ \tilde{r}_1(t) &= \nu_1 t - \frac{1}{\kappa}\ln\left[e^{-\kappa r_0} - \frac{L(1-e^{\kappa\nu_1 t})}{\nu_1}\right], \ \tilde{r}_2(t) = \nu_2 t - \frac{1}{\kappa}\ln\left[e^{-\kappa r_0} - \frac{L(1-e^{\kappa\nu_1 t})}{\nu_1}\right]. \end{aligned}$$

Now, for any given sufficiently large δ, we take the initial values p_0 and r_0 satisfying

$$p_0 = -\frac{1}{\kappa}\ln\left[e^{-\kappa r_0} + \frac{2L}{\nu_1}\right] < -\delta \text{ and } r_0 < -\frac{1}{\kappa}\ln\left[e^{\kappa\delta} + \frac{2L}{\nu_1}\right] < -\delta.$$

Then, it follows that

$$\begin{aligned}
&\lim_{t\to-\infty}(r_1(t)-p_1(t))=\lim_{t\to-\infty}(p_2(t)-r_2(t))=0,\\
&\lim_{t\to-\infty}(p_1(t)-\nu_1 t)=-\frac{1}{\kappa}\ln\left[e^{-\kappa p_0}-\frac{L}{\nu_1}\right],\\
&\lim_{t\to-\infty}(p_2(t)-\nu_2 t)=\frac{1}{\kappa}\ln\left[e^{-\kappa p_0}-\frac{L}{\nu_1}\right]+p_0+r_0,\\
&\lim_{t\to-\infty}(r_1(t)-\nu_1 t)=-\frac{1}{\kappa}\ln\left[e^{-\kappa r_0}+\frac{L}{\nu_1}\right],\\
&\lim_{t\to-\infty}(r_2(t)-\nu_2 t)=\frac{1}{\kappa}\ln\left[e^{-\kappa r_0}+\frac{L}{\nu_1}\right]+p_0+r_0,\\
&\lim_{t\to-\infty}(\tilde{p}_1(t)-\tilde{r}_1(t))=\lim_{t\to-\infty}(\tilde{p}_2(t)-\tilde{r}_2(t))=0,\\
&\lim_{t\to-\infty}(\tilde{p}_1(t)-\nu_1 t)=\lim_{t\to-\infty}(\tilde{p}_2(t)-\nu_2 t)=-\frac{1}{\kappa}\ln\left[e^{-\kappa p_0}+\frac{L}{\nu_1}\right],\\
&\lim_{t\to-\infty}(\tilde{r}_1(t)-\nu_1 t)=\lim_{t\to-\infty}(\tilde{r}_2(t)-\nu_2 t)=-\frac{1}{\kappa}\ln\left[e^{-\kappa r_0}-\frac{L}{\nu_1}\right].
\end{aligned}\tag{3.5}$$

In addition, there exists a positive constant N (dependent on L, p_0, and r_0) such that

$$\begin{aligned}
&0<r_1(t)-p_1(t)=p_2(t)-r_2(t)\le Ne^{\kappa\nu_1 t}\text{ for }t\le 0,\\
&0<\tilde{p}_1(t)-\tilde{r}_1(t)=\tilde{p}_2(t)-\tilde{r}_2(t)\le Ne^{\kappa\nu_1 t}\text{ for }t\le 0,
\end{aligned}\tag{3.6}$$

and

$$p_1(t),\,p_2(t),\,r_1(t),\,r_2(t),\,\tilde{p}_1(t),\,\tilde{p}_2(t),\,\tilde{r}_1(t),\,\tilde{r}_2(t)\le-\delta\text{ for }t\le 0.$$

Remark 3.4 In the next six subsections, the values of ν_1 and ν_2 are different for the six types of entire solutions, since s_1, s_2, and s_3 will be changed for every type. However, the above results about the solutions of Eqs. (3.2) and (3.3) are always true.

3.2 I-Type Entire Solution

In this subsection, we construct the first type of entire solution, which originates from one monostable and two bistable traveling wave solutions.

Theorem 3.5 *Assume (J1)–(J4) hold and $c_0>\hat{c}_0>c_2$. Then,* (1.1) *admits an entire solution $\mathcal{W}_1:\mathbb{R}^2\to[0,u_2^*]\times[0,v_2^*]$ satisfying*

$$\lim_{t\to-\infty}\Bigg\{\sup_{x\le\rho_1(t)}\|\mathcal{W}_1(x,t)-\Phi_2(x+c_2t-\vartheta_1)\|$$
$$+\sup_{\rho_1(t)\le x\le\rho_2(t)}\|\mathcal{W}_1(x,t)-\hat{\Phi}(x+\hat{c}_0t+\vartheta_1)\|$$

$$+\sup_{\rho_2(t)\le x}\|\mathcal{W}_1(x,t)-\Phi(x+c_0t+\vartheta_2)\|\Bigg\}=0, \tag{3.7}$$

where ϑ_1 *and* ϑ_2 *are some constants satisfying*

$$\vartheta_1:=-\frac{1}{\kappa}\ln\left[e^{-\kappa r_0}+\frac{L}{\nu_1}\right],\quad \vartheta_2:=\frac{1}{\kappa}\ln\left[e^{-\kappa r_0}+\frac{L}{\nu_1}\right]+p_0+r_0,$$

and $\rho_1,\rho_2:\mathbb{R}\to\mathbb{R}$ *are defined by*

$$\rho_1(t):=\frac{-(c_2+\hat{c}_0)t}{2},\qquad \rho_2(t):=\frac{-(\hat{c}_0+c_0)t}{2}.$$

Moreover, when (J5) holds, there are two positive constants D_1 *and* D_2 *such that for any* $(x,t)\in\mathbb{R}^2$ *and* $\eta>0$*,*

$$\|\mathcal{W}_1(x+\eta,t)-\mathcal{W}_1(x,t)\|\le D_1\eta,\quad \left\|\frac{\partial\mathcal{W}_1}{\partial t}(x+\eta,t)-\frac{\partial\mathcal{W}_1}{\partial t}\right\|\le D_2\eta. \tag{3.8}$$

The entire solution $\mathcal{W}_1(x,t)$ given by Theorem 3.5 can be regarded as a combination of the traveling wave solutions (c_2,Φ_2), $(\hat{c}_0,\hat{\Phi})$, and (c_0,Φ) with $c_0>\hat{c}_0>c_2$. When $t\to-\infty$, its profile is found in Fig. 1a. Obviously, $\mathcal{W}_1(x,t)$ is differentiable with respect to t, and (3.8) further shows that it is globally Lipschitz continuous with respect to x.

As stated in Sect. 1, the construction of $\mathcal{W}_1$ is essentially a procedure that we link the profile of Φ_2 with $\hat{\Phi}$ and link the profile of $\hat{\Phi}$ with Φ successively, and the main difficulty is to construct appropriate auxiliary functions and a pair of super- and sub-solutions. For $\mathcal{W}_1$ in Theorem 3.5, the construction procedure is quite complicated and it will be provided in the rest of this subsection. We first define the auxiliary functions as follows:

$$\begin{aligned}P(x,y,z)&=y\\&+\frac{(x-u_1^*)(u_2^*-y)z(u_2^*-y)+(u_2^*-x)y(u_2^*-z)(u_1^*-y)}{u_2^*(x-u_1^*)(u_2^*-y)+(u_2^*-u_1^*)y(u_2^*-z)},\quad (x,y,z)\in D_1\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}Q(x,y,z)&=y\\&+\frac{(x-v_1^*)(v_2^*-y)z(v_2^*-y)+(v_2^*-x)y(v_2^*-z)(v_1^*-y)}{v_2^*(x-v_1^*)(v_2^*-y)+(v_2^*-v_1^*)y(v_2^*-z)},\quad (x,y,z)\in D_2,\end{aligned} \tag{3.10}$$

where

$$D_1 := \left\{(x,y,z) \in [u_1^*, u_2^*] \times [0, u_2^*] \times [0, u_2^*] \;\middle|\; u_2^*(x-u_1^*)(u_2^*-y) + (u_2^*-u_1^*)y(u_2^*-z) > 0\right\},$$
$$D_2 := \left\{(x,y,z) \in [v_1^*, v_2^*] \times [0, v_2^*] \times [0, v_2^*] \;\middle|\; v_2^*(x-v_1^*)(v_2^*-y) + (v_2^*-v_1^*)y(v_2^*-z) > 0\right\}.$$

Lemma 3.6 *There exists a positive constant C_3 such that*

$$\begin{aligned}
&\frac{|P_{xx}(x,y,z)|}{|y|}, \frac{|P_{xx}(x,y,z)|}{|u_2^*-y|}, \frac{|P_{xx}(x,y,z)|}{|u_2^*-z|}, \frac{|P_{yy}(x,y,z)|}{|x-u_1^*|}, \frac{|P_{yy}(x,y,z)|}{|u_2^*-z|}, \frac{|P_{yy}(x,y,z)|}{|u_2^*-x|+|z|} \le C_3,\\
&\frac{|P_{zz}(x,y,z)|}{|x-u_1^*|}, \frac{|P_{zz}(x,y,z)|}{|u_2^*-y|}, \frac{|P_{zz}(x,y,z)|}{|y|}, \frac{|P_{xy}(x,y,z)|}{|u_2^*-z|}, \frac{|P_{yz}(x,y,z)|}{|x-u_1^*|} \le C_3,\\
&\frac{|P_{xz}(x,y,z)|}{|u_2^*-y|}, \frac{|P_{xz}(x,y,z)|}{|y|}, \frac{|P_{xz}(x,y,z)|}{|x-u_1^*|}, \frac{|P_{xz}(x,y,z)|}{|u_2^*-z|} \le C_3,\\
&\frac{|Q_{xx}(x,y,z)|}{|y|}, \frac{|Q_{xx}(x,y,z)|}{|v_2^*-y|}, \frac{|Q_{xx}(x,y,z)|}{|v_2^*-z|}, \frac{|Q_{yy}(x,y,z)|}{|x-v_1^*|}, \frac{|Q_{yy}(x,y,z)|}{|v_2^*-z|}, \frac{|Q_{yy}(x,y,z)|}{|v_2^*-x|+|z|} \le C_3,\\
&\frac{|Q_{zz}(x,y,z)|}{|x-v_1^*|}, \frac{|Q_{zz}(x,y,z)|}{|v_2^*-y|}, \frac{|Q_{zz}(x,y,z)|}{|y|}, \frac{|Q_{xy}(x,y,z)|}{|v_2^*-z|}, \frac{|Q_{yz}(x,y,z)|}{|x-v_1^*|} \le C_3,\\
&\frac{|Q_{xz}(x,y,z)|}{|v_2^*-y|}, \frac{|Q_{xz}(x,y,z)|}{|y|}, \frac{|Q_{xz}(x,y,z)|}{|x-v_1^*|}, \frac{|Q_{xz}(x,y,z)|}{|v_2^*-z|} \le C_3.
\end{aligned}$$

Proof Note that P can be rewritten as

$$P(x,y,z) = x + (x-u_1^*)(u_2^*-y)\frac{u_2^*(z-x)}{u_2^*(x-u_1^*)(u_2^*-y) + (u_2^*-u_1^*)y(u_2^*-z)},$$
$$P(x,y,z) = z + y(u_2^*-z)\frac{(u_2^*-u_1^*)(x-z)}{u_2^*(x-u_1^*)(u_2^*-y) + (u_2^*-u_1^*)y(u_2^*-z)}.$$

Similarly, Q can be rewritten as

$$Q(x,y,z) = x + (x-v_1^*)(v_2^*-y)\frac{v_2^*(z-x)}{v_2^*(x-v_1^*)(v_2^*-y) + (v_2^*-v_1^*)y(v_2^*-z)},$$
$$Q(x,y,z) = z + y(v_2^*-z)\frac{(v_2^*-v_1^*)(x-z)}{v_2^*(x-v_1^*)(v_2^*-y) + (v_2^*-v_1^*)y(v_2^*-z)}.$$

With the help of the above formulas, the calculations of derivatives can be simplified and we omit the details of calculations in the rest of the proof. □

In order to obtain a pair of suitable super- and sub-solutions, we denote some notations. As stated in Remark 3.4, throughout this subsection we need to choose that

$$s_1 := c_2, \; s_2 := \hat{c}_0, \; s_3 := c_0. \tag{3.11}$$

Recall that (c_2, Φ_2), $(\hat{c}_0, \hat{\Phi})$, and (c_0, Φ) are the traveling wave solutions of (1.1). For convenience, we denote

$$\begin{aligned}\boldsymbol{\phi}_1 &= (\phi_{11}, \phi_{12}) := \Phi_2 = (\varphi_2, \psi_2),\\ \boldsymbol{\phi}_2 &= (\phi_{21}, \phi_{22}) := \hat{\Phi} = (\hat{\varphi}, \hat{\psi}),\\ \boldsymbol{\phi}_3 &= (\phi_{31}, \phi_{32}) := \Phi = (\varphi, \psi).\end{aligned}$$

Then, $\boldsymbol{\phi}_i = (\phi_{i1}, \phi_{i2})$ with $i \in \{1, 2, 3\}$ satisfies

$$\begin{cases} s_i\phi_{i1}'(\xi) = J_1 * \phi_{i1}(\xi) - \phi_{i1}(\xi) - \alpha\phi_{i1}(\xi) + h(\phi_{i2}(\xi)),\\ s_i\phi_{i2}'(\xi) = J_2 * \phi_{i2}(\xi) - \phi_{i2}(\xi) - \beta\phi_{i2}(\xi) + g(\phi_{i1}(\xi)),\end{cases}$$

and

$$\begin{aligned}&\big(\boldsymbol{\phi}_1(-\infty), \boldsymbol{\phi}_1(+\infty), \boldsymbol{\phi}_2(-\infty), \boldsymbol{\phi}_2(+\infty), \boldsymbol{\phi}_3(-\infty), \boldsymbol{\phi}_3(+\infty)\big)\\ &\quad = (E_1, E_2, E_2, E_0, E_0, E_2).\end{aligned}$$

By a translation if necessary, we may assume

$$\begin{aligned}\phi_{11}(0) = \frac{u_1^* + u_2^*}{2},\ (\phi_{21}(0), \phi_{22}(0)) \geq \left(\frac{u_1^* + u_2^*}{2}, \frac{v_1^* + v_2^*}{2}\right),\ \text{and } (\phi_{31}(0), \phi_{32}(0))\\ \leq (u_1^*, v_1^*). \qquad (3.12)\end{aligned}$$

By a transformation $(u(x, t), v(x, t)) = (U(\xi, t), V(\xi, t))$ with $\xi := x + \bar{c}t$ and $\bar{c} := \frac{s_1+s_2}{2}$, (1.1) becomes

$$\begin{cases} \dfrac{\partial U(\xi, t)}{\partial t} = (J_1 * U - U)(\xi, t) - \bar{c}U_\xi(\xi, t) + f_1(U(\xi, t), V(\xi, t)),\ (\xi, t) \in \mathbb{R}^2,\\ \dfrac{\partial V(\xi, t)}{\partial t} = (J_2 * V - V)(\xi, t) - \bar{c}V_\xi(\xi, t) + f_2(U(\xi, t), V(\xi, t)),\ (\xi, t) \in \mathbb{R}^2,\end{cases} \qquad (3.13)$$

where

$$f_1(U, V) := -\alpha U + h(V), \quad f_2(U, V) := -\beta V + g(U). \qquad (3.14)$$

It is easy to check that (3.13) has three traveling wave solutions, namely

$$\boldsymbol{\phi}_1(\xi - \nu_1 t),\ \boldsymbol{\phi}_2(\xi + \nu_1 t),\ \text{and } \boldsymbol{\phi}_3(\xi + \nu_2 t),$$

where ν_1 and ν_2 are defined by (3.1) and $\nu_2 > \nu_1 > 0$. We define the super- and sub-solutions of (3.13) similarly to Definition 3.1 for (1.1).

