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Centers and Limit Cycles of Vector Fields Defined on Invariant Spheres

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Abstract
The aim of this paper is the study of the center-focus and cyclicity problems inside the class $\mathfrak{X}$ of 3-dimensional vector fields that admit a first integral that leaves invariant any sphere centered at the origin. We classify the centers of linear, quadratic homogeneous and a family of quadratic vector fields $\mathcal{F} \subset \mathfrak{X}$, restricted to one of these spheres. Moreover, we show the existence of at least 4 limit cycles in family $\mathcal{F}$.

Keywords Vector fields on invariant spheres · Integrability · Center-focus problem · Local cyclicity

Mathematics Subject Classification Primary 34C07; Secondary 34C23 · 37C27

1 Introduction

Differential equations and dynamical systems appear naturally in the description of many phenomena for which local processes are known. The central problem is then to obtain global information on these phenomena. Once the local equations are formu-

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lated in a particular context, the next usual step is to solve them. But, as in general, the evolution of these process is governed by nonlinear differential equations, it is not always simple to solve them. The basic idea behind the first works in the eighteenth and nineteenth centuries was to seek solutions that are combinations of known functions. That is why it is imperative to search for new more geometric methods for a better understanding of the behavior of the solutions of a system of differential equations. Integrability is one of them.

The integrability is an intrinsic property of a given system that imposes strong constraints on the way solutions evolve in phase space. The notion of integrability was introduced to describe the property of equations for which all local and global information can be obtained either explicitly from the solutions or implicitly from invariants. The first class of invariants is the constants of motion, conserved quantities, or first integrals. Of course, there are also other invariants such as integral invariants, integrating factors, Jacobi multipliers, or symmetries which give rise to different techniques for integrating differential equations, see, for instance, Ablowitz et al. (1980), Berrone and Giacomini (2003), Goriely (2001), Olver (1986) and references therein. We have been motivated to consider the existence of first integrals.

The importance of the existence of a nonconstant first integral lies in the fact that the trajectories of the vector field remain in the level sets of the function that defines the first integral, and hence, this is a strong constraint on the dynamical behavior. In the theory of ordinary differential equations, the existence of first integrals is important not only because they allow decreasing the dimension where the differential system is defined but also because they simplify the characterization of the phase portrait. It is important to mention that if we are working in a space of dimension n and the system of equations has $n-1$ independent first integrals, then we say that the system is completely integrable. The complete integrability means that we can obtain the trajectories just intersecting the level sets of the first integrals.

In this work, we consider the class $\mathfrak{X}$ of vector fields on $\mathbb{R}^3$ that admit the first integral $H(x, y, z) = x^2 + y^2 + z^2$, and we denote by $\mathfrak{X}_n$ when we restrict to the polynomial of degree n class. It means that $\langle \mathrm{x}, X(\mathrm{x})\rangle = 0$, for all $\mathrm{x} \in \mathbb{R}^3$, and that any sphere centered at the origin is invariant by the flow of $X \in \mathfrak{X}$. We are interested on the center-focus and cyclicity problems inside this class of systems. These problems are strongly related to the study and definition of *limit cycle* (i.e., isolated periodic orbit) for planar polynomials vector fields. This concept is due to H. Poincaré, in 1880s. At the end of the XIX century, D Hilbert presented a list of 23 problems at the International Congress of Mathematicians, in Paris. The question on the second part of the sixteenth problem about an estimation of the maximal number and relative positions of the limit cycles of a planar polynomial vector field remains unsolved. For more details on the history of this important and open problem, we refer to the interesting survey written by Ilyashenko (2002).

Throughout the work, our strategy is to consider the restriction of the system of differential equations to a 2-dimensional sphere and to use a stereographic projection to consider a planar vector field. From there, we can use all the tools that are normally used to study the dynamics of planar differential systems.

Let $X \in \mathfrak{X}_1$ be a linear vector field, we will see that it is always homogeneous and it writes in the form

$$\begin{aligned} \dot{x} &= -a_1 y - a_2 z, \\ \dot{y} &= a_1 x - a_3 z, \\ \dot{z} &= a_2 x + a_3 y. \end{aligned} \tag{1}$$

The following result provides a qualitative classification of the equilibrium points of the above differential system.

Theorem 1 *Let* $p \in \mathbb{S}^2_\rho = \{(x, y, z) : x^2 + y^2 + z^2 = \rho^2\}$ *be an equilibrium point of system* (1) *which is isolated on* $\mathbb{S}^2_\rho$*. Then,* p *is of center type. Moreover, the system is completely integrable.*

As we have commented above, if $X \in \mathfrak{X}_1$ is linear, then X is homogeneous and hence $\mathfrak{X}_1^H = \mathfrak{X}_1$, where the superscript denotes the homogeneous property and the subscript the degree of the vector field. Inspired by this fact, we study and classify the center equilibrium points in the class of homogeneous quadratic vector fields, that is in $\mathfrak{X}_2^H$.

In Sect. 3, we prove that, without loss of generality, a vector field $X \in \mathfrak{X}_2^H$ writes in the following canonical form

$$\begin{aligned} \dot{x} &= -a_4 xy - a_5 xz - (a_6 + a_7) yz - a_8 z^2, \\ \dot{y} &= a_4 x^2 + a_6 xz - a_9 z^2, \\ \dot{z} &= a_5 x^2 + a_7 xy + a_8 xz + a_9 yz. \end{aligned} \tag{2}$$

We notice that the equilibrium point is located at $(0, 1, 0)$. On the following result, we classify the equilibrium points of center type.

Theorem 2 *The equilibrium point* $(0, 1, 0)$ *of system* (2) *is a nondegenerate center if, and only if,* $a_7 \neq 0$, $a_4 = a_9$, *and* $a_4a_5a_8a_9 + a_5a_6a_7a_8 + a_5^2a_7a_9 + a_5a_8a_9^2 - a_7a_8^2a_9 = 0$.

Also, in Sect. 3, we will see that the behavior of linear and quadratic homogeneous vector fields is the same on all spheres. But this special property cannot be extended for all quadratic vector fields $\mathfrak{X}_2$. Because of the difficulty of doing a general study, we restrict our analysis to the unit sphere $\mathbb{S}^2_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. In this case, we will show that, generically, any $X \in \mathfrak{X}_2$ writes in its canonical form as

$$\begin{aligned} \dot{x} &= -a_1 y - a_2 z - a_4 xy - a_5 xz - a_{10} y^2 - (a_6 + a_7) yz - a_8 z^2, \\ \dot{y} &= a_1 x - a_3 z + a_4 x^2 + a_{10} xy + a_6 xz - a_{11} yz - a_9 z^2, \\ \dot{z} &= a_2 x + a_3 y + a_5 x^2 + a_7 xy + a_8 xz + a_{11} y^2 + a_9 yz. \end{aligned} \tag{3}$$

In the following result, we provide some center families for the above system when $a_1 + a_{10} = 0$, $a_2 + a_7 = 1$, $a_3 + a_{11} = 0$, and $a_9 = 0$. In Sect. 4, we will justify why we have restricted our analysis to this special family. Although the study of local cyclicity presented in Theorem 4 has been done for this family, when the centers of Theorem 3 are perturbed without these conditions, no more limit cycles of small amplitude are obtained. This is explained in Sect. 4.2. Finally, the monodromic property around the equilibrium point is guaranteed by $w^2 = a_6 + 1 > 0$ and it is not restrictive to study only $w > 0$.

Theorem 3 *The differential system*

$$\begin{aligned}
\dot{x} &= -a_1 y - (1 - a_7) z - a_4 x y - a_5 x z + a_1 y^2 + (1 - a_7 - w^2) y z - a_8 z^2,\\
\dot{y} &= a_1 x + a_{11} z + a_4 x^2 - a_1 x y + (w^2 - 1) x z - a_{11} y z,\\
\dot{z} &= (1 - a_7) x - a_{11} y + a_5 x^2 + a_7 x y + a_8 x z + a_{11} y^2,
\end{aligned} \tag{4}$$

has a center at the equilibrium point $(0, 1, 0)$ *if* $a_4 = 0$*, and one of the following conditions is satisfied:*

(a) $w = 1,\ a_1 a_5 + a_8 a_{11} = 0$;
(b) $a_1 = 0,\ a_8 = 0$;
(c) $a_5 = 0,\ a_{11} = 0$;
(d) $a_1 = a_8,\ a_5 = -a_{11}$;
(e) $w \neq 1,\ a_1 = \dfrac{w^2 - 1}{w^2 + 1} a_8,\ a_5 = \dfrac{w^2 + 1}{w^2 - 1} a_{11},\ a_7 = \dfrac{1}{w^2 + 1} - \dfrac{1}{(w^2 + 1)} a_8^2 - \dfrac{w^2 + 1}{(w^2 - 1)^2} a_{11}^2$.

We think that the above result provides (generically) a complete center classification for the considered family. In Sect. 4, we have checked it for $w \in \{1/2, 2, 3\}$, but the expressions to be manipulated are too big to get the proof for every w.

Finally, we have also analyzed the local cyclicity of the families in Theorem 3, studying bifurcations of small amplitude limit cycles from a weak focus on $\mathbb{S}_1^2$. Next result provides the highest number of limit cycles surrounding a monodromic equilibrium point in the quadratic family $\mathfrak{X}_2$ that we have found.

Theorem 4 *Consider the system*

$$\begin{aligned}
\dot{x} &= 2\alpha y + \frac{9}{20} z - x z - 2\alpha y^2 - \frac{89}{20} y z - \alpha z^2,\\
\dot{y} &= -2\alpha x + 2 z + 2\alpha x y + 3 x z - 2 y z,\\
\dot{z} &= -\frac{9}{20} x - 2 y + x^2 + \frac{29}{20} x y + \alpha x z + 2 y^2.
\end{aligned} \tag{5}$$

The equilibrium point $p = (0, 1, 0)$ *of* (5) *is of center type if* $\alpha = 0$*. Otherwise, it is a weak focus of order* 4 *when* $\alpha = \pm\sqrt{857/488}$*, or of order* 3 *when* $\alpha \notin A := \{0, \pm\sqrt{857/488}\}$*. Moreover, if* $\alpha = \pm\sqrt{857/488}$ *(*$\alpha \notin A$*), there exist* 4 *(resp.* 3*) small amplitude limit cycles, on* $\mathbb{S}_1^2$*, bifurcating from* p *considering a perturbation of* (5) *inside family* (4).

Remark 5 We notice that near the centers of (5) for $\alpha = 0$ only weak foci of order three exist. Consequently, the cyclicity of the center should be also three.

