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Perihelion Librations in the Secular Three-Body Problem

Gabriella Pinzari[1] (ORCID)

Abstract

A normal form theory for non-quasiperiodic systems is combined with the special properties of the partially averaged Newtonian potential pointed out in Pinzari (Celest Mech Dyn Astron 131(5):22, 2019) to prove, in the averaged, planar three-body problem, the existence of a plenty of motions where, periodically, the perihelion of the inner body affords librations about one equilibrium position and its ellipse squeezes to a segment before reversing its direction and again decreasing its eccentricity (*perihelion librations*).

Keywords Three-body problem · Normal form theory · Euler integral · Canonical coordinates · Perihelion librations

Mathematics Subject Classification Primary 34C20 · 70F10 · 37J10 · 37J15 · 37J40; Secondary 34D10 · 70F07 · 70F15 · 37J25 · 37J35

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✉ Gabriella Pinzari
  gabriella.pinzari@math.unipd.it

1 Dipartimento di Matematica T. Levi-Civita, Universitá di Padova, via Trieste 63, 35131 Padua, Italy

1 Introduction

This paper deals with certain motions of three point masses undergoing Newtonian attraction. More precisely, we study the case of two light bodies orbiting their common center of mass (a "binary asteroid system") while interacting with a heavier mass (a "planet"), whose position is external to their trajectories. As no Newtonian interaction can be neglected, there is no reason to claim that the system undergoes the "Keplerian" approximation, successfully used in the so-called planetary model[1] (Arnold 1963b; Féjoz 2004; Laskar and Robutel 1995; Pinzari 2009; Chierchia and Pinzari 2011). We think to the case that the time of mutual revolution of the lighter bodies is much shorter than the timescale of the motions of the massive body. We also assume that the ratio ϵ between the semi-axis of the ellipse of the asteroids and the position ray of the heavier one keeps to be less than $\frac{1}{2}$, so that collisions between the asteroids and the planet are not possible. Recall in fact that a body moving on a Keplerian ellipse does not go beyond twice the semimajor axis from its focus. We look at the system from a reference frame centered with one of the asteroids or with their center of mass, so as to deal with an effective two-particle system, given by the other asteroid and the planet. We shall refer as "asteroidal ellipse" the instantaneous ellipse of this asteroid, focused on the center of the reference frame. We are interested to its motions. To simplify the analysis a little bit, we introduce three assumptions. The main one is based on the belief that, as long as the difference between the timescales persists, the system is "well represented" by a certain "averaged" problem, which we call *secular problem*. We remark that such average is meant with respect to the proper time of the asteroid, so it should not be confused with the homonymous procedure often studied in the literature (e.g., Féjoz and Guardia 2016), where the Keplerian approximation is used for two particles about their common sun, and the average is done with respect to *both* their mean anomalies.

The secular system is simpler than the original problem, as we lose information concerning the position of the asteroid. In particular, collisions between the lighter particles are not observable. The degrees of freedom of the system are the motions of the eccentricity (or of the angular momentum) and of the pericenter direction of the asteroidal ellipse and the motions of the massive body. We associate with such system a certain "limiting system" which is similarly defined, but with the massive body being firm. From now on, we refer to such limiting problem as "unperturbed" and to the full secular problem as "perturbed." The terminology is here used with abuse, as we do not assume that the massive body has slow velocity in the full prob-

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[1] The planetary model is the dynamical system of $1 + n$ ($n \geq 2$) point masses undergoing Newtonian attraction, in the case when one of the masses ("sun") is much heavier than the remaining ones ("planets"), which are of comparable size.

lem. For the unperturbed system, only movements of the asteroidal ellipse occur. By Pinzari (2019), the unperturbed problem turns out to be integrable, in the sense that it possesses a complete family of independent and commuting first integrals. More importantly, it reveals a surprising property, which we named *renormalizable integrability*. Such property (recalled for completeness in Sect. 2.2) offers a remarkable shortcut to the knowledge of movements of the asteroidal ellipse. By Pinzari (2020), in the case that the interacting particles are constrained on a plane (this is actually our second assumption), and $\varepsilon < \frac{1}{2}$, there are two stable equilibria such that the pericenter direction of the asteroidal ellipse in the unperturbed problem affords small oscillations about them, while its angular momentum oscillates about zero, affording a periodic change of sign. Physically, this means that the asteroidal ellipse is highly eccentric at any time and, moreover, there are two times ("squeezing times") in a period of oscillation of the pericenter when the eccentricity is equal to one. Namely, at those times, the ellipse becomes a segment. After the squeezing times, the eccentricity of the ellipse decreases while the sense of the motion is reversed. We call "perihelion librations" such kind of motions. The question which here we address is whether perihelion librations do persist in the full perturbed problem, when the massive body moves. We are able to give a positive answer to this question under our third assumption, which consists in taking the total angular momentum of the system (which is preserved during the motion) equal to zero. Under this assumption, the symmetries of the Hamiltonian ensure that the equilibria persist also in the perturbed problem, even though the integrability is lost. On the other side, such assumption is a source of difficulty, as the angular momenta of the two particles will simultaneously vanish, and hence collisions of the heavy body with the center of the system are to be controlled. We shall formulate our result below, after we have introduced some mathematical tool.

