Journal of Nonlinear Science (2021) 31:10
https://doi.org/10.1007/s00332-020-09656-3

Traveling Wave Solutions for a Class of Discrete Diffusive SIR Epidemic Model

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Received: 29 January 2020 / Accepted: 11 November 2020 / Published online: 7 January 2021

Abstract
This paper is concerned with the conditions of existence and nonexistence of traveling wave solutions (TWS) for a class of discrete diffusive epidemic model. We find that the existence of TWS is determined by the so-called basic reproduction number and the critical wave speed: When the basic reproduction number $\Re_0 > 1$, there exists a critical wave speed $c^* > 0$, such that for each $c \geq c^*$ the system admits a nontrivial TWS and for $c < c^*$ there exists no nontrivial TWS for the system. In addition, the boundary asymptotic behavior of TWS is obtained by constructing a suitable Lyapunov functional and employing Lebesgue dominated convergence theorem. Finally, we apply our results to two discrete diffusive epidemic models to verify the existence and nonexistence of TWS.

Keywords Lattice dynamical system · Schauder's fixed point theorem · Traveling wave solutions · Diffusive epidemic model · Lyapunov functional

Mathematics Subject Classification 35C07 · 35K57 · 92D30

Communicated by Mary Pugh.

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1 Introduction

In a pioneering work, the classical susceptible-infectious-recovered (SIR) epidemic model was introduced by Kermack and McKendrick (1927) in 1927. Since then, epidemic modeling has become one of the most important tools to study spread of the disease, we refer readers to a good survey (Hethcote 2000) on this topic. In order to understand the geographic spread of infectious diseases, the spatial effect would give insights into disease spread and control. Due to this fact, epidemic models with spatial diffusion have been studied for decades. Considering spatial effects, Hosono and Ilyas 1995 proposed and studied the following SIR epidemic model with diffusion:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \Delta S(t,x) - \beta S(x,t) I(x,t), & x \in \mathbb{R},\ t > 0, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \Delta I(t,x) + \beta S(x,t) I(x,t) - \gamma I(x,t), & x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

with initial conditions

$$S(x,0) = S_0(x), \quad I(x,0) = I_0(x) > 0,$$

where $S(x,t)$ and $I(x,t)$ denote the densities of susceptible and infected individuals at position x and time t, respectively; $d_i (i = 1,2)$ are the diffusion rates of each compartment; β denotes the transmission rate between susceptible and infected individuals; γ is the remove rate. All parameters in system (1.1) are assumed to be positive. It was shown that the existence of traveling wave solutions of system (1.1) with a constant speed. Reaction–diffusion epidemic models have been investigated extensively, regarding the existence problem of traveling wave solutions, see (Bai and Zhang 2015; Ducrot and Magal 2011; Fu 2016; He and Tsai 2019; Lam et al. 2018; Li et al. 2017, 2014; Tian and Yuan 2017; Wang and Ma 2018; Zhang and Xu 2013; Zhao et al. 2017) and references therein.

However, there are relatively few works on epidemic models with discrete spatial structure. In contrast to continuous media, lattice dynamical systems are more realistic in describing the discrete diffusion (for example, patch environment (San and Wang 2019)). Lattice dynamical systems are systems of ordinary differential equations with a discrete spatial structure. Such systems arise from practical backgrounds, such as biology (Fang et al. 2010; Guo and Wu 2012; Han and Kloeden 2019; Weng et al. 2003; Wu et al. 2015; Yang and Zhang 2018), chemical reaction (Erneux and Nicolis 1993; Kapral 1991) and material science (Bates and Chmaj 1999; Brucal-Hallare and Vleck 2011). In a recent paper (Fu et al. 2016), Fu et al. studied the existence of traveling wave solutions for a lattice dynamical system arising in a discrete diffusive epidemic model:

$$\begin{cases} \dfrac{\mathrm{d}S_n(t)}{\mathrm{d}t} = [S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] - \beta S_n(t) I_n(t), & n \in \mathbb{Z}, \\ \dfrac{\mathrm{d}I_n(t)}{\mathrm{d}t} = d[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \beta S_n(t) I_n(t) - \gamma I_n(t), & n \in \mathbb{Z}, \end{cases} \tag{1.2}$$

where $S_n(t)$ and $I_n(t)$ denote the populations densities of susceptible and infectious individuals at niche n and time t, respectively; 1 and d denote the random migration coefficients for susceptible and infectious individuals, respectively; β is the transmission coefficient between susceptible and infectious individuals; γ is the recovery rate of infectious individuals. Note that system (1.2) is a spatially discrete version of system (1.1). It was proved in Fu et al. (2016) that the conditions of existence and nonexistence of traveling wave solution for system (1.2) are determined by a threshold number and the critical wave speed c^*. If the threshold number is greater than one, then there exists a traveling wave solution for any $c > c^*$ and there is no traveling wave solutions for $c < c^*$. Also, the nonexistence of traveling wave solutions for the threshold number less than 1 was derived. Furthermore, Wu 2017 studied the existence of traveling wave solutions with critical speed $c = c^*$ of system (1.2). Moreover, we refer to Zhang and Wu 2019 and Zhou et al. 2020 for the existence and nonexistence of traveling wave solutions with saturated incidence rate. By introducing the constant recruitment, Chen et al. 2017 studied the traveling wave solutions for the following discrete diffusive epidemic model:

$$\begin{cases} \dfrac{\mathrm{d}S_n(t)}{\mathrm{d}t} = [S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] + \mu - \beta S_n(t)I_n(t) - \mu S_n(t), & n \in \mathbb{Z}, \\ \dfrac{\mathrm{d}I_n(t)}{\mathrm{d}t} = d[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \beta S_n(t)I_n(t) - (\gamma + \mu)I_n(t), & n \in \mathbb{Z}, \end{cases} \tag{1.3}$$

where μ is the input rate of the susceptible population; meanwhile, the death rates of susceptible and infectious individuals are also assumed to be μ. In Chen et al. (2017), the authors showed that the existence of traveling wave solutions connects the disease-free equilibrium to the endemic equilibrium, but they do not prove that the traveling wave solutions converge to the endemic equilibrium at $+\infty$. As explained in Chen et al. (2017), the main difficulties come from the fact that (1.3) is a system and is nonlocal. In fact, the traveling wave solutions of (1.2) and (1.3) are totally different: For the system like (1.2) without constant recruitment, it can be shown that I tends to 0 as $\xi \to \pm\infty$, where $\xi = n + ct$ is the wave profile, which will be introduced in the next section; however, for the diffusive model with positive constant recruitment, it is more likely to get that $I(\xi) \to 0$ as $\xi \to -\infty$ and $I(\xi) \to I^*$ as $\xi \to +\infty$, where I^* is the positive endemic equilibrium (see Li et al. 2014 for nonlocal diffusive epidemic model; Fu 2016 for random diffusive epidemic model). Therefore, it naturally raises a question: For discrete diffusive systems, does the traveling wave solutions converge to the endemic equilibrium as $\xi \to +\infty$? This constitutes our first motivation of the present paper.

Our second motivation is the nonlinear incidence rate which plays a critical role in the epidemic modeling (Anderson and May 1991). For the discrete diffusive systems with nonlinear incidence rate, will the traveling wave solutions still converge to the endemic equilibrium as $\xi \to +\infty$? Generally, the incidence rate of an infectious disease in most of the literature is assumed to be of mass action form βSI (Anderson and May 1991). Yet the disease transmission process is generally unknown (Korobeinikov and Maini 2005), some nonlinear incidence rates have been introduced and studied, for example, the saturated incidence rate with $f(I) = \frac{I}{1+\alpha I}$ by Capasso and Serio

(1978), the saturated nonlinear incidence rate with $f(I) = \frac{I}{1+\alpha I^p}(0 < p < 1)$ by Liu et al. (1986), and so on. For more general cases, Capasso and Serio (1978) considered a more general incidence rate with the form $Sf(I)$, and the general nonlinear incidence rate could bring nontrivial challenges in analysis. Therefore, it is of great significance to study the convergence property of traveling wave solutions of the system with nonlinear incidence rate.

In this paper, we consider a discrete diffusive SIR epidemic model with general nonlinear incidence rate. The main model of this paper is formulated as the following system:

$$\begin{cases} \frac{\mathrm{d}S_n(t)}{\mathrm{d}t} = d_1[S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] + \Lambda - \beta S_n(t) f(I_n(t)) - \mu_1 S_n(t), \quad n \in \mathbb{Z}, \\ \frac{\mathrm{d}I_n(t)}{\mathrm{d}t} = d_2[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \beta S_n(t) f(I_n(t)) - \gamma I_n(t) - \mu_2 I_n(t), \quad n \in \mathbb{Z}, \\ \frac{\mathrm{d}R_n(t)}{\mathrm{d}t} = d_3[R_{n+1}(t) + R_{n-1}(t) - 2R_n(t)] + \gamma I_n(t) - \mu_1 R_n(t), \quad n \in \mathbb{Z}, \end{cases} \tag{1.4}$$

where $S_n(t)$, $I_n(t)$ and $R_n(t)$ denote the densities of susceptible, infectious and removed individuals at niche n and time t, respectively; $d_i (i = 1, 2, 3)$ is the random migration coefficients for each compartment; Λ is the input rate of susceptible individuals. The biological meaning of other parameters is the same as in model (1.3).

Since $R_n(t)$ is decoupled from other equations and denote $\mu_2 = \gamma + \mu_1$, then we only need to study the following system:

$$\begin{cases} \frac{\mathrm{d}S_n(t)}{\mathrm{d}t} = d_1[S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] + \Lambda - \beta S_n(t) f(I_n(t)) - \mu_1 S_n(t), \quad n \in \mathbb{Z}, \\ \frac{\mathrm{d}I_n(t)}{\mathrm{d}t} = d_2[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \beta S_n(t) f(I_n(t)) - \mu_2 I_n(t), \quad n \in \mathbb{Z}. \end{cases} \tag{1.5}$$

Our first goal in this paper is to study the existence and boundedness of the traveling wave solutions of model (1.4). Adopting nonlinear incidence, random and nonlocal diffusive SIR model is studied in Bai and Zhang (2015) and Zhou et al. (2018), respectively. It was shown that the existence of traveling wave solutions for each model by analyzing auxiliary system. Unlike in Bai and Zhang (2015); Zhou et al. (2018) where there is no constant recruitment, here we allow constant recruitment in model (1.4) and it is necessary to consider the boundedness of traveling wave solutions, which is different from Bai and Zhang (2015); Zhou et al. (2018). In a recent paper (Zhang et al. 2018), Zhang et al. studied a time delay nonlocal diffusive SIR model with general incidence, and they established the boundedness of the traveling wave solutions, but with an additional assumption:

(**H**) $S_0 f(I_0) - \gamma I_0 \leq 0$ for some $I_0 > 0$,

where S_0 is the disease-free steady state. This assumption has also been used in a recent literature (Zhou et al. 2020). However, we should point out here that (H) cannot be applied for some incidence, such as bilinear incidence. Hence, it comes naturally to consider the boundedness of traveling wave solution for system (1.4) without this assumption. As a result, our model could cover more special cases.

Our second goal in this paper is to study the convergence property of the traveling wave solutions of model (1.4). In proving the convergence property of traveling wave solutions for random diffusive epidemic model, the method combined with Lyapunov functional and Lebesgue dominated convergence theorem played a crucial role, see (Ducrot and Magal 2011; Fu 2016).

In general, there are two ways to construct Lyapunov functionals for the random diffusive model: (i) rewriting the random diffusive model as a system of first-order ODEs and constructing a Lyapunov function for ODE systems (Fu 2016); (ii) constructing a Lyapunov functional that contains a first derivative term (Li et al. 2015). However, the above two methods are not applicable to discrete diffusive models (1.4), and it is challenging to construct a suitable Lyapunov functional for the model with lattice structure.

In the present paper, a new Lyapunov functional will be constructed, which contains a specific functional to handle the lattice structure in the corresponding wave profile system. Additionally, the well-posedness for the Lyapunov functional will be also investigated. For the random diffusive epidemic model, the well-posedness of its Lyapunov functional could be achieved via Harnack inequality (for instance, see Li et al. 2015), but for discrete diffusive models, more deeper analysis is needed to verify the permanence of traveling wave solutions, which plays a crucial role in proving the well-posedness for the Lyapunov functional.

We make the following assumptions on function f:

(A1) $f(I) \geq 0$ and $f'(I) > 0$ for all $I \geq 0$, $f(I) = 0$ if and only if $I = 0$.
(A2) $\frac{f(I)}{I}$ is continuous and monotonously nonincreasing for all $I \geq 0$ and $\lim_{I \to 0^+} \frac{f(I)}{I}$ exists.