Theorem 1 shows that there exists some similarity between the linear case on $\mathbb{R}^2$ and the linear case on $\mathbb{S}^2$, since for both we do not have limit cycles, but instead of the many different types of equilibrium points for linear vector fields on $\mathbb{R}^2$, in $\mathfrak{X}_1$ we only have centers. For planar quadratic polynomials vector fields, N.N. Bautin proves (see Bautin 1954) that there exists at most 3 limit cycles bifurcating from a monodromic equilibrium point. But our last main theorem exhibits a weak focus of order four that unfolds 4 limit cycles in $\mathfrak{X}_2$. That is, we have at least one limit cycle more than in the classical planar case. Another final difference is obtained, considering also the case with complex coefficients. In this case, we prove that the local (complex) cyclicity of centers (e) in Theorem 3 is at least 6 perturbing inside family (4). The fact that the cyclicity is higher when complex coefficients are considered, can explain the difficulties that we have found to look for vector fields with high cyclicity values in the real coefficients study, as well as in the center classification.

This paper is structured as follows. Section 2 is devoted to recall the necessary classical results for proving our main theorems. In particular, the projection of a 3-dimensional vector field with invariant spheres and the definitions of Lyapunov constants, first integrals, inverse integrating factors, and integrability. In Sect. 3, we study the linear and quadratic homogeneous case and we also prove Theorems 1 and 2. In Sect. 4, we study the quadratic case and we prove Theorems 3 and 4. We finish studying the family considering that the coefficients are complex numbers.

2 Preliminary Results

In this section, we recall some classical concepts and bifurcation techniques that are necessary for the proofs of the results stated in the paper. Firstly, we introduce the general vector fields $X : \mathbb{R}^3 \to \mathbb{R}^3$ having $H(x, y, z) = x^2 + y^2 + z^2$ as a first integral together with some properties that they satisfy. Secondly, as the main results will be proved projecting each 3-dimensional vector field to a planar one, we recall some usual notions and definitions for planar vector fields. We mainly study the center-focus and local cyclicity problems for polynomial systems. So we need to introduce briefly the Darboux integrability concept and the computation of the Lyapunov constants. This last notion is the usual planar mechanism to distinguish when a monodromic equilibrium point is of center or of focus type.

2.1 Setting the Problem

Consider a vector field $X : \mathbb{R}^3 \to \mathbb{R}^3$ and its associated differential system $\dot{\mathrm{x}} = X(\mathrm{x})$. Assume that $\langle \mathrm{x}, X(\mathrm{x})\rangle = 0$, for all $\mathrm{x} \in \mathbb{R}^3$. That means that X admits $H(x, y, z) = x^2 + y^2 + z^2$ as a first integral. In other words, all the spheres of center at $(0, 0, 0)$ and radius ρ, $\mathbb{S}^2_\rho = \{(x, y, z) : x^2 + y^2 + z^2 = \rho^2\}$, are invariant by the flow of X. We recall that in the previous section, we have denoted by $\mathfrak{X}$ this class of vector fields and by $\mathfrak{X}_n$ when the components are polynomials of degree n.

Next results show that an orthogonal change of coordinates keeps all the spheres invariant and that it is not restrictive to assume that the equilibrium point, that always exists, can be located at $(0, \rho, 0)$.

Lemma 6 *Let M be an orthogonal matrix. If $X \in \mathfrak{X}$, then $M \cdot X(M^t) \in \mathfrak{X}$.*

Proof The differential equation $\dot{\mathrm{x}} = X(\mathrm{x})$, with the orthogonal change of coordinates $\mathrm{y} = M \cdot \mathrm{x}$, moves to $\dot{\mathrm{y}} = M \cdot \dot{\mathrm{x}} = M \cdot X(\mathrm{x}) = M \cdot X(M^t \cdot \mathrm{y}) = Y(y)$ because $M^{-1} = M^t$. The proof follows just checking that $\langle \mathrm{y}, Y(\mathrm{y})\rangle = \left\langle M \cdot \mathrm{x}, M \cdot X(M^t \cdot \mathrm{y})\right\rangle$ $= (M \cdot \mathrm{x})^t \cdot M \cdot X(M^t \cdot \mathrm{y}) = \mathrm{x}^t \cdot X(\mathrm{x}) = \langle \mathrm{x}, X(\mathrm{x})\rangle = 0$ for all $\mathrm{y} \in \mathbb{R}^3$. □

Lemma 7 *The equilibrium point of $X \in \mathfrak{X}$ can be always located at* $(0, \rho, 0)$.

Proof Let $p = (x_0, y_0, z_0) \in \mathbb{S}^2_\rho$ be an equilibrium point of X. Then, $X(p) = 0$ and $\|p\| = \rho > 0$. Consider the unit vector $\mathrm{v} = (x_0, y_0, z_0)/\rho$ and the plane

$$\mathcal{P} = \left\{(x, y, z) : \frac{x x_0}{\rho} + \frac{y y_0}{\rho} + \frac{z z_0}{\rho} = 0\right\},$$

passing through the origin, which is perpendicular to the vector v. Note that the intersection of $\mathcal{P}$ with the sphere $x^2 + y^2 + z^2 = 1$ is a circumference and we can take $\mathrm{u} = (0, -z_0, y_0)/\sqrt{y_0^2 + z_0^2}$ as a unit vector on this circumference. So, we have a new orthogonal basis $\mathcal{B} = \{\mathrm{u}, \mathrm{v}, \mathrm{w}\}$, where w is obtained by the cross product of u and v. The matrix M that changes the canonical bases to $\mathcal{B}$ is orthogonal and this change of coordinates sends the equilibrium point $p = (x_0, y_0, z_0)$ to $(0, \rho, 0)$. We notice that Lemma 6 ensures that the new vector field also is in class $\mathfrak{X}$. □

The stereographic projection with respect to the antipodal point of the equilibrium point located at $(0, \rho, 0)$ allows us to consider planar vector fields instead of 3-dimensional vector fields restricted to spheres.

Let $\pi : \mathbb{S}^2_\rho \backslash \{(0, -\rho, 0)\} \to \mathbb{R}^2$ be the stereographic projection on the plane $\{(x, y, z) \in \mathbb{R}^3 : y = \rho\}$ given by $\pi(x, y, z) = 2\rho(x, z)/(y + \rho)$. Then, the projection $Y : \mathbb{R}^2 \to \mathbb{R}^2$ of the vector field X writes as

$$Y(\mathrm{x}) = d\pi_{\pi^{-1}(\mathrm{x})} \circ X \circ \pi^{-1}(\mathrm{x}), \tag{6}$$

where $X = X|_{\mathbb{S}^2_\rho}$. Note that π preserves closed curves and contact between curves contained on its domain of definition. We say that $p \in \mathbb{S}^2_\rho$ is an equilibrium point of center type of $X|_{\mathbb{S}^2_\rho}$ if $\pi(p) = q$ and q is an equilibrium point of center type of $Y(\mathrm{x})$ defined in (6). Moreover, as $\pi(0, \rho, 0) = (0, 0)$ we can assume that $(0, 0)$ is an equilibrium point of the planar projected system (6).

There are other transformations to project a 3-dimensional vector field to a planar one, for example, the one used in Llibre and Pessoa (2006a, b). Although these works study vector fields defined on the sphere, the objectives are completely different. In particular, they study when the maximal circles are invariant for the considered vector fields. Some of the above properties are used but without proving them.

2.2 Darboux Integrability

Consider the planar differential system

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{7}$$

where P and Q are polynomials in the variables x and y with coefficients on $\mathbb{F}$, where $\mathbb{F}$ is the field of reals or complex numbers. As usual, we also denote by $\mathbb{F}[x, y]$ the ring of polynomials in the variables x and y and coefficients in $\mathbb{F}$. Let m be the maximum between the degree of P and Q. We say that (7) is *integrable* or *completely integrable* on an open subset $U \subset \mathbb{F}^2$ if there exists an analytic function $H : U \to \mathbb{F}$, nonconstant, such that

$$P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0,$$

on U. In this case, H is called a *first integral*. In addition, a vector field $X : \mathbb{R}^3 \to \mathbb{R}^3$ is *completely integrable* if it has two independent first integrals. Note that if $X \in \mathfrak{X}$, then, by definition, it has at least one first integral. We say that an analytic (and not identically zero) function $V : U \to \mathbb{F}$ is an *inverse integrating factor* of (7) on U if

$$\operatorname{div}(P/V, Q/V) = \frac{\partial(P/V)}{\partial x} + \frac{\partial(Q/V)}{\partial y} \equiv 0,$$

and, consequently, a first integral can be obtained by direct integration.

Let $f \in \mathbb{C}[x, y]$ being not identically zero. We say that $f(x, y) = 0$ is an invariant algebraic curve of (7) if there exists a *cofactor* $K \in \mathbb{C}[x, y]$ such that

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{8}$$

If the degree of the polynomial differential system is m, then the cofactor K has degree at most $m - 1$. By (8), we could see that the gradient of f is orthogonal to the vector field (7) and, because of that, the flow of (7) is tangent to the curve $f(x, y) = 0$. Thus, this curve is formed by trajectories of the vector field.

An object that also satisfies (8) is *the exponential factor* that we define on the following. Let $g, h \in \mathbb{C}[x, y]$ such that g, h are relative prime in $\mathbb{C}[x, y]$ or $h \equiv 1$. The function $\exp(g/h)$ is called an exponential factor of (7) if there exists a cofactor $K \in \mathbb{C}[x, y]$ of degree at most $m - 1$ such that

$$X\left(\exp\left(\frac{g}{h}\right)\right) = K\exp\left(\frac{g}{h}\right).$$

Note that although the function $\exp(g/h)$ satisfies (8), as it is always nonzero, an exponential factor does not define invariant curves.

The Darboux integrability theory for complex polynomial systems gives us some conditions in which the existence of invariant algebraic curves and exponential factors ensures that the vector field is integrable. In this sense, suppose that the vector field (7), of degree m, admits p irreducible invariant algebraic curves $f_i = 0$, with cofactors K_i, $i = 1, \ldots, p$ and q exponential factors $\exp(g_j/h_j)$ with cofactors K_{p+j}, $j = 1, \ldots, q$. Then, there exist λ_i, $\mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j K_{p+j} = 0$ if and only if the (multivalued) function

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{9}$$

is a first integral of system (7). Beside of that, (9) is an inverse integrating factor for system (7) if, and only if, there exist λ_i, $\mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j K_{p+j} = \operatorname{div}(P, Q)$. We emphasize that f_i and $\exp(g_j/h_j)$ could be complex but when system (7) is real, the Darbouxian function (9) is also real. See Dumortier et al. (2006) for more details.

