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Global Existence of Weak Solutions to the Incompressible Axisymmetric Euler Equations Without Swirl

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Abstract

In this paper, we consider solutions to the incompressible axisymmetric Euler equations without swirl. The main result is to prove the global existence of weak solutions if the initial vorticity w_0^θ satisfies that $\frac{w_0^\theta}{r} \in L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 1$. It is not required that the initial energy is finite, that is, the initial velocity u_0 belongs to $L^2(\mathbb{R}^3)$ here. We construct the approximate solutions by regularizing the initial data and show that the concentrations of energy do not occur in this case. The key ingredient in the proof lies in establishing the $L^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ estimates of velocity fields for some $\alpha > 0$, which is new to the best of our knowledge.

Keywords Axisymmetric · Euler equations · Global weak solutions

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1 Introduction and Main Results

In this paper, we are concerned with the three-dimensional incompressible Euler equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

in the whole space $\mathbb{R}^3$ with initial data $u(0, x) = u_0(x)$, where $u = (u_1, u_2, u_3)$ and $p = p(x, t)$ represent the velocity fields and pressure, respectively.

The mathematical study to the incompressible Euler equations takes a long history with a large amount of associated literature. For two-dimensional case, Wolibner (1933) obtained the global well-posedness of smooth solutions in 1933. Then, this work was extended by Yudovich (1963), who proved the existence and uniqueness for a certain class of weak solutions if the initial vorticity w_0 lies in $L^1 \cap L^\infty(\mathbb{R}^2)$. Later, under the assumption that $w_0 \in L^1 \cap L^p(\mathbb{R}^2)$ for some $p > 1$, DiPerna and Majda showed that the weak solutions exist globally in DiPerna and Majda (1987b). Furthermore, if w_0 is a finite Radon measure with one sign, there are also many works about the global existence of weak solutions, which can be referred to Delort (1991), Majda (1993), Evans and Müller (1994) and Liu and Xin (1995) for details. However, the global existence of smooth solutions for 3D incompressible Euler equations with smooth initial data is still an important open problem, with a large literature.

From mathematical point of view, in two-dimensional case, the corresponding vorticity $w = \partial_2 u_1 - \partial_1 u_2$ is a scalar field and satisfies the following transport equation

$$\partial_t w + u \cdot \nabla w = 0,$$

which infers that its L^p norm is conserved for all time. Nevertheless, for the three-dimensional case, w becomes a vector fields and the *vortex stretching* term $w \cdot \nabla u$ appears in the equations of vorticity

$$\partial_t w + u \cdot \nabla w = w \cdot \nabla u,$$

where $w = \nabla \times u$. The presence of *vortex stretching* term brings more difficulties to prove the global regularity, which is the main reason causing this problem open. Therefore, many mathematicians explore the flows with certain geometrical assumptions, which attempt to fill the gap between 2D and 3D flows. One typical case is the axisymmetric flows.

Whereas, even with this axisymmetric structure, it is still open to exclude possible singularities. But if the swirl component of velocity fields u_θ is trivial, i.e., so-called flows *without swirl* or *with non-swirl*, Ukhovskii and Yudovich (1968), Serfati (1994), Saint Raymond (1994) and Majda and Bertozzi (2002) proved that the weak solutions of incompressible axisymmetric Euler equations are regular for all time. It should be noted that under the assumption without swirl, the corresponding vorticity quantity $\frac{w_\theta}{r}$ is a scalar field and transported by a divergence free vector fields, which makes the problem closer to the 2D case.

However, for the incompressible axisymmetric Euler equations without swirl and vortex sheets initial data, the problem on global existence of weak solutions remains open, which is quite different from the 2D case. In the subsequent research, many mathematicians are concentrated in determining more precisely for which initial vorticity (allowed *a little more regular* than for vortex sheets), one can obtain the global existence of a weak solution. There is a large literature devoted to this subject. In 1997, D. Chae and N. Kim proved the global existence of a weak solution under the assumption that $\frac{w_0^\theta}{r} \in L^p(\mathbb{R}^3)$ for some $p > 6/5$ in Chae and Kim (1997). Later, Chae and Imanuvilov (1998) obtained the similar result by assuming $u_0 \in L^2(\mathbb{R}^3)$ and $|\frac{w_0^\theta}{r}|[1 + (\log^+|\frac{w_0^\theta}{r}|)^\alpha] \in L^1(\mathbb{R}^3)$ with $\alpha > 1/2$. Recently, Jiu et al. (2015) also obtained the global existence result under the assumptions that $u_0 \in L^2(\mathbb{R}^3)$ and $\frac{w_0^\theta}{r} \in L^1 \cap L^p(\mathbb{R}^3)$ (for some $p > 1$) by using the method of viscous approximations. It is referred to Jiu and Liu (2015), Jiu and Liu (2018), Liu (2016), Leonardi et al. (1999), Shirota and Yanagisawa (1994), Gang and Zhu (2007), Danchin (2007), Jiu and Xin (2004), Jiu and Xin (2006), Liu and Niu (2017), Jiu et al. (2018), Bronzi et al. (2015), Jiu et al. (2017), Ettinger and Titi (2009) and DiPerna and Majda (1988) for more related works. It should be noted that in Chae and Imanuvilov (1998) and Jiu et al. (2015), the initial velocity is assumed *with the finite energy*, i.e., $u_0 \in L^2(\mathbb{R}^3)$. The main reason lies in that the proof in Chae and Imanuvilov (1998) and Jiu et al. (2015) highly relies on a key estimate, that is

$$\int_0^{\mathrm{T}} \int_{\mathbb{R}^3} \frac{1}{1+z^2} \left(\frac{u_r}{r}\right)^2 \mathrm{d}x\mathrm{d}t \le C \left(\|u_0\|^2_{L^2(\mathbb{R}^3)} + \|\frac{w_0^\theta}{r}\|_{L^1(\mathbb{R}^3)} \right), \tag{1.2}$$

which is raised by Chae–Imanuvilov in Chae and Imanuvilov (1998).

Nevertheless, one very important open problem is to identify whether the weak solutions (possessing *only* locally finite kinetic energy *other than* finite kinetic energy, see Definition 1.1 for details) conserve kinetic energy or if it is possible to lose energy to the small scales of the flow, i.e., through the concentrations of energy, such as the pioneering work (DiPerna and Majda 1987b) by DiPerna and Majda, whose main point of departure is to search for the initial vorticity that generates flows conserving kinetic energy, namely, without concentrations. Motivated by this work and recent progress in this direction for helically symmetric flows without helical swirl (Jiu et al. 2017), we would like to know whether analogical phenomenon happens for the incompressible axisymmetric Euler equations without swirl.

In this paper, we give a positive answer to this question. That is, given the initial vorticity such that $\frac{w_0^\theta}{r} \in L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 1$, the incompressible axisymmetric Euler equations without swirl has at least one weak solution, which indicates that the concentrations of energy do not occur if the initial vorticity is *slightly more regular* than for vortex sheets. Moreover, we have a new observation that $\frac{w^\theta}{r} \in L^1 \cap L^p(\mathbb{R}^3)$ implies $u \in L_{\mathrm{loc}}^{\frac{2p}{2-p}}(\mathbb{R}^3)$ for $1 < p < 2$.

We construct the approximate solutions by smoothing the initial data and prove that there exists a subsequence of the approximate solutions that converge strongly in

$L^2_{\rm loc}$-space (with respect to time and space variables). In the process of proof, there are two main difficulties to be overcome. Firstly, the basic energy estimates take no effect and hence we do not have any estimates of velocity fields itself. As a matter of fact, for the incompressible axisymmetric Euler equations without swirl, whether or not $\frac{w^\theta}{r} \in L^1 \cap L^p(\mathbb{R}^3)$ $(p > 1)$ conclude $u \in L^2(\mathbb{R}^3)$, even $L^2_{\rm loc}(\mathbb{R}^3)$, is an interesting and open problem itself. To overcome them, we make the first attempt to establish the $L^p_{\rm loc}(\mathbb{R}^3)$ $(p > 1)$ estimates for the velocity fields. More precisely, we find out the explicit form of stream function in terms of vorticity and then establish the $L^p_{\rm loc}(\mathbb{R}^3)$ estimates and further $W^{1,p}_{\rm loc}(\mathbb{R}^3)$ estimates of velocity fields for any $p > 1$.

However, this is still far from resolving the original problem, because current estimates only guarantee the strong convergence of approximate solutions in $L^2(0, T; Q)$ for any $Q \subset\subset \mathbb{R}^3\backslash\{x \in \mathbb{R}^3 | r = 0\}$, other than $L^2(0, T; L^2_{\rm loc}(\mathbb{R}^3))$. As in Jiu et al. (2015), current argument is enough to conclude the global existence of weak solutions, if the following proposition introduced by Jiu and Xin (2006) is applicable.