Now, we construct

$$\begin{cases} \overline{U}(\xi,t) = P(\phi_{11}(\xi - p_1(t)), \phi_{21}(\xi + p_1(t)), \phi_{31}(\xi + p_2(t))), \\ \overline{V}(\xi,t) = Q(\phi_{12}(\xi - p_1(t)), \phi_{22}(\xi + p_1(t)), \phi_{32}(\xi + p_2(t))) \end{cases} \tag{3.15}$$

and

$$\begin{cases} \underline{U}(\xi,t) = P(\phi_{11}(\xi - r_1(t)), \phi_{21}(\xi + r_1(t)), \phi_{31}(\xi + r_2(t))), \\ \underline{V}(\xi,t) = Q(\phi_{12}(\xi - r_1(t)), \phi_{22}(\xi + r_1(t)), \phi_{32}(\xi + r_2(t))), \end{cases} \tag{3.16}$$

where $(p_1(t), p_2(t), r_1(t), r_2(t))$ is the solution of (3.2). In what follows, our purpose is to prove that $(\overline{U}(\xi,t), \overline{V}(\xi,t))$ and $(\underline{U}(\xi,t), \underline{V}(\xi,t))$ are a pair of super- and sub-solution of (3.13), and apply it to complete the proof of Theorem 3.5. We accomplish this purpose (see Lemma 3.15) by showing lots of estimates for some related functions (see Lemmas 3.7–3.14).

For convenience, we first restate the results in Theorems 2.6, 2.7, and 2.16 about the asymptotic behavior of $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, and $\boldsymbol{\phi}_3$ in a uniform format, and hence, the proof of the following lemma is omitted.

Lemma 3.7 *Assume (J1)–(J4) hold. Let*

$$\eta_1 = \min\{\lambda_1^+(c_2), \lambda_2^+(\hat{c}_0), \lambda_0^+(c_0)\},\ \ \eta_2 = \min\{\lambda_2^-(c_2), \lambda_0^-(\hat{c}_0), \lambda_2^-(c_0)\}.$$

There are three positive numbers C_0, C_1, and C_2 such that

(i) for $x \le m$,

$$\begin{aligned} &|\phi_{11}'(x)|, |\phi_{12}'(x)|, |\phi_{21}'(x)|, |\phi_{22}'(x)|, |\phi_{31}'(x)|, |\phi_{32}'(x)| \le C_0 e^{\eta_1 x}, C_1 \\ &\le \frac{|\phi_{11}'(x)|}{|\phi_{11}(x) - u_1^*|}, \frac{|\phi_{12}'(x)|}{|\phi_{12}(x) - v_1^*|}, \frac{|\phi_{21}'(x)|}{|u_2^* - \phi_{21}(x)|}, \frac{|\phi_{22}'(x)|}{|v_2^* - \phi_{22}(x)|}, \frac{|\phi_{31}'(x)|}{|\phi_{31}(x)|}, \frac{|\phi_{32}'(x)|}{|\phi_{32}(x)|} \\ &\le C_2; \end{aligned}$$

(ii) for $x \ge -m$,

$$\begin{aligned} &|\phi_{11}'(x)|, |\phi_{12}'(x)|, |\phi_{21}'(x)|, |\phi_{22}'(x)|, |\phi_{31}'(x)|, |\phi_{32}'(x)| \le C_0 e^{-\eta_2 x}, C_1 \\ &\le \frac{|\phi_{11}'(x)|}{|u_2^* - \phi_{11}(x)|}, \frac{|\phi_{12}'(x)|}{|v_2^* - \phi_{12}(x)|}, \frac{|\phi_{21}'(x)|}{|\phi_{21}(x)|}, \frac{|\phi_{22}'(x)|}{|\phi_{22}(x)|}, \frac{|\phi_{31}'(x)|}{|u_2^* - \phi_{31}(x)|}, \frac{|\phi_{32}'(x)|}{|v_2^* - \phi_{32}(x)|} \\ &\le C_2, \end{aligned}$$

where m is defined in (J4). Moreover, there is a constant $C > 0$ such that for any $x \in \mathbb{R}$ and $y \in supp(J)$,

$$\begin{aligned} &\frac{\phi_{11}(x+y) - u_1^*}{\phi_{11}(x) - u_1^*}, \frac{\phi_{12}(x+y) - v_1^*}{\phi_{12}(x) - v_1^*}, \frac{u_2^* - \phi_{11}(x+y)}{u_2^* - \phi_{11}(x)}, \frac{v_2^* - \phi_{12}(x+y)}{v_2^* - \phi_{12}(x)} \le C, \\ &\frac{u_2^* - \phi_{21}(x+y)}{u_2^* - \phi_{21}(x)}, \frac{v_2^* - \phi_{22}(x+y)}{v_2^* - \phi_{22}(x)}, \frac{\phi_{21}(x+y)}{\phi_{21}(x)}, \frac{\phi_{22}(x+y)}{\phi_{22}(x)} \le C, \end{aligned}$$

$$\frac{|\phi_{31}(x+y)|}{|\phi_{31}(x)|}, \frac{|\phi_{32}(x+y)|}{|\phi_{32}(x)|} \frac{|u_2^* - \phi_{31}(x+y)|}{|u_2^* - \phi_{31}(x)|}, \frac{|v_2^* - \phi_{32}(x+y)|}{|v_2^* - \phi_{32}(x)|} \le C.$$

Next, we give estimates for some related functions defined below. To increase the readability of the paper, their proofs are given in the Appendix at the end of the paper. Suppose that q_1 and $q_2 : \mathbb{R}^- \to \mathbb{R}^-$ are two smooth functions satisfying

$$q_2(t) \le q_1(t) \le 0 \quad \text{for any } t < 0. \tag{3.17}$$

Define four functions from $\mathbb{R} \times \mathbb{R}^-$ to $\mathbb{R}$ as follows:

$$\begin{aligned}
P^0(\xi,t) &:= P\left(\phi_{11}(\xi - q_1(t)), \phi_{21}(\xi + q_1(t)), \phi_{31}(\xi + q_2(t))\right), \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^-,\\
P^x(\xi,t) &:= P_x\left(\phi_{11}(\xi - q_1(t)), \phi_{21}(\xi + q_1(t)), \phi_{31}(\xi + q_2(t))\right), \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^-,\\
P^y(\xi,t) &:= P_y\left(\phi_{11}(\xi - q_1(t)), \phi_{21}(\xi + q_1(t)), \phi_{31}(\xi + q_2(t))\right), \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^-,\\
P^z(\xi,t) &:= P_z\left(\phi_{11}(\xi - q_1(t)), \phi_{21}(\xi + q_1(t)), \phi_{31}(\xi + q_2(t))\right), \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^-.
\end{aligned}$$

Note that all the functions P^0, P^x, P^y, and P^z are dependent on q_1, q_2, and ϕ_{i1} with $i \in \{1, 2, 3\}$. The next lemma gives some lower bounds of P^x, P^y, and P^z.

Lemma 3.8 *Assume (J1)–(J4) hold. There are some positive constants μ_1, μ_2, and μ_3 such that for any $t < 0$,*

$$\begin{aligned}
&P^x(\xi,t) \ge \mu_1 \quad \textit{when } \xi \le -q_1(t),\\
&P^y(\xi,t) \ge \mu_2 \quad \textit{when } q_1(t) \le \xi \le -q_2(t),\\
&P^z(\xi,t) \ge \mu_3 \quad \textit{when } \xi \ge -q_1(t).
\end{aligned}$$

For given $\delta \in \mathbb{R}^+$, we further assume that the functions $q_1, q_2 : \mathbb{R}^- \to \mathbb{R}^-$ satisfy

$$q_1(t) < -\delta, \;\; q_2(t) < -\delta \quad \text{for any } t \in \mathbb{R}^-. \tag{3.18}$$

For $(\xi, t) \in \mathbb{R} \times \mathbb{R}^-$, define

$$\begin{aligned}
A_1(\xi,t) := P^x(\xi,t)\phi_{11}'(\xi - q_1(t)) - P^y(\xi,t)\phi_{21}'(\xi + q_1(t))\\
+P^z(\xi,t)\phi_{31}'(\xi + q_2(t)).
\end{aligned}$$

Obviously, A_1 is bounded from above. The next lemma shows that under some appropriate conditions on q_1 and q_2, the function A_1 also has a positive bound from below.

Lemma 3.9 *Assume (J1)–(J4) hold. There exists a sufficiently large constant $\delta > 0$ such that if q_1 and q_2 satisfy (3.18) and $q_2(t) - q_1(t) < -\delta$ for any $t < 0$, then*

$$A_1(\xi,t) > 0 \;\; \textit{for } (\xi,t) \in \mathbb{R} \times \mathbb{R}^-,$$

and for any $t < 0$, we have that

(i) when $\xi \le q_1(t)$, $A_1(\xi,t) \ge \frac{1}{2} P^x(\xi,t)|\phi_{11}'(\xi - q_1(t))|$;
(ii) when $q_1(t) \le \xi \le -q_1(t)$,

$$A_1(\xi,t) \ge \frac{1}{2}\left[P^x(\xi,t)|\phi_{11}'(\xi - q_1(t))| + P^y(\xi,t)|\phi_{21}'(\xi + q_1(t))|\right];$$

(iii) when $-q_1(t) \le \xi \le -q_2(t)$,

$$A_1(\xi,t) \ge \frac{1}{2}\left[P^y(\xi,t)|\phi_{21}'(\xi + q_1(t))| + P^z(\xi,t)|\phi_{31}'(\xi + q_2(t))|\right];$$

(iv) when $\xi \ge -q_2(t)$, $A_1(\xi,t) \ge \frac{1}{2} P^z(\xi,t)|\phi_{31}'(\xi + q_2(t))|$.

For $(\xi,t) \in \mathbb{R} \times \mathbb{R}^-$, we define

$$\begin{aligned} H_1(\xi,t) :=& (J_1 * P^0(\xi,t) - P^0(\xi,t)) - P^x(\xi,t)(J_1 * \phi_{11}(\xi - q_1(t)) - \phi_{11}(\xi - q_1(t))) \\ & - P^y(\xi,t)(J_1 * \phi_{21}(\xi + q_1(t)) - \phi_{21}(\xi + q_1(t))) \\ & - P^z(\xi,t)(J_1 * \phi_{31}(\xi + q_2(t)) - \phi_{31}(\xi + q_2(t))). \end{aligned}$$

The following lemma gives some estimates for $H_1(\xi,t)/A_1(\xi,t)$.

Lemma 3.10 *Assume (J1)–(J4) hold. For the* δ *given by Lemma 3.9, if* q_1 *and* q_2 *satisfy* (3.18) *and* $q_2(t) - q_1(t) < -\delta$ *for any* $t \in \mathbb{R}^-$*, then there is a positive constant* $\tilde{M}$ *such that for any* $t < 0$,

$$\left|\frac{H_1(\xi,t)}{A_1(\xi,t)}\right| \le \begin{cases} \tilde{M}\left(e^{\eta_1 q_1} + e^{\eta_1 q_2}\right) & \text{when } \xi \le 0, \\ \tilde{M}\left(e^{\eta_2 q_1} + e^{\eta_1 (q_2 - q_1)/2}\right) & \text{when } 0 \le \xi \le \dfrac{-q_1(t) - q_2(t)}{2}, \\ \tilde{M}\left(e^{\eta_2 q_1} + e^{\eta_2 (q_2 - q_1)/2}\right) & \text{when } \xi \ge \dfrac{-q_1(t) - q_2(t)}{2}. \end{cases} \tag{3.19}$$

For $(\xi,t) \in \mathbb{R} \times \mathbb{R}^-$, we define

$$\begin{aligned} F_1(\xi,t) :=& f_1(P^0(\xi,t), Q^0(\xi,t)) - P^x(\xi,t) f_1(\phi_{11}(\xi - q_1(t)), \phi_{12}(\xi - q_1(t))) \\ & - P^y(\xi,t) f_1(\phi_{21}(\xi + q_1(t)), \phi_{22}(\xi + q_1(t))) \\ & - P^z(\xi,t) f_1(\phi_{31}(\xi + q_2(t)), \phi_{32}(\xi + q_2(t))), \end{aligned}$$

where f_1 is defined by (3.14), and

$$Q^0(\xi,t) := Q\left(\phi_{12}(\xi - q_1(t)), \phi_{22}(\xi + q_1(t)), \phi_{32}(\xi + q_2(t))\right), \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^-.$$

By a similar argument to Wu et al. (2018, Lemma 4.4), there is some positive constant C_5 such that

$$\begin{cases} F_1(\xi,t) \le C_5(\phi_{11} - u_1^* + \phi_{12} - v_1^*)(u_2^* - \phi_{21} + v_2^* - \phi_{22})[(u_2^* - \phi_{11} + v_2^* - \phi_{12}) + (\phi_{31} + \phi_{32})], \\ F_1(\xi,t) \le C_5(\phi_{21} + \phi_{22})(u_2^* - \phi_{31} + v_2^* - \phi_{32})[(u_2^* - \phi_{11} + v_2^* - \phi_{12}) + (\phi_{31} + \phi_{32})]. \end{cases} \tag{3.20}$$

With the help of (3.20), we can obtain the following result similar to Lemma 3.10, and the details of the proof are omitted.

Lemma 3.11 *Assume (J1)–(J4) hold. For the* δ *given by Lemma 3.9, if* q_1 *and* q_2 *satisfy* (3.18) *and* $q_2(t) - q_1(t) < -\delta$ *for any* $t \in \mathbb{R}^-$, *then there is a positive constant* $\bar{M}$ *such that for any* $t < 0$,

$$\left|\frac{F_1(\xi,t)}{A_1(\xi,t)}\right| \le \begin{cases} \bar{M}\left(e^{\eta_1 q_1} + e^{\eta_1 q_2}\right) \text{ when } \xi \le 0, \\ \bar{M}\left(e^{\eta_2 q_1} + e^{\eta_1 (q_2 - q_1)/2}\right) \text{ when } 0 \le \xi \le \dfrac{-q_1(t) - q_2(t)}{2}, \\ \bar{M}\left(e^{\eta_2 q_1} + e^{\eta_2 (q_2 - q_1)/2}\right) \text{ when } \xi \ge \dfrac{-q_1(t) - q_2(t)}{2}. \end{cases}$$

Note that Lemmas 3.9–3.11 provide the estimates for $A_1(\xi,t)$, $F_1(\xi,t)$, and $H_1(\xi,t)$, which are related to $P^0(\xi,t)$, $P^x(\xi,t)$, $P^y(\xi,t)$, and $P^z(\xi,t)$. For $(\xi,t) \in \mathbb{R} \times \mathbb{R}^-$, we define

$$\begin{aligned} A_2(\xi,t) &:= Q^x(\xi,t)\phi_{12}'(\xi - q_1(t)) - Q^y(\xi,t)\phi_{22}'(\xi + q_1(t)) + Q^z(\xi,t)\phi_{32}'(\xi + q_2(t)), \\ H_2(\xi,t) &:= (J_2 * Q^0(\xi,t) - Q^0(\xi,t)) - Q^x(\xi,t)(J_2 * \phi_{12}(\xi - q_1(t)) - \phi_{12}(\xi - q_1(t))) \\ &\quad - Q^y(\xi,t)(J_2 * \phi_{22}(\xi + q_1(t)) - \phi_{22}(\xi + q_1(t))) \\ &\quad - Q^z(\xi,t)(J_2 * \phi_{32}(\xi + q_2(t)) - \phi_{32}(\xi + q_2(t))), \\ F_2(\xi,t) &:= f_2(P^0(\xi,t), Q^0(\xi,t)) - Q^x(\xi,t) f_2(\phi_{11}(\xi - q_1(t)), \phi_{12}(\xi - q_1(t))) \\ &\quad - Q^y(\xi,t) f_2(\phi_{21}(\xi + q_1(t)), \phi_{22}(\xi + q_1(t))) \\ &\quad - Q^z(\xi,t) f_2(\phi_{31}(\xi + q_2(t)), \phi_{32}(\xi + q_2(t))), \end{aligned}$$

where f_2 is defined by (3.14), and

$$\begin{aligned} Q^x(\xi,t) &:= Q_x\left(\phi_{12}(\xi - q_1(t)), \phi_{22}(\xi + q_1(t)), \phi_{32}(\xi + q_2(t))\right), \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^-, \\ Q^y(\xi,t) &:= Q_y\left(\phi_{12}(\xi - q_1(t)), \phi_{22}(\xi + q_1(t)), \phi_{32}(\xi + q_2(t))\right), \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^-, \\ Q^z(\xi,t) &:= Q_z\left(\phi_{12}(\xi - q_1(t)), \phi_{22}(\xi + q_1(t)), \phi_{32}(\xi + q_2(t))\right), \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^-. \end{aligned}$$

The next three lemmas give estimates for $Q^x(\xi,t)$, $Q^y(\xi,t)$, $Q^z(\xi,t)$, $A_2(\xi,t)$, $F_2(\xi,t)$, and $H_2(\xi,t)$. The proofs of these three lemmas, which are omitted, are similar to Lemmas 3.8, 3.9, and the combination of Lemmas 3.10 and 3.11, respectively.