2.3 Lyapunov Constants and Local Cyclicity

We will recall the stability algorithm for equilibrium points of nondegenerate center-focus type having its linear part in Jordan's normal form:

$$\begin{aligned} \dot{x} &= \alpha x - \beta y + \sum_{k=2}^{n} P_k(x, y), \\ \dot{y} &= \beta x + \alpha y + \sum_{k=2}^{n} Q_k(x, y), \end{aligned} \tag{10}$$

where P_k and Q_k are homogeneous polynomials of degree k in the variables x and y. The nondegeneracity condition is $\beta \neq 0$. The above system writes, in usual polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$, as

$$\begin{aligned} \dot{r} &= \mathcal{R}(r, \theta), \\ \dot{\theta} &= \beta + \Theta(r, \theta), \end{aligned}$$

where $\mathcal{R}$ and Θ are polynomials in r and trigonometric polynomials in $\cos\theta$ and $\sin\theta$. Removing the time dependence, we consider the 1-dimensional differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathcal{R}(r, \theta)}{\beta + \Theta(r, \theta)} = R(r, \theta), \tag{11}$$

which is well defined in a small neighborhood of the origin.

Let $r(\theta, r_0)$ be the solution of (11) satisfying $r(0, r_0) = r_0$. If $\alpha \neq 0$ by the Grobman–Hartman theorem, we conclude that $(0, 0)$ is a hyperbolic focus and the sign of α provides the stability of the origin. When $\alpha = 0$, the stability depends on the higher-order terms. For r_0 sufficiently small, we expand the solution in Taylor's series and write

$$r(\theta, r_0) = r_0 + \sum_{k=2}^{\infty} r_k(\theta) r_0^k,$$

with $r_k(0) = 0$, for all $k \geq 2$. The *Poincaré first return map* is defined evaluating the above solution at 2π:

$$\Pi(r_0) = r(2\pi, r_0).$$

The corresponding solution of (10) turns around the origin. The stability of the origin depends on the sign of the displacement function

$$\Delta(r) = \Pi(r_0) - r_0,$$

for r_0 is small enough. If there exists k such that $\Delta'(0) = \cdots = \Delta^{(k-1)}(0) = 0$ and $\Delta^{(k)}(0) \neq 0$, then k is always an odd integer number and we write $k = 2K+1$, for $K \geq 0$. The K-th *Lyapunov constant* is defined as $L_K = r_{2K+1}(2\pi) = \Delta^{(2K+1)}(r_0)/(2K+1)!$ when $L_1 = \cdots = L_{K-1} = 0$ and $L_K \neq 0$. Then, we say that the origin of system (10) is a weak focus of order K if there exists $K \geq 1$ such that $L_K \neq 0$; otherwise, we say that the origin is a center.

The above described method to study the center-focus problem comes from Lyapunov. There is an alternative method due to Poincaré, also when $\alpha = 0$. It consists in looking for a function $H(x, y) = x^2 + y^2 + \mathcal{O}_3(x, y)$ which satisfies

$$\dot{H} = \frac{\mathrm{d}H}{\mathrm{d}t} = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = \sum_{k=2}^{\infty} h_{2k}\, r^{2k},$$

being $r^2 = x^2 + y^2$. The first nonvanishing coefficient of $\dot{H}$, which always has an even subscript, h_{2K}, is called a *focal value* and determines the stability of the origin of (10). In this case, H acts as a Lyapunov function. It is well known that both coefficients, h_{2K+2} and L_{2K+1}, differ only by a multiplicative nonzero constant and by the Poincaré–Lyapunov theorem, the origin is a center of (10) if, and only if, $h_{2K} = 0$ for all K and, in others words, if and only if (10) admits an analytical first integral. We notice that the Taylor series of $H(x, y) = x^2 + y^2 + \mathcal{O}_3(x, y)$ converges to an analytic first integral because the differential system also is. For more details, we refer the reader to Andronov et al. (1973), Dumortier et al. (2006) and Roussarie (1998).

From the computational point of view, it is better to consider the last method in complex variables $z = x + \mathrm{i}\, y$. In this case, if $\alpha = 0$, the system (10) writes as

$$\dot{z} = R(z, \bar{z}) = \mathrm{i}\, z + \sum_{k=2}^{n} R_k(z, \bar{z}),$$

being $R_k(z,\bar{z})$ homogeneous polynomials of degree k in z and $\bar{z}$. We only write the first equation because as (10) has real coefficients, the second one is its conjugate. In this case, we have $H(z,\bar{z}) = z\bar{z} + \mathcal{O}_3(z,\bar{z})$ and, consequently,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \dot{H} = \dot{z}\frac{\partial H}{\partial z} + \dot{\bar{z}}\frac{\partial H}{\partial \bar{z}} = \sum_{k=2}^{\infty} g_k (z\bar{z})^{2k}.$$

Thus, the focal values are the coefficients g_k.

It is well known that the Lyapunov constants and the focal values differ in multiplicative constants and that they are polynomials in the coefficients of P_k and Q_k. Moreover, each L_K is always defined modulus the previous vanish. So, independently of the used mechanism, we will denote them by L_k. The last approach provides some good algebraic properties about degree and weighted homogeneity with respect to the perturbation parameters. See them in Cima et al. (1997). For more details on the center-focus problem and related problems, we refer the reader to Andronov et al. (1973) and Romanovski and Shafer (2009).

The classical *Hopf bifurcation* occurs near $\alpha = 0$ and when the first Lyapunov constant (when $\alpha = 0$) is nonvanishing. It is not restrictive if we assume $L_1 > 0$. In this case, when $\alpha = 0$, the origin is unstable and a small amplitude stable limit cycle bifurcates from the origin when the trace parameter α becomes negative but small enough. The degenerate Hopf bifurcation occurs when the limit cycles of small amplitude bifurcate from a weak focus of higher order. It is known that at most K, limit cycles bifurcate from an order K weak focus under analytic perturbations (see Roussarie 1998), but the unfolding is not always complete when the perturbation is restricted to a polynomial family of fixed degree. As we are interested in polynomial perturbations, we need a simple condition for proving the existence of K limit cycles bifurcating from the origin. Instead of looking for weak focus of higher order and its unfolding, we will study degenerate Hopf bifurcations from centers. The key point is an interesting application of the implicit function theorem due to Chicone and Jacobs in Chicone and Jacobs (1989). Also, Han (1999) uses it. We will present the approach of Christopher (2005) that uses the Taylor developments of first order of the Lyapunov constants with respect to the perturbation parameters.

Theorem 8 (Christopher 2005) *Suppose that p is a point on the center variety and that the first k-Lyapunov constants, $L_1, \ldots, L_k$ have independent linear parts (with respect to the Taylor expansion of L_i about p), then p lies on a component of the center variety of codimension at least k and there are bifurcations which produce k limit cycles locally from the center corresponding to the parameter value p. If, furthermore, we know that p lies on a component of the center variety of codimension k, then p is a smooth point of the variety, and the cyclicity of the center for the parameter value p is exactly k. In the latter case, k is also the cyclicity of a generic point on this component of the center variety.*

We notice that in the above result, we are moving also the trace parameter α. In Christopher (2005), we can also find another result for bifurcating limit cycles of small amplitude using higher-order developments of the Lyapunov constants. These higher-order studies are better explained and developed in Giné et al. (2021) and Gouveia and Torregrosa (2021).

3 Centers for Linear and Quadratic Homogeneous Vector Fields

In this section, we classify the type of equilibrium points that a linear vector field $X \in \mathfrak{X}$ can have proving Theorem 1. As we have already commented before, when $X \in \mathfrak{X}$ is linear, then X is also homogeneous, so $X \in \mathfrak{X}_1^H$. We have used this motivation for studying the centers in $\mathfrak{X}_2^H$. They are classified in Theorem 2. Before proving these results, we give some technical lemmas about homogeneous vector fields on $\mathfrak{X}$.

Lemma 9 *The homogeneity property is invariant by an orthogonal change of coordinates.*

Proof The proof follows directly from the proof of Lemma 6, just checking that the change of variables does not break the homogeneity. □

Lemma 10 *Let $X \in \mathfrak{X}^H$. The phase portrait in each sphere is topologically equivalent to the one in the sphere of radius* 1*. Moreover, X has a straight line passing through the origin filled of equilibrium points.*

Proof The proof follows just doing the change of coordinates $\mathrm{y} = \mathrm{x}/\rho$ and a time rescaling, if necessary. □

In the above lemma, the existence of an equilibrium point in one sphere is transformed by continuity using a dilation when the vector field is homogeneous. We notice that this is not the case for a general vector field in $\mathfrak{X}$ nor for the quadratic family $\mathfrak{X}_2$.

Lemma 11 *Let $X \in \mathfrak{X}^H$. Then, the projected system Y defined in* (6) *is homogeneous if, and only if, X is linear.*

Proof First, we will study the projection of $X \in \mathfrak{X}_1^H$ defined in (1). By Lemma 7, we can suppose that any equilibrium point can be located at $(0, \rho, 0)$. This implies that $a_1 = a_3 = 0$. So, system (1) writes as

$$(X_1, X_2, X_3) = (-a_2 z, 0, a_2 x)$$

and the corresponding projected system (6) is

$$(Y_1, Y_2) = (-a_2 v, a_2 u), \tag{12}$$

which is also homogeneous and linear.

The proof finishes just checking that for degree two, for example, the projection breaks the homogeneity property. Although we will see in the proof of Theorem 2 the general case, here we consider a particular system (2) choosing, for example,

$a_4 = a_7 = 1$ and $a_5 = a_6 = a_8 = a_9 = 0$. Then, the corresponding projected system (6) writes as the nonhomogeneous cubic system

$$\dot{u} = -4\,u - 4\,v - u^3 + u^2 v + u v^2 + v^3,$$
$$\dot{v} = 4\,u - u^3 - 2\,u^2 v - u v^2.$$

□

3.1 Linear Case

We recall that if $X \in \mathfrak{X}_1$, then it writes in the canonical form (1). On the following, we prove Theorem 1.

Proof of Theorem 1 By Lemma 10, all the spheres are equivalent, so we can restrict to the sphere of radius $\rho = 1$. The proof follows directly from Lemma 11, because the projected system (12) has a center at the origin when it is an isolated equilibrium point, that is, when $a_2 \neq 0$.

Note that $H_2(x, y, z) = a_3 x - a_2 y + a_1 z - k$ is a first integral for the linear system (1). As, by definition, $H_1(x, y, z) = x^2 + y^2 + z^2$ is also a first integral, system (1) is completely integrable. □

3.2 Quadratic Homogeneous Case

The next technical result ensures that we can assume that the equilibrium point is located at $(0, 1, 0)$ and so we only work with X defined on $\mathbb{S}^2_1$. Then, we prove our second main result.

Lemma 12 *The canonical form of $X \in \mathfrak{X}_2^H$ is system* (2).