In terms of Jacobi coordinates,[2] the three-body problem Hamiltonian with masses m_0, μm_0, κm_0 is the translation-free function

$$H_1 = \frac{\|\mathbf{y}\|^2}{2m_0}\left(1 + \frac{1}{\mu}\right) + \frac{\|\mathbf{y}'\|^2}{2m_0}\left(\frac{1}{1+\mu} + \frac{1}{\kappa}\right) - \frac{\mu m_0^2}{\|\mathbf{x}\|} - \frac{\mu\kappa m_0^2}{\|\mathbf{x}' - \frac{1}{1+\mu}\mathbf{x}\|} - \frac{\kappa m_0^2}{\|\mathbf{x}' + \frac{\mu}{1+\mu}\mathbf{x}\|}.$$

Here, $(\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x}) \in (\mathbb{R}^3)^4$ (or $(\mathbb{R}^2)^4$), $\|\cdot\|$ denotes Euclidean distance and the gravity constant has been taken equal to one, by a proper choice of the units system. We rescale impulses and positions

$$\mathbf{y} \to \frac{\mu}{1+\mu}\mathbf{y}, \quad \mathbf{x} \to (1+\mu)\mathbf{x}, \quad \mathbf{y}' \to \mu\beta\mathbf{y}', \quad \mathbf{x}' \to \beta^{-1}\mathbf{x}' \tag{1}$$

---

[2] If $(\mathbf{y}_0, \mathbf{x}_0), (\mathbf{y}_1, \mathbf{x}_1), (\mathbf{y}_2, \mathbf{x}_2)$ are impulse–position coordinates of m_0, μm_0, κm_0, respectively, by "Jacobi coordinates" one usually means a linear, canonical change $(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) \in (\mathbb{R}^3)^6 \to (\mathbf{y}_c, \mathbf{y}, \mathbf{y}', \mathbf{x}_c, \mathbf{x}, \mathbf{x}') \in (\mathbb{R}^3)^6$ defined so that $\mathbf{x}_c = (\mathbf{x}_0 + \mu\mathbf{x}_1 + \kappa\mathbf{x}_2)(1+\mu+\kappa)^{-1}$ is the center of mass of the system, while $\mathbf{x}, \mathbf{x}'$ are, respectively, the mutual position of $\mathbf{x}_1$ with respect to $\mathbf{x}_0$ and the position of $\mathbf{x}_2$ with respect to the center of mass of $\mathbf{x}_0$ and $\mathbf{x}_1$: $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_0$, $\mathbf{x}' = \mathbf{x}_2 - \frac{\mu}{1+\mu}\mathbf{x}_1 - \frac{1}{1+\mu}\mathbf{x}_0$. The new impulses $(\mathbf{y}_c, \mathbf{y}, \mathbf{y}')$ are uniquely defined by the constraint of symplecticity. The new Hamiltonian turns to be $\mathbf{x}_c$-independent, due to the conservation of $\mathbf{y}_c$. The dependence on $\mathbf{y}_c$ can be eliminated choosing (as it is always possible to do) a reference frame where $\mathbf{x}_c \equiv \mathbf{0}$. See e.g., Giorgilli (2008, §5.2–5.3) for more details.

multiply the Hamiltonian by $\frac{1+\mu}{\mu}$ (by a rescaling of time) and obtain

$$H_1 = \frac{\|\mathbf{y}\|^2}{2m_0} - \frac{m_0^2}{\|\mathbf{x}\|} + \gamma \left(\frac{\|\mathbf{y}'\|^2}{2m_0} - \frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{m_0^2}{\|\mathbf{x}' - \beta\mathbf{x}\|} - \frac{\beta}{\beta + \overline{\beta}} \frac{m_0^2}{\|\mathbf{x}' + \overline{\beta}\mathbf{x}\|} \right) \quad (2)$$

with

$$\gamma = \frac{\kappa^3(1 + \mu)^4}{\mu^3(1 + \mu + \kappa)}, \quad \beta = \frac{\kappa^2(1 + \mu)^2}{\mu^2(1 + \mu + \kappa)}, \quad \overline{\beta} = \mu\beta. \quad (3)$$

Likewise, one might consider the problem written in the so-called m_0-centric[3] coordinates, and in this case the Hamiltonian is

$$H_2 = \frac{1}{2m_0} \left(1 + \frac{1}{\mu} \right) \|\mathbf{y}\|^2 + \frac{1}{2m_0} \left(1 + \frac{1}{\kappa} \right) \|\mathbf{y}'\|^2 - \frac{\mu m_0^2}{\|\mathbf{x}\|} - \frac{\kappa m_0^2}{\|\mathbf{x}'\|} - \frac{\mu\kappa m_0^2}{\|\mathbf{x} - \mathbf{x}'\|} + \frac{\mathbf{y} \cdot \mathbf{y}'}{m_0}.$$

We apply an analogue rescaling, but with

$$\gamma = \frac{\kappa^3(1 + \mu)^3}{\mu^3(1 + \kappa)}, \quad \beta = \frac{\kappa^2(1 + \mu)}{\mu^2(1 + \kappa)}, \quad \overline{\beta} = \mu\beta. \quad (4)$$

We arrive at

$$H_2 = \frac{\|\mathbf{y}\|^2}{2m_0} - \frac{m_0^2}{\|\mathbf{x}\|} + \gamma \left(\frac{\|\mathbf{y}'\|^2}{2m_0} - \frac{\beta}{\beta + \overline{\beta}} \frac{m_0^2}{\|\mathbf{x}'\|} - \frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{m_0^2}{\|\mathbf{x}' - (\beta + \overline{\beta})\mathbf{x}\|} \right) + \overline{\beta} \frac{\mathbf{y}' \cdot \mathbf{y}}{m_0}. \quad (5)$$

We remark that in the case, of our interest, that $\kappa \gg \mu \sim 1$, the above definition of Jacobi coordinates differs substantially from the usual one, because the barycentric reduction begins with one of the two lighter masses, rather than with the heavier one. A similar observation holds for the m_0-centric reduction, which here is not centered on the most massive body, contrarily to the usual convention. We look at the Hamiltonians H_i in (2) and (5). The assumptions mentioned above are:

(A_1) If ℓ is the mean anomaly associated with the Keplerian motions of the term

$$\frac{\|\mathbf{y}\|^2}{2m_0} - \frac{m_0^2}{\|\mathbf{x}\|} = -\frac{m_0^5}{2\Lambda^2}, \quad (6)$$

we replace the Hamiltonians (2) and (5) with their respective ℓ-averages

$$\overline{H}_i = -\frac{m_0^5}{2\Lambda^2} + \gamma\widehat{H}_i; \quad (7)$$

where $\widehat{H}_i$ are the ℓ-averages of the terms inside parentheses in (2), (5).