Lemma 3.12 *Assume (J1)–(J4) hold. There are some positive constants* μ_1, μ_2, *and* μ_3 *such that for any* $t < 0$,

$$\begin{aligned} Q^x(\xi,t) &\ge \mu_1 \quad \text{when } \xi \le -q_1(t), \\ Q^y(\xi,t) &\ge \mu_2 \quad \text{when } q_1(t) \le \xi \le -q_2(t), \\ Q^z(\xi,t) &\ge \mu_3 \quad \text{when } \xi \ge -q_1(t). \end{aligned}$$

Lemma 3.13 *Assume (J1)–(J4) hold. There exists a sufficiently large constant* $\delta > 0$ *such that if* q_1 *and* q_2 *satisfy* (3.18) *and* $q_2(t) - q_1(t) < -\delta$ *for any* $t < 0$, *then*

$$A_2(\xi,t)>0 \quad for\ (\xi,t)\in\mathbb{R}\times\mathbb{R}^-,$$

and for any $t<0$, we have that

(i) when $\xi\le q_1(t)$, $A_2(\xi,t)\ge\frac{1}{2}Q^x(\xi,t)|\phi_{12}'(\xi-q_1(t))|$;
(ii) when $q_1(t)\le\xi\le -q_1(t)$,

$$A_2(\xi,t)\ge\frac{1}{2}\left[Q^x(\xi,t)|\phi_{12}'(\xi-q_1(t))|+Q^y(\xi,t)|\phi_{22}'(\xi+q_1(t))|\right];$$

(iii) when $-q_1(t)\le\xi\le -q_2(t)$,

$$A_2(\xi,t)\ge\frac{1}{2}\left[Q^y(\xi,t)|\phi_{22}'(\xi+q_1(t))|+Q^z(\xi,t)|\phi_{32}'(\xi+q_2(t))|\right];$$

(iv) when $\xi\ge -q_2(t)$, $A_2(\xi,t)\ge\frac{1}{2}Q^z(\xi,t)|\phi_{32}'(\xi+q_2(t))|$.

Lemma 3.14 *Assume (J1)–(J4) hold. For the δ given by Lemma 3.13, if q_1 and q_2 satisfy (3.18) and $q_2(t)-q_1(t)<-\delta$ for any $t<0$, then there is a positive constant $\hat{M}$ such that for any $t<0$,*

$$\left|\frac{H_2(\xi,t)}{A_2(\xi,t)}\right|+\left|\frac{F_2(\xi,t)}{A_2(\xi,t)}\right|\le\begin{cases}\hat{M}\left(e^{\eta_1 q_1}+e^{\eta_1 q_2}\right)\ when\ \xi\le 0,\\ \hat{M}\left(e^{\eta_2 q_1}+e^{\eta_1(q_2-q_1)/2}\right)\ when\ 0\le\xi\le\dfrac{-q_1(t)-q_2(t)}{2},\\ \hat{M}\left(e^{\eta_2 q_1}+e^{\eta_2(q_2-q_1)/2}\right)\ when\ \xi\ge\dfrac{-q_1(t)-q_2(t)}{2}.\end{cases}$$

Finally, we can give a pair of suitable super- and sub-solutions for the first type of entire solution and complete the proof of Theorem 3.5 by applying them.

Lemma 3.15 *Assume that (J1)–(J4) hold, and $c_0>\hat{c}_0>c_2$. Let $(p_1(t),p_2(t),r_1(t),r_2(t))$ be the solution of (3.2). Then, the functions defined by (3.15) and (3.16), namely,*

$$\begin{cases}\overline{U}(\xi,t)=P(\phi_{11}(\xi-p_1(t)),\phi_{21}(\xi+p_1(t)),\phi_{31}(\xi+p_2(t))),\\ \overline{V}(\xi,t)=Q(\phi_{12}(\xi-p_1(t)),\phi_{22}(\xi+p_1(t)),\phi_{32}(\xi+p_2(t)))\end{cases}$$

and

$$\begin{cases}\underline{U}(\xi,t)=P(\phi_{11}(\xi-r_1(t)),\phi_{21}(\xi+r_1(t)),\phi_{31}(\xi+r_2(t))),\\ \underline{V}(\xi,t)=Q(\phi_{12}(\xi-r_1(t)),\phi_{22}(\xi+r_1(t)),\phi_{32}(\xi+r_2(t)))\end{cases}$$

are a pair of super- and sub-solutions of (3.13) for $t\le t_0<0$ with some $t_0<0$, when the constant L in (3.2) and (3.3) is sufficiently large. Moreover, there exists $\mu>0$ such that

$$\underline{U}(\xi,t)\le\overline{U}(\xi,t),\quad \sup_{\xi\in\mathbb{R}}\{\overline{U}(\xi,t)-\underline{U}(\xi,t)\}\le\mu e^{\kappa\nu_1 t}\ for\ \xi\in\mathbb{R},\ t\le t_0,\tag{3.21}$$

$$\underline{V}(\xi,t) \le \overline{V}(\xi,t), \quad \sup_{\xi\in\mathbb{R}}\{\overline{V}(\xi,t)-\underline{V}(\xi,t)\} \le \mu e^{\kappa\nu_1 t} \text{ for } \xi \in \mathbb{R},\ t \le t_0. \tag{3.22}$$

where ν_1 and κ are defined by (3.1) and (3.4), respectively.

Proof Let δ be a positive constant such that Lemmas 3.9 and 3.13 hold. By (3.2), it is easy to get that $p_2(t)-p_1(t)\to-\infty$ and $r_2(t)-r_1(t)\to-\infty$ as $t\to-\infty$. By (3.4), there exists a constant $t_0<0$ such that

$$p_2(t)-p_1(t)<-\delta, \quad r_2(t)-r_1(t)<-\delta \text{ when } t\le t_0,$$

and

$$\max\left\{\frac{\eta_1(p_2(t)-p_1(t))}{2}, \frac{\eta_2(p_2(t)-p_1(t))}{2}\right\} < \kappa p_1(t) < 0 \text{ when } t\le t_0, \tag{3.23}$$

$$\max\left\{\frac{\eta_1(r_2(t)-r_1(t))}{2}, \frac{\eta_2(r_2(t)-r_1(t))}{2}\right\} < \kappa r_1(t) < 0 \text{ when } t\le t_0. \tag{3.24}$$

Recall that s_1, s_2, and s_3 satisfy (3.11). For simplicity, we denote

$$\begin{aligned}
\Gamma_1(U,V)(\xi,t) &:= \frac{\partial U}{\partial t} - (J_1 * U - U) + \bar{c}U_\xi - f_1(U,V),\\
\Gamma_2(U,V)(\xi,t) &:= \frac{\partial V}{\partial t} - (J_2 * V - V) + \bar{c}V_\xi - f_2(U,V),
\end{aligned}$$

where $\xi = x+\bar{c}t$, $\bar{c}=\frac{s_1+s_2}{2}$, and f_1, f_2 are defined by (3.14). To prove the first conclusion of this lemma, it is sufficient to show that

$$\Gamma_i(\overline{U},\overline{V})(\xi,t)\ge 0 \text{ and } \Gamma_i(\underline{U},\underline{V})(\xi,t)\le 0, \text{ for any } \xi\in\mathbb{R}, t\le t_0, i=1,2.$$

Next, we only prove $\Gamma_i(\overline{U},\overline{V})(\xi,t)\ge 0$ with $i=1,2$ for $\xi\in\mathbb{R}$ and $t\le t_0$, since the proof of $\Gamma_i(\underline{U},\underline{V})(\xi,t)\le 0$ is similar. Direct calculations give that

$$\begin{aligned}
\Gamma_1(\overline{U},\overline{V}) = &-P^x\phi_{11}'(p_1'-\nu_1) + P^y\phi_{21}'(p_1'-\nu_1) + P^z\phi_{31}'(p_2'-\nu_2)\\
&-(J_1*\overline{U}-\overline{U}) + P^x(J_1*\phi_{11}-\phi_{11}) + P^y(J_1*\phi_{21}-\phi_{21})\\
&+P^z(J_1*\phi_{31}-\phi_{31})\\
&-f_1(\overline{U},\overline{V}) + P^x f_1(\phi_{11},\phi_{12}) + P^y f_1(\phi_{21},\phi_{22}) + P^z f_1(\phi_{31},\phi_{32}),
\end{aligned}$$

where ν_1 and ν_2 are defined in (3.1). By (3.2) and the definitions of the functions $A_1(\xi,t)$, $F_1(\xi,t)$, and $H_1(\xi,t)$, we get that

$$\begin{aligned}
\Gamma_1(\overline{U},\overline{V})(\xi,t) =&(P^x\phi_{11}' - P^y\phi_{21}' + P^z\phi_{31}')Le^{\kappa p_1(t)} - H_1(\xi,t) - F_1(\xi,t)\\
=&A_1(\xi,t)Le^{\kappa p_1(t)} - H_1(\xi,t) - F_1(\xi,t).
\end{aligned}$$

From Lemmas 3.10, 3.11, and (3.23), it follows that

$$|H_1(\xi,t)+F_1(\xi,t)| \le A_1(\xi,t)(\tilde{M}+\bar{M})e^{\kappa p_1(t)}, \text{ for any } \xi\in\mathbb{R}, t\le t_0.$$

By choosing $L > \tilde{M}+\bar{M}$, we have

$$\Gamma_1(\overline{U},\overline{V})(\xi,t) \ge A_1(\xi,t)[Le^{\kappa p_1(t)} - (\tilde{M}+\bar{M})e^{\kappa p_1(t)}] \ge 0, \text{ for any } \xi\in\mathbb{R}, t\le t_0.$$

By a similar argument to $\Gamma_1(\overline{U},\overline{V})(\xi,t)$ (with replacing Lemmas 3.8, 3.9, 3.10, 3.11, and (3.23) by Lemmas 3.12, 3.13, 3.14, and (3.24)), we can take $L \ge \hat{M}$ such that

$$\begin{aligned}\Gamma_2(\overline{U},\overline{V})(\xi,t) &= (Q^x\phi_{12}' - Q^y\phi_{22}' + Q^z\phi_{32}')Le^{\kappa p_1(t)} - H_2(\xi,t) - F_2(\xi,t)\\ &\ge A_2(\xi,t)(L-\hat{M})e^{\kappa p_1(t)} \ge 0, \text{ for any } \xi\in\mathbb{R}, t\le t_0.\end{aligned}$$

Finally, we prove (3.21), and the proof of (3.22) is similar to (3.21). We see that

$$\begin{aligned}&\overline{U}(\xi,t) - \underline{U}(\xi,t)\\ =&P_1(\phi_{11}(\xi-p_1),\phi_{21}(\xi+p_1),\phi_{31}(\xi+p_2)) - P_1(\phi_{11}(\xi-r_1),\phi_{21}(\xi+r_1),\phi_{31}(\xi+r_2)).\end{aligned}$$

From Newton–Leibniz formula, it follows that

$$\begin{aligned}&\overline{U}(\xi,t) - \underline{U}(\xi,t)\\ &= (r_1-p_1)\int_0^1 P_x(\phi_{11}(\xi-\theta p_1-(1-\theta)r_1),\phi_{21}(\xi+p_1),\phi_{31}(\xi+p_2))\phi_{11}'\mathrm{d}\theta\\ &\quad - (r_1-p_1)\int_0^1 P_y(\phi_{11}(\xi-r_1),\phi_{21}(\xi+\theta p_1+(1-\theta)r_1),\phi_{31}(\xi+p_2))\phi_{21}'\mathrm{d}\theta\\ &\quad + (p_2-r_2)\int_0^1 P_z(\phi_{11}(\xi-r_1),\phi_{21}(\xi+r_1),\phi_{31}(\xi+\theta p_2+(1-\theta)r_2)\phi_{31}'\mathrm{d}\theta.\end{aligned} \tag{3.25}$$

Obviously, the integral terms in (3.25) are bounded and positive. Then, by (3.6), we can easily obtain that

$$\underline{U}(\xi,t) \le \overline{U}(\xi,t) \text{ and } \sup_{\xi\in\mathbb{R}}\{\overline{U}(\xi,t)-\underline{U}(\xi,t)\} \le \mu e^{\kappa\nu_1 t}, \text{ for any } \xi\in\mathbb{R}, t\le t_0.$$

It completes the proof. □

Proof of Theorem 3.5 We define $(\overline{U},\overline{V})$ and $(\underline{U},\underline{V})$ by the same functions as in Lemma 3.15. Let $\bar{c} = (c_2+\hat{c}_0)/2$, and denote

$$\begin{aligned}\overline{\boldsymbol{\omega}}(x,t) &:= (\overline{u},\overline{v})(x,t) = (\overline{U},\overline{V})(x+\bar{c}t,t),\\ \underline{\boldsymbol{\omega}}(x,t) &:= (\underline{u},\underline{v})(x,t) = (\underline{U},\underline{V})(x+\bar{c}t,t).\end{aligned}$$

By Lemma 3.15, $\overline{\boldsymbol{\omega}}(x,t)$ and $\underline{\boldsymbol{\omega}}(x,t)$ are a pair of super- and sub-solutions of (1.1) for $x \in \mathbb{R}$ and $t \le t_0$. Now, we consider the following Cauchy problem

$$\begin{cases} \partial_t u_n(x,t) = (J_1 * u_n - u_n)(x,t) - \alpha u_n(x,t) + h(v_n(x,t)), \\ \partial_t v_n(x,t) = (J_2 * v_n - v_n)(x,t) - \beta v_n(x,t) + g(u_n(x,t)), \\ (u_n, v_n)(x,-n) = \underline{\boldsymbol{\omega}}(x,-n), \; x \in \mathbb{R}, t > -n. \end{cases} \tag{3.26}$$

By Lemma 2.10, (3.26) has a unique solution $\boldsymbol{\omega}_n(x,t) = (u_n(x,t), v_n(x,t))$ satisfying

$$E_0 \le \boldsymbol{\omega}_n(x,t) \le E_2, \text{ for any } x \in \mathbb{R}, t > -n.$$