Proof The proof follows straightforward using Lemmas 7, 9, and 10. □

Proof of Theorem 2 Using Lemma 12, we can consider system (2) restricted to the sphere $\mathbb{S}^2_1$ and its projection Y defined in (6). We restrict our attention to the equilibrium point of (2) which is located at the origin after projection. It will be of nondegenerate center-focus type if the Jacobian matrix J associated with the projected vector field Y has zero trace and positive determinant. Straightforward computations ensure that it occurs if, and only if, $a_4 - a_9 = 0$ and $a_6 a_7 + a_7^2 - a_9^2 > 0$. So under these conditions we have a weak focus at the origin and, writing $w^2 = a_6 a_7 + a_7^2 - a_9^2$, the projected system Y can be written as

$$\begin{aligned}\dot{u} &= -4a_4 u - 4\xi v - 4a_5 uv - 4a_8 v^2 - a_4 u^3 \\ &\quad - (\xi - 2a_7)u^2 v + (a_4 + 2a_9)uv^2 + \xi v^3, \\ \dot{v} &= 4a_7 u + 4a_9 v + 4a_5 u^2 + 4a_8 uv - a_7 u^3 \\ &\quad - (2a_4 + a_9)u^2 v - (2\xi - a_7)uv^2 + a_9 v^3,\end{aligned} \tag{13}$$

where $\xi = (w^2 + a_4 a_9)/a_7$. It is easy to check that the trace and determinant of J are $-4(a_4 - a_9)$ and $16w^2$, respectively. So, when $a_4 \neq a_9$, the origin is a hyperbolic focus for system (13) and $a_4 - a_9 = 0$ provides the first condition in the statement to have a weak focus. In this last case, as we have explained in Sect. 2.3, the stability depends on the computation of the Lyapunov constants corresponding to the origin of system (13). Here, we only need to compute the first one, that is

$$L_1 = \frac{16(a_7^2 + a_9^2 + w^2)C}{3(a_9^2 + w^2)^2},$$

where $C = -a_5^2 a_7 a_9 + a_5 a_7^2 a_8 - a_5 a_8 a_9^2 - a_5 a_8 w^2 + a_7 a_8^2 a_9$. As $w \neq 0$, the condition $C = 0$ is also necessary to have a center at the origin. The second condition in the statement follows substituting the value of w^2 in the above expression of C.

The proof finishes just showing that under the two conditions in the statement, system (2) is time reversible with respect to an straight line passing through the origin. We will show this property proving that there exists φ such that the transformation

$$(\hat{u}, \hat{v}) = (\cos\varphi\, u - \sin\varphi\, v, \sin\varphi\, u + \cos\varphi\, v) \tag{14}$$

changes the vector field (13) to a new one $Y_\varphi(\hat{u}, \hat{v})$ that is time reversible with respect to the new $\hat{v}$-axis, i.e., its phase portrait is invariant under reflection with respect to the $\hat{v}$-axis in the direction of time. In other words, it is invariant with respect to the change $(\hat{u}, \hat{v}, t) \mapsto (-\hat{u}, \hat{v}, -t)$, being t the time variable. Note that if a system is time reversible with respect to a line, then the equilibrium points on it are not attractors or repellers. See Lamb and Roberts (1998) for more details.

Using the rational parameterization $\sin\varphi = 2\tau/(1+\tau^2)$ and $\cos\varphi = (1-\tau^2)/(1+\tau^2)$, the vector field $Y_\varphi(\hat{u}, \hat{v})$ is time reversible with respect to the $\hat{v}$-axis if, and only if,

$$\frac{1}{a_7(\tau^2+1)^2}\left((4 + \hat{u}^2 - 3\hat{v}^2) f_1(\tau)\hat{u} + 4a_7(\tau^2+1) f_2(\tau)\hat{u}\hat{v}\right) \equiv 0,$$
$$\frac{1}{a_7(\tau^2+1)^2}\left((4 - 3\hat{u}^2 + \hat{v}^2) f_1(\tau)\hat{v} + 4a_7(\tau^2+1) f_2(\tau)\hat{u}^2\right) \equiv 0,$$

where

$$f_1(\tau) = -a_7 a_9 \tau^4 + 2(a_7^2 - a_9^2 - w^2)\tau^3 + 6a_7 a_9 \tau^2 - 2(a_7^2 - a_9^2 - w^2)\tau - a_7 a_9,$$
$$f_2(\tau) = a_5\tau^2 + 2a_8\tau - a_5.$$

Therefore, we only need to check that there exists a common root of the polynomials $f_1(\tau)$ and $f_2(\tau)$. We notice that $f_2(\tau)$ always has simple real solutions or it vanishes identically ($a_5 = a_8 = 0$). The resultant between $f_1(\tau)$ and $f_2(\tau)$ with respect to τ is $\mathrm{res}(f_1, f_2, \tau) = 16C^2$. Hence, there exists a solution for $\{f_1(\tau) = f_2(\tau) = 0\}$ if, and only if, our second condition $C = 0$ is satisfied. When $a_9 \neq 0$, as $a_7 \neq 0$, there exists a real solution because $f_1(0) f_1(1) = -(a_7 a_9)^2$, while when $a_9 = 0$, we

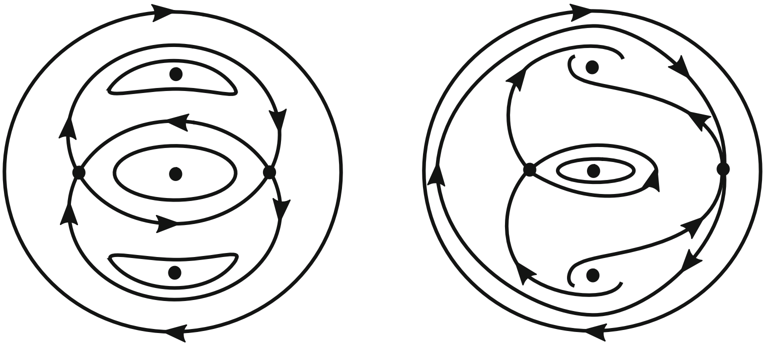

Fig. 1 Phrase portrait of (15) for $a = 0$ (left) and $a > 0$ (right)

have $C = a_5 a_8 (a_7^2 - w^2)$ and $f_1(\tau) = 2(a_7^2 - w^2)\tau(\tau^2 - 1)$ and some cases must be distinguished: $a_5 = 0$, $a_8 = 0$, or $a_7 = \pm w$. When $a_5 = 0$, we take $\tau = 0$, and when $a_8 = 0$, we take $\tau = 1$. If $a_7 = \pm w$, then $f_1(\tau) \equiv 0$ and the proof is finished because $f_2(\tau)$ always has real solutions. □

As we have proved in Lemmas 7 and 10, when $X \in \mathfrak{X}^H$, a straight line of equilibrium points always exists and it passes through $(0, 1, 0)$ and $(0, -1, 0)$. The projected vector field (6) has an equilibrium at the origin and another at infinity. Moreover, when the degree n is odd, the system is invariant by the change of coordinates $(x, y, z) \mapsto (-x, -y, -z)$ and the point $(0, 1, 0)$ moves to $(0, -1, 0)$. On the other hand, when n is even, the invariance needs an inversion of the time. So, if the origin is a center of (13), the infinity is also a center that rotates in the opposite (resp. same) direction when n is even (resp. odd). This property is exhibited in Theorem 1 for vector fields in $\mathfrak{X}_1^H$ because the unique phase portrait is a global center. For vector fields in $\mathfrak{X}_2^H$, the centers in Theorem 2 located at the origin and at infinity turn in opposite directions and we can have different global phase portraits. We can see two of them in Figure 1, where we have drawn the phase portraits of the one-parameter cubic family (15) studied in Proposition 13. We notice that the systems are time reversible with respect to the x-axis, so the origin and infinity are simultaneously centers. The other symmetric points can be centers or antisaddles, in this last case, with opposite stability.

In Caubergh et al. (2012) and Caubergh and Torregrosa (2013), we can find the classification of the global phase portraits of reversible cubic centers where the infinity and the origin rotate in the same direction.

Proposition 13 *The system*

$$\begin{aligned} \dot{u} &= -8v - 4auv - u^2 v + 2v^3, \\ \dot{v} &= 2u + 4au^2 - \frac{7}{2}uv^2 - \frac{1}{2}u^3, \end{aligned} \tag{15}$$

has centers at the origin and at the infinity simultaneously for all a. These centers rotate in opposite directions, and the only possible phase portraits are those shown in Figure 1.

Proof We notice that system (15) is (13) choosing $a_5 = a$, $a_7 = 1/2$, $a_8 = a_9 = 0$, and $w = 1$. So, by Theorem 2 the origin is a center. We have already explained that, by the symmetry of the corresponding vector field in $\mathbb{R}^3$, the infinity is also a center which rotates in an opposite direction with respect to the origin. With the change of coordinates $(u, v) \mapsto (-u, -v)$, if necessary, we can assume that $a \geq 0$. Furthermore, system (15) is time reversible with respect to the u-axis and if $a = 0$, it is also time reversible with respect to the v-axis. In particular, as the line of infinity has no equilibrium points, we do not need to use the Poincaré compactification to study the dynamics near the infinity. Only the finite real equilibrium points are necessary to be analyzed, and the nonexistence of limit cycles property. Straightforward computations show that they are $(0, 0)$, $A_{\pm} = (4a \pm 2\sqrt{4a^2 + 1}, 0)$ and $B_{\pm} = (0, \pm 2)$. Let J be the Jacobian matrix of the vector field associated to (15). Then, $A_{\pm}$ are saddles because of the time reversibility and the fact that the determinant of J at $A_{\pm}$ is $-48(2a\sqrt{4a^2 + 1} \pm 4a^2 \pm 1)^2 < 0$. The matrix J at the equilibrium points $B_{\pm}$ has positive determinant. So, the local stability is determined by the sign of the trace of J, which is $\mp 8a$. So, as $a > 0$, B_+ is an attractor and B_- is a repeller. When $a = 0$, the symmetry with respect to the v-axis proves the existence of a center at each of those points, obtaining the phase portrait depicted in Figure 1 (left). In this case, we have also the first integral

$$H(u, v) = \frac{u^2 + 4v^2}{(u^2 + v^2 + 4)^2}.$$

When $a \neq 0$, the Lie derivative of H with respect to (15) is

$$\dot{u}\frac{\partial H}{\partial u} + \dot{v}\frac{\partial H}{\partial v} = \frac{24au^4 v}{(u^2 + v^2 + 4)^2}$$

and it is always positive (resp. negative) when $v > 0$ (resp. $v < 0$). So, we can use H as a Lyapunov function on $\{(u, v) \in \mathbb{R}^2 : v > 0\}$ (resp. on $\{(u, v) \in \mathbb{R}^2 : v < 0\}$). Hence, if $a \neq 0$, we have no periodic orbits completely contained in the half planes $v > 0$ or $v < 0$. Considering this last property, together with the time symmetry and the local phase portraits of all equilibrium points, we have that the global phase portrait is the one depicted in Fig. 1 (right) when $a > 0$. □

4 Centers and Cyclicity for Quadratic Vector Fields

In this section, we fix our attention to the quadratic vector fields (3), proving our main Theorems 3 and 4. Firstly, we will prove some center characterization and secondly some results on limit cycle bifurcation near centers. Before the proofs, we will show with the next example that the qualitative behavior of a vector field in $\mathfrak{X}_2$ on a sphere of radius ρ_1 centered at the origin can be totally different from the behavior on another sphere of radius $\rho_2 \neq \rho_1$. In particular, the number of equilibrium points can change.