The conditions of Assumption (A1) and (A2) are satisfied in all the following specific incidence rates:

(i) the bilinear incidence rate with $f(I) = I$ (see Anderson and May 1991);
(ii) the saturated incidence rate with $f(I) = \frac{I}{1+\alpha I}$ (see Capasso and Serio 1978);
(iii) the saturated nonlinear incidence rate with $f(I) = \frac{I}{1+\alpha I^p}$, where $\alpha > 0$ and $0 < p < 1$ (a special case in Liu et al. (1986), see also Muroya et al. (2015));
(iv) the nonlinear incidence rate with $f(I) = \frac{I}{1+kI+\sqrt{1+2kI}}$ (see Heesterbeek and Metz 1993; Thieme 2011);
(v) the nonlinear incidence rate with $f(I) = \frac{I}{(\epsilon^\alpha+I^\alpha)^\gamma}$, where $\epsilon, \alpha, \gamma > 0$ and $\alpha\gamma < 1$ (see Thieme 2011);
(vi) the nonlinear incidence rate for pathogen transmission in infection of insects with $f(I) = k \ln\left(1 + \frac{\nu I}{k}\right)$, which could be described by epidemic model (see Briggs and Godfray 1995).

Hence, system (1.5) covers many models as special cases. Now, we introduce some results on the system (1.5) without migration, which takes the form as:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta S(t) f(I(t)) - \mu_1 S(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta S(t) f(I(t)) - \mu_2 I(t). \end{cases} \tag{1.6}$$

It is well known that the global dynamics of (1.6) is completely determined by the basic reproduction number $\Re_0 = \frac{\beta S_0 f'(0)}{\mu_2}$ (see Korobeinikov 2006): That is, if the number is less than unity, then the disease-free equilibrium $E_0 = (S_0, 0) = (\Lambda/\mu_1, 0)$ is globally asymptotically stable, while if the number is greater than unity, then a positive endemic equilibrium $E^* = (S^*, I^*)$ exists and it is globally asymptotically stable, where E^* satisfy

$$\begin{cases} \Lambda - \beta S^* f(I^*) - \mu_1 S^* = 0, \\ \beta S^* f(I^*) - \mu_2 I^* = 0. \end{cases} \tag{1.7}$$

The organization of this paper is as follows: In Sect. 2, we apply Schauder's fixed point theorem to construct a family of solutions of the truncated problem. In Sect. 3, we show the existence and boundedness of traveling wave solutions. Further, we use a Lyapunov functional to show that the convergence of traveling wave solutions at $+\infty$. In Sect. 4, we investigate the nonexistence of traveling wave solutions by using two-sided Laplace transform. At last, there is an application for our general results and a brief discussion.

2 Preliminaries

In this section, since system (1.5) does not enjoy the comparison principle, we will construct a pair upper and lower solutions and apply Schauder's fixed point theorem to investigate the existence of traveling wave solutions of system (1.5). Consider traveling wave solutions which can be expressed as bounded profiles of continuous variable $n + ct$ such that

$$S_n(t) = S(n + ct) \text{ and } I_n(t) = I(n + ct), \tag{2.1}$$

where c denotes the wave speed. Let $\xi = n + ct$, then we can rewrite system (1.5) as follows:

$$\begin{cases} cS'(\xi) = d_1 J[S](\xi) + \Lambda - \mu_1 S(\xi) - \beta S(\xi) f(I(\xi)), \\ cI'(\xi) = d_2 J[I](\xi) + \beta S(\xi) f(I(\xi)) - \mu_2 I(\xi) \end{cases} \tag{2.2}$$

for all $\xi \in \mathbb{R}$, where $J[\phi](\xi) := \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi)$. We want to find traveling wave solutions with the following asymptotic boundary conditions:

$$\lim_{\xi \to -\infty} (S(\xi), I(\xi)) = (S_0, 0), \tag{2.3}$$

and

$$\lim_{\xi \to +\infty} (S(\xi), I(\xi)) = (S^*, I^*), \tag{2.4}$$

where $(S_0, 0)$ is the disease-free equilibrium and (S^*, I^*) is the positive endemic equilibrium, which is defined in Section 1. Linearizing the second equation of system

(2.2) at disease-free equilibrium $(S_0, 0)$, we have

$$cI'(\xi) = d_2 J[I](\xi) - \mu_2 I(\xi) + \beta S_0 f'(0) I(\xi). \tag{2.5}$$

Letting $I(\xi) = e^{\lambda\xi}$ and substituting it in (2.5) yield

$$d_2[e^{\lambda} + e^{-\lambda} - 2] - c\lambda + \beta S_0 f'(0) - \mu_2 = 0.$$

Denote

$$\Delta(\lambda, c) = d_2[e^{\lambda} + e^{-\lambda} - 2] - c\lambda + \beta S_0 f'(0) - \mu_2. \tag{2.6}$$

By some calculations, we have

$$\begin{aligned} &\Delta(0, c) = \beta S_0 f'(0) - \mu_2, \quad \lim_{c \to +\infty} \Delta(\lambda, c) = -\infty, \\ &\frac{\partial \Delta(\lambda, c)}{\partial \lambda} = d_2[e^{\lambda} - e^{-\lambda}] - c, \quad \frac{\partial \Delta(\lambda, c)}{\partial c} = -\lambda < 0, \\ &\frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} = d_2[e^{\lambda} + e^{-\lambda}] > 0, \quad \left.\frac{\partial \Delta(\lambda, c)}{\partial \lambda}\right|_{(0,c)} = -c < 0, \end{aligned}$$

for $\lambda > 0$ and $c > 0$. Therefore, we have the following lemma.

Lemma 1 *Let* $\Re_0 > 1$. *There exist* $c^* > 0$ *and* $\lambda^* > 0$ *such that*

$$\left.\frac{\partial \Delta(\lambda, c)}{\partial \lambda}\right|_{(\lambda^*, c^*)} = 0 \;\; \text{and} \;\; \Delta(\lambda^*, c^*) = 0.$$

Furthermore,

(i) *if* $c = c^*$, *then* $\Delta(\lambda, c) = 0$ *has only one positive real root* λ^*;
(ii) *if* $0 < c < c^*$, *then* $\Delta(\lambda, c) > 0$ *for all* $\lambda \in (0, +\infty)$;
(iii) *if* $c > c^*$, *then* $\Delta(\lambda, c) = 0$ *has two positive real roots* λ_1, λ_2 *with* $\lambda_1 < \lambda^* < \lambda_2$.

From Lemma 1, we have

$$\Delta(\lambda, c) \begin{cases} > 0 & \text{for} \;\; \lambda < \lambda_1, \\ < 0 & \text{for} \;\; \lambda_1 < \lambda < \lambda_2, \\ > 0 & \text{for} \;\; \lambda > \lambda_2. \end{cases} \tag{2.7}$$

In the following of this section, we always fix $c > c^*$ and $\Re_0 > 1$.

2.1 Construction of Upper and Lower Solutions

Definition 1 $(S^+(\xi), I^+(\xi))$ and $(S^-(\xi), I^-(\xi))$ are called a pair upper and lower solutions of (2.2) if S^+, I^+, S^-, I^- satisfy

$$\begin{cases} d_1 J[S^+](\xi) - cS^{+'}(\xi) + \Lambda - \mu_1 S^+(\xi) - \beta S^+(\xi) f(I^-(\xi)) \leq 0, \\ d_2 J[I^+](\xi) - cI^{+'}(\xi) + \beta S^+(\xi) f(I^+(\xi)) - \mu_2 I^+(\xi) \leq 0, \\ d_1 J[S^-](\xi) - cS^{-'}(\xi) + \Lambda - \mu_1 S^-(\xi) - \beta S^-(\xi) f(I^+(\xi)) \geq 0, \\ d_2 J[I^-](\xi) - cI^{-'}(\xi) + \beta S^-(\xi) f(I^-(\xi)) - \mu_2 I^-(\xi) \geq 0. \end{cases}$$

Define the following functions:

$$\begin{cases} S^+(\xi) = S_0, \\ I^+(\xi) = e^{\lambda_1 \xi}, \end{cases} \quad \begin{cases} S^-(\xi) = \max\{S_0(1 - M_1 e^{\varepsilon_1 \xi}), 0\}, \\ I^-(\xi) = \max\{e^{\lambda_1 \xi}(1 - M_2 e^{\varepsilon_2 \xi}), 0\}, \end{cases} \tag{2.8}$$

where M_i and $\varepsilon_i (i = 1, 2)$ are some positive constants to be determined in the following lemmas. Now we show that (2.8) are a pair upper and lower solutions of (2.2).

Lemma 2 *The function* $I^+(\xi) = e^{\lambda_1 \xi}$ *satisfies*

$$cI^{+'}(\xi) = d_2 J[I^+](\xi) - \mu_2 I^+(\xi) + \beta S_0 f'(0) I^+(\xi). \tag{2.9}$$

Lemma 3 *The function* $S^+(\xi) = S_0$ *satisfies*

$$cS^{+'}(\xi) \geq d_1 J[S^+](\xi) + \Lambda - \mu_1 S^+(\xi) - \beta S^+(\xi) f(I^-(\xi)). \tag{2.10}$$

The proof of the above two lemmas is straightforward, so we omit the details.

Lemma 4 *For each sufficiently small* $0 < \varepsilon_1 < \lambda_1$ *and* $M_1 > 0$ *is large enough, the function* $S^-(\xi) = \max\{S_0(1 - M_1 e^{\varepsilon_1 \xi}), 0\}$ *satisfies*

$$cS^{-'}(\xi) \leq d_1 J[S^-](\xi) + \Lambda - \mu_1 S^-(\xi) - \beta S^-(\xi) f(I^+(\xi)) \tag{2.11}$$

with $\xi \neq \frac{1}{\varepsilon_1} \ln \frac{1}{M_1} := \mathfrak{X}_1$.

Proof If $\xi > \mathfrak{X}_1$, then inequality (2.11) holds since $S^-(\xi) = S^-(\xi + 1) = 0$ and $S^-(\xi - 1) \geq 0$. If $\xi < \mathfrak{X}_1$, then

$$S^-(\xi) = S_0(1 - M_1 e^{\varepsilon_1 \xi}), \quad S^-(\xi - 1) = S_0(1 - M_1 e^{\varepsilon_1 (\xi - 1)})$$

and

$$S^-(\xi + 1) \geq S_0(1 - M_1 e^{\varepsilon_1 (\xi + 1)}).$$

From the concavity of function $f(I)$, we have

$$f(I^+(\xi)) \leq f'(0) I^+(\xi);$$

thus,

$$
\begin{aligned}
&d_1 J[S^-](\xi) + \Lambda - \mu S^-(\xi) - \beta S^-(\xi) f(I^+(\xi)) - c S^{-'}(\xi) \\
&\quad \geq e^{\varepsilon_1 \xi} S_0 \left[-M_1 (d_1 e^{\varepsilon_1} + d_1 e^{-\varepsilon_1} - 2d_1 - \mu - c\varepsilon_1) - \beta_1 f\left(e^{\lambda_1 \xi}\right) e^{-\varepsilon_1 \xi} + \beta_1 M_1 \varepsilon_1 f\left(e^{\lambda_1 \xi}\right) \right] \\
&\quad \geq e^{\varepsilon_1 \xi} S_0 \left[-M_1 (d_1 e^{\varepsilon_1} + d_1 e^{-\varepsilon_1} - 2d_1 - \mu - c\varepsilon_1) - \beta_1 f'(0) e^{\lambda_1 \xi} e^{-\varepsilon_1 \xi} \right].
\end{aligned}
$$

Select $0 < \varepsilon_1 < \lambda_1$ small enough such that $-d_1(2 - e^{\varepsilon_1} - e^{-\varepsilon_1}) - \mu - c\varepsilon_1 < 0$ and note that $e^{(\lambda_1 - \varepsilon_1)\xi} \leq 1$ since $\xi < \mathfrak{X}_1 < 0$. Hence, we need to choose

$$
M_1 \geq \frac{\beta_1 f'(0)}{d_1(2 - e^{\varepsilon_1} - e^{-\varepsilon_1}) + \mu + c\varepsilon_1}
$$

large enough to make sure inequality (2.11) holds. This completes the proof. □

Lemma 5 *For each sufficiently small* $0 < \varepsilon_2 < \lambda_1$ *and* $M_2 > 0$ *is large enough, the function* $I^-(\xi) = \max\{e^{\lambda_1 \xi}(1 - M_2 e^{\varepsilon_2 \xi}), 0\}$ *satisfies*

$$
cI'(\xi) \leq d_2 J[I](\xi) + \beta S^-(\xi) f(I(\xi)) - \mu_2 I(\xi) \tag{2.12}
$$

with $\xi \neq \frac{1}{\varepsilon_2} \ln \frac{1}{M_2} := \mathfrak{X}_2$.