Proposition *Suppose that* $u_0 \in L^2(\mathbb{R}^3)$. *For the approximate solutions* $\{u^\epsilon\}$ *constructed in Theorem 4.1* (*see* Jiu and Xin 2006), *if there exists a subsequence* $\{u^{\epsilon_j}\} \subset \{u^\epsilon\}$ *such that, for any* $Q \subset\subset \mathbb{R}^3\backslash\{x \in \mathbb{R}^3 | r = 0\}$ *and* $\epsilon_j \to 0$,

$$u^{\epsilon_j} \to u \quad \textit{strongly} \quad \text{in} \quad L^2\left(0, T; L^2(Q)\right),$$

then there exists a further subsequence of $\{u^{\epsilon_j}\}$, *still denoted by itself, such that, as* $\epsilon_j \to 0$,

$$u^{\epsilon_j} \to u \quad \textit{strongly} \quad \text{in} \quad L^2\left(0, T; L^2_{\rm loc}(\mathbb{R}^3)\right).$$

Unfortunately, in our case, this method would not work any more due to lack of the initial assumption $u_0 \in L^2(\mathbb{R}^3)$. This brings the other difficulty in solving our problem. It is necessary to find a new way to establish the convergence of approximate solutions in the region *contains* the axis of symmetry. To this end, we try to look for some estimates of velocity fields *stronger* than $L^2_{\rm loc}(\mathbb{R}^3)$ and then establish the $L^{\frac{2p}{2-p}}_{\rm loc}(\mathbb{R}^3)$ estimates of velocity fields for $1 < p < 2$, based on delicate analysis of the axisymmetric structure of model. The obtained estimates seem *optimal*. Finally, we deduce the strong convergence of approximate solutions in $L^2(0, T; L^2_{\rm loc}(\mathbb{R}^3))$, which is sufficient to prove the global existence of weak solutions.

Before stating our main theorems, we introduce the definition of weak solutions to the system (1.1).

Definition 1.1 (*Weak solution*) A velocity fields $u(x, t) \in L^\infty(0, T; L^2_{\rm loc}(\mathbb{R}^3))$ for any $T > 0$ is a *weak solution* of the 3D incompressible Euler equations with initial data $u_0(x)$ provided that

(i) for any vector field $\varphi \in C_0^\infty([0, T); \mathbb{R}^3)$ with $\nabla \cdot \varphi = 0$,

$$\int_0^{\rm T} \int_{\mathbb{R}^3} u \cdot \varphi_t \,{\rm d}x{\rm d}t + \int_0^{\rm T} \int_{\mathbb{R}^3} u \cdot \nabla\varphi \cdot u \,{\rm d}x{\rm d}t = \int_{\mathbb{R}^3} u_0 \cdot \varphi_0 \,{\rm d}x;$$

(ii) the velocity fields $u(x,t)$ is incompressible in the weak sense, i.e., for any scalar function $\phi \in C_0^\infty([0,T);\mathbb{R}^3)$,

$$\int_0^T \int_{\mathbb{R}^3} u \cdot \nabla\phi \, \mathrm{d}x\mathrm{d}t = 0;$$

(iii) the velocity fields $u(x,t)$ belongs to $\mathrm{Lip}(0,T;H_{\mathrm{loc}}^{-L}(\mathbb{R}^3))$ for some $L > 0$ and $u(x,0) = u_0(x)$ in $H_{\mathrm{loc}}^{-L}(\mathbb{R}^3)$.

Our main results are stated as follows.

Theorem 1.1 *Suppose that $w_0^\theta = w_0^\theta(r,z)$ is a scalar axisymmetric function such that $w_0 = w(x,0) = w_0^\theta e_\theta$ and $\frac{w_0^\theta}{r} \in L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 1$. Then, for any $T > 0$, there exists at least an axisymmetric weak solution u without swirl in the sense of Definition 1.1.*

Remark 1.1 On the basis of Definition 1.1, the weak solution is a solution with *locally finite kinetic energy*. It is natural that $u_0 \in L_{\mathrm{loc}}^2(\mathbb{R}^3)$ instead of $L^2(\mathbb{R}^3)$, which is guaranteed by the initial assumptions in Theorem 1.1 and Proposition 3.3.

This paper is organized as follows. In Sect. 2, we introduce some notations and technical lemmas. In Sect. 3, we will concentrate on the a priori estimates of velocity fields. Section 4 is devoted to proving the global existence of weak solutions, i.e., the proof of Theorem 1.1.

2 Preliminary

In this section, we introduce notations and set down some basic definitions. Initially, we would like to introduce the definition of axisymmetric flow.

Definition 2.1 (*Axisymmetric flow*) A vector fields $u(x,t)$ is called axisymmetric if it can be described by the form of

$$u(x,t) = u_r(r,z,t)e_r + u_\theta(r,z,t)e_\theta + u_z(r,z,t)e_z \tag{2.1}$$

in the cylindrical coordinate, where $e_r = (\cos\theta, \sin\theta, 0)$, $e_\theta = (-\sin\theta, \cos\theta, 0)$, $e_z = (0,0,1)$. We call the components of vector fields $u_r(r,z,t),\ u_\theta(r,z,t),\ u_z(r,z,t)$ as radial, swirl and z-component, respectively.

Throughout this paper, for simplicity, we will use $u_r,\ u_\theta,\ u_z$ to denote $u_r(r,z,t)$, $u_\theta(r,z,t)$, $u_z(r,z,t)$, respectively.

Then, we set up the equations satisfied by $u_r,\ u_\theta,\ u_z$. Under the cylindrical coordinate, the gradient operator can be expressed in the form of $\nabla = e_r\partial_r + \frac{1}{r}e_\theta\partial_\theta + e_z\partial_z$.

Then, by some basic calculations, one can rewrite (1.1) as

$$\begin{cases} \partial_t u_r + \tilde{u} \cdot \tilde{\nabla} u_r + \partial_r p = \dfrac{(u_\theta)^2}{r}, \\ \partial_t u_\theta + \tilde{u} \cdot \tilde{\nabla} u_\theta = -\dfrac{u_\theta u_r}{r}, \\ \partial_t u_z + \tilde{u} \cdot \tilde{\nabla} u_z + \partial_z p = 0, \\ \partial_r (r u_r) + \partial_z (r u_z) = 0, \end{cases} \tag{2.2}$$

where $\tilde{u} = (u_r, u_z)$ and $\tilde{\nabla} = (\partial_r, \partial_z)$. In addition, by $(2.2)^2$ and some basic calculations, it is clear that the quantity ru_θ satisfies the following transport equation:

$$\partial_t (r u_\theta) + \tilde{u} \cdot \tilde{\nabla} (r u_\theta) = 0. \tag{2.3}$$

Thanks to (2.3), the following conclusion holds.

Proposition 2.1 *Assume u is a smooth solution of incompressible axisymmetric Euler equations, then the swirl component of velocity fields u_θ will be vanishing if its initial data u_0^θ be given zero.*

Proof Thanks to the incompressible condition $(2.2)^4$, by multiplying (2.3) with ru_θ and integrating on $(0, t)$, it follows that

$$\|r u_\theta(t)\|_{L^2(\mathbb{R}^3)} \leq \|r u_0^\theta\|_{L^2(\mathbb{R}^3)} = 0.$$

Then, considering that u_θ is smooth and $u_\theta|_{r=0} \equiv 0$, we can conclude that $u_\theta \equiv 0$ for any $t > 0$. □

Therefore, if $u_0^\theta = 0$, then the corresponding velocity fields become $\tilde{u}$ and its vorticity can be described as $w = w_\theta e_\theta$, where $w_\theta = \partial_z u_r - \partial_r u_z$. What is more, the scalar quantity w_θ is satisfied by the equation

$$\partial_t w_\theta + \tilde{u} \cdot \tilde{\nabla} w_\theta = \frac{u_r w_\theta}{r}, \tag{2.4}$$

and $\frac{w_\theta}{r}$ is transported by $\tilde{u}$, i.e.,

$$\partial_t \left(\frac{w_\theta}{r}\right) + \tilde{u} \cdot \tilde{\nabla} \left(\frac{w_\theta}{r}\right) = 0. \tag{2.5}$$

This means that $\frac{w_\theta}{r}$ is conserved along the particle trajectory. As a result, given the initial data smooth sufficiently, the incompressible axisymmetric Euler equations without swirl always possess a unique global solution (DiPerna and Majda 1987a; Saint Raymond 1994). Besides, by employing the incompressible condition and some basic calculations, we have the following conclusion.

Conservation laws for $\frac{w_\theta}{r}$. *Suppose that u is a smooth solution of incompressible axisymmetric Euler equations, with its initial swirl component u_0^θ vanishing, then the estimates*

$$\|\frac{w_\theta}{r}\|_{L^p(\mathbb{R}^3)} \leq \|\frac{w_0^\theta}{r}\|_{L^p(\mathbb{R}^3)} \tag{2.6}$$

hold for any $p \in [1, \infty]$, where $w_0^\theta = w^\theta(x, 0)$.

Subsequently, we will introduce the stream function, whose existence is proved in Lemma 2 of Liu and Wang (2009).

Proposition 2.2 *Let u be a smooth axisymmetric vector fields without swirl and $\nabla \cdot u = 0$, then there exists a unique scalar function $\psi = \psi(r, z)$ such that $u = \nabla \times (\psi e_\theta)$ and $\psi = 0$ on the axis of symmetry $r = 0$.*

Finally, we will collect below some useful estimates of velocity fields in terms of $\frac{w_\theta}{r}$, see Lei (2015), Jiu and Liu (2015) and Miao and Zheng (2013) for instance.