Example 14 All the spheres $x^2 + y^2 + z^2 = \rho^2$ are invariant for the quadratic system

$$(\dot{x}, \dot{y}, \dot{z}) = (-xz - yz - z^2 - z, -z^2, x^2 + xy + xz + yz + x).$$

There are two straight lines full of equilibrium points: $\{x = z = 0\}$ and $\{x + y + 1 = z = 0\}$. So, for every fixed invariant sphere of radius ρ centered at the origin, we have four equilibrium points when $\rho > 1/\sqrt{2}$, three when $\rho = 1/\sqrt{2}$, and only two when $\rho < 1/\sqrt{2}$.

In the following, we will study the existence of periodic orbits and limit cycles on the invariant spheres. Due to the above example, we will concentrate our efforts to study them fixing one sphere, describing the centers and the existence of limit cycles of small amplitude. So, we fix our attention to the unit sphere $\mathbb{S}_1^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ and, as we have explained previously, we will assume that $(0, 1, 0)$ is an equilibrium point of a quadratic vector field $X \in \mathfrak{X}_2$. This fact forces the conditions $a_{10} + a_1 = 0$ and $a_3 + a_{11} = 0$ in system (3) that we can write as

$$\begin{aligned} \dot{x} &= -a_1 y - a_2 z - a_4 xy - a_5 xz + a_1 y^2 - (a_6 + a_7) yz - a_8 z^2, \\ \dot{y} &= a_1 x + a_{11} z + a_4 x^2 - a_1 xy + a_6 xz - a_{11} yz - a_9 z^2, \\ \dot{z} &= a_2 x - a_{11} y + a_5 x^2 + a_7 xy + a_8 xz + a_{11} y^2 + a_9 yz. \end{aligned} \tag{16}$$

We notice that $(0, -1, 0)$ is an equilibrium point of the above differential system if, and only if, $a_1 = a_{11} = 0$. So, in contrary with the homogeneous case, not always exists the line of equilibrium points passing through the two antipodal points $(0, \pm 1, 0)$.

The main objective of the next subsections is to exhibit some differences between the quadratic vector fields in the plane with respect to the quadratic vector fields in the sphere $\mathbb{S}_1$, emphasizing in the number of limit cycles of small amplitude that can bifurcate from a monodromic nondegenerate equilibrium point. We will see that the weak focus order is higher in $\mathbb{S}_1^2$ than in $\mathbb{R}^2$ and also the number of limit cycles of small amplitude.

4.1 Center Characterizations

In order to have a monodromic equilibrium point at $(0, 1, 0)$, we will add some extra conditions in the projected system Y obtained doing the transformation (6) to the quadratic differential equation (16). As before, we denote by J the Jacobian matrix associated with Y at an equilibrium point. The origin is a monodromic nondegenerate equilibrium point of Y if, and only if, the trace and the determinant of J are zero and positive, respectively, that is, when $a_4 = a_9$ and $a_2 a_6 + a_6 a_7 + 2a_2 a_7 + a_2^2 + a_7^2 - a_9^2 > 0$. Due to the big number of free parameters and to simplify a little the computational difficulties, we will restrict our analysis adding two extra conditions: $a_9 = 0$ and $a_2 + a_7 = 1$. In this case, the projected vector field Y has a weak focus at the origin if, and only if, $a_4 = 0$ and $a_6 + 1 > 0$. Moreover, when the trace is zero, the matrix J is in the real Jordan normal form. Taking all into account and writing $w^2 = a_6 + 1$, with $w \neq 0$, we obtain system (4). After a reparametrization of the time, the corresponding projected system for (4) is

$$\begin{aligned}\dot{u} &= -\frac{a_4}{w}u - v - \frac{a_1}{2}u^2 - \frac{a_5}{w}uv - \frac{a_1+2a_8}{2w^2}v^2 - \frac{a_4 w}{4}u^3 + \frac{2a_7 - w^2}{4}u^2 v \\ &\quad + \frac{a_4}{4w}uv^2 + \frac{w^2+2a_7-2}{4w^2}v^3 - \frac{a_1 w^2}{8}u^4 - \frac{a_{11}w}{4}u^3 v - \frac{a_{11}}{4w}uv^3 + \frac{a_1}{8w^2}v^4, \\ \dot{v} &= u + \frac{(2a_5 - a_{11})w}{2}u^2 + a_8 uv - \frac{a_{11}}{2w}v^2 - \frac{(2a_7-1)w^2}{4}u^3 - \frac{a_4 w}{2}u^2 v \\ &\quad - \frac{2w^2+2a_7-3}{4}uv^2 + \frac{w^3 a_{11}}{8}u^4 - \frac{w^2 a_1}{4}u^3 v - \frac{a_1}{4}uv^3 - \frac{a_{11}}{8w}v^4.\end{aligned} \tag{17}$$

The following result shows that when we fix the value of the determinant of the Jacobian matrix, or equivalently w, the center-focus problem can be completely solved. The proof of case (a) in Theorem 3 follows directly from it.

Proposition 15 *Consider system* (4) *with* $w = 1$. *Then,* $(0, 1, 0)$ *is a center on* $\mathbb{S}_1^2$ *if, and only if,* $a_4 = 0$ *and* $a_1 a_5 + a_8 a_{11} = 0$.

Proof Instead of working with system (4), we will work with the equivalent projected planar differential system (17). It is easy to check that if $a_4 \neq 0$, then $(0, 0)$ is a hyperbolic focus. So $a_4 = 0$ is the first center condition in the statement. The second center condition detailed in the statement, $a_1 a_5 + a_8 a_{11} = 0$, follows using the algorithm described in Sect. 2.3 for computing the Lyapunov constants for providing the stability of the origin. In fact, it appears as a common factor in the first three:

$$\begin{aligned}L_1 &= -\frac{2}{3}(a_1 a_5 + a_8 a_{11}), \\ L_2 &= \frac{1}{15}(a_1 a_5 + a_8 a_{11})(3a_1^2 - 6a_1 a_8 + a_5^2 - 18a_5 a_{11} + 51a_{11}^2 - 7a_8^2 - 6a_7), \\ L_3 &= -\frac{1}{630}(a_1 a_5 + a_8 a_{11})(3a_1^4 - 6a_1^3 a_8 + 195a_1^2 a_5^2 - 858a_1^2 a_5 a_{11} - 186a_1^2 a_{11}^2 \\ &\quad + 543a_1^2 a_8^2 + 478a_1 a_5^2 a_8 - 832a_1 a_5 a_8 a_{11} - 7182a_1 a_8 a_{11}^2 + 798a_1 a_8^3 - 18a_5^4 \\ &\quad + 242a_5^3 a_{11} - 509a_5^2 a_{11}^2 + 72a_5^2 a_8^2 - 6242a_5 a_{11}^3 + 2146a_5 a_8 a_{11}^2 + 14267a_{11}^4 \\ &\quad - 8041a_8^2 a_{11}^2 + 282a_8^4 - 276a_1^2 a_7 + 600a_1 a_7 a_8 - 84a_5^2 a_7 + 1752a_5 a_7 a_{11} \\ &\quad - 6276a_7 a_{11}^2 + 516a_7 a_8^2 - 108a_1^2 - 324a_1 a_8 + 24a_5^2 - 324a_5 a_{11} + 540a_{11}^2 \\ &\quad + 144a_7^2 - 168a_8^2 - 72a_7).\end{aligned}$$

The next step is to prove that under these two conditions, the origin is a time-reversible center with respect to a straight line. It follows using the same idea as in the proof of Theorem 2. That is, from the change (14) again with the rational parameterization $\sin\varphi = 2\tau/(1+\tau^2)$ and $\cos\varphi = (1-\tau^2)/(1+\tau^2)$. Straightforward computations show that the transformed vector field (with $a_4 = 0$) is invariant with respect to the change $(\hat{u}, \hat{v}, t) \mapsto (-\hat{u}, \hat{v}, -t)$, being $(\hat{u}, \hat{v})$ the new variables, if and only if

$$\hat{u}\hat{v}((\hat{u}^2+\hat{v}^2)f_1(\tau)-4f_2(\tau))\equiv 0,$$
$$(\hat{u}^2+\hat{v}^2)(\hat{u}^2-\hat{v}^2-4)f_1(\tau)-8f_2(\tau)\hat{u}^2\equiv 0,$$

where $f_1(\tau)=-a_{11}\tau^2+2a_1\tau+a_{11}$ and $f_2(\tau)=a_5\tau^2+2a_8\tau-a_5$. The proof finishes because, when the second condition $a_1a_5+a_{11}a_8=0$ holds, f_1 and f_2 have a common real root, which provides the symmetry line. □

Now, we prove the remaining center families in our third main result.

Proof of Theorem 3 As in the above proof, we will start with the projected system (17). Moreover, we assume $a_4=0$; otherwise, we have a hyperbolic focus at the origin.

The case (a) follows directly from Theorem 15 and cases (b) and (c) because the corresponding systems are time reversible with respect to the u-axis and v-axis, respectively. The system in the fourth case (d) is Darboux integrable, having the next rational first integral

$$H(u,v)=\frac{(a_{11}wu-a_8v-2a_7+1)(w^2u^2+v^2)-(w^2-1)v^2-4(a_7-1)}{(w^2u^2+v^2+4)^2}.$$

The remaining case (e) is also Darboux integrable. But the proof of the existence of an inverse integrating factor is more intricate. We will prove the existence of four (complex) invariant straight lines for the projected system (17), and then, we will provide an inverse integrating factor of the form (9).