Since $\underline{\boldsymbol{\omega}}(x,t)$ is also a sub-solution of (3.26), we have

$$\underline{\boldsymbol{\omega}}(x,-n) = \boldsymbol{\omega}_n(x,-n) \le \boldsymbol{\omega}_{n+1}(x,-n) \le E_2, \text{ for any } x \in \mathbb{R}, t > -n.$$

It follows from Lemma 3.2 that

$$E_0 \le \underline{\boldsymbol{\omega}}(x,t) \le \boldsymbol{\omega}_n(x,t) \le \boldsymbol{\omega}_{n+1}(x,t) \le \min\{E_2, \overline{\boldsymbol{\omega}}(x,t)\}, \text{ for any } x \in \mathbb{R}, t > -n.$$

Therefore, $\{\boldsymbol{\omega}_n(x,t)\}_{n=1}^{\infty}$ is bounded and nondecreasing with respect to n for any $(x,t) \in \mathbb{R} \times (-n, +\infty)$. Then, there exists a function $\boldsymbol{\omega}(x,t) = (u(x,t), v(x,t))$ with $E_0 \le \boldsymbol{\omega}(x,t) \le E_2$ such that for any $(x,t) \in \mathbb{R}^2$,

$$\lim_{n\to\infty} (u_n(x,t), v_n(x,t)) = (u(x,t), v(x,t)).$$

Note that for any given $t_0 \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $t_0 > -n$ and

$$\boldsymbol{\omega}_n(x,t) = \mathbf{P}(t)[\boldsymbol{\omega}_n(\cdot,t_0)](x) + \int_{t_0}^{t} \mathbf{P}(t-s)[\mathbf{F}(\boldsymbol{\omega}_n(\cdot,s))](x)\mathrm{d}s, \quad x \in \mathbb{R}, \; t \ge t_0.$$

By Lebesgue dominated convergence theorem, we have that

$$\boldsymbol{\omega}(x,t) = \mathbf{P}(t)[\boldsymbol{\omega}(\cdot,t_0)](x) + \int_{t_0}^{t} \mathbf{P}(t-s)[\mathbf{F}(\boldsymbol{\omega}(\cdot,s))](x)\mathrm{d}s.$$

We easily check that $\boldsymbol{\omega}(x,t)$ is continuous and differentiable with respect to t. Thus, we obtain

$$\begin{aligned} u_t(x,t) = {} & \frac{\partial}{\partial t} P_1(t)[u(\cdot,t_0)](x) + \int_{t_0}^{t} \frac{\partial}{\partial t} P_1(t-s)[-\alpha u(\cdot,s) + h(v(\cdot,s))](x)\mathrm{d}s \\ & + [-\alpha u(x,t) + h(v(x,t))]. \end{aligned}$$

From (2.17), it follows that

$$
\begin{aligned}
u_t(x,t) &= -P_1(t)[u(\cdot,t_0)](x) + J_1 * P_1(t)[u(\cdot,t_0)](x) + [-\alpha u(x,t) + h(v(x,t))] \\
&\quad + \int_{t_0}^{t} \{-P_1(t)[-\alpha u(\cdot,s) + h(v(\cdot,s))](x) + J_1 * P_1(t) \\
&\quad [-\alpha u(\cdot,s) + h(v(\cdot,s))](x)\}\,\mathrm{d}s \\
&= J_1 * u(x,t) - u(x,t) - \alpha u(x,t) + h(v(x,t)).
\end{aligned}
$$

Similarly, we have that

$$
v_t(x,t) = J_2 * v(x,t) - v(x,t) - \beta v(x,t) + g(u(x,t)).
$$

Thus, $\boldsymbol{\omega}(x,t) = (u(x,t), v(x,t))$ is an entire solution of (1.1). The asymptotic behavior and the smooth property of $\boldsymbol{\omega}(x,t)$ can be obtained by similar arguments to Chen et al. (2018, Theorem 4.3) and Zhang et al. (2016, Theorem 1.8), respectively, and we only give an outline of the proof here. According to Lemma 3.7, (3.5), (3.21), and (3.22), we can prove (3.7). By applying Lemma 3.3, Arzela–Ascoli theorem, a diagonal extraction process, and the uniqueness of the solution of (3.26), we can obtain the last assertion of this theorem. Therefore, the proof is completed. □

3.3 II-Type Entire Solution

In this subsection, we construct the second type of entire solution, which originates from one monostable and two bistable traveling wave solutions, namely $(\hat{c}_1, \hat{\Phi}_1)$, (c_0, Φ), and $(\hat{c}_0, \hat{\Phi})$ with $\hat{c}_1 < c_0 < \hat{c}_0$. We see Fig. 1b for the profile of this entire solution as $t \to -\infty$. As stated in Sect. 3.2, the difficulty in the study of entire solutions is how to construct appropriate auxiliary functions and super- and sub-solutions for different types of entire solutions. Hence, we only provide the construction of auxiliary functions and super- and sub-solutions and omit the other details, which are similar to Theorem 3.5. The second type of entire solution is considered in the following theorem.

Theorem 3.16 *Assume (J1)–(J4) hold and $\hat{c}_0 > c_0 > \hat{c}_1$. Then,* (1.1) *admits an entire solution $\mathcal{W}_2(x,t) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ satisfying*

$$
\begin{aligned}
\lim_{t\to-\infty} \Bigg\{ & \sup_{x \le \rho_1(t)} \|\mathcal{W}_2(x,t) - \hat{\Phi}_1(x + \hat{c}_1 t - \vartheta_3)\| \\
& + \sup_{\rho_1(t) \le x \le \rho_2(t)} \|\mathcal{W}_2(x,t) - \Phi(x + c_0 t + \vartheta_3)\| \\
& + \sup_{\rho_2(t) \le x} \|\mathcal{W}_2(x,t) - \hat{\Phi}(x + \hat{c}_0 t + \vartheta_4)\| \Bigg\} = 0,
\end{aligned}
$$

where ϑ_3 and ϑ_4 are some constants satisfying

$$\vartheta_3 := -\frac{1}{\kappa}\ln\left[e^{-\kappa p_0} - \frac{L}{\nu_1}\right], \quad \vartheta_4 := \frac{1}{\kappa}\ln\left[e^{-\kappa p_0} - \frac{L}{\nu_1}\right] + p_0 + r_0,$$

and $\rho_1, \rho_2 : \mathbb{R} \mapsto \mathbb{R}$ are defined by

$$\rho_1(t) := \frac{-(\hat{c}_1 + c_0)t}{2}, \qquad \rho_2(t) := \frac{-(c_0 + \hat{c}_0)t}{2}.$$

Moreover, when (J5) holds, there are two positive constants D_1 and D_2 such that for any $(x, t) \in \mathbb{R}^2$ and $\eta > 0$,

$$\|\mathcal{W}_2(x+\eta, t) - \mathcal{W}_2(x, t)\| \le D_1\eta, \quad \left\|\frac{\partial \mathcal{W}_2}{\partial t}(x+\eta, t) - \frac{\partial \mathcal{W}_2}{\partial t}\right\| \le D_2\eta.$$

Now, we show the construction of auxiliary functions and super- and sub-solutions for $\mathcal{W}_2$. Throughout this subsection, choose the values of s_1, s_2, and s_3 as follows:

$$s_1 := \hat{c}_1, \;\; s_2 := c_0, \;\; s_3 := \hat{c}_0.$$

For convenience, we denote

$$\boldsymbol{\phi}_1(\xi) := \hat{\Phi}_1(\xi), \;\; \boldsymbol{\phi}_2(\xi) := \Phi(\xi), \;\; \boldsymbol{\phi}_3(\xi) := \hat{\Phi}(\xi),$$

and assume that

$$\phi_{11}(0) = \frac{u_1^*}{2}, \;\; (\phi_{21}(0), \phi_{22}(0)) \ge \left(\frac{u_1^* + u_2^*}{2}, \frac{v_1^* + v_2^*}{2}\right), \text{ and } (\phi_{31}(0), \phi_{32}(0)) \ge \left(\frac{u_1^* + u_2^*}{2}, \frac{v_1^* + v_2^*}{2}\right).$$

Corresponding to (3.9) and (3.10) for the first type of entire solution, two auxiliary functions for this case to link traveling wave solutions are defined as follows:

$$P(x, y, z) = y + \frac{(u_1^* - x)y(u_2^* - z)(-y) + x(u_2^* - y)z(u_1^* - y)}{(u_1^* - x)yu_2^* + u_1^*(u_2^* - y)z}, \quad (x, y, z) \in D_1$$

and

$$Q(x, y, z) = y + \frac{(v_1^* - x)y(v_2^* - z)(-y) + x(v_2^* - y)z(v_1^* - y)}{(v_1^* - x)yu_2^* + v_1^*(v_2^* - y)z}, \quad (x, y, z) \in D_2,$$

where

$$D_1 := \left\{(x, y, z) \in [0, u_1^*] \times [0, u_2^*] \times [0, u_2^*] \middle| (u_1^* - x)yu_2^* + u_1^*(u_2^* - y)z > 0\right\}$$

and

$$D_2 := \left\{(x, y, z) \in [0, v_1^*] \times [0, v_2^*] \times [0, v_2^*] \middle| (v_1^* - x)yu_2^* + v_1^*(v_2^* - y)z > 0\right\}.$$

The following lemma constructs a pair of super- and sub-solutions, and by this lemma we can prove Theorem 3.16. We omit the proofs, which are similar to Lemma 3.15 and Theorem 3.5.

Lemma 3.17 *Assume (J1)–(J4) hold and $\hat{c}_0 > c_0 > \hat{c}_1$. Let $(p_1(t), p_2(t), r_1(t), r_2(t))$ be the solution of (3.2), and let $\bar{c} := (\hat{c}_1 + c_0)/2$. Then, the functions defined by*

$$\begin{cases} \overline{u}(x,t) = P(\phi_{11}(x + \bar{c}t - r_1(t)), \phi_{21}(x + \bar{c}t + r_1(t)), \phi_{31}(x + \bar{c}t + r_2(t))), \\ \overline{v}(x,t) = Q(\phi_{12}(x + \bar{c}t - r_1(t)), \phi_{22}(x + \bar{c}t + r_1(t)), \phi_{32}(x + \bar{c}t + r_2(t))) \end{cases}$$

and

$$\begin{cases} \underline{u}(x,t) = P(\phi_{11}(x + \bar{c}t - p_1(t)), \phi_{21}(x + \bar{c}t + p_1(t)), \phi_{31}(x + \bar{c}t + p_2(t))), \\ \underline{v}(x,t) = Q(\phi_{12}(x + \bar{c}t - p_1(t)), \phi_{22}(x + \bar{c}t + p_1(t)), \phi_{32}(x + \bar{c}t + p_2(t))) \end{cases}$$

are a pair of super- and sub-solutions of (1.1) for $(x,t) \in \mathbb{R} \times (-\infty, t_1]$ with some $t_1 < 0$. Moreover, (3.21) and (3.22) hold for $(\overline{u}(x,t), \overline{v}(x,t))$ and $(\underline{u}(x,t), \underline{v}(x,t))$.

3.4 III-Type Entire Solution

The third type of entire solution originates from two monostable and one bistable traveling wave solutions, namely (c_1, Φ_1), $(\hat{c}_1, \hat{\Phi}_1)$, and (c_0, Φ) with $c_1 < \hat{c}_1 < c_0$. We see Fig. 1c for the profile of this entire solution as $t \to -\infty$.

Theorem 3.18 *Assume (J1)–(J4) hold and $c_0 > \hat{c}_1 > c_1$. Then, (1.1) admits an entire solution $\mathcal{W}_3(x,t) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ satisfying*

$$\lim_{t \to -\infty} \left\{ \sup_{x \le \rho_1(t)} \|\mathcal{W}_3(x,t) - \Phi_1(x + c_1 t - \vartheta_1)\| + \sup_{\rho_1(t) \le x \le \rho_2(t)} \|\mathcal{W}_3(x,t) - \hat{\Phi}_1(x + \hat{c}_1 t + \vartheta_1)\| + \sup_{\rho_2(t) \le x} \|\mathcal{W}_3(x,t) - \Phi(x + c_0 t + \vartheta_2)\| \right\} = 0,$$

where ϑ_1 and ϑ_2 are some constants satisfying

$$\vartheta_1 := -\frac{1}{\kappa} \ln\left[e^{-\kappa r_0} + \frac{L}{\nu_1}\right], \quad \vartheta_2 := \frac{1}{\kappa} \ln\left[e^{-\kappa r_0} + \frac{L}{\nu_1}\right] + p_0 + r_0,$$

and $\rho_1, \rho_2 : \mathbb{R} \mapsto \mathbb{R}$ *are defined by*

$$\rho_1(t) := \frac{-(c_1 + \hat{c}_1)t}{2}, \qquad \rho_2(t) := \frac{-(\hat{c}_1 + c_0)t}{2}.$$

Moreover, when (J5) holds, there are two positive constants D_1 *and* D_2 *such that for any* $(x, t) \in \mathbb{R}^2$ *and* $\eta > 0$,

$$\|\mathcal{W}_3(x+\eta, t) - \mathcal{W}_3(x, t)\| \leq D_1\eta, \quad \left\| \frac{\partial \mathcal{W}_3}{\partial t}(x+\eta, t) - \frac{\partial \mathcal{W}_3}{\partial t} \right\| \leq D_2\eta.$$

Similar to the case of $\mathcal{W}_2$, we only provide the construction of auxiliary functions and super- and sub-solutions for $\mathcal{W}_3$ and omit the other details in the proof of Theorem 3.18. Throughout this subsection, we choose the values of s_1, s_2, and s_3 as follows:

$$s_1 := c_1, \; s_2 := \hat{c}_1, \; s_3 := c_0.$$

Denote

$$\boldsymbol{\phi}_1(\xi) := \Phi_1(\xi), \; \boldsymbol{\phi}_2(\xi) := \hat{\Phi}_1(\xi), \; \boldsymbol{\phi}_3(\xi) := \Phi(\xi),$$

and assume that

$$(\phi_{11}(0), \phi_{21}(0)) \geq \left(\frac{u_1^*}{2}, \frac{v_1^*}{2}\right), \; \phi_{21}(0) = \frac{u_1^*}{2}, \text{ and } (\phi_{31}(0), \phi_{32}(0)) \leq \left(\frac{u_1^*}{4}, \frac{v_1^*}{4}\right).$$

The two auxiliary functions are defined as follows:

$$P(x, y, z) = y + \frac{x(u_1^* - y)z(u_2^* - y) + (u_1^* - x)y(u_2^* - z)(-y)}{x(u_1^* - y)u_2^* + u_1^* y(u_2^* - z)}, \quad (x, y, z) \in D_1$$

and

$$Q(x, y, z) = y + \frac{x(v_1^* - y)z(v_2^* - y) + (v_1^* - x)y(v_2^* - z)(-y)}{x(v_1^* - y)v_2^* + v_1^* y(v_2^* - z)}, \quad (x, y, z) \in D_2,$$

where

$$D_1 := \left\{(x, y, z) \in [0, u_1^*] \times [0, u_1^*] \times [0, u_2^*] \middle| x(u_1^* - y)u_2^* + u_1^* y(u_2^* - z) > 0\right\}$$

and

$$D_2 := \left\{(x, y, z) \in [0, v_1^*] \times [0, v_1^*] \times [0, v_2^*] \middle| x(v_1^* - y)v_2^* + v_1^* y(v_2^* - z) > 0\right\}.$$

The super- and sub-solutions are given by the following lemma.