Lemma 2.1 *Let ψ be as in Proposition 2.2, it holds that*

$$\begin{aligned}&\|\partial_r^2\left(\frac{\psi}{r}\right)\|_{L^p(\mathbb{R}^3)} + \|\frac{1}{r}\partial_r\left(\frac{\psi}{r}\right)\|_{L^p(\mathbb{R}^3)} + \|\partial_{rz}^2\left(\frac{\psi}{r}\right)\|_{L^p(\mathbb{R}^3)} + \|\partial_z^2\left(\frac{\psi}{r}\right)\|_{L^p(\mathbb{R}^3)}\\ &\leq C\|\frac{w_\theta}{r}\|_{L^p(\mathbb{R}^3)}\end{aligned}$$

for any $p > 1$, where C is an absolute constant. In particular,

$$\|\partial_r\left(\frac{u_r}{r}\right)\|_{L^p(\mathbb{R}^3)} + \|\partial_z\left(\frac{u_r}{r}\right)\|_{L^p(\mathbb{R}^3)} \leq C\|\frac{w_\theta}{r}\|_{L^p(\mathbb{R}^3)}. \tag{2.7}$$

Lemma 2.2 *Suppose that u is a smooth solution of incompressible axisymmetric Euler equations without swirl, then there holds*

$$\|\frac{u_r}{r}\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)} \leq C\|\frac{w_\theta}{r}\|_{L^p(\mathbb{R}^3)} \qquad \forall p \in (1, 3), \tag{2.8}$$

where C is an absolute constant.

3 A Priori Estimates of Velocity Fields

3.1 $W^{1,p}_{loc}(\mathbb{R}^3)$ ($p > 1$) Estimates

In this section, we will focus on the $W^{1,p}_{loc}(\mathbb{R}^3)$ estimates of velocity fields. Firstly, Proposition 2.2 together with $\nabla \cdot u = 0$ and $w = \nabla \times u = w_\theta e_\theta$ tells us that

$$-\Delta(\psi e_\theta) = w_\theta e_\theta.$$

Then, by the elliptic theory, we have

$$\psi\,(r_x, z_x)\, e_{\theta_x} = \int_{\mathbb{R}^3} G(X, Y) w_\theta(r_y, z_y) e_{\theta_y} \mathrm{d}Y, \tag{3.1}$$

where $X = (r_x, \theta_x, z_x)$ and $G(X, Y) = |X - Y|^{-1}$ stands for the three-dimensional Green's function in the whole space. Regarding the Green's function $G(X, Y)$, it is well known that the following two properties hold

(i)

$$|D_X^k G(X, Y)| \le C_k |X - Y|^{-1-k}, \tag{3.2}$$

(ii)

$$G\left(\bar{X}, Y\right) = G\left(X, \bar{Y}\right), \quad \partial_r G\left(\bar{X}, Y\right) = \partial_r G\left(X, \bar{Y}\right), \quad \partial_z G\left(\bar{X}, Y\right) = \partial_z G(X, \bar{Y}), \tag{3.3}$$

for all $(X, Y) \in \mathbb{R}^3$, $\bar{X} = (-x, -y, z)$ and $k = 0, 1, 2$.

Until now, we have established the formulation (3.1). However, in order to find out the explicit form of $\psi(r_x, z_x)$, we need to fix the value of θ_x. Therefore, by making use of the rotational invariance and putting $\theta_x = 0$ in (3.1), we derive the explicit form of ψ in terms of w_θ

$$\psi\,(r_x, z_x) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\pi}^{\pi} G(X, Y) w^\theta \cos\theta_y r_y \mathrm{d}\theta_y \mathrm{d}r_y \mathrm{d}z_y, \tag{3.4}$$

where $X = (r_x, 0, z_x)$.

On this basis, we intend to utilize the stream function to establish the $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ estimates of velocity fields. And we would like to introduce the following lemma, which is the cornerstone of this paper.

Lemma 3.1 *Assume u and ψ be as in Lemma 2.2, $w = \nabla \times u = w_\theta e_\theta$, then there holds that*

$$|\psi\,(r_x, z_x)\,| \le C \int_{\mathbb{R}^3} \min\left(1, \frac{r_x}{|X - Y|}\right) \frac{|w^\theta|}{|X - Y|} \mathrm{d}Y \tag{3.5}$$

and

$$|\partial_r \psi\,(r_x, z_x)\,| + |\partial_z \psi\,(r_x, z_x)\,| \le C \int_{\mathbb{R}^3} \min\left(1, \frac{r_x}{|X - Y|}\right) \frac{|w^\theta|}{|X - Y|^2} \mathrm{d}Y, \tag{3.6}$$

where C is an absolute constant and $X = (r_x, 0, z_x)$.

Proof First of all, we do the estimate of $|\partial_r\psi|$. From (3.4), we have

$$\partial_r\psi=\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\pi}^{\pi}\partial_r G(X,Y)w^{\theta}\cos\theta_y r_y \mathrm{d}\theta_y\mathrm{d}r_y\mathrm{d}z_y,$$

which together with (3.3) yields that

$$\partial_r\psi=\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\partial_r G(X,Y)-\partial_r G(\bar{X},Y))w^{\theta}\cos\theta_y r_y \mathrm{d}\theta_y\mathrm{d}r_y\mathrm{d}z_y.$$

Thus, to prove (3.6), it suffices to verify that

$$\begin{aligned}H&\triangleq\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\partial_r G(X,Y)-\partial_r G(\bar{X},Y)\right)w^{\theta}\cos\theta_y\mathrm{d}\theta_y\\&\le C\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\min\left(1,\frac{r_x}{|X-Y|}\right)\frac{|w^{\theta}|}{|X-Y|^2}\mathrm{d}\theta_y.\end{aligned}$$

Without loss of generality, we assume θ^* to be the unique real number $\theta_y\in[0,\frac{\pi}{2}]$ such that $|X-Y|=r_x$ and split the integral H into $H=\mathrm{I}+\mathrm{II}+\mathrm{III}$, with

$$\mathrm{I}=\int_{-\frac{\pi}{2}}^{-\theta^*}\mathrm{d}\theta_y,\quad \mathrm{II}=\int_{-\theta^*}^{\theta^*}\mathrm{d}\theta_y,\quad \mathrm{III}=\int_{\theta^*}^{\frac{\pi}{2}}\mathrm{d}\theta_y,$$

where $|X-Y|>r_x$ for I, III and $|X-Y|\le r_x$ for II. Otherwise, $|X-Y|>r_x$ or $|X-Y|<r_x$ for all $\theta_y\in[-\frac{\pi}{2},\frac{\pi}{2}]$. For these two cases, one can prove them along the same lines with estimating I or II.

Because $|X-Y|\le|\bar{X}-Y|$ for all $|\theta_y|\le\frac{\pi}{2}$ and the interval $[-\theta^*,\theta^*]$ corresponds to those θ_y for which $|X-Y|\le r_x$, one can conclude that II satisfies the desired estimate easily.

Regarding the first and third terms, to start with, we fix some angle $\theta_y\in[\theta^*,\frac{\pi}{2}]$ and denote $X_\beta=(r\cos\beta,r\sin\beta,z)$ for $\beta\in[-\pi,0]$. Besides, for the function $f(x,y,z)=f(r\cos\theta,r\sin\theta,z)$, it is clear that $\partial_\theta f=r\partial_h f\cdot e_\theta$, where $\partial_h=(\partial_x,\partial_y,0)$. Therefore, by the fundamental theorem of calculus, it follows that

$$\partial_r G(X,Y)-\partial_r G(\bar{X},Y)=\pi r_x\int_{-\pi}^{0}\partial_h\partial_r G\left(X_\beta,Y\right)\cdot e_\beta\mathrm{d}\beta.$$

Then, by employing the fact $|X-Y|\le|X_\beta-Y|$ for all $\beta\in[-\pi,0]$ and (3.2), it holds that

$$|\partial_r G(X,Y)-\partial_r G(\bar{X},Y)|\le Cr_x|X-Y|^{-3}.$$

Thus, we have obtained the estimate of III, that is

$$\mathrm{III} \le C r_x \int_{\theta^*}^{\frac{\pi}{2}} |X - Y|^{-3} |w^\theta| \mathrm{d}\theta_y.$$

What is more, the estimate of I can be treated by the same arguments with III. Thus, by adding up all the estimates, one can derive the estimate of $|\partial_r \psi|$. As for $|\psi|$ and $|\partial_z \psi|$, one can estimate it in the similar way and we will omit it here. □

Thanks to Lemma 3.1, we can then derive the upper bounds of $\frac{\psi}{r}$, $\frac{\partial_r \psi}{r}$, $\frac{\partial_z \psi}{r}$ in terms of $\frac{w_\theta}{r}$.

Corollary 3.1 *Under the assumptions of Lemma 3.1, it further holds that*

$$\left| \frac{\psi(r_x, z_x)}{r_x} \right| \le C \int_{\mathbb{R}^3} \frac{|w^\theta|}{r_y |X - Y|} \mathrm{d}Y \tag{3.7}$$

and

$$\left| \frac{\partial_r \psi(r_x, z_x)}{r_x} \right| + \left| \frac{\partial_z \psi(r_x, z_x)}{r_x} \right| \le C \int_{\mathbb{R}^3} \frac{|w^\theta|}{r_y |X - Y|^2} \mathrm{d}Y, \tag{3.8}$$

where C is an absolute constant and $X = (r_x, 0, z_x)$.