We denote by (Y_1,Y_2) the vector field (17) and by $F=au+bv+1$ a generic straight line. We recall that if F is invariant, its respective cofactor K will be a polynomial of degree 3. Equating the coefficients in u and v of the identity

$$Y_1\frac{\partial F}{\partial u}+Y_2\frac{\partial F}{\partial v}=F\,K,$$

we can obtain four possible pairs of values $(a,b)\neq 0$ that depend on the (complex) roots of the polynomial

$$p_A(Z)=Z^4+A_3Z^3+A_2Z^2+A_1Z^4+A_0, \tag{18}$$

where $A_0=\frac{(w^2-1)^2}{16(w^2+1)^2}$, $A_1=\frac{w^2-1}{2(w^2+1)^2}a_8$, $A_2=\frac{1}{(w^2-1)^2}a_{11}^2+\frac{1}{(w^2+1)^2}a_8^2-\frac{w^2-1}{2(w^2+1)}$, and $A_3=\frac{-2}{w^2+1}a_8$. Note that if z_1 and z_2 are roots of (18), then $z_3=\frac{1-w^2}{4z_1(w^2+1)}$ and $z_4=\frac{1-w^2}{4z_2(w^2+1)}$ also are. So, writing the parameters a_8 and a_{11} of (17) in terms of the two roots z_1 and z_2, that is, $a_8=\frac{(z_1+z_2)(4z_1z_2(w^2+1)-w^2+1)}{8z_1z_2}$ and $a_{11}=\frac{\mathrm{i}(z_1-z_2)(w^2-1)(4z_1z_2(w^2+1)+w^2-1)}{8z_1z_2(w^2+1)}$, we have a complete factorization of the polynomial $p_A(Z)=\prod_{i=1}^{4}(Z-z_i)$. We notice that we are assuming that all the coefficients are real, so we have that $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$. Hence, we can write the four invariant straight lines and its respective cofactors as

$$
\begin{aligned}
F_1 &= 1 + \mathrm{i}\, z_1 w u + z_1 v, \\
F_2 &= 1 - \mathrm{i}\, z_2 w u + z_2 v, \\
F_3 &= 1 - \frac{(w^2-1)(v + \mathrm{i}\, w u)}{4(w^2+1) z_1}, \\
F_4 &= 1 - \frac{(w^2-1)(v - \mathrm{i}\, w u)}{4(w^2+1) z_2},
\end{aligned}
$$

and

$$
\begin{aligned}
K_1 = {} & \frac{1}{32 z_2 w (w^2+1)}((-w^3+w)u + \mathrm{i}((w^2-1)v - 4z_1(w^2+1)) \\
& w^2(4z_2^2(w^2+1) - w^2+1)u^2 + (4z_2^2(w^2+1) - w^2+1)v^2 \\
& + 8z_2 w^2 v + 8\,\mathrm{i}\, z_2 w u), \\
K_2 = {} & \frac{1}{32 z_1 w (w^2+1)}((w^3-w)u + \mathrm{i}((w^2-1)v - 4z_2(w^2+1)) \\
& (w^2(-4z_1^2(w^2+1) + w^2-1)u^2 - (4z_1^2(w^2+1) - w^2+1)v^2 \\
& - 8z_1 w^2 v + 8\,\mathrm{i}\, z_1 w u), \\
K_3 = {} & \frac{(-z_1 w u + \mathrm{i}(z_1 v + 1))(w^2-1)}{z_1((-w^3+w)u + \mathrm{i}((w^2-1)v - 4z_1(w^2+1)))} K_1, \\
K_4 = {} & \frac{(z_2 w u + \mathrm{i}(z_2 v + 1))(w^2-1)}{z_2((w^3-w)u + \mathrm{i}((w^2-1)v - 4z_2(w^2+1)))} K_2.
\end{aligned}
$$

The proof finishes showing that $\prod_{i=1}^{4} F_i^{\lambda_i}$ is an inverse integrating factor of the system. This last property holds because taking

$$
\begin{aligned}
\lambda_1 &= \frac{4(w^2+1)z_1^2 + 2(w^2-1)}{4(w^2+1)z_1^2 + w^2 - 1}, \quad & \lambda_2 &= \frac{4(w^2+1)z_2^2 + 2(w^2-1)}{4(w^2+1)z_2^2 + w^2 - 1}, \\
\lambda_3 &= \frac{8(w^2+1)z_1^2 + w^2 - 1}{4(w^2+1)z_1^2 + w^2 - 1}, & \lambda_4 &= \frac{8(w^2+1)z_2^2 + w^2 - 1}{4(w^2+1)z_2^2 + w^2 - 1},
\end{aligned}
$$

the condition

$$
\sum_{i=1}^{4} \lambda_i K_i = \operatorname{div}(Y_1, Y_2)
$$

is satisfied. Moreover, not all λ_i, for $i = 1, \ldots, 4$, are zero. □

Remark 16 We notice that the above proof is also valid when all the parameters are complex. But in this case, we will have no relation between the two main roots z_1 and z_2.

If we fix the value of the special parameter w, we conclude that there are no other centers cases different from the ones detailed in Theorem 3. But the expressions that appear are very big to provide a complete proof for an arbitrary w. This is why in the next result we have chosen some explicit values.

Proposition 17 *Consider the differential system* (3) *with* $w \in \{1/2, 2, 3\}$. *Then,* $(0, 1, 0)$ *is a center if, and only if,* $a_4 = 0$ *and one of the conditions (b), (c), (d), or (e) in Theorem* 3 *is satisfied.*

Proof From the proof of Theorem 3, we know that all the families detailed in the statement are centers. Hence, we only need to check that there are no others. We present only the proof for the case $w = 2$. The key point is the computation of enough center conditions as it was done for Proposition 15, but in this case we need to compute more Lyapunov constants using the method explained in Sect. 2.3 for system (17). We have needed six to finish the proof. In the proof, we will denote by $\mathcal{S}_L = \{L_1 = L_2 = \cdots = L_6 = 0\}$ the system of equations needed to be solved. The other cases $w = 3$ and $w = 1/2$ are completely analogous.

So, we fix $w = 2$ and the first Lyapunov constant writes as

$$L_1 = -a_1 \frac{37a_5 - 15a_{11}}{48} + a_8 \frac{15a_5 - 11a_{11}}{24}.$$

Clearly, it vanishes when $a_1 = a_8 = 0$ which is Family (b), so we have centers in this case. So, if necessary, we can assume that a_1 and a_8 do not vanish simultaneously. The same conclusion holds for $a_5 = a_{11} = 0$ (Family (c)). Consequently, we have two possibilities $a_1 = 15a_5 - 11a_{11} = 0$ or $a_8 = \frac{a_1(37a_5 - 15a_{11})}{2(15a_5 - 11a_{11})}$.

The first case follows easily computing the next Lyapunov constants using that a_5 and a_8 do not vanish; otherwise, we obtain the previous studied families. From $L_2 = 0$, we obtain

$$a_7 = -\frac{507a_5^2 + 1452a_8^2 + 15367}{13915},$$

and then, $L_i = a_5 a_8 \mathcal{L}_i(a_5, a_8)$, for $i = 3, 4, 5$, being $\mathcal{L}_i$ polynomials of degree 4, 6, and 8, respectively. We have no other families in this case because the system of equations $\{\mathcal{L}_3 = \mathcal{L}_4 = \mathcal{L}_5 = 0\}$ reduces to $\{225a_5^2 - 363 = 225a_8^2 + 9408 = 0\}$, which has no real solutions.

For the second case, we can assume that $a_8 = \frac{a_1(37a_5-15a_{11})}{2(15a_5-11a_{11})}$ and $15a_5 - 11a_{11} \neq 0$. Straightforward computations show that $L_1 = 0$ and

$$
\begin{aligned}
L_2 =& \frac{a_1(a_5+a_{11})}{320(15a_5-11a_{11})^3}\big(196(2a_5-a_{11})(8a_5-5a_{11})(42a_5-23a_{11})a_1^2 \\
&+ (672a_5^3 - 1124a_5^2a_{11} + 624a_5a_{11}^2 \\
&- 115a_{11}^3 - 543a_5 + 576a_{11})(15a_5-11a_{11})^2 \\
&+ 15(106a_5 - 67a_{11})(15a_5-11a_{11})^2a_7\big).
\end{aligned} \tag{19}
$$

When $106a_5 - 67a_{11} = 0$, the numerator of the above expression, up to a nonzero rational factor, writes as $a_1a_5^4(1149184a_1^2 + 33856a_5^2 + 59367025)$ and we have no new center families when L_2 vanishes. When $a_5 + a_{11} = 0$,we have $a_1 = a_8$ which is Family (d) in Theorem 3. Therefore, on the following we write a_7, from (19), as a rational function of (a_1, a_5, a_{11}). The next Lyapunov constants write as

$$
L_i = \frac{a_1(a_5+a_{11})(3a_5-5a_{11})\,\mathcal{L}_i(a_1,a_5,a_{11})}{(106a_5-67a_{11})^{i-1}(15a_5-11a_{11})^{2i-1}} \tag{20}
$$

for $i = 3, 4, 5, 6$, being $\mathcal{L}_i$ polynomials with rational coefficients of degrees $5(i-1)$ and 50, 120, 235, 406 monomials, respectively. We do not write them here because of their size. The common factor of the above expressions gets, respectively, the Families (b), (d), and (e) in Theorem 3.

The proof finishes checking that the noncommon factors $\mathcal{L}_i$ do not vanish simultaneously in $\mathbb{R}^3$ in new families, different from the ones detailed in the statement. We will show this fact computing some crossing resultants between them and proving that the unique new possible intersection points have complex coordinates.

We start removing the parameter a_1 computing the crossing resultants with respect to the first factor: $R_k = \text{res}(\mathcal{L}_3, \mathcal{L}_{k+3}, a_1)$, for $k = 1, 2, 3$, are polynomials in (a_5, a_{11}) of degrees 30, 40, and 50, respectively, which decompose in some irreducible factors with some natural powers (multiplicity). As the powers are not essential for solving the equations, we can remove all, and defining $\hat{R}_i$ from R_i, having the same factors but with multiplicity one. They have the common factor

$$
\begin{aligned}
\hat{\mathcal{L}}_b =& (1200a_5^2 - 1110a_5a_{11} + 225a_{11}^2 - 169)(2a_5 - a_{11})(11a_5 - 9a_{11}) \\
&(8a_5 - 5a_{11})(5a_5 + a_{11})(15a_5 - 11a_{11})(106a_5 - 67a_{11}).
\end{aligned}
$$

The analysis of each new factor in $\hat{\mathcal{L}}_b$ follows analogously to the ones described before, $15a_5 - 11a_{11} = 0$ or $106a_5 - 67a_{11} = 0$. When $2a_5 - a_{11} = 0$, $11a_5 - 9a_{11} = 0$, $8a_5 - 5a_{11} = 0$, or $5a_5 + a_{11} = 0$, we always obtain Family (c). The condition $1200a_5^2 - 1110a_5a_{11} + 225a_{11}^2 - 169 = 0$ needs a more accurate analysis. Assuming it, the new real solutions of $\mathcal{S}_L$ should satisfy $a_1^2 = (\pm 19712a_5\sqrt{a_5^2+1} - 20288a_5^2 - 7744)/225$ and $a_{11} = (37a_5 \pm 13\sqrt{a_5^2+1})/15$, but as $\pm 19712a_5\sqrt{a_5^2+1} - 20288a_5^2 - 7744 \leq 0$, for each a_5 we have no new center families because, additionally, we are assuming $a_1 \neq 0$.