Proof If $\xi > \mathfrak{X}_2$, then inequality (2.12) holds since $I^-(\xi) = I^-(\xi - 1) = 0$ and $I^-(\xi + 1) \geq 0$. If $\xi < \mathfrak{X}_2$, then

$$
I^-(\xi) = e^{\lambda_1 \xi}(1 - M_2 e^{\varepsilon_2 \xi}), \quad I^-(\xi - 1) = e^{\lambda_1(\xi - 1)}(1 - M_2 e^{\varepsilon_2(\xi - 1)})
$$

and

$$
I^-(\xi + 1) \geq e^{\lambda_1(\xi + 1)}(1 - M_2 e^{\varepsilon_2(\xi + 1)}).
$$

Inequality (2.12) is equivalent to the following inequality:

$$
\begin{aligned}
&\beta S_0 f'(0) I^-(\xi) - \beta S^-(\xi) f(I^-(\xi)) \\
&\quad \leq d_2 J[I^-](\xi) - \mu_2 I^-(\xi) - c I^{-'}(\xi) + \beta S_0 f'(0) I^-(\xi).
\end{aligned} \tag{2.13}
$$

Note that $\frac{f(I)}{I}$ is nonincreasing on $(0, \infty)$ in Assumption (A2). Then, for any $\epsilon \in (0, f'(0))$, there exists a small positive number $\delta > 0$ such that

$$
\frac{f(I)}{I} \geq f'(0) - \epsilon, \quad 0 < I < \delta.
$$

For $0 < I < \delta$, we have

$$
\beta S_0 f'(0) I^-(\xi) - \beta S^-(\xi) f(I^-(\xi)) = \left(\beta S_0 - \beta S^-(\xi) \frac{f(I^-(\xi))}{I^-(\xi)} \right) I^-(\xi)
$$

$$\leq \left(\frac{\beta S_0 - \beta S^-(\xi) \frac{f(I^-(\xi))}{I^-(\xi)} + I^-(\xi)}{2} \right)^2$$
$$\leq \left(\beta S_0 - \beta S^-(\xi)(f'(0) - \epsilon) + I^-(\xi)\right)^2. \tag{2.14}$$

Recall that $\xi < \mathfrak{X}_2$, we can choose M_2 large enough such that

$$0 < I^-(\xi) < \delta \ \text{ and } \ S^-(\xi) \to S_0.$$

Since (2.14) is valid for any ϵ, we have

$$\beta S_0 f'(0) I^-(\xi) - \beta S^-(\xi) f(I^-(\xi)) \leq [I^-(\xi)]^2.$$

Furthermore, the right hand of (2.13) satisfies

$$\begin{aligned}&d_2 J[I^-](\xi) - \mu_2 I^-(\xi) - c{I^-}'(\xi) + \beta S_0 f'(0) I^-(\xi)\\ &\geq e^{\lambda_1 \xi} \Delta(\lambda_1, c) - M_2 e^{\lambda_1 + \varepsilon_2} \Delta(\lambda_1 + \varepsilon_2, c).\end{aligned}$$

Using the definition of $\Delta(\lambda, c)$ and $[I^-(\xi)]^2 \leq e^{2\lambda_1 \xi}$, noticing that $\Delta(\lambda_1 + \varepsilon_2, c) < 0$ for small $\varepsilon_2 > 0$ by (2.7), then it is suffice to show that

$$e^{(\lambda_1 - \varepsilon_2)\xi} \leq -M_2 \Delta(\lambda_1 + \varepsilon_2, c).$$

The above inequality holds for M_2 large enough, since the left-hand side vanishes and the right-hand side tends to infinity as $M_2 \to +\infty$. This ends the proof. □

Hence, functions (2.8) are a pair upper and lower solutions of (2.2) by Definition 1.

2.2 Truncated Problem

Let $X > \max\{-\mathfrak{X}_1, -\mathfrak{X}_2\}$. Define the following set:

$$\Gamma_X := \left\{ (\phi, \psi) \in C([-X, X], \mathbb{R}^2) \,\middle|\, \begin{array}{l} S^-(\xi) \leq \phi(\xi) \leq S^+(\xi),\ \phi(-X) = S^-(-X), \\ I^-(\xi) \leq \psi(\xi) \leq I^+(\xi),\ \ \psi(-X) = I^-(-X), \\ \forall \xi \in [-X, X]. \end{array} \right\}.$$

It is clear that Γ_X is a nonempty bounded closed convex set in $C([-X, X], \mathbb{R}^2)$. For any $(\phi, \psi) \in C([-X, X], \mathbb{R}^2)$, extend it as

$$\hat{\phi}(\xi) = \begin{cases} \phi(X), & \text{for } \xi > X, \\ \phi(\xi), & \text{for } \xi \in [-X, X], \\ S^-(\xi), & \text{for } \xi < -X, \end{cases} \qquad \hat{\psi}(\xi) = \begin{cases} \psi(X), & \text{for } \xi > X, \\ \psi(\xi), & \text{for } \xi \in [-X, X], \\ I^-(\xi), & \text{for } \xi < -X. \end{cases}$$

Denote

$$\begin{cases} d_1\hat{\phi}(\xi+1)+d_1\hat{\phi}(\xi-1)+\Lambda+\alpha\phi(\xi)-\beta\phi(\xi)f(\psi(\xi)):=H_1(\phi,\psi),\\ d_2\hat{\psi}(\xi+1)+d_2\hat{\psi}(\xi-1)+\beta\phi(\xi)f(\psi(\xi)):=H_2(\phi,\psi). \end{cases}$$

Consider the following truncated initial problem:

$$\begin{cases} cS'(\xi)+(2d_1+\mu_1+\alpha)S(\xi)=H_1(\phi,\psi),\\ cI'(\xi)+(2d_2+\mu_2)I(\xi)=H_2(\phi,\psi),\\ (S,I)(-X)=(S^-,I^-)(-X), \end{cases} \tag{2.15}$$

where $(\phi,\psi)\in\Gamma_X$ and α is a constant large enough such that $H_1(\phi,\psi)$ is nondecreasing on $\phi(\xi)$. By the ordinary differential equation theory, system (2.15) has a unique solution $(S_X(\xi),I_X(\xi))$ satisfying $(S_X(\xi),I_X(\xi))\in C^1([-X,X],\mathbb{R}^2)$. Then, we define an operator

$$\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2):\Gamma_X\to C^1\left([-X,X],\mathbb{R}^2\right)$$

by

$$S_X(\xi)=\mathcal{A}_1(\phi,\psi)(\xi)\ \text{ and }\ I_X(\xi)=\mathcal{A}_2(\phi,\psi)(\xi).$$

Next we show the operator $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2)$ has a fixed point in Γ_X by Schauder's fixed point theorem (see Chang 2005, Corollary 2.3.10).

Lemma 6 *The operator $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2)$ maps Γ_X into itself.*

Proof Firstly, we show that $S^-(\xi)\le S_X(\xi)$ for any $\xi\in[-X,X]$. If $\xi\in(\mathfrak{X}_1,X)$, it follows that $S^-(\xi)=0$ and it is a lower solution of the first equation of (2.15). If $\xi\in(-X,\mathfrak{X}_1)$, then $S^-(\xi)=S_0(1-M_1e^{\varepsilon_1\xi})$, from Lemma 4, we have

$$\begin{aligned} &cS^{-\prime}(\xi)+(2d_1+\mu_1+\alpha)S^-(\xi)-d_1(\hat{\phi}(\xi+1)\\ &\quad+\hat{\phi}(\xi-1))-\Lambda-\alpha\phi(\xi)+\beta\phi(\xi)f(\psi(\xi))\\ &\quad\le cS^{-\prime}(\xi)-d_1J[S^-](\xi)-\Lambda+\mu_1S^-(\xi)+\beta S^-(\xi)f(I^+(\xi))\\ &\quad\le 0, \end{aligned}$$

which implies that $S^-(\xi)=S_0(1-M_1e^{\varepsilon_1\xi})$ is a lower solution of the first equation of (2.15). Thus, $S^-(\xi)\le S_X(\xi)$ for any $\xi\in[-X,X]$.

Secondly, we show that $S_X(\xi)\le S^+(\xi)=S_0$ for any $\xi\in[-X,X]$. In fact,

$$\begin{aligned} &cS^{+\prime}(\xi)+(2d_1+\mu_1+\alpha)S^+(\xi)-d_1\hat{\phi}(\xi+1)\\ &\quad-d_1\hat{\phi}(\xi-1)-\Lambda-\alpha\phi(\xi)+\beta\phi(\xi)f(\psi(\xi))\\ &\quad\ge\beta S_0f(I^-(\xi))\\ &\quad\ge 0; \end{aligned}$$

thus, $S^+(\xi) = S_0$ is an upper solution to the first equation of (2.15), which gives us $S_X(\xi) \le S_0$ for any $\xi \in [-X, X]$.

Similarly, we can show that $I^-(\xi) \le I_X(\xi) \le I^+(\xi)$ for any $\xi \in [-X, X]$. This completes the proof. □

Lemma 7 *The operator $\mathcal{A}$: $\Gamma_X \to \Gamma_X$ is completely continuous.*

Proof Suppose $(\phi_i(\xi), \psi_i(\xi)) \in \Gamma_X,\ i = 1, 2$. Denote

$$S_{X,i}(\xi) = \mathcal{A}_1(\phi_i(\xi), \psi_i(\xi)) \ \text{ and } \ I_{X,i}(\xi) = \mathcal{A}_2(\phi_i(\xi), \psi_i(\xi)).$$

We show that the operator $\mathcal{A}$ is continuous. By direct calculation, we have

$$S_X(\xi) = S^-(-X)e^{-\frac{2d_1+\mu_1+\alpha}{c}(\xi+X)} + \frac{1}{c}\int_{-X}^{\xi} e^{\frac{2d_1+\mu_1+\alpha}{c}(\tau+X)} H_1(\phi, \psi)(\tau)\mathrm{d}\tau,$$

and

$$I_X(\xi) = I^-(-X)e^{-\frac{2d_2+\mu_2}{c}(\xi+X)} + \frac{1}{c}\int_{-X}^{\xi} e^{\frac{2d_2+\mu_2}{c}(\tau+X)} H_2(\phi, \psi)(\tau)\mathrm{d}\tau,$$

where $H_i(\phi, \psi)(i = 1, 2)$ are defined in (2.15). For any $(\phi_i, \psi_i) \in \Gamma_X, i = 1, 2$, we have

$$\begin{aligned}
&|\phi_1(\xi) f(\psi_1(\xi)) - \phi_2(\xi) f(\psi_2(\xi))| \\
&\quad \le |\phi_1(\xi) f(\psi_1(\xi)) - \phi_1(\xi) f(\psi_2(\xi))| + |\phi_1(\xi) f(\psi_2(\xi)) - \phi_2(\xi) f(\psi_2(\xi))| \\
&\quad \le S_0 f'(0) \max_{\xi\in[-X,X]} |\psi_1(\xi) - \psi_2(\xi)| + f'(0)e^{\lambda_1 X} \max_{\xi\in[-X,X]} |\phi_1(\xi) - \phi_2(\xi)|.
\end{aligned}$$

Since S_X and I_X are class of $C^1([-X, X])$, note that

$$\begin{aligned}
&\left|c(S'_{X,1}(\xi) - S'_{X,2}(\xi)) + (2d_1 + \mu_1)(S_{X,1}(\xi) - S_{X,2}(\xi))\right| \\
&\quad \le d_1|(\hat{\phi}_1(\xi + 1) - \hat{\phi}_2(\xi + 1))| + d_1|(\hat{\phi}_1(\xi - 1) - \hat{\phi}_2(\xi - 1))| \\
&\qquad + \beta|\phi_1(\xi) f(\psi_1(\xi)) - \phi_2(\xi) f(\psi_2(\xi))| \\
&\quad \le \beta S_0 f'(0) \max_{\xi\in[-X,X]} |\psi_1(\xi) - \psi_2(\xi)| \\
&\qquad + \left(2d_1 + \beta f'(0)e^{\lambda_1 X}\right) \max_{\xi\in[-X,X]} |\phi_1(\xi) - \phi_2(\xi)|.
\end{aligned}$$

Same arguments on I_X. Thus, it is easy to see that the operator $\mathcal{A}$ is continuous. Next, it follows from (2.15), we can obtain that S'_X and I'_X are bounded. Hence, the operator $\mathcal{A}$ is compact and is completely continuous. This ends the proof. □

Applying Schauder's fixed point theorem, we have the following lemma.

Lemma 8 *There exists* $(S_X, I_X) \in \Gamma_X$ *such that*

$$(S_X(\xi), I_X(\xi)) = \mathcal{A}(S_X, I_X)(\xi)$$

for $\xi \in [-X, X]$.