Lemma 3.19 *Assume (J1)–(J4) hold and* $c_0 > \hat{c}_1 > c_1$. *Let* $(p_1(t), p_2(t), r_1(t), r_2(t))$ *be the solution of* (3.2), *and* $\bar{c} := (c_1 + \hat{c}_1)/2$. *Then, the functions defined by*

$$\begin{cases} \overline{u}(x,t) = P(\phi_{11}(x+\bar{c}t - p_1(t)), \phi_{21}(x+\bar{c}t + p_1(t)), \phi_{31}(x+\bar{c}t + p_2(t))), \\ \overline{v}(x,t) = Q(\phi_{12}(x+\bar{c}t - p_1(t)), \phi_{22}(x+\bar{c}t + p_1(t)), \phi_{32}(x+\bar{c}t + p_2(t))) \end{cases}$$

and

$$\begin{cases} \underline{u}(x,t) = P(\phi_{11}(x+\bar{c}t - r_1(t)), \phi_{21}(x+\bar{c}t + r_1(t)), \phi_{31}(x+\bar{c}t + r_2(t))), \\ \underline{v}(x,t) = Q(\phi_{12}(x+\bar{c}t - r_1(t)), \phi_{22}(x+\bar{c}t + r_1(t)), \phi_{32}(x+\bar{c}t + r_2(t))) \end{cases}$$

are a pair of super- and sub-solutions of (1.1) *for* $(x,t) \in \mathbb{R} \times (-\infty, t_2]$ *with some* $t_2 < 0$. *Moreover,* (3.21) *and* (3.22) *hold for* $(\overline{u}(x,t), \overline{v}(x,t))$ *and* $(\underline{u}(x,t), \underline{v}(x,t))$.

3.5 IV-Type Entire Solution

The fourth type of entire solution originates from two monostable and one bistable traveling wave solutions, namely $(\hat{c}_2, \hat{\Phi}_2)$, $(\hat{c}_1, \hat{\Phi}_1)$, and (c_0, Φ) with $\hat{c}_2 < \hat{c}_1 < c_0$. We see Fig. 1d for the profile of this entire solution as $t \to -\infty$.

Theorem 3.20 *Assume (J1)–(J4) hold and* $c_0 > \hat{c}_1 > \hat{c}_2$. *Then* (1.1) *admits an entire solution* $\mathcal{W}_4(x,t) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ *satisfying*

$$\lim_{t\to-\infty} \Bigg\{ \sup_{x \le \rho_1(t)} \|\mathcal{W}_4(x,t) - \hat{\Phi}_2(x + \hat{c}_2 t - \vartheta_5)\| + \sup_{\rho_1(t) \le x \le \rho_2(t)} \|\mathcal{W}_4(x,t) - \hat{\Phi}_1(x + \hat{c}_1 t + \vartheta_5)\| + \sup_{\rho_2(t) \le x} \|\mathcal{W}_4(x,t) - \Phi(x + c_0 t + \vartheta_5)\| \Bigg\} = 0,$$

where ϑ_5 *is a constant satisfying*

$$\vartheta_5 := -\frac{1}{\kappa} \ln\left[e^{-\kappa r_0} - \frac{L}{\nu_1} \right],$$

and $\rho_1, \rho_2 : \mathbb{R} \mapsto \mathbb{R}$ *are defined by*

$$\rho_1(t) := \frac{-(\hat{c}_2 + \hat{c}_1)t}{2}, \qquad \rho_2(t) := \frac{-(\hat{c}_1 + c_0)t}{2}.$$

Moreover, when (J5) holds, there are two positive constants D_1 *and* D_2 *such that for any* $(x,t) \in \mathbb{R}^2$ *and* $\eta > 0$,

$$\|\mathcal{W}_4(x+\eta,t) - \mathcal{W}_4(x,t)\| \le D_1 \eta, \quad \left\| \frac{\partial \mathcal{W}_4}{\partial t}(x+\eta,t) - \frac{\partial \mathcal{W}_4}{\partial t} \right\| \le D_2 \eta.$$

Now, we give the construction of auxiliary functions and super- and sub-solutions for $\mathcal{W}_4$, which is the main difficulty, and the other details in the proof of Theorem 3.20 are omitted. Throughout this subsection, we choose the values of s_1, s_2, and s_3 as follows:

$$s_1 := \hat{c}_2 \le \hat{c}_2^* < 0,\ s_2 := \hat{c}_1 \ge \hat{c}_1^* > 0,\ s_3 := c_0.$$

Denote

$$\boldsymbol{\phi}_1(\xi) := \hat{\Phi}_2(\xi),\ \boldsymbol{\phi}_2(\xi) := \hat{\Phi}_1(\xi),\ \boldsymbol{\phi}_3(\xi) := \Phi(\xi)$$

and assume that

$$\phi_{11}(0) = \frac{u_1^*}{2},\ (\phi_{21}(0), \phi_{22}(0)) \ge \left(\frac{u_1^* + u_2^*}{2}, \frac{v_1^* + v_2^*}{2}\right),\ \text{and}\ (\phi_{31}(0), \phi_{32}(0)) \ge \left(\frac{u_1^* + u_2^*}{2}, \frac{v_1^* + v_2^*}{2}\right).$$

The two auxiliary functions are defined as follows:

$$P(x, y, z) = y + (u_2^* - y)\frac{(u_2^* - x)(u_1^* - y)z + (x - u_1^*)y(u_2^* - z)}{(u_2^* - x)(u_1^* - y)u_2^* + (u_2^* - u_1^*)y(u_2^* - z)}, \quad (x, y, z) \in D_1$$

and

$$Q(x, y, z) = y + (v_2^* - y)\frac{(v_2^* - x)(v_1^* - y)z + (x - v_1^*)y(v_2^* - z)}{(v_2^* - x)(v_1^* - y)v_2^* + (v_2^* - v_1^*)y(v_2^* - z)}, \quad (x, y, z) \in D_2,$$

where

$$D_1 := \left\{(x, y, z) \in [u_1^*, u_2^*] \times [0, u_1^*] \times [0, u_2^*] \middle| x(u_1^* - y)u_2^* + u_1^* y(u_2^* - z) > 0\right\}$$

and

$$D_2 := \left\{(x, y, z) \in [v_1^*, v_2^*] \times [0, v_1^*] \times [0, v_2^*] \middle| x(v_1^* - y)v_2^* + v_1^* y(v_2^* - z) > 0\right\}.$$

The super- and sub-solutions are given by the following lemma.

Lemma 3.21 *Assume (J1)–(J4) hold and $c_0 > \hat{c}_1 > \hat{c}_2$. Let $(\tilde{p}_1(t), \tilde{p}_2(t), \tilde{r}_1(t), \tilde{r}_2(t))$ be the solution of (3.3). Then, the functions defined by*

$$\begin{cases} \overline{u}(x,t) = P(\phi_{11}(x + \bar{c}t - \tilde{p}_1(t)), \phi_{21}(x + \bar{c}t + \tilde{r}_1(t)), \phi_{31}(x + \bar{c}t + \tilde{p}_2(t))), \\ \overline{v}(x,t) = Q(\phi_{12}(x + \bar{c}t - \tilde{p}_1(t)), \phi_{22}(x + \bar{c}t + \tilde{r}_1(t)), \phi_{32}(x + \bar{c}t + \tilde{p}_2(t))) \end{cases}$$

and

$$\begin{cases} \underline{u}(x,t) = P(\phi_{11}(x+\bar{c}t-\tilde{r}_1(t)), \phi_{21}(x+\bar{c}t+\tilde{p}_1(t)), \phi_{31}(x+\bar{c}t+\tilde{r}_2(t))), \\ \underline{v}(x,t) = Q(\phi_{12}(x+\bar{c}t-\tilde{r}_1(t)), \phi_{22}(x+\bar{c}t+\tilde{p}_1(t)), \phi_{32}(x+\bar{c}t+\tilde{r}_2(t))) \end{cases}$$

are a pair of super- and sub-solutions of (1.1) *for* $(x,t) \in \mathbb{R} \times (-\infty, t_3]$ *with some* $t_3 < 0$. *Moreover,* (3.21) *and* (3.22) *hold for* $(\overline{u}(x,t), \overline{v}(x,t))$ *and* $(\underline{u}(x,t), \underline{v}(x,t))$.

3.6 V-Type Entire Solution

The fifth type of entire solution originates from two monostable and one bistable traveling wave solutions, namely (c_1, Φ_1), (c_2, Φ_2), and $(\hat{c}_0, \hat{\Phi})$ with $c_1 < c_2 < \hat{c}_0$. We see Fig. 1e for the profile of this entire solution as $t \to -\infty$.

Theorem 3.22 *Assume (J1)–(J4) hold and* $\hat{c}_0 > c_2 > c_1$. *Then,* (1.1) *admits an entire solution* $\mathcal{W}_5(x,t) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ *satisfying*

$$\lim_{t\to-\infty} \left\{ \sup_{x \le \rho_1(t)} \|\mathcal{W}_5(x,t) - \Phi_1(x + c_1 t - \vartheta_6)\| + \sup_{\rho_1(t) \le x \le \rho_2(t)} \|\mathcal{W}_5(x,t) - \Phi_2(x + c_2 t + \vartheta_5)\| + \sup_{\rho_2(t) \le x} \|\mathcal{W}_5(x,t) - \hat{\Phi}(x + \hat{c}_0 t + \vartheta_6)\| \right\} = 0,$$

where ϑ_5 *and* ϑ_5 *are some constants satisfying*

$$\vartheta_5 := -\frac{1}{\kappa} \ln\left[e^{-\kappa r_0} - \frac{L}{\nu_1}\right], \quad \vartheta_6 := -\frac{1}{\kappa} \ln\left[e^{-\kappa p_0} + \frac{L}{\nu_1}\right],$$

and $\rho_1, \rho_2 : \mathbb{R} \mapsto \mathbb{R}$ *are defined by*

$$\rho_1(t) := \frac{-(c_1 + c_2)t}{2}, \qquad \rho_2(t) := \frac{-(c_2 + \hat{c}_0)t}{2}.$$

Moreover, when (J5) holds, there are two positive constants D_1 *and* D_2 *such that for any* $(x,t) \in \mathbb{R}^2$ *and* $\eta > 0$,

$$\|\mathcal{W}_5(x+\eta,t) - \mathcal{W}_5(x,t)\| \le D_1\eta, \quad \left\| \frac{\partial \mathcal{W}_5}{\partial t}(x+\eta,t) - \frac{\partial \mathcal{W}_5}{\partial t} \right\| \le D_2\eta.$$

Similar to other types of entire solutions, we focus only on the construction of auxiliary functions and super- and sub-solutions for $\mathcal{W}_5$ and omit the other details in the proof of Theorem 3.22. Throughout this subsection, we choose the values of s_1, s_2, and s_3 as follows:

$$s_1 := c_1, \; s_2 := c_2, \; s_3 := \hat{c}_0.$$

Denote

$$\boldsymbol{\phi}_1(\xi) := \Phi_1(\xi),\ \boldsymbol{\phi}_2(\xi) := \Phi_2(\xi),\ \boldsymbol{\phi}_3(\xi) := \hat{\Phi}(\xi)$$

and assume that

$$\phi_{11}(0) = \frac{u_1^*}{2},\ \phi_{21}(0) = \frac{u_1^* + u_2^*}{2},\ \text{and } (\phi_{31}(0), \phi_{32}(0)) \geq (u_1^*, v_1^*).$$

The two auxiliary functions are defined as follows:

$$P(x, y, z) = y + (-y)\frac{x(y - u_1^*)(u_2^* - z) + (u_1^* - x)(u_2^* - y)z}{x(y - u_1^*)u_2^* + u_1^*(u_2^* - y)z}, \quad (x, y, z) \in D_1$$

and

$$Q(x, y, z) = y + (-y)\frac{x(y - v_1^*)(v_2^* - z) + (v_1^* - x)(v_2^* - y)z}{x(y - v_1^*)v_2^* + v_1^*(v_2^* - y)z}, \quad (x, y, z) \in D_2,$$

where

$$D_1 := \left\{(x, y, z) \in [0, u_1^*] \times [u_1^*, u_2^*] \times [0, u_2^*] \middle| x(y - u_1^*)u_2^* + u_1^*(u_2^* - y)z > 0\right\}$$

and

$$D_2 := \left\{(x, y, z) \in [0, v_1^*] \times [v_1^*, v_2^*] \times [0, v_2^*] \middle| x(y - v_1^*)v_2^* + v_1^*(v_2^* - y)z > 0\right\}.$$

The super- and sub-solutions are given by the following lemma.

Lemma 3.23 *Assume (J1)–(J4) hold and* $\hat{c}_0 > c_2 > c_1$. *Let* $(\tilde{p}_1(t), \tilde{p}_2(t), \tilde{r}_1(t), \tilde{r}_2(t))$ *be the solution of* (3.3)*, and* $\bar{c} := (c_1 + c_2)/2$. *Then, the functions defined by*

$$\begin{cases} \overline{u}(x, t) = P(\phi_{11}(x + \bar{c}t - \tilde{r}_1(t)), \phi_{21}(x + \bar{c}t + \tilde{p}_1(t)), \phi_{31}(x + \bar{c}t + \tilde{r}_2(t))), \\ \overline{v}(x, t) = Q(\phi_{12}(x + \bar{c}t - \tilde{r}_1(t)), \phi_{22}(x + \bar{c}t + \tilde{p}_1(t)), \phi_{32}(x + \bar{c}t + \tilde{r}_2(t))) \end{cases}$$

and

$$\begin{cases} \underline{u}(x, t) = P(\phi_{11}(x + \bar{c}t - \tilde{p}_1(t)), \phi_{21}(x + \bar{c}t + \tilde{r}_1(t)), \phi_{31}(x + \bar{c}t + \tilde{p}_2(t))), \\ \underline{v}(x, t) = Q(\phi_{12}(x + \bar{c}t - \tilde{p}_1(t)), \phi_{22}(x + \bar{c}t + \tilde{r}_1(t)), \phi_{32}(x + \bar{c}t + \tilde{p}_2(t))) \end{cases}$$

are a pair of super- and sub-solutions of (1.1) *for* $(x, t) \in \mathbb{R} \times (-\infty, t_4]$ *with some* $t_4 < 0$. *Moreover,* (3.21) *and* (3.22) *hold for* $(\overline{u}(x, t), \overline{v}(x, t))$ *and* $(\underline{u}(x, t), \underline{v}(x, t))$.

3.7 VI-Type Entire Solution

The sixth type of entire solution originates from two monostable and one bistable traveling wave solutions, namely $(\hat{c}_2, \hat{\Phi}_2)$, (c_2, Φ_2), and $(\hat{c}_0, \hat{\Phi})$ with $\hat{c}_2 < c_2 < \hat{c}_0$. We see Fig. 1f for the profile of this entire solution as $t \to -\infty$.