Proof Initially, if $Y \in \mathbb{R}^3$ are such that $|X - Y| \le r_x$ for any r_x, then one has $r_y \le r_x + |r_x - r_y| \le r_x + |X - Y| \le 2r_x$, which together with (3.5) and (3.6) implies that

$$\left| \frac{\psi(r_x, z_x)}{r_x} \right| \le C \int_{\mathbb{R}^3} \frac{1}{r_x} \frac{|w^\theta|}{|X - Y|} \mathrm{d}Y \le 2C \int_{\mathbb{R}^3} \frac{1}{r_y} \frac{|w^\theta|}{|X - Y|} \mathrm{d}Y$$

and

$$\left| \frac{\partial_r \psi(r_x, z_x)}{r_x} \right| + \left| \frac{\partial_z \psi(r_x, z_x)}{r_x} \right| \le C \int_{\mathbb{R}^3} \frac{1}{r_x} \frac{|w^\theta|}{|X - Y|^2} \mathrm{d}Y \le 2C \int_{\mathbb{R}^3} \frac{1}{r_y} \frac{|w^\theta|}{|X - Y|^2} \mathrm{d}Y.$$

Otherwise, if $|X - Y| > r_x$, it is clear that $\frac{r_y}{|X-Y|} \le \frac{r_x + |r_x - r_y|}{|X-Y|} \le \frac{r_x + |X-Y|}{|X-Y|} \le 2$. Then, we can get that

$$\left| \frac{\psi(r_x, z_x)}{r_x} \right| \le C \int_{\mathbb{R}^3} \frac{1}{|X - Y|} \frac{|w^\theta|}{|X - Y|} \mathrm{d}Y \le 2C \int_{\mathbb{R}^3} \frac{1}{r_y} \frac{|w^\theta|}{|X - Y|} \mathrm{d}Y$$

and

$$\left| \frac{\partial_r \psi(r_x, z_x)}{r_x} \right| + \left| \frac{\partial_z \psi(r_x, z_x)}{r_x} \right| \le C \int_{\mathbb{R}^3} \frac{1}{|X - Y|} \frac{|w^\theta|}{|X - Y|^2} \mathrm{d}Y$$

$$\leq 2C \int_{\mathbb{R}^3} \frac{1}{r_y} \frac{|w^\theta|}{|X-Y|^2} \mathrm{d}Y.$$

Thus, the proof is finished. □

Remark 3.1 The proof of Lemma 3.1 and Corollary 3.1 borrows some ideas from Shirota and Yanagisawa (1994) and Danchin (2007). In Danchin (2007), the author used the explicit form of $|\frac{\partial_z \psi}{r}|$ in (3.8) to establish the $L^\infty(\mathbb{R}^3)$ estimate of $\frac{u_r}{r}$. Here, we discover more applications of stream functions in establishing some estimates of velocity fields, which will be shown in the following content.

With the help of Lemma 3.1 and Corollary 3.1, we can then derive the following $L^p_{loc}(\mathbb{R}^3)$ estimates of velocity fields, which is the first key contribution of our work.

Proposition 3.1 ($L^p_{loc}(\mathbb{R}^3)$ estimates) *Given u as a smooth axisymmetric velocity fields without swirl satisfying $\nabla \cdot u = 0$, then there holds*

$$\|u\|_{L^p(B_R \times [-R,R])} \leq C_R \|\frac{w^\theta}{r}\|_{L^1 \cap L^p(\mathbb{R}^3)}$$

for any $p \in (1, \infty)$. Here $B_R = B_R(0) \subset \mathbb{R}^2$ be a 2D ball and the constant C_R depends only on R.

Proof According to Lemma 2.2, for the smooth axisymmetric velocity fields u with zero swirl component, there exists a unique stream function ψ such that

$$u = u_r e_r + u_z e_z = \nabla \times (\psi e_\theta).$$

This implies that $u_r = -\partial_z \psi, \ \ u_z = \partial_r \psi + \frac{\psi}{r}$ and therefore $|u| \leq |\partial_z \psi| + |\partial_r \psi| + |\frac{\psi}{r}|$. Then, by Lemma 3.1 and Corollary 3.1, it follows that

$$\begin{aligned}
|u| &\leq C \int_{\mathbb{R}^3} \frac{|w^\theta|}{r_y|X-Y|} \mathrm{d}Y + C \int_{\mathbb{R}^3} \frac{|w^\theta|}{|X-Y|^2} \mathrm{d}Y \\
&\leq C \int_{|X-Y|\leq 1} \frac{|w^\theta|}{r_y|X-Y|} \mathrm{d}Y + C \int_{|X-Y|>1} \frac{|w^\theta|}{r_y|X-Y|} \mathrm{d}Y \\
&\quad + C \int_{|X-Y|\leq 1} \frac{|w^\theta|}{|X-Y|^2} \mathrm{d}Y + C \int_{|X-Y|>1} \frac{|w^\theta|}{|X-Y|^2} \mathrm{d}Y \\
&\leq C \int_{|X-Y|\leq 1} \frac{|w^\theta|}{r_y|X-Y|} \mathrm{d}Y + C \int_{|X-Y|>1} \frac{|w^\theta|}{r_y|X-Y|} \mathrm{d}Y \\
&\quad + C r_x \int_{|X-Y|\leq 1} \frac{|w^\theta|}{r_y|X-Y|^2} \mathrm{d}Y + C \int_{|X-Y|\leq 1} \frac{|w^\theta||r_x - r_y|}{r_y|X-Y|^2} \mathrm{d}Y \\
&\quad + C r_x \int_{|X-Y|>1} \frac{|w^\theta|}{r_y|X-Y|^2} \mathrm{d}Y + C \int_{|X-Y|>1} \frac{|w^\theta||r_x - r_y|}{r_y|X-Y|^2} \mathrm{d}Y \\
&\leq C \int_{|X-Y|\leq 1} \frac{|w^\theta|}{r_y|X-Y|} \mathrm{d}Y + C \int_{|X-Y|>1} \frac{|w^\theta|}{r_y|X-Y|} \mathrm{d}Y
\end{aligned}$$

$$
\begin{aligned}
&+Cr_x \int_{|X-Y|\le 1} \frac{|w^\theta|}{r_y|X-Y|^2}\mathrm{d}Y + C\int_{|X-Y|\le 1} \frac{|w^\theta|}{r_y|X-Y|}\mathrm{d}Y\\
&+Cr_x \int_{|X-Y|> 1} \frac{|w^\theta|}{r_y|X-Y|^2}\mathrm{d}Y + C\int_{|X-Y|> 1} \frac{|w^\theta|}{r_y|X-Y|}\mathrm{d}Y\\
\le\ & 2C\int_{|X-Y|\le 1} \frac{|w^\theta|}{r_y|X-Y|}\mathrm{d}Y + Cr_x\int_{|X-Y|\le 1} \frac{|w^\theta|}{r_y|X-Y|^2}\mathrm{d}Y\\
&+2C\int_{|X-Y|> 1} \frac{|w^\theta|}{r_y|X-Y|}\mathrm{d}Y + Cr_x\int_{|X-Y|> 1} \frac{|w^\theta|}{r_y|X-Y|^2}\mathrm{d}Y\\
=\ & \sum_{i=1}^{4} \mathrm{I}^i,
\end{aligned}
\tag{3.9}
$$

where we used the fact $|r_x - r_y| \le |X-Y|$ in above inequalities. Therefore, by using of Young's inequality for convolutions, it holds that

$$
\begin{aligned}
&\|\mathrm{I}^1\|_{L^p(B_R\times[-R,R])} + \|\mathrm{I}^2\|_{L^p(B_R\times[-R,R])}\\
&\le C\|\frac{\chi_{\{|x|\le 1\}}}{|x|}\|_{L^1(\mathbb{R}^3)}\|\frac{w^\theta}{r}\|_{L^p(\mathbb{R}^3)} + CR\|\frac{\chi_{\{|x|\le 1\}}}{|x|^2}\|_{L^1(\mathbb{R}^3)}\|\frac{w^\theta}{r}\|_{L^p(\mathbb{R}^3)}\\
&\le C(R+1)\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}
\end{aligned}
\tag{3.10}
$$

for any $p \in (1,\infty)$ and cut-off function χ_A with compact support set A.

Regarding the left terms, by applying Hölder inequality and Young's inequality for convolutions, it follows that

$$
\begin{aligned}
&\|\mathrm{I}^3\|_{L^p(B_R\times[-R,R])} + \|\mathrm{I}^4\|_{L^p(B_R\times[-R,R])}\\
&\le CR^2\|\mathrm{I}^3\|_{L^{3p}(B_R\times[-R,R])} + CR\|\mathrm{I}^4\|_{L^{\frac{3p}{2}}(B_R\times[-R,R])}\\
&\le CR^2\|\frac{\chi_{\{|x|>1\}}}{|x|}\|_{L^{3p}(\mathbb{R}^3)}\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)} + CR^2\|\frac{\chi_{\{|x|>1\}}}{|x|^2}\|_{L^{\frac{3p}{2}}(\mathbb{R}^3)}\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)}\\
&\le CR^2\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)}.
\end{aligned}
\tag{3.11}
$$

Finally, by summing up (3.9)–(3.11), one can finish all the proof. □

Subsequently, we get to establish the $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ estimates of ∇u in terms of w_θ. According to Proposition 2.20 in Majda and Bertozzi (2002), the gradient of velocity fields can be expressed in terms of its vorticity by

$$
[\nabla u]h = [\mathcal{P}w]h + \frac{1}{3}w \times h. \tag{3.12}
$$

Here $\mathcal{P}$ is a singular integral operator of Calderón–Zygmund type which is generated by a homogeneous kernel of degree -3 (see Kato 1972) and h is a vector fields. Moreover, the explicit form of $[\mathcal{P}w]h$ is

$$[\mathcal{P}w]h = -P.V. \int_{\mathbb{R}^3} \left(\frac{1}{4\pi} \frac{w(y) \times h}{|x-y|^3} + \frac{3}{4\pi} \frac{\{[(x-y) \times w(y)] \otimes (x-y)\}\, h}{|x-y|^5} \right) \mathrm{d}y. \tag{3.13}$$

Therefore, with the help of (3.12) and (3.13), we are in the position to build up the following estimates.