In the following, we show some prior estimates for (S_X, I_X). Define

$$C^{1,1}([-X, X]) = \{u \in C^1([-X, X]) \mid u, u' \text{are Lipschitz continuous}\}$$

with the norm

$$\|u\|_{C^{1,1}([-X,X])} = \max_{x\in[-X,X]} |u| + \max_{x\in[-X,X]} |u'| + \sup_{\substack{x,y\in[-X,X]\\ x\neq y}} \frac{|u'(x) - u'(y)|}{|x - y|}.$$

Lemma 9 *There exists a constant* $C(Y) > 0$ *such that*

$$\|S_X\|_{C^{1,1}([-Y,Y])} \le C(Y) \text{ and } \|I_X\|_{C^{1,1}([-Y,Y])} \le C(Y)$$

for $0 < Y < X$ *and* $X > \max\{-\mathfrak{X}_1, -\mathfrak{X}_2\}$.

Proof Recall that (S_X, I_X) is the fixed point of the operator $\mathcal{A}$, then

$$\begin{cases} cS_X'(\xi) = d_1\hat{S}_X(\xi+1) + d_1\hat{S}_X(\xi-1) - (2d_1+\mu_1)S_X(\xi) + \Lambda - \beta S_X(\xi) f(I_X(\xi)), \\ cI_X'(\xi) = d_2\hat{I}_X(\xi+1) + d_2\hat{I}_X(\xi-1) - (2d_2+\mu_2)I_X(\xi) + \beta S_X(\xi) f(I_X(\xi)), \end{cases} \tag{2.16}$$

where

$$\hat{S}_X(\xi) = \begin{cases} S_X(X), & \text{for } \xi > X, \\ S_X(\xi), & \text{for } \xi \in [-X, X], \\ S^-(\xi), & \text{for } \xi < -X, \end{cases} \quad \hat{I}_X(\xi) = \begin{cases} I_X(X), & \text{for } \xi > X, \\ I_X(\xi), & \text{for } \xi \in [-X, X], \\ I^-(\xi), & \text{for } \xi < -X. \end{cases}$$

Since $0 \le S_X(\xi) \le S_0$ and $0 \le I_X(\xi) \le e^{\lambda_1 Y}$ for all $\xi \in [-Y, Y]$, from (2.16) we have

$$|S_X'(\xi)| \le \frac{4d_1+\mu_1}{c} S_0 + \frac{\Lambda}{c} + \frac{\beta S_0 f'(0)}{c} e^{\lambda_1 Y},$$

and

$$|I_X'(\xi)| \le \frac{4d_2+\mu_2+\beta S_0 f'(0)}{c} e^{\lambda_1 Y}.$$

Thus, there exists some constant $C_1(Y) > 0$ such that

$$\|S_X\|_{C^1([-Y,Y])} \le C_1(Y) \text{ and } \|I_X\|_{C^1([-Y,Y])} \le C_1(Y).$$

For any $\xi, \eta \in [-Y, Y]$, it follows from Zhang and Wu (2019) that

$$
\begin{aligned}
&|\hat{S}_X(\xi+1) - \hat{S}_X(\eta+1)| \\
&= \begin{cases}
|S_X(Y) - S_X(Y)| = 0, & \text{for } \xi+1, \eta+1 > Y, \\
|S_X(\xi+1) - S_X(\eta+1)| \le C_1(Y)|\xi - \eta|, & \text{for } \xi+1, \eta+1 < Y, \\
|S_X(\xi+1) - S_X(Y)| \le C_1(Y)(Y - \xi - 1) \le C_1(Y)|\xi - \eta|, & \text{for } \xi+1 < Y, \eta+1 > Y, \\
|S_X(X) - S_X(\eta+1)| \le C_1(Y)(Y - \eta - 1) \le C_1(Y)|\xi - \eta|, & \text{for } \xi+1 > Y, \eta+1 < Y.
\end{cases}
\end{aligned}
$$

Then, $|\hat{S}_X(\xi+1) - \hat{S}_X(\eta+1)| \le C_1(Y)|\xi - \eta|$ for all $\xi, \eta \in [-Y, Y]$. Similarly, we have

$$|\hat{S}_X(\xi-1) - \hat{S}_X(\eta-1)| \le C_1(Y)|\xi - \eta|$$

for all $\xi, \eta \in [-Y, Y]$. Furthermore,

$$
\begin{aligned}
&|\beta S_X(\xi) f(I_X(\xi)) - \beta S_X(\eta) f(I_X(\eta))| \\
&\le |\beta S_X(\xi) f(I_X(\xi)) - \beta S_X(\xi) f(I_X(\eta))| + |\beta S_X(\xi) f(I_X(\eta)) - \beta S_X(\eta) f(I_X(\eta))| \\
&\le \beta f'(0) C_1(Y) \left(|S_X(\xi) - S_X(\eta)| + |I_X(\xi) - I_X(\eta)|\right)
\end{aligned}
$$

for all $\xi, \eta \in [-Y, Y]$. Hence, there exists some constant $C(Y) > 0$ such that

$$\|S_X\|_{C^{1,1}([-Y,Y])} \le C(Y).$$

Similarly,

$$\|I_X\|_{C^{1,1}([-Y,Y])} \le C(Y)$$

for any $Y < X$. This completes the proof. □

3 Existence of Traveling Wave Solutions

We first state the main results of this section as follows.

Theorem 1 *For any wave speed $c > c^*$, system (1.5) admits a nontrivial traveling wave solution $(S(\xi), I(\xi))$ satisfying*

$$S^- \le S(\xi) \le S^+ \text{ and } I^- \le I(\xi) \le I^+ \text{ in } \mathbb{R}.$$

Furthermore,

$$\lim_{\xi \to -\infty} (S(\xi), I(\xi)) = (S_0, 0) \text{ and } \lim_{\xi \to +\infty} (S(\xi), I(\xi)) = (S^*, I^*).$$

The proof of Theorem 1 is divided into the following several steps.

Step 1 We show that system (1.5) admits a nontrivial traveling wave solution $(S(\xi), I(\xi))$ in $\mathbb{R}$ and satisfying $\lim_{\xi \to -\infty}(S(\xi), I(\xi)) = (S_0, 0)$.

Let $\{X_n\}_{n=1}^{+\infty}$ be an increasing sequence such that $X_n > -\mathfrak{X}_2$, $X_n > Y$ and $X_n \to +\infty$ as $n \to +\infty$ for all $n \in \mathbb{N}$, where Y is from Lemma 9. Denote $(S_n, I_n) \in \Gamma_{X_n}$ be the solution of system (2.15). For any $N \in \mathbb{N}$, since the function $I^+(\xi)$ is bounded in $[-X_N, X_N]$, then the sequences

$$\{S_n\}_{n \geq N} \text{ and } \{I_n\}_{n \geq N}$$

are uniformly bounded in $[-X_N, X_N]$. Then, by (2.15), we can obtain that

$$\{S_n'\}_{n \geq N} \text{ and } \{I_n'\}_{n \geq N}$$

are also uniformly bounded in $[-X_N, X_N]$. Again with (2.15), we can express $S_n''(\xi)$ and $I_n''(\xi)$ in terms of $S_n(\xi)$, $I_n(\xi)$, $S_n(\xi \pm 1)$, $I_n(\xi \pm 1)$, $S_n(\xi \pm 2)$ and $I_n(\xi \pm 2)$, which give us

$$\{S_n''\}_{n \geq N} \text{ and } \{I_n''\}_{n \geq N}$$

are uniformly bounded in $[-X_N + 2, X_N - 2]$. By the Arzela–Ascoli theorem (see Rudin 1991, Theorem A5), we can use a diagonal process to extract a subsequence, denoted by $\{S_{n_k}\}_{k \in \mathbb{N}}$ and $\{I_{n_k}\}_{k \in \mathbb{N}}$ such that

$$S_{n_k} \to S, \ I_{n_k} \to I, \ S_{n_k}' \to S' \text{ and } I_{n_k}' \to I' \text{ as } k \to +\infty$$

uniformly in any compact subinterval of $\mathbb{R}$, for some functions S and I in $C^1(\mathbb{R})$. Thus, $(S(\xi), I(\xi))$ is a solution of system (2.2) with

$$S^-(\xi) \leq S(\xi) \leq S^+(\xi) \text{ and } I^-(\xi) \leq I(\xi) \leq I^+(\xi) \text{ in } \mathbb{R}.$$

Furthermore, by the definition of (2.8), it follows that

$$\lim_{\xi \to -\infty} (S(\xi), I(\xi)) = (S_0, 0).$$

Step 2 We claim that the functions $S(\xi)$ and $I(\xi)$ satisfy $0 < S(\xi) < S_0$ and $I(\xi) > 0$ in $\mathbb{R}$.

We first show that $S(\xi) > 0$ for all $\xi \in \mathbb{R}$. Assume reversely, that is, assume that if there exists some real number ξ_0 such that $S(\xi_0) = 0$, then $S'(\xi_0) = 0$ and $J[S](\xi_0) \geq 0$. By the first equation of (2.2), we have

$$0 = d_1 J[S](\xi_0) + \Lambda > 0,$$

which is a contradiction. Thus, $S(\xi) > 0$ for all $\xi \in \mathbb{R}$.

Next, we show that $I(\xi) > 0$ for all $\xi \in \mathbb{R}$. By way of contradiction, we assume that if there exists ξ_1 such that $I(\xi_1) = 0$ and $I(\xi) > 0$ for all $\xi < \xi_1$. From the second equation of (2.2), we have

$$I(\xi_1 + 1) + I(\xi_1 - 1) = 0.$$

Consequently, $I(\xi_1+1)=I(\xi_1-1)=0$ since $I(\xi)\geq 0$ in $\mathbb{R}$, which is a contradiction to the definition of ξ_1.

To show that $S(\xi)<S_0$ for all $\xi\in\mathbb{R}$, we assume that if there exists ξ_2 such that $S(\xi_2)=S_0$, it is easy to obtain

$$0=d_1J[S](\xi_2)-\beta S(\xi_2)f(I(\xi_2))<0.$$

This contradiction leads to $S(\xi)<S_0$ for all $\xi\in\mathbb{R}$.

Step 3 Boundedness of traveling wave solutions $S(\xi)$ and $I(\xi)$ in $\mathbb{R}$.

We need to consider two cases of the nonlinear incidence function $f(x)$. In fact, the function $f(x)$ satisfying Assumptions (A1) and (A2) has two possibilities: (i) $\lim\limits_{x\to+\infty} f(x)$ exists; (ii) $\lim\limits_{x\to+\infty} f(x)=+\infty$. For example, the saturated incidence with $f(x)=\frac{bx}{1+cx}$ satisfies (i) since $\lim\limits_{x\to+\infty}\frac{bx}{1+cx}=\frac{b}{c}$ and the bilinear incidence with $f(x)=bx$ satisfies (ii).

Case 1. $\lim\limits_{x\to+\infty} f(x)$ exists. Without losing generality, we assume that $\lim\limits_{x\to+\infty} f(x)=\bar{f}<+\infty$, then it is easy to verify that $\frac{\Lambda}{\mu_1+\beta\bar{f}}$ is a lower solution of $S(\xi)$ and $\frac{\beta S_0\bar{f}}{\mu_2}$ is an upper solution of $I(\xi)$. Then, we obtain

$$\frac{\Lambda}{\mu_1+\beta\bar{f}}\leq S(\xi)<S_0 \text{ and } 0<I(\xi)\leq\frac{\beta S_0\bar{f}}{\mu_2} \text{ for all } \xi\in\mathbb{R}. \tag{3.1}$$

Case 2. $\lim\limits_{x\to+\infty} f(x)=+\infty$. In this case, we have the following lemmas.

Lemma 10 *The functions* $\frac{I(\xi\pm1)}{I(\xi)}$ *and* $\frac{I'(\xi)}{I(\xi)}$ *are bounded in* $\mathbb{R}$.