Theorem 3.24 *Assume (J1)–(J4) hold and $\hat{c}_0 > c_2 > \hat{c}_2$. Then, (1.1) admits an entire solution $\mathcal{W}_6(x,t) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ satisfying*

$$\lim_{t\to-\infty} \Bigg\{ \sup_{x\le\rho_1(t)} \|\mathcal{W}_6(x,t) - \hat{\Phi}_2(x + \hat{c}_2 t - \vartheta_3)\| + \sup_{\rho_1(t)\le x\le\rho_2(t)} \|\mathcal{W}_6(x,t) - \Phi_2(x + c_2 t + \vartheta_3)\| + \sup_{\rho_2(t)\le x} \|\mathcal{W}_6(x,t) - \hat{\Phi}(x + \hat{c}_0 t + \vartheta_4)\| \Bigg\} = 0,$$

where ϑ_3 and ϑ_4 are some constants satisfying

$$\vartheta_3 := -\frac{1}{\kappa} \ln\left[e^{-\kappa p_0} - \frac{L}{\nu_1}\right], \quad \vartheta_4 := \frac{1}{\kappa} \ln\left[e^{-\kappa p_0} - \frac{L}{\nu_1}\right] + p_0 + r_0,$$

and $\rho_1, \rho_2 : \mathbb{R} \mapsto \mathbb{R}$ are defined by

$$\rho_1(t) := \frac{-(\hat{c}_2 + c_2)t}{2}, \qquad \rho_2(t) := \frac{-(c_2 + \hat{c}_0)t}{2}.$$

Moreover, when (J5) holds, there are two positive constants D_1 and D_2 such that for any $(x,t) \in \mathbb{R}^2$ and $\eta > 0$,

$$\|\mathcal{W}_6(x+\eta,t) - \mathcal{W}_6(x,t)\| \le D_1\eta, \quad \left\| \frac{\partial \mathcal{W}_6}{\partial t}(x+\eta,t) - \frac{\partial \mathcal{W}_6}{\partial t} \right\| \le D_2\eta.$$

Now, the construction of auxiliary functions and super- and sub-solutions for $\mathcal{W}_6$ is provided, and the other details in the proof of Theorem 3.24, which are similar to other types of entire solutions, are omitted. Throughout this subsection, we choose the values of s_1, s_2, and s_3 as follows:

$$s_1 := \hat{c}_2, \; s_2 := c_2, \; s_3 := \hat{c}_0.$$

Denote

$$\boldsymbol{\phi}_1(\xi) := \hat{\Phi}_2(\xi), \; \boldsymbol{\phi}_2(\xi) := \Phi_2(\xi), \; \boldsymbol{\phi}_3(\xi) := \hat{\Phi}(\xi)$$

and assume that

$$(\phi_{11}(0), \phi_{12}(0)) < \left(\frac{u_1^* + u_2^*}{2}, \frac{v_1^* + v_2^*}{2}\right), \; \phi_{21}(0) = \frac{u_1^* + u_2^*}{2}, \; (\phi_{31}(0), \phi_{32}(0))$$

$$> \left(\frac{u_1^* + u_2^*}{2}, \frac{v_1^* + v_2^*}{2}\right).$$

The two auxiliary functions are defined as follows:

$$P(x,y,z) = y + \frac{(u_2^* - x)(y - u_1^*)(u_2^* - z)(-y) + (x - u_1^*)(u_2^* - y)z(u_2^* - y)}{(u_2^* - x)(y - u_1^*)u_2^* + (u_2^* - u_1^*)(u_2^* - y)z}, \quad (x,y,z) \in D_1$$

and

$$Q(x,y,z) = y + \frac{(v_2^* - x)(y - v_1^*)(v_2^* - z)(-y) + (x - v_1^*)(v_2^* - y)z(v_2^* - y)}{(v_2^* - x)(y - v_1^*)v_2^* + (v_2^* - v_1^*)(v_2^* - y)z}, \quad (x,y,z) \in D_2,$$

where

$$D_1 := \left\{(x,y,z) \in [u_1^*, u_2^*] \times [u_1^*, u_2^*] \times [0, u_2^*] \middle| (u_2^* - x)(y - u_1^*)u_2^* + (u_2^* - u_1^*)(u_2^* - y)z > 0\right\}$$

and

$$D_2 := \left\{(x,y,z) \in [v_1^*, v_2^*] \times [v_1^*, v_2^*] \times [0, v_2^*] \middle| (v_2^* - x)(y - v_1^*)v_2^* + (v_2^* - v_1^*)(v_2^* - y)z > 0\right\}.$$

The super- and sub-solutions are given by the following lemma.

Lemma 3.25 *Assume (J1)–(J4) hold and $\hat{c}_0 > c_2 > \hat{c}_2$. Let $(p_1(t), p_2(t), r_1(t), r_2(t))$ be the solution of (3.2), and $\bar{c} := (\hat{c}_2 + c_2)/2$. Then, the functions defined by*

$$\begin{cases} \overline{u}(x,t) = P\left(\phi_{11}(x + \bar{c}t - r_1(t)), \phi_{21}(x + \bar{c}t + r_1(t)), \phi_{31}(x + \bar{c}t + r_2(t))\right), \\ \overline{v}(x,t) = Q\left(\phi_{12}(x + \bar{c}t - r_1(t)), \phi_{22}(x + \bar{c}t + r_1(t)), \phi_{32}(x + \bar{c}t + r_2(t))\right) \end{cases}$$

and

$$\begin{cases} \underline{u}(x,t) = P\left(\phi_{11}(x + \bar{c}t - p_1(t)), \phi_{21}(x + \bar{c}t + p_1(t)), \phi_{31}(x + \bar{c}t + p_2(t))\right), \\ \underline{v}(x,t) = Q\left(\phi_{12}(x + \bar{c}t - p_1(t)), \phi_{22}(x + \bar{c}t + p_1(t)), \phi_{32}(x + \bar{c}t + p_2(t))\right) \end{cases}$$

are a pair of super- and sub-solutions of (1.1) for $(x,t) \in \mathbb{R} \times (-\infty, t_5]$ with some $t_5 < 0$. Moreover, (3.21) and (3.22) hold for $(\overline{u}(x,t), \overline{v}(x,t))$ and $(\underline{u}(x,t), \underline{v}(x,t))$.

3.8 Nonexistence of *N*-Wave Entire Solutions for $N \geq 5$

In this subsection, we show that when $c_0 \neq \hat{c}_0$, there is no entire solution originating from more than four traveling wave solutions. When we say $c_0 = \hat{c}_0$, it means that the nondecreasing bistable traveling wave solution $\Phi(x + c_0 t)$ and the nonincreasing bistable traveling wave solution $\hat{\Phi}(x + \hat{c}_0 t)$ have the same propagating direction and the same speed. For the special case where the nondecreasing bistable traveling wave solution and the nonincreasing bistable traveling wave solution are symmetric, we write not $c_0 = \hat{c}_0$, but $c_0 = -\hat{c}_0$. Obviously, the assumption $c_0 \neq \hat{c}_0$ covers most cases of bistable spreading speeds. However, when $c_0 = \hat{c}_0$, we cannot prove whether or not there exists N-wave entire solution with $N \geq 5$.

We first introduce some definitions. Let $(c_j, \boldsymbol{\phi}_j)$ with $j = 1, 2, \ldots, N$ be N traveling wave solutions of (1.1). We say that $S_N = \{(c_1, \boldsymbol{\phi}_1), (c_2, \boldsymbol{\phi}_2), \ldots, (c_N, \boldsymbol{\phi}_N)\}$ is a generative sequence, if there is an entire solution (u, v) of (1.1) originating from S_N, in the sense that, (u, v) and S_N satisfy that

$$c_1 \leq c_2 \leq \ldots \leq c_N, \tag{3.27}$$

and there exist constants ϑ_j with $j = 1, 2, \ldots, N$ such that (1.4) holds. For example, the sequences corresponding to $\mathcal{W}_1$–$\mathcal{W}_6$ in Sects. 3.2–3.7 are generative, which are given by

$$\{(c_2, \Phi_2), (\hat{c}_0, \hat{\Phi}), (c_0, \Phi)\}, \quad \{(\hat{c}_1, \hat{\Phi}_1), (c_0, \Phi), (\hat{c}_0, \hat{\Phi})\}, \quad \{(c_1, \Phi_1), (\hat{c}_1, \hat{\Phi}_1), (c_0, \Phi)\},$$
$$\{(\hat{c}_2, \hat{\Phi}_2), (\hat{c}_1, \hat{\Phi}_1), (c_0, \Phi)\}, \quad \{(c_1, \Phi_1), (c_2, \Phi_2), (\hat{c}_0, \hat{\Phi})\}, \quad \{(\hat{c}_2, \hat{\Phi}_2), (c_2, \Phi_2), (\hat{c}_0, \hat{\Phi})\}.$$

For any $j = 1, 2, \ldots, N-1$, when $x = \rho_{j+1}(t)$, it follows from (1.4) that

$$\begin{aligned}\lim_{t \to -\infty} (u, v)(\rho_{j+1}(t), t) &= \lim_{t \to -\infty} \boldsymbol{\phi}_j(\rho_{j+1}(t) + c_j t + \vartheta_j) \\ &= \lim_{t \to -\infty} \boldsymbol{\phi}_{j+1}(\rho_{j+1}(t) + c_{j+1} t + \vartheta_{j+1}),\end{aligned}$$

namely,

$$\lim_{t \to -\infty} \boldsymbol{\phi}_j \left(\frac{(c_j - c_{j+1})t}{2} + \vartheta_j \right) = \lim_{t \to -\infty} \boldsymbol{\phi}_{j+1} \left(\frac{(c_{j+1} - c_j)t}{2} + \vartheta_{j+1} \right).$$

By (3.27), we have that

$$\boldsymbol{\phi}_j(+\infty) = \boldsymbol{\phi}_{j+1}(-\infty) \text{ for any } j = 1, 2, \ldots, N-1. \tag{3.28}$$

Therefore, (3.27) and (3.28) are necessary for that S_N is a generative sequence. By (3.28), when S_N is a generative sequence, we can denote a sequence

$$P_N = \{p_0, p_1, \ldots, p_N\},$$

where

$$p_0 \triangleq \boldsymbol{\phi}_1(-\infty),\ p_j \triangleq \boldsymbol{\phi}_j(+\infty) = \boldsymbol{\phi}_{j+1}(-\infty),\ j \in \{1, \ldots, N-1\},\ \text{and}\ p_N \triangleq \boldsymbol{\phi}_N(+\infty).$$

We say that $S_N = \{(c_1, \boldsymbol{\phi}_1), (c_2, \boldsymbol{\phi}_2), \ldots, (c_N, \boldsymbol{\phi}_N)\}$ is a non-generative sequence, if there is no entire solution originating from S_N. Obviously, S_N is non-generative if $\boldsymbol{\phi}_j(+\infty) \neq \boldsymbol{\phi}_{j+1}(-\infty)$ for some $j \in \{1, 2, \ldots, N-1\}$ or there exist $i, j \in \{1, 2, \ldots, N\}$ such that $c_i > c_j$ and $i < j$.

Lemma 3.26 *In the following two cases, the sequence $S_N = \{(c_1, \boldsymbol{\phi}_1), (c_2, \boldsymbol{\phi}_2), \ldots, (c_N, \boldsymbol{\phi}_N)\}$ is non-generative.*

(i) *When $c_0 \neq \hat{c}_0$, there exist $i, j \in \{1, 2, \ldots, N\}$ with $i < j$ such that*

$$(c_i, \boldsymbol{\phi}_i) = (c_j, \boldsymbol{\phi}_j) = (c_0, \Phi) \quad or \quad (c_i, \boldsymbol{\phi}_i) = (c_j, \boldsymbol{\phi}_j) = (\hat{c}_0, \hat{\Phi}).$$

(ii) *There exist $i, j \in \{1, 2, \ldots, N\}$ with $i < j$ such that*

$$\boldsymbol{\phi}_i(-\infty) = \boldsymbol{\phi}_j(+\infty) = E_1.$$

Proof (i) We only consider the case where there exist $i, j \in \{1, 2, \ldots, N\}$ with $i < j$ such that $(c_i, \boldsymbol{\phi}_i) = (c_j, \boldsymbol{\phi}_j) = (c_0, \Phi)$, and the proof for the case $(c_i, \boldsymbol{\phi}_i) = (c_j, \boldsymbol{\phi}_j) = (\hat{c}_0, \hat{\Phi})$ is similar. Suppose, by contradiction, that S_N is generative. Obviously, it holds that $i < j-1$; otherwise, $\boldsymbol{\phi}_i(+\infty) = E_2 \neq \boldsymbol{\phi}_j(-\infty) = E_0$ with $i = j-1$. Note that $p_i = \boldsymbol{\phi}_i(+\infty) = E_2$ and $p_{j-1} = \boldsymbol{\phi}_j(-\infty) = E_0$. Consider the sequence $\{p_i, \ldots, p_{j-1}\}$, and choose i_0 with $i \leq i_0 < j$ satisfying that p_{i_0} is the last E_2 in $\{p_i, \ldots, p_{j-1}\}$. Next, consider the sequence $\{p_{i_0}, \ldots, p_{j-1}\}$ and choose j_0 with $i_0 < j_0 \leq j$ satisfying that p_{j_0-1} is the first E_0 in $\{p_{i_0}, \ldots, p_{j-1}\}$. Let $S = \{p_{i_0}, \ldots, p_{j_0-1}\}$. Then, we have either

$$S = \{p_{i_0}, p_{j_0-1}\} = \{E_2, E_0\} \text{ with } i_0 = j_0 - 2. \tag{3.29}$$

or

$$S = \{p_{i_0}, p_{i_0+1} = p_{j_0-2}, p_{j_0-1}\} = \{E_2, E_1, E_0\} \text{ with } i_0 = j_0 - 3. \tag{3.30}$$

When (3.29) holds, we have that $\boldsymbol{\phi}_{i_0+1} = \hat{\Phi}$ and $c_{i_0+1} = \hat{c}_0$. Then, (3.27) and $c_i = c_j = c_0$ imply that $c_{i_0+1} = c_0 = \hat{c}_0$. It is a contradiction with $c_0 \neq \hat{c}_0$. When (3.30) holds, we have that $\boldsymbol{\phi}_{i_0+1} = \hat{\Phi}_2$ and $\boldsymbol{\phi}_{i_0+2} = \hat{\Phi}_1$ with $c_{i_0+1} = \hat{c}_2$ and $c_{i_0+2} = \hat{c}_1$. From (3.27) and $c_i = c_j = c_0$, it follows that $\hat{c}_2 = \hat{c}_1 = c_0$, which is a contradiction with $\hat{c}_2 \leq c_R < c_L \leq \hat{c}_1$ in (2.3).

(ii) By Theorems 2.1 and 2.2 , $\boldsymbol{\phi}_i$ must be $\hat{\Phi}_1$ or Φ_2 (which implies that $c_i \geq c_L$), and $\boldsymbol{\phi}_j$ must be Φ_1 or $\hat{\Phi}_2$ (which implies that $c_j \leq c_R$). From (2.3), it follows that $c_i \geq c_L > c_R \geq c_j$. Hence, S_N is non-generative. □

Theorem 3.27 *When $c_0 \neq \hat{c}_0$, any sequence $S_N = \{(c_1, \boldsymbol{\phi}_1), (c_2, \boldsymbol{\phi}_2), \ldots, (c_N, \boldsymbol{\phi}_N)\}$ with $N \geq 5$ is non-generative.*

Proof Let S_N be a generative sequence. Obviously, there are at most two bistable traveling wave solutions in S_N (otherwise, Lemma 3.26 (i) happens). Now, we prove that there are at most two monostable traveling wave solutions in S_N. We claim that E_1 appears at most one time in P_N. Indeed suppose, by contradiction, that E_1 appears at least two times in P_N. Then, any two E_1 in P_N must be nonadjacent since there is no traveling wave solution $(c, \boldsymbol{\phi}_c)$ satisfying that $\boldsymbol{\phi}_c(\pm\infty) = E_1$. Let $i, j \in \{0, 1, \ldots, N\}$ satisfy $p_i = p_j = E_1$ and $i + 1 < j$. Then, we have that

$$\boldsymbol{\phi}_{i+1}(-\infty) = p_i = E_1, \quad \boldsymbol{\phi}_j(+\infty) = p_j = E_1.$$

From Lemma 3.26 (ii), we get that S_N is non-generative, which is a contradiction. Therefore, E_1 appears at most one time in P_N, which implies that there are at most two monostable traveling wave solutions in S_N. Finally, we get that if S_N be a generative sequence, then $N \leq 4$, namely, there exist at most two monostable and two bistable traveling wave solutions in S_N. □

Discussions. These twelve types of three-wave entire solutions in Figs. 1 and 2 contain all possibilities of entire solutions originating from three traveling wave solutions $(c_j, \boldsymbol{\phi}_j)_{j=1,2,3}$. By (3.28) in Sect. 3.8, the profiles must satisfy $\boldsymbol{\phi}_j(+\infty) = \boldsymbol{\phi}_{j+1}(-\infty)$ for $j = 1, 2$. The twelve types in Figs. 1 and 2 show all possibilities of $(c_j, \boldsymbol{\phi}_j)_{j=1,2,3}$ satisfying this condition. Moreover, the wave speeds $\{c_j\}_{j=1,2,3}$ also satisfy $c_1 \leq c_2 \leq c_3$, and otherwise, the two adjacent traveling wave solutions with $c_j > c_{j+1}$ for some $j \in \{1, 2\}$ must intersect at some negative time. When the speeds of bistable and monostable traveling wave solutions are nonzero, the existence results in Sect. 3 show all possibilities of three-wave entire solutions satisfying $c_1 < c_2 < c_3$. But for the case $c_1 = c_2$ or $c_2 = c_3$, it remains open whether or not the entire solution of (1.1) originating from $(c_j, \boldsymbol{\phi}_j)_{j=1,2,3}$ exists.