Proposition 3.2 ($\|\nabla u\|_{L^p_{\mathrm{loc}}(\mathbb{R}^3)}$ estimates) *Assume that u is a smooth axisymmetric velocity fields with divergence free and zero swirl component, then for any $p \in (1, \infty)$, there holds*

$$\|\nabla u\|_{L^p(B_R \times [-R,R])} \leq C_R \|\frac{w^\theta}{r}\|_{L^1 \cap L^p(\mathbb{R}^3)},$$

where $B_R = B_R(0) \subset \mathbb{R}^2$ be a 2D ball and the constant C_R depends only on R.

Proof Thanks to (3.12), it is clear that $\|\nabla u\|_{L^p(\mathbb{R}^3)} \simeq \sum_i \|[\nabla u]e_i\|_{L^p(\mathbb{R}^3)}$ holds for any $p \in (1, \infty)$, where $e_i (i = r, \theta, z)$ is the orthogonal basis in (2.1). Then, by setting $\chi(r, z)$ be a smooth cut-off function such that $\chi(r, z) = 1$ in $B_{2R} \times [-2R, 2R]$, and $\operatorname{supp} \chi \subset B_{3R} \times [-3R, 3R]$, we can split $[\nabla u]e_i$ into three parts as

$$\begin{aligned} [\nabla u]e_i &= [\mathcal{P}(\chi w)]\, e_i + [\mathcal{P}\{(1-\chi)w\}]\, e_i + \frac{1}{3} w \times e_i \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Because $\mathcal{P}$ is a singular operator of Calderón–Zygmund type, by the Calderón–Zygmund inequality for $p \in (1, \infty)$, it is clear that

$$\begin{aligned} &\|\mathrm{I}\|_{L^p(B_R \times [-R,R])} + \|\mathrm{III}\|_{L^p(B_R \times [-R,R])} \\ &\leq C\| [\mathcal{P}(\chi w)] \|_{L^p(\mathbb{R}^3)} + C\|w\|_{L^p(B_R \times [-R,R])} \\ &\leq C\|w_\theta\|_{L^p(B_{2R} \times [-2R,2R])} \\ &\leq CR\|\frac{w_\theta}{r}\|_{L^p(\mathbb{R}^3)}. \end{aligned} \tag{3.14}$$

As for the second term, by (3.13), we have

$$\mathrm{II} = -P.V. \int_{\mathbb{R}^3} \left(\frac{1}{4\pi} \frac{g(y) \times e_i}{|x-y|^3} + \frac{3}{4\pi} \frac{\{[(x-y) \times g(y)] \otimes (x-y)\}\, e_i}{|x-y|^5} \right) \mathrm{d}y,$$

where $g(y) = (1 - \chi(y))w(y)$. In addition, as $\operatorname{supp}(1 - \chi(y)) \subset \mathbb{R}^3 \setminus B_{2R} \times [-2R, 2R]$, it is clear that $|x - y| \geq |y| - |x| \geq R$ for $x \in B_R \times [-R, R]$ and $y \in \mathbb{R}^3 \setminus B_{2R} \times [-2R, 2R]$. Therefore, for $x \in B_R \times [-R, R]$, there holds

$$
\begin{aligned}
|\mathrm{II}| &\leq C \int_{|x-y|\geq R} \frac{|w_\theta(y)|}{|x-y|^3} \mathrm{d}y \\
&\leq C r_x \int_{|x-y|\geq R} \frac{|w_\theta(y)|}{r_y|x-y|^3} \mathrm{d}y + C \int_{|x-y|\geq R} \frac{|w_\theta(y)||r_x - r_y|}{r_y|x-y|^3} \mathrm{d}y \\
&\leq C r_x \int_{|x-y|\geq R} \frac{|w_\theta(y)|}{r_y|x-y|^3} \mathrm{d}y + C \int_{|x-y|\geq R} \frac{|w_\theta(y)|}{r_y|x-y|^2} \mathrm{d}y \\
&\leq \frac{C}{R^2} \|\frac{w_\theta}{r}\|_{L^1(\mathbb{R}^3)},
\end{aligned}
$$

which further implies, after utilizing some basic calculations, that

$$
\|\mathrm{II}\|_{L^p(B_R\times[-R,R])} \leq CR\|\frac{w_\theta}{r}\|_{L^1(\mathbb{R}^3)}. \tag{3.15}
$$

Thus, we can finish the proof by adding up (3.14) and (3.15). □

3.2 $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ ($p > 2$) Estimates

As stated in the introduction, to prove the global existence of weak solutions, we need the strong convergence of approximate solutions in $L^2(0, T; L^2_{\mathrm{loc}}(\mathbb{R}^3))$. Although we have built up the $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^3)$ ($p > 1$) estimates of velocity fields, it only implies the strong convergence of approximate solutions in $L^2(0, T; Q)$ for any $Q \subset\subset \mathbb{R}^3\backslash\{x \in \mathbb{R}^3 | r = 0\}$, other than $L^2(0, T; L^2_{\mathrm{loc}}(\mathbb{R}^3))$.

To solve this gap, we will focus on establishing the estimates of velocity fields *stronger* than $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. The first step is to achieve the $L^p_{\mathrm{loc}}(\mathbb{R}^2_+)$ ($p > 1$) estimates for $\tilde{u}$, which is a *new* ingredient in this paper.

Lemma 3.2 ($\|\tilde{u}\|_{L^p_{\mathrm{loc}}(\mathbb{R}^2_+)}$ estimates) *Suppose $u = u_r(r, z, t)e_r + u_z(r, z, t)e_z$ is a smooth axisymmetric velocity fields without swirl satisfying $\nabla \cdot u = 0$ and let $\tilde{u} = (u_r, u_z)$, then the estimates*

$$
\|\tilde{u}\|_{L^p([0,R]\times[-R,R])} \leq C_R\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}
$$

hold for any $p \in (1, \infty)$ and the constant C_R depending only on R.

Proof Firstly, with the help of the estimates of $\|\frac{u_r}{r}\|_{L^p(B_R\times[-R,R])}$ in Proposition 3.2 and noticing $p > 1$, it is clear that

$$
\begin{aligned}
&\|u_r\|_{L^p([0,R]\times[-R,R])} \\
&= \left[\frac{1}{2\pi}\int_{-R}^{R}\int_0^R\int_{-\pi}^{\pi} |\frac{u_r}{r}|^p r^{p-1} r \mathrm{d}\theta \mathrm{d}r \mathrm{d}z\right]^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq CR^{1-\frac{1}{p}}\|\frac{u_r}{r}\|_{L^p(B_R\times[-R,R])}\\
&\leq CR^{1-\frac{1}{p}}\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}. \qquad (3.16)
\end{aligned}
$$

Regarding the estimates of $\|u_z\|_{L^p([0,R]\times[-R,R])}$, by Proposition 2.2, there holds that $|u_z|\leq|\partial_r\psi|+|\frac{\psi}{r}|$. Then, we will estimate the two terms by different ways. For the first term, by similar skills as in (3.16) and Corollary 3.1, it follows that

$$
\|\partial_r\psi\|_{L^p([0,R]\times[-R,R])}\leq CR^{1-\frac{1}{p}}\|\frac{\partial_r\psi}{r}\|_{L^p(B_R\times[-R,R])}
$$

and

$$
\begin{aligned}
|\frac{\partial_r\psi}{r}| &\leq C\int_{\mathbb{R}^3}\frac{|w^\theta|}{r_y|X-Y|^2}\mathrm{d}Y\\
&\leq C\int_{|X-Y|\leq 1}\frac{|w^\theta|}{r_y|X-Y|^2}\mathrm{d}Y + C\int_{|X-Y|>1}\frac{|w^\theta|}{r_y|X-Y|^2}\mathrm{d}Y\\
&= I_1+I_2. \qquad (3.17)
\end{aligned}
$$