Proof From the second equation of (2.2), one has that

$$cI'(\xi)+(2d_2+\mu_2)I(\xi)=d_2I(\xi+1)+d_2I(\xi-1)+\beta S(\xi)f(I(\xi))>0.$$

Denote $U(\xi)=e^{\nu\xi}I(\xi)$, where $\nu=(2d_2+\mu_2)/c$. It follows that

$$cU'(\xi)=e^{\nu\xi}(cI'(\xi)+(2d_2+\mu_2)I(\xi))>0,$$

thus $U(\xi)$ is increasing on ξ. Then, $U(\xi-1)<U(\xi)$, that is,

$$\frac{I(\xi-1)}{I(\xi)}<e^{\nu} \text{ for all } \xi\in\mathbb{R}.$$

Note that

$$\begin{aligned}\left[e^{\nu\xi}I(\xi)\right]'&=\frac{1}{c}e^{\nu\xi}\left[d_2I(\xi+1)+d_2I(\xi-1)+\beta S(\xi)f(I(\xi))\right]\\&>\frac{d_2}{c}e^{\nu\xi}I(\xi+1).\end{aligned} \tag{3.2}$$

Integrating (3.2) over $[\xi, \xi+1]$ and using the fact that $e^{\nu\xi} I(\xi)$ is increasing, we have

$$\begin{aligned} e^{\nu(\xi+1)} I(\xi+1) &> e^{\nu\xi} I(\xi) + \frac{d_2}{c}\int_{\xi}^{\xi+1} e^{\nu s} I(s+1)\mathrm{d}s \\ &> e^{\nu\xi} I(\xi) + \frac{d_2}{c}\int_{\xi}^{\xi+1} e^{\nu(\xi+1)} I(\xi+1) e^{-\nu}\mathrm{d}s \\ &= e^{-\nu}\left[I(\xi) + \frac{d_2}{c} I(\xi+1)\right]. \end{aligned}$$

By (3.2), we obtain

$$\left[e^{\nu\xi} I(\xi)\right]' > \left(\frac{d_2}{c}\right)^2 e^{-2\nu} e^{\nu(\xi+1)} I(\xi+1). \tag{3.3}$$

Integrating (3.3) over $[\xi - \frac{1}{2}, \xi]$ yields

$$\begin{aligned} e^{\nu\xi} I(\xi) &> \left(\frac{d_2}{c}\right)^2 e^{-2\nu} \int_{\xi-\frac{1}{2}}^{\xi} e^{\nu(s+1)} I(s+1)\mathrm{d}s \\ &> \left(\frac{d_2}{c}\right)^2 \frac{e^{-2\nu}}{2} e^{\nu(\xi+\frac{1}{2})} I\left(\xi + \frac{1}{2}\right), \end{aligned}$$

that is

$$\frac{I\left(\xi+\frac{1}{2}\right)}{I(\xi)} < 2\left(\frac{c}{d_2}\right)^2 e^{\frac{3}{2}\nu} \text{ for all } \xi \in \mathbb{R}.$$

Similarly, integrating (3.3) over $[\xi, \xi + \frac{1}{2}]$, we have

$$\frac{I(\xi+1)}{I\left(\xi+\frac{1}{2}\right)} < 2\left(\frac{c}{d_2}\right)^2 e^{\frac{3}{2}\nu} \text{ for all } \xi \in \mathbb{R}.$$

Thus,

$$\frac{I(\xi+1)}{I(\xi)} = \frac{I\left(\xi+\frac{1}{2}\right)}{I(\xi)} \frac{I(\xi+1)}{I\left(\xi+\frac{1}{2}\right)} < 4\left(\frac{c}{d_2}\right)^4 e^{3\nu} \text{ for all } \xi \in \mathbb{R}.$$

By the second equation of (2.2), it follows that

$$\begin{aligned} c\frac{I'(\xi)}{I(\xi)} &= \frac{I(\xi+1)}{I(\xi)} + \frac{I(\xi-1)}{I(\xi)} + \beta S(\xi)\frac{f(I(\xi))}{I(\xi)} - (2d_2+\mu_2) \\ &\le \frac{I(\xi+1)}{I(\xi)} + \frac{I(\xi-1)}{I(\xi)} + \beta S_0 f'(0) - (2d_2+\mu_2), \end{aligned}$$

which gives us $\frac{I'(\xi)}{I(\xi)}$ is bounded in $\mathbb{R}$. The proof is end. □

Lemma 11 *Let $\{c_k, S_k, I_k\}$ be a sequence of traveling wave solutions of (1.5) with speed $\{c_k\}$ in a compact subinterval of $(0, \infty)$. If there is a sequence $\{\xi_k\}$ such that $I(\xi_k) \to +\infty$ as $k \to +\infty$, then $S(\xi_k) \to 0$ as $k \to +\infty$.*

Proof Assume that there is a subsequence of $\{\xi_k\}_{k\in\mathbb{N}}$ again denoted by ξ_k, such that $I_k(\xi_k) \to +\infty$ as $k \to +\infty$ and $S_k(\xi_k) \geq \varepsilon$ in $\mathbb{R}$ for all $k \in \mathbb{N}$ with some positive constant ε. From the first equation of (2.2), we have

$$S_k'(\xi) \leq \frac{2S_0 + \Lambda}{\tilde{c}} := \delta_0 \ \text{ in } \ \mathbb{R},$$

where $\tilde{c}$ is a positive lower bound of $\{c_k\}$. It follows that

$$S_k(\xi) \geq \frac{\varepsilon}{2}, \quad \forall \xi \in [\xi_k - \delta, \xi_k]$$

for all $k \in \mathbb{N}$, where $\delta = \frac{\varepsilon}{\delta_0}$. By Lemma 10, we can assume that $\left|\frac{I_k'(\xi)}{I_k(\xi)}\right| < C_0$ for some $C_0 > 0$. Then,

$$\frac{I_k(\xi_k)}{I_k(\xi)} = \exp\left\{\int_{\xi}^{\xi_k} \frac{I_k'(s)}{I_k(s)} \mathrm{d}s\right\} \leq e^{C_0\delta}, \quad \forall \xi \in [\xi_k - \delta, \xi_k]$$

for all $k \in \mathbb{N}$. Thus,

$$\min_{\xi\in[\xi_k-\delta,\xi_k]} I_k(\xi) \geq e^{-C_0\delta} I_k(\xi_k),$$

which give us

$$\min_{\xi\in[\xi_k-\delta,\xi_k]} I_k(\xi) \to +\infty \ \text{ as } \ k \to +\infty$$

since $I_k(\xi_k) \to +\infty$ as $k \to +\infty$. Recalling we assumed that $\lim_{x\to+\infty} f(x) = +\infty$ in this case, one has that

$$\max_{\xi\in[\xi_k-\delta,\xi_k]} S_k'(\xi) \leq \delta_0 - \frac{\beta\varepsilon}{2} f(\varpi_k) \to -\infty \ \text{ as } \ k \to +\infty.$$

where $\varpi_k := \min_{\xi\in[\xi_k-\delta,\xi_k]} I_k(\xi)$. Moreover, there exists some $K > 0$ such that

$$S_k'(\xi) \leq -\frac{2S_0}{\delta}, \quad \forall k \geq K \ \text{ and } \ \xi \in [\xi_k - \delta, \xi_k].$$

Note that $S_k < S_0$ in $\mathbb{R}$ for each $k \in \mathbb{N}$. Hence, $S_k(\xi_k) \leq -S_0$ for all $k \geq K$, which reduces to a contradiction since $S_k(\xi_k) \geq \varepsilon$ in $\mathbb{R}$ for all $k \in \mathbb{N}$ with some positive constant ε. This completes the proof. □

Lemma 12 *If* $\limsup_{\xi\to+\infty} I(\xi) = +\infty$, *then* $\lim_{\xi\to+\infty} I(\xi) = +\infty$.

The proof of Lemma 12 is similar to that of (Chen et al. 2017, Lemma 3.4), so we omit the details.

Lemma 13 *The function* $I(\xi)$ *is bounded in* $\mathbb{R}$.

Proof Assume that $\limsup_{\xi\to+\infty} I(\xi) = +\infty$, then we have $\lim_{\xi\to+\infty} S(\xi) = 0$ by Lemma 11 and Lemma 12. Set $\theta(\xi) = \frac{I'(\xi)}{I(\xi)}$, from the second equation of (2.2), we have

$$c\theta(\xi) = d_2 e^{\int_\xi^{\xi+1}\theta(s)\mathrm{d}s} + d_2 e^{\int_\xi^{\xi-1}\theta(s)\mathrm{d}s} - (2d_2 + \mu_2) + B(\xi),$$

where

$$B(\xi) = \beta S(\xi)\frac{f(I(\xi))}{I(\xi)}.$$

It is easy to have that $\lim_{\xi\to+\infty} B(\xi) = 0$ since $\frac{f(I(\xi))}{I(\xi)} \le f'(0)$ and $\lim_{\xi\to+\infty} S(\xi) = 0$. By using (Chen and Guo 2003, Lemma 3.4), $\theta(\xi)$ has a finite limit ω at $+\infty$ and satisfies the following equation:

$$h(\omega, c) := d_2\left(e^{\omega} + e^{-\omega} - 2\right) - c\omega - \mu_2 = 0.$$

By some calculations, we obtain

$$h(0, c) < 0, \quad \left.\frac{\partial h(\omega, c)}{\partial\omega}\right|_{\omega=0} < 0, \quad \frac{\partial^2 h(\omega, c)}{\partial\omega^2} > 0 \ \text{ and } \ \lim_{\omega\to+\infty} h(\omega, c) = -\infty.$$

Thus, there exists a unique positive real root ω_0 of $h(\omega, c) = 0$. Recall that λ_1 is the smaller real root of (2.6) and λ_2 is the larger real root of (2.6). From Lemma 1, one has

$$d_2\left(e^{\lambda_2} + e^{-\lambda_2} - 2\right) - c\lambda_2 - \mu_2 = -\beta S_0 f'(0) < 0;$$

thus, we have $\lambda_2 < \omega_0$. Since $\lim_{\xi\to+\infty} \theta(\xi) = \omega_0$, there exists some $\tilde{\xi}$ large enough such that

$$I(\xi) \ge C\exp\left\{\left(\frac{\lambda_2 + \omega_0}{2}\right)\xi\right\} \ \text{ for all } \ \xi \ge \tilde{\xi}$$

with some constant C. This is a contradiction since $I(\xi) \le e^{\lambda_1\xi}$ in $\mathbb{R}$ and $\lambda_1 < \omega_0$. This ends the proof. □

By the above lemmas, we know that $f(I(\xi))$ is bounded from the above since $I(\xi)$ is bounded, then Proposition (3.1) follows. Hence, we obtained that $S(\xi)$ and $I(\xi)$ are bounded from the above and $S(\xi)$ has a strictly positive lower bound in $\mathbb{R}$. In the following, we will show $I(\xi)$ could not approach zero.

Lemma 14 *Let* $0 < c_1 \le c_2$ *be given and* $(S(\xi), I(\xi))$ *be a solution of system* (2.2) *with speed* $c \in [c_1, c_2]$ *satisfying* $0 < S(\xi) < S_0$ and $I(\xi) > 0$ in $\mathbb{R}$. *Then, there exists some small enough constant* $\varepsilon_0 > 0$, *such that* $I'(\xi) > 0$ *provided that* $I(\xi) \le \varepsilon_0$ *for all* $\xi \in \mathbb{R}$.

Proof Assume by way of contradiction that there is no such ε_0, that is there exists some sequence $\{\xi_k\}_{k\in\mathbb{N}}$ with speed $c_k \in (\underline{c}, \overline{c})$ such that $I(\xi_k) \to 0$ as $k \to +\infty$ and $I'(\xi_k) \le 0$, where $\underline{c}$ and $\overline{c}$ are two given positive real numbers. Denote

$$S_k(\xi) := S(\xi_k + \xi) \text{ and } I_k(\xi) := I(\xi_k + \xi).$$

Thus, we have $I_k(0) \to 0$ as $k \to +\infty$ and $I_k(\xi) \to 0$ locally uniformly in $\mathbb{R}$ as $k \to +\infty$. As a consequence, there also holds that $I_k'(\xi) \to 0$ locally uniformly in $\mathbb{R}$ as $k \to +\infty$ by the second equation of (2.2). From the proof of (Chen et al. 2017, Lemma 3.8), we can obtain that $S_\infty = S_0$. Let $\Psi_k(\xi) := \frac{I_k(\xi)}{I_k(0)}$. In the view of

$$\Psi_k'(\xi) = \frac{I_k'(\xi)}{I_k(0)} = \frac{I_k'(\xi)}{I_k(\xi)}\Psi_k(\xi),$$

we have that $\Psi_k(\xi)$ and $\Psi_k'(\xi)$ are also locally uniformly in $\mathbb{R}$ as $k \to +\infty$. Letting $k \to +\infty$, thus

$$c_\infty \Psi_\infty'(\xi) = d_2 J[\Psi_\infty](\xi) + \beta S_0 f'(0)\Psi_\infty(\xi) - \mu_2 \Psi_\infty(\xi).$$

We claim that $\Psi_\infty(\xi) > 0$ in $\mathbb{R}$. In fact, if there exists some ξ_0 such that $\Psi_\infty(\xi_0) = 0$, we also have $\Psi_\infty'(\xi_0) = 0$ since $\Psi_\infty(\xi) \ge 0$, then

$$0 = d_2(\Psi_\infty(\xi_0 + 1) + \Psi_\infty(\xi_0 - 1)).$$

Thus, $\Psi_\infty(\xi_0 + 1) = \Psi_\infty(\xi_0 - 1) = 0$, and it follows that $\Psi_\infty(\xi_0 + m) = 0$ for all $m \in \mathbb{Z}$. Recall that $c_\infty \Psi_\infty'(\xi) \ge -\mu_2 \Psi_\infty(\xi)$, then the map $\xi \mapsto \Psi_\infty(\xi) e^{\frac{\mu_2 \xi}{c_\infty}}$ is nondecreasing. Since it vanishes at $\xi_0 + m$ for all $m \in \mathbb{Z}$ and $e^{\frac{\mu_2 \xi}{c_\infty}}$ is increasing, one can concluded that $\Psi_\infty = 0$ in $\mathbb{R}$, which is a contradicts with $\Psi_\infty(0) = 1$.