Note that four-wave entire solutions have not been considered in this paper, and it is interesting to show whether or not there exist four-wave entire solutions of (1.1). The main difficulty in the proof of the existence is how to construct some appropriate auxiliary functions. We will consider this problem in future research.

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Data Availability The data that support the findings of this study are available from the authors on reasonable request.

Appendix A. Proofs of Some Lemmas in Section 3.2

A.1 Proof of Lemma 3.8

We denote

$$\epsilon_1 := \min\left\{\frac{\phi_{11}(0)}{u_1^*}, \frac{\phi_{12}(0)}{v_1^*}\right\}, \quad \epsilon_2 := \max\left\{\frac{\phi_{11}(0)}{u_2^*}, \frac{\phi_{12}(0)}{v_2^*}\right\},$$

$$\epsilon_3 := \min\left\{\frac{\phi_{21}(0)}{u_1^*}, \frac{\phi_{22}(0)}{v_1^*}\right\}, \quad \epsilon_4 := \max\left\{\frac{\phi_{21}(0)}{u_2^*}, \frac{\phi_{22}(0)}{v_2^*}\right\},$$
$$\epsilon_5 := \min\left\{\frac{\phi_{31}(0)}{u_1^*}, \frac{\phi_{32}(0)}{v_1^*}\right\}, \quad \epsilon_6 := \max\left\{\frac{\phi_{31}(0)}{u_2^*}, \frac{\phi_{32}(0)}{v_2^*}\right\}.$$

It follows that $\epsilon_1, \epsilon_3 > 1, \epsilon_2, \epsilon_4, \epsilon_5, \epsilon_6 \in (0, 1)$ and

$$\begin{aligned} E_1 \ll \epsilon_1 E_1 \le \boldsymbol{\phi}_1(0) \le \epsilon_2 E_2 \ll E_2,\\ E_0 \ll \epsilon_3 E_1 \le \boldsymbol{\phi}_2(0) \le \epsilon_4 E_2 \ll E_2,\\ E_0 \ll \epsilon_5 E_1 \le \boldsymbol{\phi}_3(0) \le \epsilon_6 E_2 \ll E_2. \end{aligned}$$

In what follows, we give the upper and lower bounds of $\boldsymbol{\phi}_1(\xi - q_1)$, $\boldsymbol{\phi}_2(\xi + q_2)$, and $\boldsymbol{\phi}_3(\xi + q_2)$ in four cases. The calculations are not complicated and we omit them. First, when $\xi \le q_1(t)$, it holds that

$$E_1 \le \boldsymbol{\phi}_1(\xi - q_1) \le \epsilon_2 E_2, \ \epsilon_3 E_1 \le \boldsymbol{\phi}_2(\xi + q_2) \le E_2, \ E_0 \le \boldsymbol{\phi}_3(\xi + q_2) \le \epsilon_6 E_2.$$

Second, when $q_1(t) \le \xi \le -q_1(t)$, we have

$$\epsilon_1 E_1 \le \boldsymbol{\phi}_1(\xi - q_1) \le E_2, \ \epsilon_3 E_1 \le \boldsymbol{\phi}_2(\xi + q_2) \le E_2, \ E_0 \le \boldsymbol{\phi}_3(\xi + q_2) \le \epsilon_6 E_2.$$

Third, when $-q_1(t) \le \xi \le -q_2(t)$, it follows that

$$\epsilon_1 E_1 \le \boldsymbol{\phi}_1(\xi - q_1) \le E_2, \ E_0 \le \boldsymbol{\phi}_2(\xi + q_2) \le \epsilon_4 E_2, \ E_0 \le \boldsymbol{\phi}_3(\xi + q_2) \le \epsilon_6 E_2.$$

Fourth, when $\xi \ge -q_2(t)$, we can get

$$\epsilon_1 E_1 \le \boldsymbol{\phi}_1(\xi - q_1) \le E_2, \ E_0 \le \boldsymbol{\phi}_2(\xi + q_2) \le \epsilon_4 E_2, \ \epsilon_5 E_1 \le \boldsymbol{\phi}_3(\xi + q_2) \le E_2.$$

Some calculations imply that

$$\begin{aligned} 0 &< u_1^* u_2^* \min\{\epsilon_3(1 - \epsilon_6)(u_2^* - u_1^*), (\epsilon_1 - 1)(1 - \epsilon_4)u_2^*\}\\ &\le u_2^*[\phi_{11}(\xi - q_1) - u_1^*][u_2^* - \phi_{21}(\xi + q_1)]\\ &\quad + (u_2^* - u_1^*)\phi_{21}(\xi + q_1)[u_2^* - \phi_{31}(\xi + q_2)]\\ &\le 2u_2^{*2}(u_2^* - u_1^*) \end{aligned}$$

and

$$\begin{aligned} 0 &< v_1^* v_2^* \min\{\epsilon_3(1 - \epsilon_6)(v_2^* - v_1^*), (\epsilon_1 - 1)(1 - \epsilon_4)v_2^*\}\\ &\le v_2^*\left[\phi_{12}(\xi - q_1) - v_1^*\right][v_2^* - \phi_{22}(\xi + q_1)]\\ &\quad + (v_2^* - v_1^*)\phi_{22}(\xi + q_1)[v_2^* - \phi_{32}(\xi + q_2)]\\ &\le 2v_2^{*2}(v_2^* - v_1^*) \end{aligned}$$

for any $\xi \in \mathbb{R}$ and q_2, q_1 satisfying (3.17). Then, the second and third inequalities in Lemma 3.8 can be easily obtained. For the first inequality, we get from (3.12) and $\boldsymbol{\phi}_2' < 0$ that

$$(\phi_{21}(\xi+q_1), \phi_{22}(\xi+q_1)) \geq (\phi_{21}(0), \phi_{22}(0)) \geq \left(\frac{u_1^*+u_2^*}{2}, \frac{v_1^*+v_2^*}{2}\right) \text{ when } \xi \leq q_1(t).$$

It follows that

$$\begin{aligned}
&u_2^*[\phi_{31}(\xi+q_2)-u_1^*][u_2^*-\phi_{21}(\xi+q_1)]+\phi_{21}(\xi+q_1)[u_2^*-\phi_{31}(\xi+q_2)](u_2^*-u_1^*)\\
&\quad\geq u_2^*[\phi_{31}(\xi+q_2)-u_1^*]\left(u_2^*-\frac{u_1^*+u_2^*}{2}\right)+\frac{u_1^*+u_2^*}{2}[u_2^*-\phi_{31}(\xi+q_2)](u_2^*-u_1^*)\\
&\quad\geq \frac{u_2^*}{2}(u_2^*-u_1^*)^2,
\end{aligned}$$

and then

$$\begin{aligned}
P^x(\xi,t)&=\phi_{21}(u_2^*-\phi_{31})\frac{(u_2^*-u_1^*)[u_2^*(\phi_{31}-u_1^*)(u_2^*-\phi_{21})+\phi_{21}(u_2^*-\phi_{31})(u_2^*-u_1^*)]}{[u_2^*(\phi_{11}-u_1^*)(u_2^*-\phi_{21})+(u_2^*-u_1^*)\phi_{21}(u_2^*-\phi_{31})]^2}\\
&\geq \frac{\epsilon_3(1-\epsilon_6)u_1^*u_2^*(u_2^*-u_1^*)\frac{u_2^*}{2}(u_2^*-u_1^*)^2}{[2u_2^{*2}(u_2^*-u_1^*)]^2}=\frac{\epsilon_3(1-\epsilon_6)u_1^*(u_2^*-u_1^*)}{8u_2^{*2}}.
\end{aligned}$$

It completes the proof.

A.2 Proof of Lemma 3.9

When $\xi \leq -q_1(t)$, we have that

$$\phi_{31}(\xi+q_2) \leq \phi_{31}(0) \leq u_1^* \leq \phi_{11}(\xi-q_1), \quad \text{and } P^y(\xi,t) \geq 0.$$

By $P^z, \phi_{11}', \phi_{31}' \geq 0$ and $\phi_{21}' \leq 0$, we have that when $\xi \leq q_1(t)$,

$$A_1(\xi,t)=P^x(\xi,t)\phi_{11}'-P^y(\xi,t)\phi_{21}'+P^z(\xi,t)\phi_{31}' \geq P^x(\xi,t)\phi_{11}' \geq \frac{1}{2}P^x(\xi,t)|\phi_{11}'|,$$

and when $\xi \leq -q_1(t)$,

$$A_1(\xi,t) \geq \frac{1}{2}\left[P^x(\xi,t)|\phi_{11}'|+P^y(\xi,t)|\phi_{21}'|\right].$$

When $-q_1(t) \leq \xi \leq -q_2(t)$, we have that P^y, $P^z \geq 0$. Then, we compute that

$$\begin{aligned}
&A_1(\xi,t)-\frac{1}{2}\left[P^y(\xi,t)|\phi_{21}'(\xi+q_1)|+P^z(\xi,t)|\phi_{31}'(\xi+q_2)|\right]\\
&\quad=P^x(\xi,t)\phi_{11}'+\frac{1}{2}P^y(\xi,t)|\phi_{21}'|+\frac{1}{2}P^z(\xi,t)\phi_{31}' \geq P^x(\xi,t)\phi_{11}'+\frac{1}{2}P^y(\xi,t)|\phi_{21}'|
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\phi_{21}(u_2^* - \phi_{31})(u_2^* - u_1^*)[u_2^*(\phi_{31} - u_1^*)(u_2^* - \phi_{21}) + \phi_{21}(u_2^* - \phi_{31})(u_2^* - u_1^*)]}{[u_2^*(\phi_{11} - u_1^*)(u_2^* - \phi_{21}) + (u_2^* - u_1^*)\phi_{21}(u_2^* - \phi_{31})]^2}\phi_{11}' \\
&\quad + \frac{\rho\phi_{21}(\phi_{11} - u_1^*)(u_2^* - \phi_{31})u_2^{*2}(u_2^* - u_1^*)(\phi_{11} - \phi_{31})}{2[u_2^*(\phi_{11} - u_1^*)(u_2^* - \phi_{21}) + (u_2^* - u_1^*)\phi_{21}(u_2^* - \phi_{31})]^2} \\
&\geq \frac{\phi_{21}(u_2^* - u_1^*)u_1^*u_2^{*3}}{[2u_2^*(u_2^* - u_1^*)]^2}\left[-C_0e^{-2\eta_2\delta} + \frac{\rho}{2}(\varepsilon_1 - 1)(1 - \varepsilon_6)(\phi_{11}(0) - \phi_{31}(0))\right] \geq 0
\end{aligned}
$$

for δ sufficiently large. When $\xi \geq -q_2(t)$, we have that $P^z \geq 0$. Similarly, we obtain that

$$
\begin{aligned}
&A_1(\xi, t) - \frac{1}{2}P^z(\xi, t)|\phi_{31}'(\xi + q_2)| = P^x(\xi, t)\phi_{11}' + P^y(\xi, t)|\phi_{21}'| \\
&\quad + \frac{1}{2}P^z(\xi, t)\phi_{31}' \geq \frac{u_2^* - \phi_{31}}{[2u_2^*(u_2^* - u_1^*)]^2}\left\{-(u_2^* - u_1^*)u_1^*u_2^{*3}C_0e^{-2\eta_2\delta}\right. \\
&\quad \left.-(u_2^* - u_1^*)^3u_2^{*2}C_0e^{-\eta_2\delta} + \frac{\rho}{2}[(\varepsilon_1 - 1)u_1^*(1 - \varepsilon_4)u_2^{*2}]^2\right\} \geq 0
\end{aligned}
$$

for δ sufficiently large. It completes the proof.

A.3 Proof of Lemma 3.10

For given $(\xi, t) \in \mathbb{R} \times \mathbb{R}^-$ and $s \in \mathbb{R}$, when there is no confusion, we simply write

$$
\begin{aligned}
\hat{\phi}_{11}(\theta) &:= \phi_{11}(\xi - q_1(t) - \theta s),\ \hat{\phi}_{21}(\theta) := \phi_{21}(\xi + q_1(t) - \theta s),\ \text{and } \hat{\phi}_{31}(\theta) \\
&:= \phi_{31}(\xi + q_2(t) - \theta s),
\end{aligned}
$$

where $\theta \in [0, 1]$. Then, $H_1(\xi, t)$ can be represented as follows:

$$
\begin{aligned}
H_1(\xi, t) = &\int_{\mathbb{R}} J_1(s)[P(\hat{\phi}_{11}(1), \hat{\phi}_{21}(1), \hat{\phi}_{31}(1)) - P(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \hat{\phi}_{31}(0))]\mathrm{d}s \\
&- P_x(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \hat{\phi}_{31}(0))\int_{\mathbb{R}} J_1(s)[\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)]\mathrm{d}s \\
&- P_y(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \hat{\phi}_{31}(0))\int_{\mathbb{R}} J_1(s)[\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)]\mathrm{d}s \\
&- P_z(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \hat{\phi}_{31}(0))\int_{\mathbb{R}} J_1(s)[\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)]\mathrm{d}s.
\end{aligned}
$$

We denote

$$
\begin{aligned}
\mathcal{I}_1 &= P_x(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \hat{\phi}_{31}(0))[\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)], \\
\mathcal{I}_2 &= P_y(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \hat{\phi}_{31}(0))[\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)], \\
\mathcal{I}_3 &= P_z(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \hat{\phi}_{31}(0))[\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)].
\end{aligned}
$$

By mean value theorem, there exist θ_1, θ_2, and θ_3 in (0, 1) such that

$$P(\hat{\phi}_{11}(1), \hat{\phi}_{21}(1), \hat{\phi}_{31}(1)) - P(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \hat{\phi}_{31}(0)) = \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6,$$

where

$$\begin{aligned}
\mathcal{I}_4 &= P_x(\theta_1\hat{\phi}_{11}(1) + (1-\theta_1)\hat{\phi}_{11}(0), \hat{\phi}_{21}(1), \hat{\phi}_{31}(1))[\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)],\\
\mathcal{I}_5 &= P_y(\hat{\phi}_{11}(0), \theta_2\hat{\phi}_{21}(1) + (1-\theta_2)\hat{\phi}_{21}(0), \hat{\phi}_{31}(1))[\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)],\\
\mathcal{I}_6 &= P_z(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \theta_3\hat{\phi}_{31}(1) + (1-\theta_3)\hat{\phi}_{31}(0))[\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)].
\end{aligned}$$

Then, we can get that

$$H_1(\xi, t) = \int_{\mathbb{R}} J_1(s)[\mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6 - \mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3]\mathrm{d}s.$$