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[3] If $(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ are impulse–position coordinates of $m_0, \mu m_0, \kappa m_0$, respectively, the m_0-centric coordinates $(\mathbf{Y}_0, \mathbf{y}, \mathbf{y}', \mathbf{X}_0, \mathbf{x}, \mathbf{x}')$ are defined via $\mathbf{X}_0 = \mathbf{x}_0$, $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_0$, $\mathbf{x}' = \mathbf{x}_2 - \mathbf{x}_0$ and by the symplecticity constraint.

(A_2) The coordinates $\mathbf{x}$, $\mathbf{x}'$ and the impulses $\mathbf{y}$, $\mathbf{y}'$ are constrained on the plane $\mathbb{R}^2$;

(A_3) The total angular momentum $\mathbf{C} = \mathbf{x}' \times \mathbf{y}' + \mathbf{x} \times \mathbf{y}$ of the system vanishes.

As mentioned above, the main assumption is (A_1). It allows us to exploit facts highlighted in Pinzari (2019, 2020), as now we describe.

Since the $\overline{\mathrm{H}}_i$'s in (7) are ℓ-independent, Λ is a first integral; hence, the term $-\frac{m_0^5}{2\Lambda^2}$ may be neglected. After a further rescaling of time $t \to \gamma t$, we are led to look at the Hamiltonians $\widehat{\mathrm{H}}_i$ in (7), which are given by

$$\widehat{\mathrm{H}}_1 := \frac{\|\mathbf{y}'\|^2}{2m_0} - \frac{m_0^2 \overline{\beta}}{\beta + \overline{\beta}} \mathrm{U}_\beta - \frac{m_0^2 \beta}{\beta + \overline{\beta}} \mathrm{U}_{-\overline{\beta}}$$

$$\widehat{\mathrm{H}}_2 := \frac{\|\mathbf{y}'\|^2}{2m_0} - \frac{m_0^2 \overline{\beta}}{\beta + \overline{\beta}} \mathrm{U}_{\beta+\overline{\beta}} - \frac{m_0^2 \beta}{\beta + \overline{\beta}} \frac{1}{\|\mathbf{x}'\|} \qquad (8)$$

where

$$\mathrm{U}_\beta := \frac{1}{2\pi} \int_0^{2\pi} \frac{d\ell}{\|\mathbf{x}' - \beta \mathbf{x}(\ell)\|} \qquad (9)$$

is the ℓ-average of the Newtonian potential. Remark that $\mathbf{y}(\ell)$ has vanishing ℓ-average,[4] so that the last term in (5) does not survive. In the case of the planar problem, after the reduction in rotation invariance, the Hamiltonians $\widehat{\mathrm{H}}_i$ have two degrees of freedom. We use the following canonical coordinates:

$$\mathrm{R} = \frac{\mathbf{y}' \cdot \mathbf{x}'}{\|\mathbf{y}'\|}, \quad \mathrm{r} = \|\mathbf{x}'\|$$

$$\mathrm{G} = \|\mathbf{x} \times \mathbf{y}\|, \quad \mathrm{g} = \text{ anomaly of } \mathbf{P} \text{ with respect to a fixed direction } \mathbf{a}$$

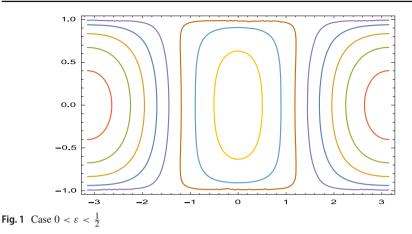where $\mathbf{P}$ is the perihelion of (6), and the direction $\mathbf{a}$, orthogonal to $\mathbf{x} \times \mathbf{y}$, will be specified later. Note the coordinates above are fit to describe motions of Keplerian elements for $(\mathbf{y}, \mathbf{x})$, but not of $(\mathbf{y}', \mathbf{x}')$. If $\mathbf{y}'$ is set to zero, the $\widehat{\mathrm{H}}_i$'s reduce to sums of averaged Newtonian potentials, which are integrable, as do not depend on R. The function $\mathrm{U}_\beta|_{\beta=1}$ has been thoroughly studied in Pinzari (2019). Its phase portrait in the plane (g, G), while the ratio $\varepsilon = a/\mathrm{r}$ (where $a = \Lambda^2/m_0^3$ is the semimajor axis associated with (6)) varies, is as follows.

(i) Case $0 < \varepsilon < \frac{1}{2}$. There exist two centers, $(0, 0)$ and $(0, \pi)$, surrounded by librational motions (Fig. 1).

---

[4] For the unperturbed Keplerian motions, it is $\mathbf{y} = m\dot{\mathbf{x}}$, so

$$\frac{1}{2\pi} \int_0^{2\pi} \mathbf{y} d\ell = \frac{1}{2\pi} \int_0^{2\pi} m_0 \dot{\mathbf{x}} d\ell = \frac{1}{T} \int_0^T m_0 \dot{\mathbf{x}} dt = 0$$

where $T = \frac{2\pi}{\omega_K}$ and $\omega_K = \frac{m_0^5}{\Lambda^3}$ is the Keplerian frequency.

Fig. 1 Case $0 < \varepsilon < \frac{1}{2}$

(ii) Case $\frac{1}{2} < \varepsilon < 1$. The equilibrium $(0,0)$ becomes a saddle, with its own separatrix (the light blue curve), while $(0, \pi)$ is still stable. Two more equilibria appear on the G-axis (Fig. 2).

(iii) Case $\varepsilon > 1$. The equilibria on the G-axis and the saddle persist. There is the birth of rotational motions (Fig. 3).