Denoting $\mathcal{Z}(\xi) := \frac{\Psi_\infty'(\xi)}{\Psi_\infty(\xi)}$, it is easy to verify that $\mathcal{Z}(\xi)$ satisfies

$$c_\infty \mathcal{Z}(\xi) = d_2 e^{\int_\xi^{\xi+1} \mathcal{Z}(s)\mathrm{d}s}\mathrm{d}y + d_2 e^{\int_\xi^{\xi-1} \mathcal{Z}(s)\mathrm{d}s}\mathrm{d}y - 2d_2 + \beta S_0 f'(0) - \mu_2, \quad (3.4)$$

Thanks to (Chen and Guo 2003, Lemma 3.4), $\mathcal{Z}(\xi)$ has finite limits $\omega_\pm$ as $\xi \to \pm\infty$, where $\omega_\pm$ are roots of

$$c_\infty \omega_\pm = d_2\left(e^{\omega_\pm} + e^{-\omega_\pm} - 2\right) + \beta S_0 f'(0) - \mu_2.$$

By the analogous arguments in Lemma 1, we have $\omega_{\pm} > 0$. Thus, $\Psi'_\infty(\xi)$ is positive at $\pm\infty$ by the definition of $\mathcal{Z}(\xi)$. Moreover, $\Psi'_\infty(\xi) > 0$ for all $\xi \in \mathbb{R}$. Indeed, if there exists some ξ^* such that $\mathcal{Z}(\xi^*) = \inf_{\mathbb{R}} \mathcal{Z}(\xi)$, then $\mathcal{Z}(\xi^*) = 0$. Differentiating (3.4) gives us

$$c_\infty \mathcal{Z}'(\xi) = d_2(\mathcal{Z}(\xi+1) - \mathcal{Z}(\xi))\frac{\Psi_\infty(\xi+1)}{\Psi_\infty(\xi)} + d_2(\mathcal{Z}(\xi-1) - \mathcal{Z}(\xi))\frac{\Psi_\infty(\xi-1)}{\Psi_\infty(\xi)},$$

it follows that

$$\mathcal{Z}(\xi^*) = \mathcal{Z}(\xi^*+1) = \mathcal{Z}(\xi^*-1).$$

Hence, $\mathcal{Z}(\xi^*) = \mathcal{Z}(\xi^*+m)$ for all $m \in \mathbb{Z}$. Then, there is

$$\inf_{\mathbb{R}} \mathcal{Z}(\xi) \geq \min\{\mathcal{Z}(+\infty), \mathcal{Z}(-\infty)\} > 0.$$

So $\Psi'_\infty(\xi) > 0$. From the definition of $\Psi_\infty(\xi)$, we have

$$0 < \Psi'_\infty(0) = \lim_{k\to+\infty} \Psi'_k(0) = \lim_{k\to+\infty} \frac{I'_k(0)}{I_k(0)}.$$

Thus, $I'(\xi_k) = I'_k(0) > 0$, which is a contradiction. This completes the proof. □

Step 4 Convergence of the traveling wave solutions as $\xi \to +\infty$. The key point is to construct a suitable Lyapunov functional.

Let $g(x) = x - 1 - \ln x$ for $x > 0$, it is easy to check $g(x) \geq 0$ since $g(x)$ has the global minimum value 0 only at $x = 1$. Define the following Lyapunov functional:

$$L(S, I)(\xi) = W_1(S, I)(\xi) + d_1 S^* W_2(S, I)(\xi) + d_2 I^* W_3(S, I)(\xi),$$

where

$$W_1(S, I)(\xi) = cS^* g\left(\frac{S(\xi)}{S^*}\right) + cI^* g\left(\frac{I(\xi)}{I^*}\right),$$

$$W_2(S, I)(\xi) = \int_0^1 g\left(\frac{S(\xi-\theta)}{S^*}\right) \mathrm{d}\theta - \int_{-1}^0 g\left(\frac{S(\xi-\theta)}{S^*}\right) \mathrm{d}\theta$$

and

$$W_3(S, I)(\xi) = \int_0^1 g\left(\frac{I(\xi-\theta)}{I^*}\right) \mathrm{d}\theta - \int_{-1}^0 g\left(\frac{I(\xi-\theta)}{I^*}\right) \mathrm{d}\theta.$$

Thanks to the boundedness of $S(\xi)$ and $I(\xi)$ (see Step 3), we have $W_1(S, I)(\xi)$ and $W_2(S, I)(\xi)$ are well defined and bounded from below. Since $\lim_{\xi\to-\infty} I(\xi) = 0$, we need to consider the process of ξ approaching negative infinity for $W_3(S, I)(\xi)$. For the ε_0 in Lemma 14, define $\xi^* = \min\{\xi \in \mathbb{R} | I(\xi) = \varepsilon_0\}$, then $I(\xi)$ is increasing in $(-\infty, \xi^*]$. By the properties of function g, we have $W_3(S, I)(\xi) \geq 0$ for $\xi \in (-\infty, \xi^*]$. Thus, the Lyapunov functional $L(S, I)(\xi)$ is well defined and bounded from below.

Next we show that the map $\xi \mapsto L(S, I)(\xi)$ is nonincreasing. The derivative of $W_1(S, I)(\xi)$ along the solution of (2.2) is calculated as follows:

$$\begin{aligned}\left.\frac{\mathrm{d}W_1(S, I)(\xi)}{\mathrm{d}\xi}\right|_{(2.2)} &= \left(1 - \frac{S^*}{S(\xi)}\right) c\frac{\mathrm{d}S(\xi)}{\mathrm{d}\xi} + \left(1 - \frac{I^*}{I(\xi)}\right) c\frac{\mathrm{d}I(\xi)}{\mathrm{d}\xi} \\ &= \left(1 - \frac{S^*}{S(\xi)}\right)(d_1 J[S](\xi) + \Lambda - \mu_1 S(\xi) - \beta S(\xi) f(I(\xi))) \\ &\quad + \left(1 - \frac{I^*}{I(\xi)}\right)(d_2 J[I](\xi) + \beta S(\xi) f(I(\xi)) - \mu_2 I(\xi)) \\ &= \left(1 - \frac{S^*}{S(\xi)}\right) d_1 J[S](\xi) + \left(1 - \frac{I^*}{I(\xi)}\right) d_2 J[I](\xi) + \Theta(\xi),\end{aligned}$$

where

$$\begin{aligned}\Theta(\xi) = &\left(1 - \frac{S^*}{S(\xi)}\right)(\Lambda - \mu_1 S(\xi) - \beta S(\xi) f(I(\xi))) \\ &+ \left(1 - \frac{I^*}{I(\xi)}\right)(\beta S(\xi) f(I(\xi)) - \mu_2 I(\xi)).\end{aligned}$$

Noticing that the endemic equilibrium (S^*, I^*) of system (1.5) satisfying (1.7) and $\mu_1 = \mu + \alpha$. By some calculations, we obtain that

$$\begin{aligned}\Theta(\xi) = &\mu_1 S^* \left(2 - \frac{S^*}{S(\xi)} - \frac{S(\xi)}{S^*}\right) - \beta S^* f(I^*) \left[g\left(\frac{S^*}{S(\xi)}\right) + g\left(\frac{I^* S(\xi) f(I(\xi))}{I(\xi) S^* f(I^*)}\right)\right] \\ &- \beta S^* f(I^*) \left[g\left(\frac{I(\xi)}{I^*}\right) - g\left(\frac{f(I(\xi))}{f(I^*)}\right)\right].\end{aligned}$$

For $W_2(S, I)(\xi)$, one has that

$$\begin{aligned}\left.\frac{\mathrm{d}W_2(S, I)(\xi)}{\mathrm{d}\xi}\right|_{(2.2)} &= \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\int_0^1 g\left(\frac{S(\xi - \theta)}{S^*}\right)\mathrm{d}\theta - \int_{-1}^0 g\left(\frac{S(\xi - \theta)}{S^*}\right)\mathrm{d}\theta\right] \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\xi} g\left(\frac{S(\xi - \theta)}{S^*}\right)\mathrm{d}\theta - \int_{-1}^0 \frac{\mathrm{d}}{\mathrm{d}\xi} g\left(\frac{S(\xi - \theta)}{S^*}\right)\mathrm{d}\theta \\ &= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} g\left(\frac{S(\xi - \theta)}{S^*}\right)\mathrm{d}\theta + \int_{-1}^0 \frac{\mathrm{d}}{\mathrm{d}\theta} g\left(\frac{S(\xi - \theta)}{S^*}\right)\mathrm{d}\theta \\ &= 2g\left(\frac{S(\xi)}{S^*}\right) - g\left(\frac{S(\xi - 1)}{S^*}\right) - g\left(\frac{S(\xi + 1)}{S^*}\right).\end{aligned}$$

Similarly,

$$\left.\frac{\mathrm{d}W_3(S,I)(\xi)}{\mathrm{d}\xi}\right|_{(2.2)} = 2g\left(\frac{I(\xi)}{I^*}\right) - g\left(\frac{I(\xi-1)}{I^*}\right) - g\left(\frac{I(\xi+1)}{I^*}\right).$$

It can be shown that

$$\left(1-\frac{S^*}{S(\xi)}\right)d_1 J[S](\xi) + S^*\left.\frac{\mathrm{d}W_2(S,I)(\xi)}{\mathrm{d}\xi}\right|_{(2.2)} = -d_1 S^*\left[g\left(\frac{S(\xi-1)}{S(\xi)}\right) + g\left(\frac{S(\xi+1)}{S(\xi)}\right)\right],$$

and

$$\left(1-\frac{I^*}{I(\xi)}\right)d_2 J[I](\xi) + I^*\left.\frac{\mathrm{d}W_3(S,I)(\xi)}{\mathrm{d}\xi}\right|_{(2.2)} = -d_2 I^*\left[g\left(\frac{I(\xi-1)}{I(\xi)}\right) + g\left(\frac{I(\xi+1)}{I(\xi)}\right)\right].$$

Thus,

$$\begin{aligned}
\left.\frac{\mathrm{d}L(S,I)(\xi)}{\mathrm{d}\xi}\right|_{(2.2)} =& - d_1 S^*\left[g\left(\frac{S(\xi-1)}{S(\xi)}\right) + g\left(\frac{S(\xi+1)}{S(\xi)}\right)\right] \\
& - d_2 I^*\left[g\left(\frac{I(\xi-1)}{I(\xi)}\right) + g\left(\frac{I(\xi+1)}{I(\xi)}\right)\right] \\
& - \beta_1 S^* f(I^*)\left[g\left(\frac{S^*}{S(\xi)}\right) + g\left(\frac{S(\xi)f(I(\xi))I^*}{S^* f(I^*)I(\xi)}\right) + g\left(\frac{I(\xi)}{I^*}\right) - g\left(\frac{f(I(\xi))}{f(I^*)}\right)\right] \\
& + \mu S^*\left(2 - \frac{S^*}{S(\xi)} - \frac{S(\xi)}{S^*}\right).
\end{aligned}$$

Since the arithmetic mean of nonnegative real numbers is greater than or equal to the geometric mean of the same list, then we have

$$2 - \frac{S^*}{S(\xi)} - \frac{S(\xi)}{S^*} \le 0.$$

From Assumption (A2), we can conclude that

$$\left(1 - \frac{f(I^*)}{f(I)}\right)\left(\frac{f(I)}{f(I^*)} - \frac{I}{I^*}\right) \le 0.$$

Then, we have

$$\begin{aligned}
g\left(\frac{f(I(\xi))}{f(I^*)}\right) - g\left(\frac{I(\xi)}{I^*}\right) =& \frac{f(I)}{f(I^*)} - \frac{I}{I^*} + \ln\left(\frac{If(I^*)}{I^* f(I)}\right) \\
\le& \frac{f(I)}{f(I^*)} - \frac{I}{I^*} + \frac{If(I^*)}{I^* f(I)} - 1 \\
=& \left(1 - \frac{f(I^*)}{f(I)}\right)\left(\frac{f(I)}{f(I^*)} - \frac{I}{I^*}\right) \\
\le& 0.
\end{aligned}$$