Also by mean value theorem, there exist θ_4, θ_5, and θ_6 in (0, 1) such that

$$\begin{aligned}
\mathcal{I}_4 - \mathcal{I}_1 &= P_{xx}(\theta_4\hat{\phi}_{11}(1) + (1-\theta_4)\hat{\phi}_{11}(0), \hat{\phi}_{21}(1), \hat{\phi}_{31}(1))\theta_1[\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)]^2\\
&\quad + P_{xy}(\hat{\phi}_{11}(0), \theta_5\hat{\phi}_{21}(1) + (1-\theta_5)\hat{\phi}_{21}(0), \hat{\phi}_{31}(1))[\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)][\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)]\\
&\quad + P_{xz}(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \theta_6\hat{\phi}_{31}(1) + (1-\theta_6)\hat{\phi}_{31}(0))[\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)][\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)]\\
&= \theta_1 G_1(\xi, t, s; \theta_4) + G_2(\xi, t, s; \theta_5) + G_3(\xi, t, s; \theta_6),
\end{aligned}$$

where

$$\begin{aligned}
G_1(\xi, t, s; \theta_4) &= P_{xx}(\theta_4\hat{\phi}_{11}(1) + (1-\theta_4)\hat{\phi}_{11}(0), \hat{\phi}_{21}(1), \hat{\phi}_{31}(1))[\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)]^2,\\
G_2(\xi, t, s; \theta_5) &= P_{xy}(\hat{\phi}_{11}(0), \theta_5\hat{\phi}_{21}(1) + (1-\theta_5)\hat{\phi}_{21}(0), \hat{\phi}_{31}(1))\\
&\quad [\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)][\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)],\\
G_3(\xi, t, s; \theta_6) &= P_{xz}(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \theta_6\hat{\phi}_{31}(1) + (1-\theta_6)\hat{\phi}_{31}(0))[\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)]\\
&\quad [\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)].
\end{aligned}$$

There exist θ_7 and θ_8 in (0, 1) such that

$$\begin{aligned}
\mathcal{I}_5 - \mathcal{I}_2 &= P_{yy}(\hat{\phi}_{11}(0), \theta_7\hat{\phi}_{21}(1) + (1-\theta_7)\hat{\phi}_{21}(0), \hat{\phi}_{31}(1))\theta_2[\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)]^2\\
&\quad - P_{yz}(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \theta_8\hat{\phi}_{31}(1) + (1-\theta_8)\hat{\phi}_{31}(0))\\
&\quad [\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)][\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)]\\
&= \theta_2 G_4(\xi, t, s; \theta_7) + G_5(\xi, t, s; \theta_8),
\end{aligned}$$

where

$$\begin{aligned}
G_4(\xi, t, s; \theta_7) &= P_{yy}(\hat{\phi}_{11}(0), \theta_7\hat{\phi}_{21}(1) + (1-\theta_7)\hat{\phi}_{21}(0), \hat{\phi}_{31}(1))[\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)]^2,\\
G_5(\xi, t, s; \theta_8) &= P_{yz}(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \theta_6\hat{\phi}_{31}(1) + (1-\theta_6)\hat{\phi}_{31}(0))[\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)]\\
&\quad [\hat{\phi}_{21}(1) - \hat{\phi}_{21}(0)].
\end{aligned}$$

There is a constant $\theta_9 \in (0,1)$ such that

$$\begin{aligned}\mathcal{I}_6 - \mathcal{I}_3 &= P_{zz}(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \theta_9\hat{\phi}_{31}(1) + (1-\theta_9)\hat{\phi}_{31}(0))\theta_3[\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)]^2 \\ &= \theta_3 G_6(\xi, t, s; \theta_9),\end{aligned}$$

where

$$G_6(\xi, t, s; \theta_9) := P_{zz}(\hat{\phi}_{11}(0), \hat{\phi}_{21}(0), \theta_9\hat{\phi}_{31}(1) + (1-\theta_9)\hat{\phi}_{31}(0))[\hat{\phi}_{31}(1) - \hat{\phi}_{31}(0)]^2.$$

Based on the above formulas, we can get that

$$\begin{aligned}H_1(\xi, t) = \int_{\mathbb{R}} J_1(s)[&\theta_1 G_1(\xi, t, s; \theta_4) + G_2(\xi, t, s; \theta_5) + G_3(\xi, t, s; \theta_6) \\ &+ \theta_2 G_4(\xi, t, s; \theta_7) + G_5(\xi, t, s; \theta_8) + \theta_3 G_6(\xi, t, s; \theta_9)]\mathrm{d}s.\end{aligned} \tag{3.31}$$

In what follows, we provide the detailed estimations only for $|G_1(\xi, t, s; \theta_4)/A_1(\xi, t)|$, and for $|G_i(\xi, t, s; \theta_{i+3})/A_1(\xi, t)|$ with $i = 2, \ldots, 6$, the methods are similar. We get from the mean value theorem that

$$\begin{aligned}|\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)| &= |\phi_{11}(\xi - q_1 - s) - \phi_{11}(\xi - q_1)| = |\phi_{11}'(\xi - q_1 - \theta_{10}s)s| \\ &= |\hat{\phi}_{11}'(\theta_{10})s|,\end{aligned}$$

where θ_{10} is some constant in $[0, 1]$. In addition, for the C_3 given by Lemma 3.6, there is a sufficiently large constant C_4 such that

$$C_3|\phi_{11}(\xi - q_1 - s) - \phi_{11}(\xi - q_1)| < C_4 \quad \text{for any } (\xi, t) \in \mathbb{R} \times \mathbb{R}^-, s \in \mathbb{R}.$$

Now, we consider the following six cases. Recall that the constant m, which appears below, is defined in assumption (J4).

Case 1: When $\xi \le q_1(t)$, based on Lemmas 3.6, 3.7, 3.8, and 3.9 (i), we have that for $s \in \mathrm{supp}(J_1)$, there exists a positive constant $\tilde{M}_1$ such that

$$\begin{aligned}\left|\frac{G_1(\xi, t, s; \theta_4)}{A_1(\xi, t)}\right| &\le \frac{2C_4|u_2^* - \hat{\phi}_{21}(1)| \cdot |\hat{\phi}_{11}(1) - \hat{\phi}_{11}(0)|}{P^x(\xi, t)|\phi_{11}'(\xi - q_1)|} \\ &\le \frac{2C_4|u_2^* - \hat{\phi}_{21}(1)| \cdot |\hat{\phi}_{11}'(\theta_{10})s|}{\mu_1|\phi_{11}'(\xi - q_1)|} \le \frac{2C_4 m}{\mu_1}|\phi_{21}'(\xi + q_1 - s)| \\ &\quad \cdot \frac{|u_2^* - \phi_{21}(\xi + q_1 - s)|}{|\phi_{21}'(\xi + q_1 - s)|} \cdot \frac{|\phi_{11}'(\xi - q_1 - \theta_{10}s)|}{|\phi_{11}(\xi - q_1 - \theta_{10}s) - u_1^*|} \\ &\quad \cdot \frac{|\phi_{11}(\xi - q_1 - \theta_{10}s) - u_1^*|}{|\phi_{11}(\xi - q_1) - u_1^*|} \cdot \frac{|\phi_{11}(\xi - q_1) - u_1^*|}{|\phi_{11}'(\xi - q_1)|} \\ &\le \frac{2C_4 m}{\mu_1} C_0 e^{\eta_1(\xi + q_1 - s)} \frac{1}{C_1} C_2 C \frac{1}{C_1} \le \tilde{M}_1 e^{\eta_1 q_1}.\end{aligned}$$

Case 2: When $q_1(t) \le \xi \le 0$, based on Lemmas 3.6, 3.7, 3.8, and 3.9 (ii), we have that for $s \in \mathrm{supp}(J_1)$, there exists a positive constant $\tilde{M}_2$ such that

$$\begin{aligned}
\left|\frac{G_1(\xi,t,s;\theta_4)}{A_1(\xi,t)}\right| &\le \frac{2C_4|u_2^* - \hat{\phi}_{21}(1)|\cdot|\hat{\phi}_{11}(1)-\hat{\phi}_{11}(0)|}{P^x(\xi,t)|\phi_{11}'(\xi-q_1)| + P^y(\xi,t)|\phi_{21}'(\xi+q_1)|}\\
&\le \frac{2C_4|u_2^* - \hat{\phi}_{21}(1)|\cdot|\hat{\phi}_{11}'(\theta_{10})s|}{\mu_1|\phi_{11}'(\xi-q_1)|} \le \frac{2C_4 m}{\mu_1}|\phi_{21}'(\xi+q_1-s)|\\
&\quad\cdot\frac{|u_2^*-\phi_{21}(\xi+q_1-s)|}{|\phi_{21}'(\xi+q_1-s)|}\cdot\frac{|\phi_{11}'(\xi-q_1-\theta_{10}s)|}{|u_2^*-\phi_{11}(\xi-q_1-\theta_{10}s)|}\\
&\quad\cdot\frac{|u_2^*-\phi_{11}(\xi-q_1-\theta_{10}s)|}{|u_2^*-\phi_{11}(\xi-q_1)|}\cdot\frac{|u_2^*-\phi_{11}(\xi-q_1)|}{|\phi_{11}'(\xi-q_1)|}\\
&\le \frac{2C_4 m}{\mu_1}C_0 e^{\eta_1(\xi+q_1-s)}\frac{1}{C_1}C_2 C\frac{1}{C_1} \le \tilde{M}_2 e^{\eta_1 q_1}.
\end{aligned}$$

Case 3: When $0 \le \xi \le -q_1(t)$, based on Lemmas 3.6, 3.7, 3.8, and 3.9 (ii), we have that for $s \in \mathrm{supp}(J_1)$, there exists a positive constant $\tilde{M}_3$ such that

$$\begin{aligned}
\left|\frac{G_1(\xi,t,s;\theta_4)}{A_1(\xi,t)}\right| &\le \frac{2C_4|u_2^* - \hat{\phi}_{21}(1)|\cdot|\hat{\phi}_{11}(1)-\hat{\phi}_{11}(0)|}{P^x(\xi,t)|\phi_{11}'(\xi-q_1)| + P^y(\xi,t)|\phi_{21}'(\xi+q_1)|}\\
&\le \frac{2C_4|u_2^* - \hat{\phi}_{21}(1)|\cdot|\hat{\phi}_{11}'(\theta_{10})s|}{\mu_2|\phi_{21}'(\xi+q_1)|} \le \frac{2C_4 m}{\mu_2}|\phi_{11}'(\xi-q_1-\theta_{10}s)|\\
&\quad\cdot\frac{|u_2^*-\phi_{21}(\xi+q_1-s)|}{|u_2^*-\phi_{21}(\xi+q_1)|}\cdot\frac{|u_2^*-\phi_{21}(\xi+q_1)|}{|\phi_{21}'(\xi+q_1)|}\\
&\le \frac{2C_4 m}{\mu_2}C_0 e^{-\eta_2(\xi-q_1-\theta_{10}s)}C\frac{1}{C_1} \le \tilde{M}_3 e^{\eta_2 q_1}.
\end{aligned}$$

Case 4: When $-q_1(t)v \le \xi \le (-q_1(t)-q_2(t))/2$, based on Lemma 3.6, 3.7, 3.8, and 3.9(iii), we have that for $s \in \mathrm{supp}(J_1)$, there exists a positive constant $\tilde{M}_4$ such that

$$\begin{aligned}
\left|\frac{G_1(\xi,t,s;\theta_4)}{A_1(\xi,t)}\right| &\le \frac{2C_4|\hat{\phi}_{21}(1)|\cdot|\hat{\phi}_{11}(1)-\hat{\phi}_{11}(0)|}{P^y(\xi,t)|\phi_{21}'(\xi+q_1)| + P^z(\xi,t)|\phi_{31}'(\xi+q_2)|}\\
&\le \frac{2C_4|\hat{\phi}_{21}(1)|\cdot|\hat{\phi}_{11}'(\theta_{10})s|}{\mu_2|\phi_{21}'(\xi+q_1)|} \le \frac{2C_4 m}{\mu_2}|\phi_{11}'(\xi-q_1-\theta_{10}s)|\\
&\quad\cdot\frac{|\phi_{21}(\xi+q_1-s)|}{|\phi_{21}(\xi+q_1)|}\cdot\frac{|\phi_{21}(\xi+q_1)|}{|\phi_{21}'(\xi+q_1)|}\\
&\le \frac{2C_4 m}{\mu_2}C_0 e^{-\eta_2(\xi-q_1-\theta_{10}s)}C\frac{1}{C_1} \le \tilde{M}_4 e^{\eta_2 q_1}.
\end{aligned}$$

Case 5: When $(-q_1(t)-q_2(t))/2 \le \xi \le -q_2(t)$, based on Lemmas 3.6, 3.7, 3.8, and 3.9 (iii), we have that for $s \in \mathrm{supp}(J_1)$, there exists a positive constant $\tilde{M}_5$ such that

$$\left|\frac{G_1(\xi,t,s;\theta_4)}{A_1(\xi,t)}\right| \le \frac{2C_4|\hat{\phi}_{21}(1)|\cdot|\hat{\phi}_{11}(1)-\hat{\phi}_{11}(0)|}{P^y(\xi,t)|\phi'_{21}(\xi+q_1)|+P^z(\xi,t)|\phi'_{31}(\xi+q_2)|}$$
$$\le \frac{2C_4|\hat{\phi}_{21}(1)|\cdot|\hat{\phi}'_{11}(\theta_{10})s|}{\mu_2|\phi'_{21}(\xi+q_1)|} \le \frac{2C_4m}{\mu_2}|\phi'_{11}(\xi-q_1-\theta_{10}s)|$$
$$\cdot\frac{|\phi_{21}(\xi+q_1-s)|}{|\phi_{21}(\xi+q_1)|}\cdot\frac{|\phi_{21}(\xi+q_1)|}{|\phi'_{21}(\xi+q_1)|}$$
$$\le \frac{2C_4m}{\mu_2}C_0e^{-\eta_2(\xi-q_1-\theta_{10}s)}C\frac{1}{C_1} \le \tilde{M}_5e^{\eta_2q_1}.$$

Case 6: When $\xi \ge -q_2(t)$, based on Lemmas 3.6, 3.7, 3.8, and 3.9 (iv), we have that for $s \in \mathrm{supp}(J_1)$, there exists a positive constant $\tilde{M}_6$ such that

$$\left|\frac{G_1(\xi,t,s;\theta_4)}{A_1(\xi,t)}\right| \le \frac{2C_4|u_2^*-\hat{\phi}_{31}(1)|\cdot|\hat{\phi}_{11}(1)-\hat{\phi}_{11}(0)|}{P^z(\xi,t)|\phi'_{31}(\xi+q_2)|}$$
$$\le \frac{2C_4|u_2^*-\hat{\phi}_{31}(1)|\cdot|\hat{\phi}'_{11}(\theta_{10})s|}{\mu_3|\phi'_{31}(\xi+q_2)|} \le \frac{2C_4m}{\mu_3}|\phi'_{11}(\xi-q_1-\theta_{10}s)|$$
$$\cdot\frac{|u_2^*-\phi_{31}(\xi+q_2-s)|}{|u_2^*-\phi_{31}(\xi+q_2)|}\cdot\frac{|u_2^*-\phi_{31}(\xi+q_2)|}{|\phi'_{31}(\xi+q_2)|}$$
$$\le \frac{2C_4m}{\mu_3}C_0e^{-\eta_2(\xi-q_1-\theta_{10}s)}C\frac{1}{C_1} \le \tilde{M}_6e^{\eta_2q_1}.$$

With the above estimates, we can get (3.19) from (3.31) and $\int_{\mathbb{R}} J_1(x)dx = 1$, immediately. It completes the proof.

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