Then, by making use of Young's inequality for convolutions, we finally deduce that

$$
\begin{aligned}
&\|\partial_r\psi\|_{L^p([0,R]\times[-R,R])}\\
&\leq CR^{1-\frac{1}{p}}\|I_1\|_{L^p(B_R\times[-R,R])}+CR\|I_2\|_{L^{\frac{3p}{2}}(B_R\times[-R,R])}\\
&\leq CR^{1-\frac{1}{p}}\|\frac{\chi_{\{|x|\leq 1\}}}{|x|^2}\|_{L^1(\mathbb{R}^3)}\|\frac{w^\theta}{r}\|_{L^p(\mathbb{R}^3)}+CR\|\frac{\chi_{\{|x|>1\}}}{|x|^2}\|_{L^{\frac{3p}{2}}(\mathbb{R}^3)}\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)}\\
&\leq C(R+1)\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)} \qquad (3.18)
\end{aligned}
$$

for any $p\in(1,\infty)$ and cut-off function χ_A with compact support set A. As for the other term, by using the notation $\tilde{X}=(r_x,z_x)$ and Corollary 3.1, we firstly obtain

$$
\begin{aligned}
|\frac{\psi}{r}| &\leq C\int_{\mathbb{R}^3}\frac{|w^\theta|}{r_y|X-Y|}\mathrm{d}Y\\
&= C\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\pi}^{\pi}\frac{|w^\theta|}{\sqrt{r_x^2+r_y^2-2r_xr_y\cos\theta_y+\left(z_x-z_y\right)^2}}\mathrm{d}r_y\mathrm{d}\theta\mathrm{d}z_y\\
&\leq 2\pi C\int_{-\infty}^{\infty}\int_0^{\infty}\frac{|w^\theta|}{\sqrt{\left(r_x-r_y\right)^2+\left(z_x-z_y\right)^2}}\mathrm{d}r_y\mathrm{d}z_y\\
&= 2\pi C\int_{-\infty}^{\infty}\int_0^{\infty}\frac{|w^\theta|}{|\tilde{X}-\tilde{Y}|}\mathrm{d}r_y\mathrm{d}z_y
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_{-\infty}^{\infty}\int_{0}^{2R}\frac{|w^\theta|}{|\tilde X-\tilde Y|}\mathrm{d}r_y\mathrm{d}z_y + C\int_{-\infty}^{\infty}\int_{2R}^{\infty}\frac{|w^\theta|}{|\tilde X-\tilde Y|}\mathrm{d}r_y\mathrm{d}z_y\\
&\le C\int_{\mathbb{R}^2}\frac{|w^\theta|\chi_{\{0<r_y<2R\}}}{|\tilde X-\tilde Y|}\mathrm{d}\tilde Y + C\int_{-\infty}^{\infty}\int_{2R}^{\infty}\frac{|w^\theta|}{|\tilde X-\tilde Y|}\mathrm{d}r_y\mathrm{d}z_y\\
&\le C\int_{|\tilde X-\tilde Y|\le 1}\frac{|w^\theta|\chi_{\{0\le r_y<2R\}}}{|\tilde X-\tilde Y|}\mathrm{d}\tilde Y + C\int_{|\tilde X-\tilde Y|>1}\frac{|w^\theta|\chi_{\{0\le r_y<2R\}}}{|\tilde X-\tilde Y|}\mathrm{d}\tilde Y\\
&+C\int_{-\infty}^{\infty}\int_{2R}^{\infty}\frac{|w^\theta|}{|\tilde X-\tilde Y|}\mathrm{d}r_y\mathrm{d}z_y\\
&= I_3+I_4+I_5, \qquad (3.19)
\end{aligned}
$$

where we used the fact that $w_\theta = 0$ on the axis of symmetry $r = 0$ in the fourth inequality. Then, for any $0 \le r_x < R$ and $r_y > 2R$, it clear holds $|\tilde X - \tilde Y| > R$ and then $I_5 \le \frac{C}{R}\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)}$. Thus, by applying Young's inequality for convolutions, we have

$$
\begin{aligned}
&\|\frac{\psi}{r}\|_{L^p([0,R]\times[-R,R])}\\
&\le C\|I_3\|_{L^p([0,R]\times[-R,R])} + CR^{\frac{1}{p}}\|I_4\|_{L^{2p}([0,R]\times[-R,R])} + C\|I_5\|_{L^p([0,R]\times[-R,R])}\\
&\le C\|I_3\|_{L^p(\mathbb{R}^2)} + CR^{\frac{1}{p}}\|I_4\|_{L^{2p}(\mathbb{R}^2)} + CR\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)}\\
&\le C\|\frac{\chi_{\{|x|\le 1\}}}{|x|}\|_{L^1(\mathbb{R}^2)}\|w^\theta\chi_{\{0<r<2R\}}\|_{L^p(\mathbb{R}^2)}\\
&+CR^{\frac{1}{p}}\|\frac{\chi_{\{|x|>1\}}}{|x|}\|_{L^{2p}(\mathbb{R}^2)}\|w^\theta\chi_{\{0<r<2R\}}\|_{L^1(\mathbb{R}^2)} + CR\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)}\\
&\le CR^{1-\frac{1}{p}}\|\frac{w^\theta}{r}\|_{L^p(\mathbb{R}^3)} + CR^{\frac{1}{p}}\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)} + CR\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)}\\
&\le C(R+1)\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}, \qquad (3.20)
\end{aligned}
$$

which together with (3.18) further implies

$$
\|u_z\|_{L^p([0,R]\times[-R,R])} \le C(R+1)\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}. \qquad (3.21)
$$

In the end, we can finish all the proof by adding up (3.16) and (3.21). □

Thanks to Lemma 3.3 and by fully exploiting the structure of axisymmetric flows without swirl, we then build up the following estimates *stronger* than $L^2_{\mathrm{loc}}(\mathbb{R}^3)$.

Proposition 3.3 ($\|u\|_{L^{\frac{2p}{2-p}}_{\mathrm{loc}}(\mathbb{R}^3)}$ estimates) *Let u be a smooth axisymmetric velocity fields without swirl as in Lemma 3.2, then the estimates*

$$
\|u\|_{L^{\frac{2p}{2-p}}(B_R\times[-R,R])} \le C_R\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}
$$

hold for any $1 < p < 2$. *Here* $B_R = B_R(0) \subset \mathbb{R}^2$ *is a 2D ball and the constant* C_R *depending only on* R.

Proof **Step 1:** $\mathbf{u_r} \in \mathbf{L}^{\frac{2p}{2-p}}_{\mathrm{loc}}(\mathbb{R}^3)$ Thanks to the Sobolev embedding inequality $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2_+) \hookrightarrow L^{\frac{2p}{2-p}}_{\mathrm{loc}}(\mathbb{R}^2_+)$ for any $1 < p < 2$, and the equality

$$\|r^{\frac{2-p}{2p}} u_r\|_{L^{\frac{2p}{2-p}}([0,R]\times[-R,R])} = 2\pi^{-\frac{2-p}{2p}} \|u_r\|_{L^{\frac{2p}{2-p}}(B_R\times[-R,R])},$$

to prove $u_r \in L^{\frac{2p}{2-p}}_{\mathrm{loc}}(\mathbb{R}^3)$, it suffices to verify $r^{\frac{2-p}{2p}} u_r \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2_+)$. First of all, we certify $r^{\frac{2-p}{2p}} u_r \in L^p([0,R]\times[-R,R])$. Through some basic calculations and Proposition 3.2, it clearly follows that

$$\begin{aligned}
&\|r^{\frac{2-p}{2p}} u_r\|_{L^p([0,R]\times[-R,R])} = \left[\frac{1}{2\pi}\int_{-R}^{R}\int_0^R\int_{-\pi}^{\pi} |\frac{u_r}{r}|^p r^{\frac{p}{2}} r \mathrm{d}\theta \mathrm{d}r \mathrm{d}z\right]^{\frac{1}{p}} \\
&\le C R^{\frac{1}{2}} \|\frac{u_r}{r}\|_{L^p(B_R\times[-R,R])} \le C_R \|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}.
\end{aligned} \tag{3.22}$$

In the second stage, we demonstrate $\partial_r\big(r^{\frac{2-p}{2p}} u_r\big) \in L^p([0,R]\times[-R,R])$. To achieve this goal, we decompose it into two terms by $\partial_r\big(r^{\frac{2-p}{2p}} u_r\big) = \partial_r\big(\frac{u_r}{r} r^{\frac{2+p}{2p}}\big) = \partial_r\big(\frac{u_r}{r}\big) r^{\frac{2+p}{2p}} + \frac{2+p}{2p}\big(\frac{u_r}{r}\big) r^{\frac{2-p}{2p}}$ and estimate them separately. Again by some basic calculations and borrowing (2.7) in Lemma 2.1, we have

$$\begin{aligned}
&\|r^{\frac{2+p}{2p}} \partial_r\left(\frac{u_r}{r}\right)\|_{L^p([0,R]\times[-R,R])} = \left[\frac{1}{2\pi}\int_{-R}^{R}\int_0^R\int_{-\pi}^{\pi} |\partial_r\left(\frac{u_r}{r}\right)|^p r^{\frac{p}{2}} r \mathrm{d}\theta \mathrm{d}r \mathrm{d}z\right]^{\frac{1}{p}} \\
&\le C R^{\frac{1}{2}} \|\partial_r\left(\frac{u_r}{r}\right)\|_{L^p(B_R\times[-R,R])} \\
&\le C R^{\frac{1}{2}} \|\frac{w^\theta}{r}\|_{L^p(\mathbb{R}^3)}.
\end{aligned} \tag{3.23}$$

The other term can be estimated by Hölder inequality and Lemma 2.2, that is

$$\begin{aligned}
&\|\frac{2+p}{2p}\left(\frac{u_r}{r}\right) r^{\frac{2-p}{2p}}\|_{L^p([0,R]\times[-R,R])} \le \left[\frac{1}{2\pi}\int_{-R}^{R}\int_0^R\int_{-\pi}^{\pi} |\frac{u_r}{r}|^p r^{-\frac{p}{2}} r \mathrm{d}\theta \mathrm{d}r \mathrm{d}z\right]^{\frac{1}{p}} \\
&\le \left[\frac{1}{2\pi}\int_{-R}^{R}\int_0^R\int_{-\pi}^{\pi} |\frac{u_r}{r}|^{\frac{3p}{3-p}} r \mathrm{d}\theta \mathrm{d}r \mathrm{d}z\right]^{\frac{3-p}{3}} \left[\frac{1}{2\pi}\int_{-R}^{R}\int_0^R\int_{-\pi}^{\pi} r^{-\frac{3}{2}} r \mathrm{d}\theta \mathrm{d}r \mathrm{d}z\right]^{\frac{1}{3}} \\
&\le C R^{\frac{1}{3}} \|\frac{u_r}{r}\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)} \left[\int_0^R r^{-\frac{1}{2}} dr\right]^{\frac{1}{3}} \\
&\le C R^{\frac{1}{2}} \|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}.
\end{aligned} \tag{3.24}$$

Regarding the term $\partial_z\big(r^{\frac{2-p}{2p}}u_r\big)$, due to $\partial_z\big(r^{\frac{2-p}{2p}}u_r\big)=\partial_z\big(\frac{u_r}{r}\big)r^{\frac{2+p}{2p}}$, the way to estimate it would be along the same line with $\partial_r\big(\frac{u_r}{r}\big)r^{\frac{2+p}{2p}}$ in (3.23) and we will omit it here to avoid repetition.