Here, we use $\frac{If(I^*)}{I^*f(I)} - 1 - \ln\left(\frac{If(I^*)}{I^*f(I)}\right) \geq 0$. Hence, the map $\xi \mapsto L(S, I)(\xi)$ is nonincreasing. Consider an increasing sequence $\{\xi_k\}_{k\geq 0}$ with $\xi_k > 0$ such that $\xi_k \to +\infty$ when $k \to +\infty$ and denote

$$\{S_k(\xi) = S(\xi + \xi_k)\}_{k\geq 0}, \quad \{I_k(\xi) = I(\xi + \xi_k)\}_{k\geq 0}.$$

Since the functions S and I are bounded, the system (2.2) give us that the functions S and I have bounded derivatives. Then, by Arzela–Ascoli theorem, the functions $\{S_k(\xi)\}$ and $\{I_k(\xi)\}$ converge in $C^\infty_{loc}(\mathbb{R})$ as $k \to +\infty$, and up to extraction of a subsequence, one may assume that the sequences of $\{S_k(\xi)\}$ and $\{I_k(\xi)\}$ convergence toward some nonnegative C^∞ functions S_∞ and I_∞. Furthermore, since $L(S, I)(\xi)$ is nonincreasing on ξ and bounded from below, there exists a constant C_0 and large k such that

$$C_0 \leq L(S_k, I_k)(\xi) = L(S, I)(\xi + \xi_k) \leq L(S, I)(\xi).$$

Therefore, there exists some $\delta \in \mathbb{R}$ such that $\lim_{k\to\infty} L(S_k, I_k)(\xi) = \delta$ for any $\xi \in \mathbb{R}$. By Lebesgue's dominated convergence theorem (see (Rudin 1976, Theorem 11.32)), we have

$$\lim_{k\to+\infty} L(S_k, I_k)(\xi) = L(S_\infty, I_\infty)(\xi), \ \xi \in \mathbb{R}.$$

Thus,

$$L(S_\infty, I_\infty)(\xi) = \delta.$$

Note that $\frac{\mathrm{d}L}{\mathrm{d}\xi} = 0$ if and only if $S(\xi) \equiv S^*$ and $I(\xi) \equiv I^*$, it follows that

$$(S_\infty, I_\infty) \equiv (S^*, I^*).$$

Hence, we complete the proof of Theorem 1. Next, we give the following theorem on the critical traveling wave solution.

Theorem 2 *For the wave speed $c = c^*$, system (1.5) admits a nontrivial traveling wave solution $(S(\xi), I(\xi))$ satisfying*

$$S^- \leq S(\xi) \leq S^+ \ \text{ and } \ I^- \leq I(\xi) \leq I^+ \ \text{ in } \ \mathbb{R}.$$

where $\xi = n + c^*t$. Furthermore,

$$\lim_{\xi\to-\infty}(S(\xi), I(\xi)) = (S_0, 0) \ \text{ and } \ \lim_{\xi\to+\infty}(S(\xi), I(\xi)) = (S^*, I^*).$$

Proof This theorem could be obtained by an approximation technique used in (Chen et al. 2017, Section 4), and we will give key sketch for the sake of completeness.

Consider a sequence $\{c_k\}$ such that $c_k \in (c^*, c^* + 1]$ for each $k \in \mathbb{N}$, and $c_k \to c^*$ as $k \to +\infty$. Let (S_k, I_k) be a traveling wave solution with wave speed c_k.

We first show that $\liminf_{k\to+\infty} \|I_k\|_{L^\infty(\mathbb{R})} > 0$. On the contrary, up to extraction of a subsequence, assuming $\|I_k\|_{L^\infty(\mathbb{R})} \to 0$ as $k \to +\infty$. Since I_k is bounded, then $I_k(+\infty)$ exists. Thanks to (Chen et al. 2017, Lemma 3.9), we can obtain that $I_k(+\infty) = I^* > 0$, which is a contradiction.

Secondly, we prove that $\limsup_{k\to+\infty} \|I_k\|_{L^\infty(\mathbb{R})} < +\infty$. By way of contradiction, up to extraction of a subsequence, we assume that $\|I_k\|_{L^\infty(\mathbb{R})} \to +\infty$ as $k \to +\infty$. There exists $\xi_k \in \mathbb{R}$ such that

$$I_k(\xi_k) \geq \left(1 - \frac{1}{k+1}\right) \|I_k\|_{L^\infty(\mathbb{R})}.$$

It follows from the second equation of (2.2) that

$$I_k{}'(\xi) \geq \frac{d_2}{c^*} I_k(\xi + 1) - \frac{2d_2 + \mu_2}{c^*} I_k(\xi)$$

for all $\xi \in \mathbb{R}$ and $k \in \mathbb{N}$. Hence, $\frac{I_k(\xi \pm 1)}{I_k(\xi)}$ is bounded by Lemma 10. Define

$$\Phi_k(\xi) := S_k(\xi + \xi_k) \quad \text{and} \quad \Psi_k(\xi) = \frac{\psi_k(\xi + \xi_k)}{\psi_k}.$$

By Lemma 11, we can conclude that $\Phi_k(\xi) \to 0$ as $k \to +\infty$ locally uniformly in $\mathbb{R}$ since $I_k(\xi + \xi_k) \to +\infty$ as $k \to +\infty$. According to Chen et al. (2017), we can obtain that $\Psi_k(\xi)$ is locally bounded and Ψ_k converges in $C^1_{loc}(\mathbb{R})$ to Ψ_∞. Moreover, Ψ_∞ satisfies

$$c^* \Psi_\infty{}' = d_2 J[\Psi_\infty] - \mu_2 \Psi_\infty \tag{3.5}$$

in $\mathbb{R}$. The proof of Lemma 14 give us Ψ_∞ is nonnegative and $\Psi_\infty(0) = \lim_{k\to+\infty} \Psi_k(0) = 1$. It follows from Chen et al. (2017) that $\Psi_\infty{}'(0) = 0$ and $J[\Psi_\infty](0) = 0$. Hence, there is a contradiction with Eq. (3.5) since $\mu_2 > 0$.

Finally, with the help of the above *priori* bound, passing the limit as $k \to +\infty$ (see (Chen et al. 2017, page 2350-2351), we can obtain the existence of critical traveling wave solution for $c = c^*$, which satisfying asymptotic boundary conditions (2.3). Recalling that the Lyapunov functional is independent on c, we can also have that $(S(n + c^*t), I(n + c^*t))$ satisfying the asymptotic boundary conditions (2.4). The proof is completed. □

4 Nonexistence of Traveling Wave Solutions

In this section, we study the nonexistence of traveling wave solutions. Firstly, we show that $c > 0$ if there exists a nontrivial positive solution $(S(\xi), I(\xi))$ of system (2.2) satisfying the asymptotic boundary conditions (2.3) and (2.4).

Lemma 15 *Assume that* $\Re_0 > 1$ *and there exists a nontrivial solution* $(S(\xi), I(\xi))$ *of system* (1.5) *satisfying the asymptotic boundary conditions* (2.3) *and* (2.4). *Then,* $c > 0$*, where c is defined in* (2.1).

Proof Assume that $c \le 0$. Since $S(\xi) \to S_0$ and $I(\xi) \to 0$ as $\xi \to -\infty$, there exists a $\xi^* < 0$ such that

$$\begin{aligned} cI'(\xi) \ge{}& d_2[I(\xi+1) + I(\xi-1) - 2I(\xi)] \\ &+ \frac{\beta S_0 f'(0) + \mu_2}{2f'(0)}(f'(0) - \epsilon)I(\xi) - \mu_2 I(\xi); \end{aligned} \tag{4.1}$$

here, we used the condition $\Re_0 > 1$. Note that inequality (4.1) is valid for any $\epsilon \in (0, f'(0))$, and then, for $\xi < \xi^*$, we have

$$cI'(\xi) \ge d_2[I(\xi+1) + I(\xi-1) - 2I(\xi)] + \frac{\beta S_0 f'(0) - \mu_2}{2} I(\xi). \tag{4.2}$$

Denote $\omega = \frac{\beta S_0 f'(0) - \mu_2}{2}$ and $Q(\xi) = \int_{-\infty}^{\xi} I(y)\mathrm{d}y$ for $\xi \in \mathbb{R}$, note that $\omega > 0$ since $\Re_0 > 1$. Integrating inequality (4.1) from $-\infty$ to ξ and using $I(-\infty) = 0$, one has that

$$cI(\xi) \ge d_2[Q(\xi+1) + Q(\xi-1) - 2Q(\xi)] + \omega Q(\xi) \ \text{ for } \ \xi < \xi^*. \tag{4.3}$$

Again, integrating inequality (4.1) from $-\infty$ to ξ yields

$$cQ(\xi) \ge d_2\left(\int_{\xi}^{\xi+1} Q(\tau)\mathrm{d}\tau - \int_{\xi-1}^{\xi} Q(\tau)\mathrm{d}\tau\right) + \omega\int_{-\infty}^{\xi} Q(\tau)\mathrm{d}\tau \ \text{ for } \ \xi < \xi^*. \tag{4.4}$$

Since $Q(\xi)$ is strictly increasing in $\mathbb{R}$ and $c \le 0$, we can conclude that

$$0 \ge cQ(\xi) \ge d_2\left(\int_{\xi}^{\xi+1} Q(\tau)\mathrm{d}\tau - \int_{\xi-1}^{\xi} Q(\tau)\mathrm{d}\tau\right) + \omega\int_{-\infty}^{\xi} Q(\tau)\mathrm{d}\tau > 0,$$

which is a contradiction. Hence, $c > 0$. The proof is finished. □

Now, we are in position to show the nonexistence of traveling wave solutions, and we will use two-sided Laplace to prove it (see Bai and Zhang 2015; Yang et al. 2013; Zhou et al. 2020)

Theorem 3 *Assume that* $\Re_0 > 1$ *and* $c < c^*$*. Then, there is no nontrivial solution* $(S(\xi), I(\xi))$ *of system* (1.5) *satisfying the asymptotic boundary conditions* (2.3) *and* (2.4).

Proof By way of contradiction, assume that there exists a nontrivial positive solution $(S(\xi), I(\xi))$ of system (1.5) satisfying the asymptotic boundary condition (2.3) and (2.4). Then, $c > 0$ by Lemma 15 and

$$S(\xi) \to S_0 \text{ and } I(\xi) \to 0 \text{ as } \xi \to -\infty.$$

Let $\omega = \frac{\beta S_0 f'(0) - \mu_2}{2}$ and $Q(\xi) = \int_{-\infty}^{\xi} I(y)\mathrm{d}y$ for $\xi \in \mathbb{R}$. It follows from the proof of Lemma 15, there exists a $\xi^* < 0$ such that

$$cQ(\xi) \geq d_2\left(\int_{\xi}^{\xi+1} Q(\tau)\mathrm{d}\tau - \int_{\xi-1}^{\xi} Q(\tau)\mathrm{d}\tau\right) + \omega\int_{-\infty}^{\xi} Q(\tau)\mathrm{d}\tau \text{ for } \xi < \xi^*.$$

Recalling that $Q(\xi)$ is strictly increasing in $\mathbb{R}$, one has that

$$d_2\left(\int_{\xi}^{\xi+1} Q(\tau)\mathrm{d}\tau - \int_{\xi-1}^{\xi} Q(\tau)\mathrm{d}\tau\right) > 0.$$

Thus,

$$cQ(\xi) \geq \omega\int_{-\infty}^{\xi} Q(\tau)\mathrm{d}\tau \text{ for } \xi < \xi^*. \tag{4.5}$$

Hence, there exists some constant $\delta > 0$ such that

$$\omega\delta Q(\xi - \delta) \leq cQ(\xi) \text{ for } \xi < \xi^*. \tag{4.6}$$

Moreover, there exists a $\nu > 0$ is large enough and $\epsilon_0 \in (0, 1)$ such that

$$Q(\xi - \nu) \leq \epsilon_0 Q(\xi) \text{ for } \xi < \xi^*. \tag{4.7}$$

Set

$$\mu_0 := \frac{1}{\nu}\ln\frac{1}{\epsilon_0} \text{ and } V(\xi) := Q(\xi)e^{-\mu_0\xi}.$$