Hence, from now on we can assume $\hat{\mathcal{L}}_b \neq 0$ and we can take $\tilde{R}_i = \hat{R}_i/\hat{\mathcal{L}}_b$. Removing the parameter a_5 from the resultants $R_{12} = \text{res}(\tilde{R}_1, \tilde{R}_2, a_5)$ and $R_{13} = \text{res}(\tilde{R}_1, \tilde{R}_3, a_5)$, we obtain two polynomials on a_{11} with degrees 353 and 479, respectively. As above, we define $\hat{R}_{12}$ and $\hat{R}_{13}$ leaving only one factor if there are multiplicity higher than one after the irreducible factorization in the rationals field. Here, the common factor is

$$\hat{\mathcal{L}}_c = a_{11}(10443a_{11}^2 + 6875)(48a_{11}^2 + 1331)(1375a_{11}^2 - 81)(5125a_{11}^2 - 507)(15a_{11}^2 - 1).$$

As only $a_{11} = 0$ gets a real solution of $\mathcal{S}_L$, providing Family (c), we can assume that $\hat{\mathcal{L}}_c \neq 0$.

The proof finishes because the last resultant $R_{123} = \text{res}(\hat{R}_{12}/\hat{\mathcal{L}}_c, \hat{R}_{13}/\hat{\mathcal{L}}_c, a_{11})$ is a nonvanishing rational number. □

In an attempt to conclude the general case, i.e., to prove that without fixing w the only centers families are the ones stated in Theorem 3, we have followed the same scheme than in the previous proof but adding the next Lyapunov constant, L_7, of (6). The first steps follow easily. From $L_1 = 0$ and $L_2 = 0$, we write a_8 and a_7 as rational functions on (a_1, a_5, a_{11}, w) and we can define the corresponding $\mathcal{L}_i(a_1, a_5, a_{11}, w)$ functions, after removing all the common factors as in (20). The first crossing resultants, $R_k = \text{res}(\mathcal{L}_3, \mathcal{L}_{k+3}, a_1)$, for $k = 1, \ldots, 4$, are big polynomials with rational coefficients on (a_5, a_{11}, w) of degrees 119, 163, 207, and 251, and the next $R_{1k} = \text{res}(\tilde{R}_1, \tilde{R}_k, a_5)$, for $k = 2, 3, 4$ are even big polynomials of degrees 2454, 3438, and 4422, respectively, on (a_{11}, w). Unfortunately, we are unable to conclude the proof of the general case because we cannot get the next crossing resultants (removing a_{11}) R_{123}, R_{124} and the last (removing w) R_{1234}, due to the huge memory requirements to do all these computations.

4.2 Bifurcation of Limit Cycles of Small Amplitude

As in many problems of degenerate Hopf bifurcation, the cyclicity of centers is usually less than the cyclicity of some special systems having a weak focus of very high order. This is also the case in our families in $\mathfrak{X}_2$. Theorem 4 provides a weak focus of order four, but we have no such points near the centers in Theorem 3. We have studied the local cyclicity of some of them, and next proposition provides our highest bifurcation result (near centers) studying Family (e) in Theorem 3. The other families have even less local cyclicity. We remark that we have also considered the perturbation of some centers in Theorem 3 without choosing the conditions detailed in Sect. 4.1, but the local cyclicity is not higher than the presented in the following results. So we have decided to do not to write such results here.

Proposition 18 *Consider the coefficients of* (4) *satisfying (e) in Theorem* 3*. Then, there exists a perturbation in* $\mathfrak{X}_2$ *such that at least* 3 *small amplitude limit cycles bifurcate from the equilibrium point* (0, 1, 0) *on* $\mathbb{S}_1^2$.

Proof We take the parameter values $(\mathsf{a}_1, \mathsf{a}_5, \mathsf{a}_7, \mathsf{a}_8, \mathsf{a}_{11}, \mathsf{w})$ satisfying Theorem 3.(e), and we consider $(a_1, a_5, a_7, a_8, a_{11}, w) = (\mathsf{a}_1 + \varepsilon_1, \mathsf{a}_5 + \varepsilon_2, \mathsf{a}_7 + \varepsilon_3, \mathsf{a}_8 + \varepsilon_4, \mathsf{a}_{11} + \varepsilon_5, \mathsf{w}+\varepsilon_6)$ in the projected system (17). We denote by $L_i(\varepsilon)$, with $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_6)$, the corresponding Lyapunov constants, computed with the method explained in Sect. 2.3. Clearly, when $\varepsilon = 0$, the origin is of center type, so $L_i(0) = 0$ for all i. Then, we write the Taylor series of first order with respect to ε as $L_i(\varepsilon) = L_i^{[1]}(\varepsilon) + \mathcal{O}_2(\varepsilon)$. The proof follows because the linear terms of the first three Lyapunov constants have rank 3 with respect to ε, and adding the trace parameter and using Theorem 8 or the implicit function theorem, we can get 3 limit cycles of small amplitude bifurcating from the origin. □

Remark 19 We notice that we have no more limit cycles up to first-order study because the rank does not increase considering more Lyapunov constants. The second-order study does not generate more either.

Before proving our last main result, we observe that the system (5) comes taking $a_1 = -2\alpha$, $a_4 = 0$, $a_5 = 1$, $a_7 = 29/20$, $a_8 = \alpha$, $a_{11} = 2$, and $w = 2$ in (4).

Proof of Theorem 4 As previously, we will work with the projected system (17) corresponding to system (5). With the algorithm described in Sect. 2.3, the Lyapunov constants are $L_1 = L_2 = 0$, $L_3 = 248832\alpha(488\alpha^2 - 857)/5$ and $L_4 = -5971968\alpha(361920\alpha^4 + 513328\alpha^2 - 3529237)/5$. As $L_3 = 0$, if, and only if, $\alpha = 0$ or $\alpha = \pm\sqrt{857/488}$, the proof of the weak focus order and center statements is clear. When $\alpha = 0$, we have a center since we are in Family (b) of Theorem 3. In the other case, when $\alpha = \pm\sqrt{857/488}$, the origin is a weak focus of order 4 because $L_4 = 6717957338234880\alpha/3721 \neq 0$. For other values of α, the weak focus order is only 3.

The cyclicity statement part follows considering the partial perturbation $a_1 = -2\alpha+\varepsilon_1$, $a_7 = 29/20+\varepsilon_2$, and $a_8 = \alpha+\varepsilon_3$. The other parameters remain unchanged. Then, we compute the Taylor series of the Lyapunov constants up to first order, writing $L_i(\varepsilon) = L_i^{[1]}(\varepsilon) + \mathcal{O}_2(\varepsilon)$ with

$$
\begin{aligned}
L_1^{[1]}(\varepsilon) &= -7\varepsilon_1 - 14\varepsilon_3,\\
L_2^{[1]}(\varepsilon) &= \frac{48}{5}(2920\alpha^2 + 3809)\varepsilon_1 - 51840\alpha\varepsilon_2 + \frac{24}{5}(2920\alpha^2 + 3809)\varepsilon_3,\\
L_3^{[1]}(\varepsilon) &= \frac{248832}{5}\alpha(488\alpha^2 - 857) - \frac{72}{5}(3886080\alpha^4 - 1845344\alpha^2 - 380679)\varepsilon_1\\
&\quad + 62208\alpha(2272\alpha^2 + 1909)\varepsilon_2\\
&\quad - \frac{36}{5}(3886080\alpha^4 + 3214240\alpha^2 - 3342471)\varepsilon_3.
\end{aligned}
$$

When $\alpha = \pm\sqrt{857/488}$, we have proved above that $L_3^{[1]}(0) = 0$. So, as the 3×3 matrix formed with the coefficients of $(L_1^{[1]}, L_2^{[1]}, L_3^{[1]})$ with respect to $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ has nonzero determinant, we get (adding the trace parameter and using the implicit function theorem) four limit cycles bifurcating from the origin. When $\alpha \notin A$, as $L_3^{[1]}(0) \neq 0$, only three limit cycles can bifurcate from the origin. The proof finishes just taking $\varepsilon_3 = 0$ and checking that the matrix formed with the coefficients of $L_1^{[1]}$, $L_2^{[1]}$ with respect to $(\varepsilon_1, \varepsilon_2)$ has rank two. □

4.3 Cyclicity on Complex Systems

Usually, the cyclicity problem is considered when the coefficients of the vector field are real numbers. But as, in fact, it depends on the study of the zeros of a polynomial, this problem can be studied also when the values of the parameters are complex numbers. In fact the existence of complex solutions increases the difficulties in finding the center classification, and sometimes, we need to go further in the Lyapunov constants because of this fact. Of course, they do not provide neither real weak foci nor real centers. As we have shown in almost all the proofs, there are complex solutions such that the orders of the *complex weak foci* are higher than the associated with real solutions and some difficulties can appear in the discussions. Moreover, the unfoldings can have more hyperbolic zeros that could be considered as complex limit cycles. The aim of the last result is to describe this phenomenon in our problem, showing why a carefully study is very important to distinguish the existence or not of (real) centers, as, for example, in Proposition 17, or which is the highest (real) weak focus order. In particular, we show the existence of six (complex) limit cycles near the origin for family (e) in Theorem 3, instead of the three that appear in Proposition 18 when we consider the problem in the reals. We notice that we have not found any point in the corresponding (real) center variety having more than three (real) limit cycles. In fact these three limit cycles have been appeared generically up to a first-order analysis. Instead of study higher order, we use the technique detailed in Giné et al. (2021) that explains that the local cyclicity changes moving parameters inside the center variety. This technique allows us to increase the number of (complex) limit cycles but not the real ones.

Proposition 20 *Consider system* (4) *with* $a_1 = \dfrac{w^2-1}{w^2+1}a_8$, $a_5 = \dfrac{w^2+1}{w^2-1}a_{11}$ *and* $a_7 = \dfrac{1}{w^2+1} - \dfrac{1}{(w^2+1)}a_8^2 - \dfrac{w^2+1}{(w^2-1)^2}a_{11}^2$ *for every fixed* (a_8, a_{11}, w) *such that* $w^2 \neq 1$, *and* $a_8 a_{11} \neq 0$. *Then, it has a center at* $p = (0, 1, 0)$ *and there exist* $(\hat{a}_8, \hat{a}_{11}, \hat{w}) \in \mathbb{C}^3$ *and complex perturbations inside the class* $\mathfrak{X}_2$ *such that at least* 6 *(complex) small amplitude limit cycles bifurcate from* p.