The purpose of this paper is to prove that motions of the kind (i) do persist in $\widehat{H}_i$, when $\mathbf{y}' \neq 0$. Note that we shall not require $\|\mathbf{y}'\|$ small.

To state the result, we introduce the following quantities, which will be used as mass parameters, at the place of μ and κ:

$$
\beta_* := \begin{cases} \dfrac{\beta\overline{\beta}}{\beta + \overline{\beta}} & \text{if } i = 1 \\[2ex] \overline{\beta} & \text{if } i = 2 \end{cases} \qquad \beta^* := \begin{cases} \max\{\beta, \overline{\beta}\} & \text{if } i = 1 \\[1ex] \beta + \overline{\beta} & \text{if } i = 2 \end{cases} \tag{10}
$$

where β and $\overline{\beta}$ are as in (3), (4), respectively. Observe that $\beta_* < \beta^*$ and the case, of our interest, $\kappa \gg \mu \sim 1$ corresponds to $\beta_* \sim \beta^*/2 \gg 1$.

We shall prove the following result.

Theorem 1.1 (Perihelion librations about $(0,0)$) *Fix an arbitrary neighborhood* U_0 *of* $(0,0)$ *and an arbitrary neighborhood* V_0 *of an unperturbed curve* $\gamma_0(t) = (G_0(t), g_0(t)) \in U_0$ *in Fig.* 1. *Then, it is possible to find six numbers* $0 < c < 1$, $0 < \beta_- < \beta_+, 0 < \alpha_- < \alpha_+ \; T > 0$, *such that, for any* $\beta_- < \beta_* \leq \beta^* < \beta_+$ *the projections* $\Gamma_0(t) = (G(t), g(t))$ *of all the orbits* $\Gamma(t) = (R(t), G(t), r(t), g(t))$ *of* $\overline{H}_1, \overline{H}_2$ *with initial datum* $(R_0, r_0, G_0, g_0) \in [\frac{1}{\sqrt{c\alpha_+}}, \frac{1}{\sqrt{c\alpha_-}}] \times [c\alpha_-, \alpha_+] \times U_0$ *belong to* V_0 *for all* $0 \leq t \leq T$. *Moreover, the angle* $\gamma(t)$ *between the position ray of* $\Gamma_0(t)$ *and the g-axis affords a variation larger than* 2π *during the time* T.

A similar statement concerning perihelion librations about $(0, \pi)$ holds true. The statement of Theorem 1.1 deserves two remarks. The former regards the motions
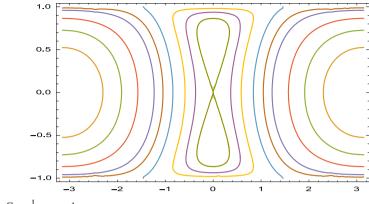
Fig. 2 Case $\frac{1}{2} < \varepsilon < 1$

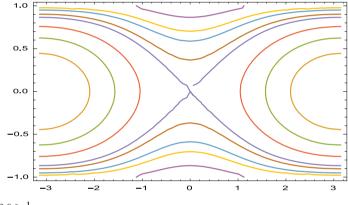**Fig. 3** Case $\varepsilon > 1$

involved, which are quasi-rectilinear, hence, close to be collisional. Generally speaking, for a three-body system composed of two asteroids and one planet, three kinds of collisions are possible: (1) collisions between the two asteroids; (2) collision between one of the asteroids and the planet; and (3) triple collision. The system under investigation is, as stressed above, an averaged problem derived from the full above problem. For this, averaged problem collisions of kind (1) or (3) do not exist, as the position of the asteroids is treated only in averaged meaning. Collisions of the kind (2) may exist, but they are to be intended as collisions between the planet and the average ellipse, rather than with a single particle on this ellipse. They are prevented by the assumption that the orbit of the planet is sufficiently far from the orbit of the asteroids, namely, with a careful choice of the domain of the coordinates. Under this assumption, the averaged Hamiltonians $\widehat{H}_i$ keep finite. Incidentally, their regularity is studied in Proposition 3.3. During the proof of Theorem 1.1, in Sect. A, the trajectory of the massive planet is controlled to keep outside the trajectory of the asteroid for all the time of a perihelion libration.

The second remark concerns the thesis of Theorem 1.1. It holds in an *open* subset of phase space. In a sense, it recalls the statement of Nekhoroshev's theorem (Nehorošev 1977). However, differently from it, Theorem 1.1 is not an application of perturbation theory, nor it uses trapping arguments. The reason is the following. In Sect. 4, we shall see that the manifolds

$$\mathcal{M}_0 := \{(R, G, r, g) : (G, g) = (0, 0)\}, \quad \mathcal{M}_\pi := \{(R, G, r, g) : (G, g) = (0, \pi)\} \tag{11}$$

are in fact invariant for $\widehat{H}_i$. On such invariant manifold, by A_3, we have

$$\|\mathbf{x}' \times \mathbf{y}'\| = \| - \mathbf{x} \times \mathbf{y}\| = G, \tag{12}$$

so

$$\|\mathbf{y}'\|^2 \Big|_{\mathcal{M}_0, \mathcal{M}_\pi} = R^2 + \frac{G^2}{r^2} \Big|_{\mathcal{M}_0, \mathcal{M}_\pi} = R^2.$$

Moreover, the functions U_β, $U_{-\bar{\beta}}$, $U_{\beta+\bar{\beta}}$ in (8) are asymptotic (as $\varepsilon \to 0$) to $\frac{1}{r}$. Hence, the motion of the coordinates (R, r) on $\mathcal{M}_0$ and $\mathcal{M}_\pi$ is ruled by an Hamiltonian asymptotic to

$$\frac{R^2}{2m_0} - \frac{m_0^2}{r}.$$

This Hamiltonian generates unbounded (hence, non-quasiperiodic) motions, for both positive and negative energies: For positive energies, both R and r are unbounded; for negative energies, only R is so. In any case, these motions are not quasiperiodic and hence we cannot apply the machinery of perturbation theory. In Sect. 4.1, we develop a theory suited to the case (see Fortunati and Wiggins 2016 for a result of the same kind). In this theory, no small denominators will arise, which is the reason why no trapping argument is needed.