We have

$$V(\xi - \nu) = Q(\xi - \nu)e^{-\mu_0(\xi-\nu)} < \epsilon_0 Q(\xi)e^{-\mu_0(\xi-\nu)} = V(\xi) \text{ for } \xi < \xi^*,$$

which implies that $V(\xi)$ is bounded as $\xi \to -\infty$. Since $\int_{-\infty}^{\infty} I(\xi)\mathrm{d}\xi < \infty$, we obtain that

$$\lim_{\xi\to\infty} V(\xi) = \lim_{\xi\to\infty} Q(\xi)e^{-\mu_0\xi} = 0.$$

From the second equation of (2.2), we have

$$cI'(\xi) \leq d_2[I(\xi+1)+I(\xi-1)-2I(\xi)]+\beta S_0 f'(0)I(\xi)-\mu_2 I(\xi);$$

integrating over $(-\infty, \xi)$ gives us

$$cI(\xi) \leq d_2[Q(\xi+1)+Q(\xi-1)-2Q(\xi)]+\beta S_0 f'(0)Q(\xi)-\mu_2 Q(\xi).$$

Hence, we can obtain that

$$\sup_{\xi\in\mathbb{R}}\left\{I(\xi)e^{-\mu_0\xi}\right\} < +\infty \quad \text{and} \quad \sup_{\xi\in\mathbb{R}}\left\{I'(\xi)e^{-\mu_0\xi}\right\} < +\infty.$$

For $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \mu_0$, define the following two-sided Laplace transform of $I(\cdot)$ by

$$\mathcal{L}(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda\xi} I(\xi)\mathrm{d}\xi.$$

Note that

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-\lambda\xi}[I(\xi+1)+I(\xi-1)]\mathrm{d}\xi \\
&= e^{\lambda}\int_{-\infty}^{\infty} e^{-\lambda(\xi+1)}I(\xi+1)\mathrm{d}\xi + e^{-\lambda}\int_{-\infty}^{\infty} e^{-\lambda(\xi-1)}I(\xi-1)\mathrm{d}\xi \\
&= \left(e^{\lambda}+e^{-\lambda}\right)\mathcal{L}(\lambda)
\end{aligned}$$

and

$$\int_{-\infty}^{\infty} e^{-\lambda\xi} I'(\xi)\mathrm{d}\xi = e^{\lambda}I(\xi)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} I(\xi)\mathrm{d}e^{-\lambda\xi} = \lambda\mathcal{L}(\lambda).$$

From the second equation of (2.2), we have

$$\begin{aligned}
&d_2 J[I](\xi)+\beta S_0 f'(0)I(\xi)-\mu_2 I(\xi)-cI'(\xi) \\
&\quad = \beta S_0 f'(0)I(\xi)-\beta S(\xi)f(I(\xi)).
\end{aligned} \tag{4.8}$$

Taking two-sided Laplace transform on (4.8) gives us

$$\Delta(\lambda, c)\mathcal{L}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\xi}\left[\beta S_0 f'(0)I(\xi)-\beta S(\xi)f(I(\xi))\right]\mathrm{d}\xi. \tag{4.9}$$

It follows from the proof in Lemma 5, as $\xi \to -\infty$, we have

$$\left[\beta S_0 f'(0)I(\xi)-\beta S(\xi)f(I(\xi))\right]e^{-2\mu_0\xi} \leq I^2(\xi)e^{-2\mu_0\xi}$$

$$\leq \left(\sup_{\xi \in \mathbb{R}} \left\{ I(\xi) e^{-\mu_0 \xi} \right\} \right)^2$$
$$\leq +\infty.$$

Thus, we can obtain that

$$\sup_{\xi \in \mathbb{R}} \left[\beta S_0 f'(0) I(\xi) - \beta S(\xi) f(I(\xi)) \right] e^{-2\mu_0 \xi} < +\infty. \tag{4.10}$$

By the property of Laplace transform (Widder 1941), either there exist a real number μ_0 such that $\mathcal{L}(\lambda)$ is analytic for $\lambda \in \mathbb{C}$ with $0 < \text{Re}\lambda < \mu_0$ and $\lambda = \mu_0$ is singular point of $\mathcal{L}(\lambda)$, or $\mathcal{L}(\lambda)$ is well defined for $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > 0$. Furthermore, the two Laplace integrals can be analytically continued to the whole right half line; otherwise, the integral on the left of (4.9) has singularity at $\lambda = \mu_0$ and it is analytic for all $\lambda < \mu_0$. However, it follows from (4.10) that the integral on the right of (4.9) is actually analytic for all $\lambda \leq 2\mu_0$, a contradiction. Thus, (4.9) holds for all $\text{Re}\lambda > 0$. From Lemma 1, $\Delta(\lambda, c) > 0$ for all $\lambda > 0$ and by the definition of $\Delta(\lambda, c)$ in (2.6), we know that $\Delta(\lambda, c) \to \infty$ as $\lambda \to \infty$, which is a contradiction with Eq. (4.9). This ends the proof. □

5 Application and Discussion

As an application, we consider the following two discrete diffusive epidemic models. The first one is a model with saturated incidence rate which has been wildly used in epidemic modeling (see, for example, (Li et al. 2014; Zhang and Wu 2019; Xu and Ma 2009; Zhang et al. 2018; Xu and Guo 2019)).

Example 1 Discrete diffusive epidemic model with saturated incidence rate:

$$\begin{cases} \dfrac{\mathrm{d}S_n(t)}{\mathrm{d}t} = d_1[S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] + \Lambda - \dfrac{\beta S_n(t) I_n(t)}{1 + \alpha I_n(t)} - \mu_1 S_n(t), \\ \dfrac{\mathrm{d}I_n(t)}{\mathrm{d}t} = d_2[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \dfrac{\beta S_n(t) I_n(t)}{1 + \alpha I_n(t)} - \gamma I_n(t) - \mu_1 I_n(t), \end{cases} \tag{5.1}$$

where $\beta I_n(t)$ is the force of infection and $\frac{1}{1+\alpha I_n(t)}$ measures the inhibition effect which is dependent on the infected individuals.

Setting $f(I_n(t)) = \frac{\beta I_n(t)}{1+\alpha I_n(t)}$ in the original system (1.5), we can easily see that (5.1) is a special case of (1.5). In fact, it is obvious that $f(I_n(t))$ satisfies Assumptions (A1) and (A2). The disease-free equilibrium of system (5.1) is similar to the original one, which is $\tilde{E}_0 = (S_0, 0)$. Moreover, we obtain the basic reproduction number of system (5.1) as $\Re_1 = \frac{\beta S_0}{\gamma + \mu_1}$ and there exists a positive equilibrium $\tilde{E}^* = (\tilde{S}^*, \tilde{I}^*)$ if $\Re_1 > 1$, where

$$\tilde{S}^* = \frac{\alpha\Lambda + \gamma + \mu_1}{\beta + \alpha\mu_1} \quad \text{and} \quad \tilde{I}^* = \frac{\Lambda\beta - \mu_1(\gamma + \mu_1)}{(\gamma + \mu_1)(\beta + \alpha\mu_1)}.$$

Hence, from Theorems 1, 2 and 3, we obtain the following corollary.

Corollary 1 *Assume that $\Re_1 > 1$. Then, there exists some $c^* > 0$ such that for any $c \geq c^*$, system (5.1) admits a traveling wave solution $(S(\xi), I(\xi))$ satisfying*

$$\lim_{\xi \to -\infty} (S(\xi), I(\xi)) = (S_0, 0) \text{ and } \lim_{\xi \to +\infty} (S(\xi), I(\xi)) = (\tilde{S}^*, \tilde{I}^*). \tag{5.2}$$

Furthermore, system (5.1) admits no traveling wave solutions satisfying (5.2) when $c < c^*$.

The next example was studied in Chen et al. (2017), and our results will solve the open problem proposed in Chen et al. (2017), which is the traveling wave solutions converging to the endemic equilibrium as $\xi \to +\infty$ for discrete diffusive system (1.3).

Example 2 Discrete diffusive epidemic model with mass action infection mechanism:

$$\begin{cases} \dfrac{\mathrm{d}S_n(t)}{\mathrm{d}t} = d_1[S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] + \Lambda - \beta S_n(t)I_n(t) - \mu_1 S_n(t), \\ \dfrac{\mathrm{d}I_n(t)}{\mathrm{d}t} = d_2[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \beta S_n(t)I_n(t) - \gamma I_n(t) - \mu_1 I_n(t). \end{cases} \tag{5.3}$$

Setting $f(I_n(t)) = \beta I_n(t)$ in the original system (1.5), we can easily see that (5.3) is a special case of (1.5) and this model has been studied in Chen et al. (2017). The disease-free equilibrium of system (5.3) is $\bar{E}_0 = (S_0, 0)$. Moreover, we obtain the basic reproduction number of system (5.3) the same with (5.1) as $\Re_1 = \frac{\beta S_0}{\gamma+\mu_1}$, and there exists a positive equilibrium $\bar{E}^* = (\bar{S}^*, \bar{I}^*)$ if $\Re_1 > 1$, where

$$\bar{S}^* = \frac{\gamma + \mu_1}{\beta} \quad \text{and} \quad \bar{I}^* = \frac{\Lambda - \mu_1\bar{S}^*}{\beta\bar{S}^*}.$$

Then, from Theorems 1, 2 and 3, we obtain the following corollary.

Corollary 2 *Assume that $\Re_1 > 1$. Then, there exists some $c^* > 0$ such that for any $c \geq c^*$, system (5.3) admits a traveling wave solution $(S(\xi), I(\xi))$ satisfying*

$$\lim_{\xi \to -\infty} (S(\xi), I(\xi)) = (S_0, 0) \text{ and } \lim_{\xi \to +\infty} (S(\xi), I(\xi)) = (\bar{S}^*, \bar{I}^*). \tag{5.4}$$

Furthermore, system (5.3) admits no traveling wave solutions satisfying (5.4) when $c < c^*$.

Note that Corollary 2 could answer the open problem proposed in Chen et al. (2017), that is, the traveling wave solutions for system (1.4) converges to the endemic equilibrium at $+\infty$.

Next, we show that how the parameters affect wave speed. Suppose $(\hat{\lambda}, \hat{c})$ be a zero root of $\Delta(\lambda, c)$ which defined in (2.6), a direct calculation yields

$$\frac{\mathrm{d}\hat{c}}{\mathrm{d}\beta} = \frac{S_0 f'(0)}{\lambda} > 0, \quad \frac{\mathrm{d}\hat{c}}{\mathrm{d}d_2} = \frac{e^{\lambda} + e^{-\lambda} - 2}{\lambda} > 0 \text{ and } \frac{\mathrm{d}\hat{c}}{\mathrm{d}\Re_0} = \frac{\mu_2}{\lambda} > 0.$$

that is, $\hat{c}$ is an increasing function on β, d_2 and $\Re_0$. Biologically, this means that the diffusion and infection ability of infected individuals can accelerate the speed of disease spreading.

Now, we are in a position to make the following summary:

In this paper, a discrete diffusive epidemic model with nonlinear incidence rate has been investigated. When the basic reproduction number $\Re_0 > 1$, we proved that there exists a critical wave speed $c^* > 0$, such that for each $c \geq c^*$ the system (1.5) admits a nontrivial traveling wave solution. Moreover, we used a Lyapunov functional to establish the convergence of traveling wave solutions at $+\infty$. We also showed the nonexistence nontrivial traveling wave solutions when $\Re_0 > 1$ and $c < c^*$. As special example of the model (1.5), we considered two different discrete diffusive epidemic model and apply our general results to show the conditions of existence and nonexistence of traveling wave solutions for the model (5.1). One of the example is studied in Chen et al. (2017), and our result solved the open problem proposed in Chen et al. (2017), which is the traveling wave solutions converge to the endemic equilibrium as $\xi \to +\infty$ for discrete diffusive system (1.3).

Here, we mention some functions $f(I)$ considered in the literature that do not satisfy Assumptions (A1) and (A2). For example, the incidence rates with media impact $f(I) = Ie^{-mI}$ in Cui et al. (2008); the specific incidence rate $f(I) = \frac{kI}{1+\alpha I^2}$ in Xiao and Ruan (2007); and the nonmonotone incidence rate $f(I) = \frac{kI}{1+\beta I+\alpha I^2}$ in Xiao and Zhou (2006). In a recent paper, Shu et al. (2019) studied a SIR model with nonmonotone incidence rates and without constant recruitment, and they investigated the existence and nonexistence of traveling wave solutions. What is the condition of existence and nonexistence of traveling wave solution for our model (1.5) with nonmonotone incidence rates will be an interesting question, and we leave this for future work.

Acknowledgements The authors would like to thank the editor and anonymous reviewers for their valuable comments and suggestions which led to a significant improvement of this work. R Zhang and S Liu were supported by Natural Science Foundation of China (11871179; 11771374), J. Wang was supported by National Natural Science Foundation of China (nos. 12071115, 11871179), Natural Science Foundation of Heilongjiang Province (nos. LC2018002, LH2019A021) and Heilongjiang Provincial Key Laboratory of the Theory and Computation of Complex Systems.

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