Step 2: $\mathbf{u_z}\in \mathbf{L}_{\mathrm{loc}}^{\frac{2p}{2-p}}(\mathbb{R}^3)$ Through recalling Proposition 2.2, it is clear that

$$u_z=\partial_r\psi+\frac{\psi}{r}=r\partial_r\left(\frac{\psi}{r}\right)+\frac{2\psi}{r} \tag{3.25}$$

and we will deal with the two terms by different methods. For the term $\frac{\psi}{r}$, we will estimate it by straightforward calculations. According to Corollary 3.1, it yields

$$\begin{aligned}|\frac{\psi}{r}|&\le C\int_{\mathbb{R}^3}\frac{|w^\theta|}{r_y|X-Y|}\mathrm{d}Y\\ &\le C\int_{|X-Y|\le 1}\frac{|w^\theta|}{r_y|X-Y|}\mathrm{d}Y+C\int_{|X-Y|>1}\frac{|w^\theta|}{r_y|X-Y|}\mathrm{d}Y\\ &=I_1+I_2,\end{aligned} \tag{3.26}$$

which further implies, after making use of Hölder inequality in bounded domain $B_R\times[-R,R]$ and Young's inequality for convolutions, that

$$\begin{aligned}\|\frac{\psi}{r}\|_{L^{\frac{2p}{2-p}}(B_R\times[-R,R])}&\le C\|I_1\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^3)}+CR^{\frac{6-3p}{4p}}\|I_2\|_{L^{\frac{4p}{2-p}}(\mathbb{R}^3)}\\ &\le C\|\frac{\chi_{\{|x|\le1\}}}{|x|}\|_{L^2(\mathbb{R}^3)}\|\frac{w^\theta}{r}\|_{L^p(\mathbb{R}^3)}+C(R+1)\|\frac{\chi_{\{|x|>1\}}}{|x|}\|_{L^{\frac{4p}{2-p}}(\mathbb{R}^3)}\|\frac{w^\theta}{r}\|_{L^1(\mathbb{R}^3)}\\ &\le C(R+1)\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}\end{aligned} \tag{3.27}$$

for $1<p<2$. In the above inequalities, we have used $\frac{1}{4}<\frac{6-3p}{4p}<\frac{3}{4}$ and $\frac{4p}{2-p}>4$. As for the other term $r\partial_r\big(\frac{\psi}{r}\big)$, our strategy is to testify $r\partial_r\big(\frac{\psi}{r}\big)\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2_+)$, which is based on the inequality

$$\|r\partial_r\left(\frac{\psi}{r}\right)\|_{L_{\mathrm{loc}}^{\frac{2p}{2-p}}(\mathbb{R}^3)}\le C\|r\partial_r\left(\frac{\psi}{r}\right)\|_{L_{\mathrm{loc}}^{\frac{2p}{2-p}}(\mathbb{R}_2^+)}$$

and the Sobolev embedding inequality $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2_+)\hookrightarrow L_{\mathrm{loc}}^{\frac{2p}{2-p}}(\mathbb{R}^2_+)$ for any $1<p<2$. To start with, we recall (3.25) that $r\partial_r\big(\frac{\psi}{r}\big)=u_z-\frac{2\psi}{r}$. Effectively, in Lemma 3.2, we have proved $u_z\in L^p_{\mathrm{loc}}(\mathbb{R}^2_+)$. Besides, the $L^p_{\mathrm{loc}}(\mathbb{R}^2_+)$ estimates of $\frac{\psi}{r}$ have been established in (3.20), that can be summarized in the following estimates

$$\|r\partial_r\left(\frac{\psi}{r}\right)\|_{L^p([0,R]\times[-R,R])}\le C(R+1)\|\frac{w^\theta}{r}\|_{L^1\cap L^p(\mathbb{R}^3)}. \tag{3.28}$$

In the next stage, to prove $\tilde{\nabla}\big[r\partial_r\big(\frac{\psi}{r}\big)\big] \in L^p_{\rm loc}(\mathbb{R}^2_+)$, we will do some decompositions, which thereby make Lemma 2.1 effective. More precisely, we will prove $\partial_r\big(r\partial_r\big(\frac{\psi}{r}\big)\big) = r\partial_r^2\big(\frac{\psi}{r}\big) + \partial_r\big(\frac{\psi}{r}\big),\ \partial_z\big(r\partial_r\big(\frac{\psi}{r}\big)\big) = r\partial_{rz}^2\big(\frac{\psi}{r}\big) \in L^p_{\rm loc}(\mathbb{R}^2_+)$. To this end, we first list the inequality

$$\|f\|_{L^p_{\rm loc}(\mathbb{R}^2_+)} \leq C\|\frac{f}{r}\|_{L^p_{\rm loc}(\mathbb{R}^3)}$$

that holds for any function $f = f(r, z, t)$. This means that it suffices to verify $\frac{1}{r}\partial_r\big(r\partial_r\big(\frac{\psi}{r}\big)\big) = \partial_r^2\big(\frac{\psi}{r}\big) + \frac{1}{r}\partial_r\big(\frac{\psi}{r}\big),\ \frac{1}{r}\partial_z\big(r\partial_r\big(\frac{\psi}{r}\big)\big) = \partial_{rz}^2\big(\frac{\psi}{r}\big) \in L^p_{\rm loc}(\mathbb{R}^3)$, which certainly holds according to Lemma 2.1. Thus, we finish all the proof. □

Thus, for $1 < p < 2$, we have established the $L^{\frac{2p}{2-p}}_{\rm loc}(\mathbb{R}^3)$ estimates of velocity fields. When $p \geq 2$, it is well known that the Sobolev embedding $W^{1,p}_{\rm loc}(\mathbb{R}^3) \hookrightarrow L^6_{\rm loc}(\mathbb{R}^3)$ holds, which also helps us deriving the following conclusion.

Lemma 3.3 *Let $u = u_r(r, z, t)e_r + u_z(r, z, t)e_z$ be a smooth axisymmetric velocity fields without swirl, $\frac{w_0^\theta}{r} \in L^1 \cap L^p(\mathbb{R}^3)$ with some $p > 1$, then there exists an $\alpha > 0$ depending only on p such that $u \in L^{2+\alpha}_{\rm loc}(\mathbb{R}^3)$.*

4 Global Existence of Weak Solutions

This section is devoted to the global existence of weak solutions. The first step is to construct a family of approximate solutions. To begin with, we would like to introduce the standard mollifier ρ_ϵ, which can be described by

$$\rho_\epsilon(x) = \frac{1}{\epsilon^3}\rho\left(\frac{|x|}{\epsilon}\right),$$

where $\rho \in C_0^\infty(\mathbb{R}^3),\ \rho \geq 0,\ \operatorname{supp}\rho \subset \{|x| \leq 1\}$ and $\int_{\mathbb{R}^3}\rho\,\mathrm{d}x = 1$. Then, we define a cut-off function χ_ϵ by

$$\chi_\epsilon(x) = \chi\left(\frac{|x|}{\epsilon}\right),$$

where $\chi \in C_0^\infty(\mathbb{R}^3),\ 0 \leq \chi \leq 1$, and $\chi(x) = 1$ on $\{|x| \leq 1\}$, $\chi(x) = 0$ on $\{|x| \geq 2\}$. Through borrowing these definitions, we then drive the following theorem.

Theorem 4.1 *Given an initial data $w_0 = w_0^\theta e_\theta$ such that $\frac{w_0^\theta}{r} \in L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 1$, then there exists a family of smooth axisymmetric solutions u^ϵ with zero swirl component and initial data u_0^ϵ for any $T > 0$. Here, $w_0^\epsilon(x) = \rho_\epsilon * w_0(x)$ and $u_0^\epsilon = \nabla \times (-\Delta)^{-1}w_0^\epsilon$. In addition, it holds that*

$$\|u^\epsilon\|_{W^{1,p}(B_R\times[-R,R])} \leq C_R \tag{4.1}$$

and

$$\|u^\epsilon\|_{L^{2+\alpha}(B_R\times[-R,R])} \le C_R, \tag{4.2}$$

where $B_R = B_R(0) \subset \mathbb{R}^2$ is a 2D ball, α be as in Lemma 3.3 and C_R is the constant depending only on R.