Before switching to full statements and proofs, we quote three open questions arising from the present setting.

(Q$_1$) Let us consider the cases $\frac{1}{2} < \varepsilon < 1$ or $\varepsilon > 1$ (Figs. 2, 3, respectively). Does the separatrix split so as to produce chaotic dynamics in the partially averaged planar problem?

(Q$_2$) Again in the cases above, let us consider the *full* three-body problem. It has three degrees of freedom. Does the separatrix split so as to produce Arnold instability (Arnol'd 1964; Delshams et al. 2019)?

(Q$_3$) What is the scenario in the case of the spatial problem?

This paper is organized as follows. In Sect. 2, we provide a review of the main results of Pinzari (2019, 2020). In particular, we recall the mathematical formulation of the mentioned renormalizable integrability and we carry from Pinzari (2020) a set of action-angle like coordinates suited to our needs. In Sect. 3, we refine the analysis of

Pinzari (2019) to the case of the *planar* secular problem, in the region of phase space where $0 < \varepsilon < \frac{1}{2}$. In this case, we are able to obtain simpler formulae compared to Pinzari (2019) and hence to study the regularity region of $\widehat{H}_i$ completely. In Sect. 4, we state a normal form theory without small divisors (Theorem 4.1), suited for aperiodic systems. The proof of Theorem 4.1 is deferred to Appendix A. In Sect. 5, we provide the proof of Theorem 1.1, as well as of a more precise version of it (Theorem 5.1), as an application of Theorem 4.1.

2 Review of the Results of Pinzari (2019, 2020)

2.1 $\mathcal{K}$ Coordinates

We describe canonical coordinates suited to our problem.

We fix an arbitrary orthonormal frame

$$F_0 : \quad \mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

in $\mathbb{R}^3$, that we call *inertial frame*.

For given $m_0 > 0$, fix a region of phase space (i.e., a set of values of $(\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x})$) where the Kepler Hamiltonian (6) takes negative values. Consider the motion generated by (6) with initial datum $(\mathbf{y}, \mathbf{x})$, and denote:

- a the semimajor axis;
- $\mathbf{P}$, with $\|\mathbf{P}\| = 1$, the direction of perihelion, assuming the ellipse is not a circle;
- ℓ: the mean anomaly, defined, mod 2π, as the area of the elliptic sector spanned by $\mathbf{x}$ from $\mathbf{P}$, normalized to 2π.

Denote also:

$$\mathbf{M} = \mathbf{x} \times \mathbf{y}, \quad \mathbf{M}' = \mathbf{x}' \times \mathbf{y}', \quad \mathbf{C} = \mathbf{M}' + \mathbf{M},$$

where "$\times$" denotes skew-product in $\mathbb{R}^3$. Observe the following relations:

$$\mathbf{x}' \cdot \mathbf{C} = \mathbf{x}' \cdot (\mathbf{M} + \mathbf{M}') = \mathbf{x}' \cdot \mathbf{M}, \quad \mathbf{P} \cdot \mathbf{M} = 0. \tag{13}$$

Let

$$\mathbf{i}_1 := \mathbf{k} \times \mathbf{C}, \quad \mathbf{i}_2 := \mathbf{C} \times \mathbf{x}', \quad \mathbf{i}_3 := \mathbf{x}' \times \mathbf{M}, \quad \mathbf{i}_4 := \mathbf{M} \times \mathbf{P} \tag{14}$$

and assume

$$\mathbf{i}_j \neq 0 \quad j = 1, 2, 3, 4.$$

Given three vectors $\mathbf{i}$, $\mathbf{i}'$ and $\mathbf{k}$, with $\mathbf{i}, \mathbf{i}' \perp \mathbf{k}$, we denote as $\alpha_{\mathbf{k}}(\mathbf{i}, \mathbf{i}')$ the oriented angle from $\mathbf{i}$ to $\mathbf{i}'$ relatively to the positive orientation established by $\mathbf{k}$.

We define the coordinates

$$\mathcal{K} = (Z, C, R, \Lambda, G, \Theta, z, \gamma, r, \ell, g, \vartheta)$$

as

$$
\begin{cases}
Z = \mathbf{C} \cdot \mathbf{k} \\
C = \|\mathbf{C}\| \\
R = \dfrac{\mathbf{y}' \cdot \mathbf{x}'}{\|\mathbf{x}'\|} \\
\Lambda = \sqrt{m_0^3 a} \\
G = \|\mathbf{M}\| \\
\Theta = \dfrac{\mathbf{M} \cdot \mathbf{x}'}{\|\mathbf{x}'\|}
\end{cases}
\qquad
\begin{cases}
z = \alpha_{\mathbf{k}}(\mathbf{i}, \mathbf{i}_1) \\
\gamma = \alpha_{\mathbf{C}}(\mathbf{i}_1, \mathbf{i}_2) \\
r = \|\mathbf{x}'\| \\
\ell = \text{mean anomaly of } \mathbf{x} \text{ on } \mathbb{E} \\
g = \alpha_{\mathbf{M}}(\mathbf{i}_3, \mathbf{i}_4) \\
\vartheta = \alpha_{\mathbf{x}'}(\mathbf{i}_2, \mathbf{i}_3)
\end{cases}
\tag{15}
$$

The canonical character of $\mathcal{K}$ has been discussed in Pinzari (2019), based on Pinzari (2013).