Proof The center property follows directly from Theorem 3.(e) because the proof does not change if we consider complex coefficients instead of reals.

As the previous proofs, we will work with the projected (perturbed) system (17) corresponding to system (4) but with $(a_1+\varepsilon_1, a_5+\varepsilon_2, a_7+\varepsilon_3, a_8+\varepsilon_4, a_{11}+\varepsilon_5, w+\varepsilon_6)$ instead of $(a_1, a_5, a_7, a_8, a_{11}, w)$. The algorithm explained in Sect. 2.3 allows us to compute the first 6 Lyapunov constants, and the first-order Taylor series write them as $L_k(\varepsilon) = L_k^{[1]}(\varepsilon) + \mathcal{O}_2(\varepsilon)$, for $k = 1, \ldots, 6$, being $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_6)$. We notice that, from the center property, $L_k(0) = L_k^{[1]}(0) = 0$. Straightforward computations show that the 3×6 matrix generated by the coefficients of $(L_1^{[1]}(\varepsilon), L_2^{[1]}(\varepsilon), L_3^{[1]}(\varepsilon))$ with respect to ε has rank 3. In fact the 6×6 matrix obtained from the first 6 has also rank three. Hence, adding the trace parameter, a first-order study only provides (generically) 3 small limit cycles using Theorem 8. Now, we will use in a different way the implicit function theorem to prove the statement. We will closely follow the scheme developed in Giné et al. (2021).

Considering only the first two, instead of the first three, we can make an analytic change of coordinates in the parameter space, changing $(\varepsilon_1, \varepsilon_2)$ to (u_1, u_2) in order that $L_1 = u_1$ and $L_2 = u_2$. Then, we can use these new coordinates to continue our analysis under the condition $u_1 = u_2 = 0$. Under this assumption, we can obtain

$$L_k^{[1]} = \frac{a_8 a_{11}\, M_k(a_8, a_{11}, w)}{w^{4k-7}(w^2+1)^{2k-3}(w^2-1)^{2k-2} N(a_8, a_{11}, w)} U_3(a_8, a_{11}, w, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6),$$

for $k = 3, \ldots, 6$, being

$$\begin{aligned} U_3 &= -(w^2+1)^2(w^2-1)^3\varepsilon_3 - 2a_8(w^2+1)(w^2-1)^3\varepsilon_4 \\ &\quad - 2a_{11}(w^2-1)(w^2+1)^3\varepsilon_5 \\ &\quad + 2w((w^2+3)(w^2+1)^2 a_{11}^2 + (w^2-1)^3 a_8^2 - (w^2-1)^3)\varepsilon_6, \\ N &= (2(6w^6+25w^4+20w^2+5))(w^2-1)^3 a_8^2 \\ &\quad + 2w^2(10w^8+49w^6+81w^4+43w^2 \\ &\quad + 9)(w^2+1)^2 a_{11}^2 - 3w^2(w^2-1)^3(w^2+1)^4 \end{aligned}$$

and M_k polynomials with rational coefficients of degrees 30, 46, 62, 78 with 74, 185, 369, 640 monomials, respectively. We only show the first one because of the size of them.

$$\begin{aligned} M_3 = &-(3w^{12}+30w^{10}-51w^8+420w^6+749w^4+350w^2+35)(w^2+1)^4 a_8^4 \\ &- (35w^{16}+350w^{14}+752w^{12}+450w^{10}-102w^8+450w^6+752w^4 \\ &+ 350w^2+35)(w^4-1)^2 a_8^2 a_{11}^2 - w^4(35w^{12}+350w^{10}+749w^8+420w^6-51w^4 \\ &+ 30w^2+3)(w^2+1)^4 a_{11}^4 - (6w^{16}+60w^{14}-3w^{12}+390w^{10}+1799w^8 \\ &+ 2240w^6+1267w^4+350w^2+35)(w^2-1)^4 a_8^2 - w^4(35w^{16}+350w^{14} \\ &+ 1267w^{12}+2240w^{10}+1799w^8+390w^6-3w^4+60w^2+6)(w^4-1)^2 \\ &a_{11}^2 - 3w^4(w^4+8w^2+1)(w^2+1)^4(w^2-1)^6. \end{aligned}$$

We notice that as $a_8 a_{11} \neq 0$ and $w^2 \neq 1$, we can change ε_3 by u_3 isolating from $u_3 = U_3$. Clearly, when M_3 is nonzero, we have rank 3 and no more limit cycles, up to first-order exist, because the next L_k vanish if L_3 does.

The proof ends checking that there exist values $\eta = (\hat{a}_8, \hat{a}_{11}, \hat{w}) \in \mathbb{C}^3$ such that $M_3(\eta) = M_4(\eta) = M_5(\eta) = 0$, $M_6(\eta) \neq 0$, $N(\eta) \neq 0$, and the determinant of the

Jacobian matrix, $\mathcal{J}$, of (M_3, M_4, M_5) does not vanish at η. Hence, for each transversal solution η, there exists a small neighborhood such that the implicit function theorem allows us to write, for $\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0$,

$$L_k = \hat{v}_k u_3 + \sum_{i=2}^{\infty} \tilde{L}_{k,i}(\hat{v}_3, \hat{v}_4, \hat{v}_5) u_3^i, \text{ for } k = 3, 4, 5,$$

and

$$L_6 = \tilde{v}_6 u_3 + \sum_{i=2}^{\infty} \tilde{L}_{6,i}(\hat{v}_3, \hat{v}_4, \hat{v}_5) u_3^i,$$

with $\tilde{v}_6 \in \mathbb{C}\backslash\{0\}$ and $\tilde{L}_{k,i}$ analytic functions. The next step needs again the implicit function theorem, dividing each equation by u_3, to write $L_k = v_k u_3$ for $k = 3, 4, 5$. Finally, for u_3 small enough, we have a (complex) weak focus of order 6 that unfolds 6 (complex) limit cycles of small amplitude. We must use the trace parameter together with u_1, u_2 to have the complete versal unfolding.

We will finish now showing how the transversal intersection together with the nonzero conditions holds. From the scheme used in the proof of Proposition 17, we start computing the crossing resultants $R_4 = \text{res}(M_3, M_4, a_8)$ and $R_5 = \text{res}(M_3, M_5, a_8)$ and after removing the common factors, we get that one of the factors of $R_{45} = \text{res}(\hat{R}_4, \hat{R}_5, a_{11})$ is the polynomial

$$\begin{aligned} \mathfrak{p}(w) = {} & 2358125w^{36} + 21253750w^{34} + 74700325w^{32} + 418209680w^{30} \\ & + 2696915172w^{28} + 11471299432w^{26} + 32469933620w^{24} \\ & + 65296240368w^{22} + 97631971158w^{20} + 111336439620w^{18} \\ & + 97631971158w^{16} + 65296240368w^{14} + 32469933620w^{12} \\ & + 11471299432w^{10} + 2696915172w^{8} + 418209680w^{6} \\ & + 74700325w^{4} + 21253750w^{2} + 2358125, \end{aligned}$$

which has degree 18 in w^2 and has no real roots. Straightforward computations provide that the solutions of

$$\{M_3(a_8, a_{11}, w) = M_4(a_8, a_{11}, w) = M_5(a_8, a_{11}, w) = \mathfrak{p}(w) = 0\}$$

can be written as $\eta = (\hat{a}_8, \hat{a}_{11}, \hat{w}) = (\gamma/393216, \beta/393216, \alpha)$ with $\mathfrak{p}(\alpha) = 0$ and

$$\begin{aligned} \gamma^2 = {} & (-2109840270437897288946799375\alpha^{34} - 1771387300580941402800709187 5\alpha^{32} \\ & - 55845814488941782868405995600\alpha^{30} - 339226787759422100359736546240\alpha^{28} \\ & - 2202132815857798644636420812396\alpha^{26} - 8895632031878776983832830622252\alpha^{24} \\ & - 23501844430016397280367284455712\alpha^{22} - 43681378432973931255571646316816\alpha^{20} \\ & - 59799471686092166179474533669890\alpha^{18} - 61675400652851000448826407685450\alpha^{16} \\ & - 47993652204362508680940846437904\alpha^{14} - 27604081780746071222684026969568\alpha^{12} \end{aligned}$$

$$
\begin{aligned}
&- 112075195671777791801702673976\,28\alpha^{10} - 2966440603081008995602043288284\alpha^{8} \\
&- 468074157329937609377913199360\alpha^{6} - 660697953909631800086788244 00\alpha^{4} \\
&- 2412999359292109298944639437 5\alpha^{2} - 3181668269212383369855156875)/9396192821225170, \\
\beta^2 = {} &(4024803641202156633404377 0625\alpha^{34} + 3481486589274059327698171706 25\alpha^{32} \\
&+ 11393905775209774956821591812 00\alpha^{30} + 66521557632507168760277854420 80\alpha^{28} \\
&+ 43419594803251066283846291084532\alpha^{26} + 17870003480856237326547582134 7460\alpha^{24} \\
&+ 48053184534356717949299005826 4000\alpha^{22} + 90698446696091674369077896926 3248\alpha^{20} \\
&+ 12580335831635505396757251869 99550\alpha^{18} + 13130151792215329243717516900 40910\alpha^{16} \\
&+ 10334251103883243680129292437 20848\alpha^{14} + 60137437774500167463859185130 2720\alpha^{12} \\
&+ 24729104968623085738994001020 7140\alpha^{10} + 66539552467687537839070855101012\alpha^{8} \\
&+ 10700521536131995853057121609120\alpha^{6} + 1483387556836472011860839676560\alpha^{4} \\
&+ 538998995971563271556998588905\alpha^{2} + 697370607508104755605366253 05)/32886674874288095.
\end{aligned}
$$

Using the above relations, we can check that, for each α, simple root of the polynomial $\mathfrak{p}$, we have that $M_3(\eta)$, $M_4(\eta)$, and $M_5(\eta)$ vanish and $M_6(\eta) = \mathfrak{p}_1(\alpha)$, $N(\eta) = \mathfrak{p}_2(\alpha)$, and $\mathcal{J}(\eta) = \alpha\beta\gamma\mathfrak{p}_3(\alpha)$ do not, because the polynomials $\mathfrak{p}_i(\alpha)$, for $i = 1, 2, 3$, which are of degree 17 in α^2 and with rational coefficients, have nonzero resultants with the polynomial $\mathfrak{p}$.

□

We remark that, in the above proof, the coefficients of M_3 with respect to (a_8, a_{11}) are negative for all $w \in \mathbb{R}$, so the described bifurcation mechanism does not provide more than the three limit cycles in Proposition 18.

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