Proof Initially, we construct

$$w_0^\epsilon = \chi_\epsilon(x)\,(\rho_\epsilon * w_0)\,(x).$$

According to our construction for initial data, it is clear that w_0^ϵ is axisymmetric. Then, we denote by u_0^ϵ the corresponding velocity fields determined by the *Biot–Savart law*, namely $u_0^\epsilon = \nabla\times(-\Delta)^{-1}w_0^\epsilon$. Again by our assumptions on the initial data, $\nabla\times u_0^\epsilon = w_0^\epsilon$ has only swirl component $w_\theta^\epsilon(x,0)$ such that $w_0^\epsilon = w_\theta^\epsilon(x,0)e_\theta$. Therefore, it is clear to conclude that u_0^ϵ has zero swirl component, i.e., $u_\theta^\epsilon(x,0) = 0$. Moreover, $u_0^\epsilon \in C^\infty(\mathbb{R}^3)$ and belongs to the space $V = \{u \in H^3(\mathbb{R}^3)|\,\nabla\cdot u = 0\}$.

Subsequently, by Majda and Bertozzi (2002), there exists a unique global smooth solution u^ϵ. What is more, considering that u_0^ϵ is axisymmetric, the Euler equations keep invariant under the rotation and translation transformations and the uniqueness of solutions, it is obvious that the velocity fields u^ϵ is still axisymmetric. Besides, the swirl component u_θ^ϵ is also vanishing due to its initial data $u_{0,\theta}^\epsilon$ given zero.

Finally, we recall a well-known conclusion that

$$\|\frac{w_0^\epsilon}{r}\|_{L^p(\mathbb{R}^3)} \le \|\frac{\rho_\epsilon * w_0{}^\theta}{r}\|_{L^p(\mathbb{R}^3)} \le C\|\frac{w_0^\theta}{r}\|_{L^p(\mathbb{R}^3)}, \quad \forall p\in[1,\infty], \tag{4.3}$$

whose proof can be referred to Lemma A.1 in Ben Ameur and Danchin (2002). Thus, through evoking the transport Eq. (2.5) satisfied by $\frac{w_\theta^\epsilon}{r}$, applying (2.6) and (4.3), we can conclude that $\|\frac{w_\theta^\epsilon}{r}\|_{L^1\cap L^p(\mathbb{R}^3)} \le C$. This together with Proposition 3.1–3.3 leads to (4.1) and (4.2). □

As discussed in the introduction, to prove the main theorem, it suffices to build up the strong convergence of approximate solutions in the space $L^2(0,T;L^2_{\text{loc}}(\mathbb{R}^3))$. Based on it, for the approximate solutions we constructed, one can then take the limit in the sense of Definition 1.1, which is essential in establishing the global existence of weak solutions. In the end, with the help of *a priori estimates* in Proposition 3.1–3.3, we get to prove our main theorem as follow.

Proof of Theorem 1.1 As stated in the introduction, for any $p > 1$, the $W^{1,p}_{\text{loc}}(\mathbb{R}^3)$ estimates of velocity fields cannot guarantee the strong convergence of approximate solutions in $L^2(0,T;L^2_{\text{loc}}(\mathbb{R}^3))$, but in $L^2(0,T;Q)$ for any $Q \subset\subset \mathbb{R}^3\backslash\{x\in\mathbb{R}^3|r=0\}$. Hence, we will verify the strong convergence by dividing any local domain of $\mathbb{R}^3$ into two parts: the region near the axis of symmetry, and the region away from it. On the one hand, thanks to Lemma 3.3, for the approximate solutions constructed in Theorem 4.1, there exists u such that

$$u^{\epsilon} \rightharpoonup u \quad \text{in} \quad L^{\infty}\left(0, T; L^{2+\alpha}\left(B_R \times [-R, R]\right)\right). \tag{4.4}$$

On the other hand, for the region $C_R \times [-R, R] = \{(x, y) \in \mathbb{R}^2 | \frac{1}{R} \leq \sqrt{x^2 + y^2} \leq R\} \times [-R, R]$, it clearly holds $\|u^{\epsilon}\|_{L^{\infty}(0,T;W^{1,p}(C_R \times [-R,R]))} \leq C_R$ by Theorem 4.1. Then, by using Eq. (1.1)[1], it further holds $\|\partial_t u^{\epsilon}\|_{L^{\infty}(0,T;W^{-1,p^*}(C_R \times [-R,R]))} \leq C_R$, where $p^* = \frac{p}{p-1}$. Then, by noticing that $|u|$ is a function of variables r, z and t, one can conclude that

$$\|u^{\epsilon}\|_{L^{\infty}\left(0,T;W^{1,p}\left([\frac{1}{R},R]\times[-R,R];\mathrm{d}r\mathrm{d}z\right)\right)} + \|\partial_t u^{\epsilon}\|_{L^{\infty}\left(0,T;W^{-1,p^*}\left([\frac{1}{R},R]\times[-R,R];\mathrm{d}r\mathrm{d}z\right)\right)} \leq C_R.$$

Next, by applying the Aubin–Lions lemma and Sobolev compact embedding $W^{1,p}([\frac{1}{R}, R] \times [-R, R]) \hookrightarrow L^2([\frac{1}{R}, R] \times [-R, R])$ for any $p > 1$, we can then find a subsequence u^{ϵ_j} (depending on R) such that

$$u^{\epsilon_j} \to \bar{u} \quad \text{in} \quad L^2\left(0, T; \left([\frac{1}{R}, R] \times [-R, R]; \mathrm{d}r\mathrm{d}z\right)\right).$$

Then, by the diagonal selection process, one can then extract a subsequence of u^{ϵ_j} independent of R (still denoted by u^{ϵ_j}) such that

$$\|u^{\epsilon_j} - \bar{u}\|_{L^2\left(0,T;\left([\frac{1}{R},R]\times[-R,R];\mathrm{d}r\mathrm{d}z\right)\right)} \to 0 \quad \text{as} \quad \epsilon_j \to 0,$$

which also implies that

$$\|u^{\epsilon_j} - \bar{u}\|_{L^2(0,T;C_R\times[-R,R])} \to 0 \quad \text{as} \quad \epsilon_j \to 0.$$

This means $u^{\epsilon_j} \to \bar{u}$ in $L^2(0, T; Q)$, for any $Q \subset\subset B_R \times [-R, R] \backslash \{x \in \mathbb{R}^3 | r = 0\}$. Then by considering the uniqueness of limits and (4.4), we actually have derived

$$u^{\epsilon_j} \to u \quad \text{in} \quad L^2(0, T; Q). \tag{4.5}$$

Now, it suffices to verify the strong convergence of velocity fields in $L^2(0, T; B_R \times [-R, R])$. For any $\epsilon > 0$, we firstly take $Q \subset\subset B_R \times [-R, R] \backslash \{x \in \mathbb{R}^3 | r = 0\}$ such that the measure $\mu(B_R \times [-R, R] \backslash Q) < \left(\frac{\epsilon}{4\sqrt{2T}C_R}\right)^{\frac{4+2\alpha}{\alpha}}$ for $\alpha > 0$ in Lemma 3.3. Then, according to (4.5), there exists a constant M such that when $j > M$, $\|u^{\epsilon_j} - u\|_{L^2(0,T;Q)} < \frac{\epsilon}{2}$. Thus, by employing Hölder inequality, (4.2) and (4.4), for $j > M$, one further has

$$\begin{aligned} &\left[\int_0^T \int_{B_R\times[-R,R]} |u^{\epsilon_j} - u|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}} \\ &\leq \left[\int_0^T \int_{B_R\times[-R,R]\backslash Q} |u^{\epsilon_j} - u|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}} + \left[\int_0^T \int_Q |u^{\epsilon_j} - u|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{2T}\left[\int_{B_R\times[-R,R]\backslash Q}|u^{\epsilon_j}|^2\mathrm{d}x+\int_{B_R\times[-R,R]\backslash Q}|u|^2\mathrm{d}x\right]^{\frac{1}{2}}+\frac{\epsilon}{2}\\
&\leq \sqrt{2T}\left[\|u^{\epsilon_j}\|_{L^{2+\alpha}(B_R\times[-R,R])}+\|u\|_{L^{2+\alpha}(B_R\times[-R,R])}\right]\\
&\quad\times[\mu(B_R\times[-R,R]\backslash Q)]^{\frac{\alpha}{4+2\alpha}}+\frac{\epsilon}{2}\\
&<\epsilon.
\end{aligned}
$$

Until now, we actually have proved that there exists an axisymmetric velocity fields u without swirl, such that

$$
u^{\epsilon_j} \to u \quad \text{strongly} \quad \text{in} \quad L^2\left(0, T; L^2_{\mathrm{loc}}(\mathbb{R}^3)\right).
$$

The last step is to pass limit in the equations (1.1) satisfied by u^ϵ. As a matter of fact, it suffices to show the convergence of nonlinear term. Considering that $u^{\epsilon_j} \to u$ strongly in $L^2(0, T; L^2_{\mathrm{loc}}(\mathbb{R}^3))$, it is not hard to infer that

$$
\int_0^{\mathrm{T}}\int_{\mathbb{R}^3} u^{\epsilon_j}\cdot\nabla\varphi\cdot u^{\epsilon_j}\,\mathrm{d}x\mathrm{d}t \to \int_0^{\mathrm{T}}\int_{\mathbb{R}^3} u\cdot\nabla\varphi\cdot u\,\mathrm{d}x\mathrm{d}t
$$

for any $\varphi \in C_0^\infty([0, T); \mathbb{R}^3)$ with $\nabla\cdot\varphi = 0$. This shows that u is a weak solution of incompressible axisymmetric Euler equations without swirl in the sense of Definition 1.1. □

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