Using the formulae in the previous section, we provide the expressions of the following functions:

$$
U = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\ell}{\|\mathbf{x}'_{\mathcal{K}} - \mathbf{x}_{\mathcal{K}}\|}, \qquad
E = \|\mathbf{x}_{\mathcal{K}} \times \mathbf{y}_{\mathcal{K}}\|^2 - m_0^3 e_{\mathcal{K}} \mathbf{P}_{\mathcal{K}} \cdot \mathbf{x}'_{\mathcal{K}}
$$

(where $\mathbf{x}_{\mathcal{K}} := \mathbf{x} \circ \mathcal{K}$, etc.) which will be mentioned in the next section. They are:

$$
U(\Lambda, G, \Theta, r, \ell, g) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\ell}{\sqrt{r^2 + 2ra(\Lambda)\sqrt{1 - \frac{\Theta^2}{G^2}} p(\Lambda, G, \ell, g) + a(\Lambda)^2 \varrho(\Lambda, G, g)^2}}
$$

$$
E(\Lambda, G, \Theta, r, \ell, g) = G^2 + m_0^3 r \sqrt{1 - \frac{\Theta^2}{G^2}} \sqrt{1 - \frac{G^2}{\Lambda^2}} \cos g
\tag{16}
$$

where $a = a(\Lambda)$, the *semimajor axis*; $e = e(\Lambda, G)$, the *eccentricity* of the ellipse; $\varrho = \varrho(\Lambda, G, \ell)$; $p = p(\Lambda, G, \ell, g)$ are defined as

$$
a(\Lambda) = \frac{\Lambda^2}{m_0^3}
$$

$$
e(\Lambda, G) := \sqrt{1 - \frac{G^2}{\Lambda^2}}
$$

$$
\varrho(\Lambda, G, \ell) := 1 - e(\Lambda, G) \cos \xi(\Lambda, G, \ell)
$$

$$
p(\Lambda, G, \ell, g) := (\cos \xi(\Lambda, G, \ell) - e(\Lambda, G)) \cos g - \frac{G}{\Lambda} \sin \xi(\Lambda, G, \ell),
\tag{17}
$$

with $\xi = \xi(\Lambda, G, \ell)$ the *eccentric anomaly*, defined as the solution of *Kepler equation*

$$
\xi - e(\Lambda, G) \sin \xi = \ell.
\tag{18}
$$

These formulae have been discussed in Pinzari (2020).

2.2 Renormalizable Integrability

We recall some results concerning the functions U and E in (16). We refer to Pinzari (2019) for full details.

Definition 2.1 (Pinzari 2019, Definition 1) Let f, g be two functions of the form

$$f(p, q, y, x) = \widehat{f}(\mathrm{I}(p, q), y, x), \quad g(p, q, y, x) = \widehat{g}(\mathrm{I}(p, q), y, x) \qquad (19)$$

where

$$(p, q, y, x) \in \mathcal{D} := \mathcal{B} \times U \qquad (20)$$

with $U \subset \mathbb{R}^2$, $\mathcal{B} \subset \mathbb{R}^{2n}$ open and connected, $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ conjugate coordinates with respect to the two-form $\omega = dy \wedge dx + \sum_{i=1}^{n} dp_i \wedge dq_i$ and $\mathrm{I}(p, q) = (\mathrm{I}_1(p, q), \ldots, \mathrm{I}_n(p, q))$, with

$$\mathrm{I}_i : \ \mathcal{B} \to \mathbb{R}, \quad i = 1, \ldots, n$$

pairwise Poisson commuting:

$$\{\mathrm{I}_i, \mathrm{I}_j\} = 0 \quad \forall \ 1 \le i < j \le n \ \ i = 1, \ldots, n. \qquad (21)$$

We say that f is *renormalizably integrable by g via $\widetilde{f}$* (or *renormalizably integrable by g*, or simply *renormalizably integrable*), if there exists a function

$$\widetilde{f} : \ \mathrm{I}(\mathcal{B}) \times g(U) \to \mathbb{R},$$

such that

$$f(p, q, y, x) = \widetilde{f}(\mathrm{I}(p, q), \widehat{g}(\mathrm{I}(p, q), y, x)) \qquad (22)$$

for all $(p, q, y, x) \in \mathcal{D}$.

Proposition 2.1 (Pinzari 2019, Proposition 4) *If f is renormalizably integrable by g, then:*

(i) $\mathrm{I}_1, \ldots, \mathrm{I}_n$ *are first integrals to f and g;*
(ii) *f and g Poisson commute.*

Observe that, if f is renormalizably integrable via g, then, generically, their respective time laws for the coordinates (y, x) are the same, up to rescaling the time. Formally:

Proposition 2.2 (Pinzari 2019, Proposition 5) *Let f be renormalizably integrable via g. Fix a value* I_0 *for the integrals* I *and look at the motion of* (y, x) *under f and g, on the manifold* $I = I_0$. *For any fixed initial datum* (y_0, x_0), *let* $g_0 := g(I_0, y_0, x_0)$. *If* $\omega(I_0, g_0) := \partial_g \tilde{f}(I, g)|_{(I_0, g_0)} \neq 0$, *the motion* $(y^f(t), x^f(t))$ *with initial datum* (y_0, x_0) *under f is related to the corresponding motion* $(y^g(t), x^g(t))$ *under g via*

$$y^f(t) = y^g(\omega(I_0, g_0)t), \quad x^f(t) = x^g(\omega(I_0, g_0)t).$$

In particular, under this condition, all the fixed points of g in the plane (y, x) *are fixed points to f. Values of* (I_0, g_0) *for which* $\omega(I_0, g_0) = 0$ *provide, in the plane* (y, x), *curves of fixed points for f (which are not necessarily curves of fixed points to g).*

We observe that U and E have the form in (19), with $I = (I_1, I_2, I_3) = (r, \Lambda, \Theta)$ verifying (21) and $(y, x) = (G, g)$.

Proposition 2.3 (Pinzari 2019, Proposition 6) U *is renormalizably integrable via* E. *Namely, there exists a function* F *such that*

$$U(\Lambda, G, \Theta, G, g, r) = F\big(\Lambda, \Theta, r, E(\Lambda, G, \Theta, G, g, r)\big). \tag{23}$$

The phase portrait of E in the planar case is shown in Figs. 1, 2 and 3, accordingly to the values of ε.

2.3 Asymptotic Action-Angle Coordinates

In this section, we focus on the planar case, i.e., when $\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x} \in \mathbb{R}^2$. In that case, the following 8-dimensional diffeomorphism replaces $\mathcal{K}$ in (15):

$$\mathcal{K}_0 : \begin{cases} C = \|\mathbf{C}\| \\ G = \|\mathbf{M}\| \\ R = \dfrac{\mathbf{y}' \cdot \mathbf{x}'}{\|\mathbf{x}'\|} \\ \Lambda = m\sqrt{\mathcal{M}a} \end{cases} \quad \begin{cases} \gamma = \alpha_{\mathbf{k}}(\mathbf{i}, \mathbf{x}') + \dfrac{\pi}{2} \\ g = \alpha_{\mathbf{k}}(\mathbf{x}', \mathbf{P}) + \pi \\ r = \|\mathbf{x}'\| \\ \ell = \text{mean anomaly of } \mathbf{x} \text{ in } \mathbb{E} \end{cases} \tag{24}$$

$\mathcal{K}_0$ may be regarded as the natural limit of $\mathcal{K}$, once Θ and ϑ are fixed to $(0, 0)$, $(0, \pi)$ (which are the values they take in the planar case), respectively, and (Z, z) are neglected. The functions U and E in (16) become

$$U(\Lambda, G, g, r) := \frac{1}{2\pi} \int_0^{2\pi} \frac{d\ell}{\sqrt{r^2 + 2a(\Lambda)rp(\Lambda, G, \ell, g) + a(\Lambda)^2 \varrho(\Lambda, G, \ell)^2}},$$

$$E(\Lambda, G, g, r) = G^2 + m_0^3 r\sqrt{1 - \frac{G^2}{\Lambda^2}} \cos g \tag{25}$$

where $a(\Lambda)$, $\varrho(\Lambda, G, \ell)$, $p(\Lambda, G, \ell, g)$ are given in (17).

Unfortunately, the action-angle coordinates associated with E are not explicit, since they are defined via inversion of elliptic integrals. However, it is possible to define, explicitly, action-angle coordinates associated with the leading part of E in the case of large r:

$$E_1 := m_0^3 r \sqrt{1 - \frac{G^2}{\Lambda^2}} \cos g.$$

As discussed in Pinzari (2020), these coordinates, denoted as[5] $(\mathcal{G}, \gamma)$, are defined via the canonical[6] change

$$\begin{cases} G = \sqrt{\Lambda^2 - \mathcal{G}^2} \cos \gamma \\ g = -\tan^{-1}\left(\frac{\Lambda}{\mathcal{G}}\sqrt{1 - \frac{\mathcal{G}^2}{\Lambda^2}} \sin \gamma\right) + k\pi \\ \text{with} \quad k = \begin{cases} 0 & \text{if } 0 < \mathcal{G} < \Lambda \\ 1 & \text{if } -\Lambda < \mathcal{G} < 0 \end{cases} \end{cases} \tag{26}$$

for any fixed value of Λ. Observe that positive values of $\mathcal{G}$ (hence, $k = 0$) provide coordinates with the image (G, g) in a neighborhood of (0, 0); negative values ($k = 1$) are for (G, g) in a neighborhood of $(0, \pi)$. Using these "approximate" coordinates, one obtains the expression of E as a close-to-be-integrable system for large r:

$$E = m_0^3 r \frac{\mathcal{G}}{\Lambda} + (\Lambda^2 - \mathcal{G}^2) \cos^2 \gamma \tag{27}$$

The coordinates $(\mathcal{G}, \gamma)$ will be used in Sect. 5.

3 A Deeper Look at the Planar Case

In the planar case, the relation (23) becomes very special.

Instead of U and E, it is convenient to switch to the functions

$$\widehat{U}_\varepsilon(\Lambda, G, g) := \frac{1}{2\pi} \int_0^{2\pi} \frac{d\ell}{\sqrt{1 + 2\varepsilon p(\Lambda, G, \ell, g) + \varepsilon^2 \varrho(\Lambda, G, \ell)^2}}$$

---

[5] Beware not to confuse the coordinate γ in (26) with its homonymous in (15). The latter is a cyclic coordinate for the Hamiltonians H_i in (2) and (5) and hence has no rôle in the paper.

[6] The change of coordinates discussed in Pinzari (2020) is a little more general than (26), since it is a four-dimensional map composed by (26) and

$$\begin{cases} \Lambda = \mathcal{L} \\ \ell = \lambda + \arg\left(\cos \gamma, \frac{\mathcal{L}}{|\mathcal{G}|} \sin \gamma\right) \end{cases}.$$

In this paper, we shall only use the two-dimensional projection (26).

$$\widehat{E}_\varepsilon(\Lambda, G, g) := \sqrt{1 - \frac{G^2}{\Lambda^2}} \cos g + \varepsilon \frac{G^2}{\Lambda^2}, \tag{28}$$

which are related to the previous ones via

$$U(\Lambda, G, g, r) := \frac{1}{r} \widehat{U}_{\varepsilon(\Lambda, r)}(\Lambda, G, g), \quad E(\Lambda, G, g, r) := m_0^3 r \, \widehat{E}_{\varepsilon(\Lambda, r)}(\Lambda, G, g) \tag{29}$$

with

$$\varepsilon(\Lambda, r) := \frac{\Lambda^2}{m_0^3 r} = \frac{a(\Lambda)}{r}$$

if $a = a(\Lambda)$ is as in (15). We rewrite relation (23) as

$$\widehat{U}_{\varepsilon(\Lambda, r)}(\Lambda, G, g) = \widehat{F}_{\varepsilon(\Lambda, r)}\big(\widehat{E}_{\varepsilon(\Lambda, r)}(\Lambda, G, g)\big). \tag{30}$$

Here, we have used that $\widehat{F}_\varepsilon$ does not depend explicitly on Λ, since both U and E depend on Λ only via $\frac{G}{\Lambda}$. We claim that

Proposition 3.1 *In the planar problem, if* $|\varepsilon| < \frac{1}{2}$ *and* $|\widehat{E}_\varepsilon| \le 1$, *(30) holds with*

$$\widehat{F}_\varepsilon(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - \cos\xi)d\xi}{\sqrt{1 - 2\varepsilon(1 - \cos\xi)t + \varepsilon^2(1 - \cos\xi)^2}}. \tag{31}$$

To prove Proposition 3.1, we need to recall the following result from Pinzari (2019).

Proposition 3.2 (Pinzari 2019, Theorem 4 and Remark 3) *Let* $f(P, y, x)$ *and* $g(P, y, x)$ *Poisson commute. Assume that, for any fixed* P*, the level sets* $\{(y, x) : g(P, y, x) = \mathcal{G}\}$ *are union of graphs*

$$y = g_{1,i}(P, x, \mathcal{G}) \quad \text{or} \quad x = g_{2,j}(P, y, \mathcal{G}).$$

Then, f *renormalizably integrable by* g *through* $\widetilde{f}$, *and* $\widetilde{f}$ *can be chosen to be*

$$\widetilde{f}(P, \mathcal{G}) = f(P, y_j, g_{2,j}(y_j, \mathcal{G})) = f(P, g_{1,i}(x_i, \mathcal{G}), x_i). \tag{32}$$

for some fixed x_i, y_j.

Proof of Proposition 3.1 We apply Proposition 3.2 to the functions $\widehat{U}_{\varepsilon(\Lambda, r)}(\Lambda, G, g)$, $\widehat{E}_{\varepsilon(\Lambda, r)}(\Lambda, G, g)$ in (29). Such two functions do commute since $U(\Lambda, G, g, r)$ and

$E(\Lambda, G, g, r)$ do, and they both commute with r. Moreover, the level sets $\{(\Lambda, G, g) : \widehat{E}_{\varepsilon(\Lambda, r)}(\Lambda, G, g) = t\}$ are graphs

$$g = \pm \cos^{-1}\left(\frac{t - \varepsilon(\Lambda, r)\frac{G^2}{\Lambda^2}}{\sqrt{1 - \frac{G^2}{\Lambda^2}}}\right) + 2j\pi =: g_j^{\pm}(r, \Lambda, G, t), \quad j \in \mathbb{Z}.$$

We use the formula in (32) with $P = (\Lambda, r)$, $g_{2,j} = g_j^{\pm}(r, \Lambda, G, t)$, $f = U$ and $y_j = G_j = 0$ for all j. When $G = 0$, the functions $g_j^{\pm}$ take the value

$$g_j^{\pm}|_{G=0} = \pm \cos^{-1} t + 2j\pi \quad \forall r, \ \Lambda$$

which is well defined as $|t| = |\widehat{E}_{\varepsilon}| \leq 1$. Then, by (32),

$$\begin{aligned}
\widehat{F}_{\varepsilon}(t) &= \widehat{U}_{\varepsilon}(0, \pm\cos^{-1} t + 2j\pi) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - \cos\xi)d\xi}{\sqrt{1 - 2\varepsilon(1 - \cos\xi)t + \varepsilon^2(1 - \cos\xi)^2}}.
\end{aligned}$$

$\square$

A first consequence of the formula in (31) is underlined in the following.

Remark 3.1 Equation (31) can also be used to provide an expansion of $\widehat{U}_{\varepsilon}$ about the equilibria $(0, 0)$ and $(0, \pi)$. Indeed, $\widehat{E}_{\varepsilon}(\Lambda, G, g)$ takes the value $+1$ at $(\Lambda, G, g) = (0, 0)$, and the value -1 at $(\Lambda, G, g) = (0, \pi)$. Therefore, an expansion of $\widehat{F}_{\varepsilon}(t)$ about ± 1 provides, via (30), an expansion $\widehat{U}_{\varepsilon}$ about the corresponding equilibrium. On the other hand, the value of $\widehat{F}_{\varepsilon}(t)$ and of its derivatives at $t = +1$ or $t = -1$ can be explicitly computed, using the residue theorem. For example, for $0 < \varepsilon < \frac{1}{2}$,

$$\widehat{F}_{\varepsilon}(1) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - \cos\xi)d\xi}{1 - \varepsilon(1 - \cos\xi)} = \frac{2}{\sqrt{1 - 2\varepsilon}(1 + \sqrt{1 - 2\varepsilon})}. \tag{33}$$

Another consequence of (31) is:

Proposition 3.3 *Let $|\varepsilon| < \frac{1}{2}$. The complex function $(\varepsilon, t) \to \widehat{F}_{\varepsilon}(t)$ loses its holomorphy if and only if*

$$4\varepsilon^2 - 4\varepsilon t + 1 = 0. \tag{34}$$

Proof We equivalently write

$$\widehat{F}_{\varepsilon}(t) = \frac{1}{\pi}\int_0^{\pi} \frac{(1 - \cos\xi)d\xi}{\sqrt{1 - 2\varepsilon(1 - \cos\xi)t + \varepsilon^2(1 - \cos\xi)^2}}.$$

Then, we change variable, letting $x = 1 - \cos \xi$. The integral becomes

$$\widehat{F}_\varepsilon(t) = \frac{1}{\pi} \int_0^2 \frac{x \, dx}{\sqrt{x(2-x)} \sqrt{1 - 2\varepsilon x t + \varepsilon^2 x^2}}.$$

The only possibility it diverges is that there are two coinciding roots of the denominator on the real interval $[0, 2]$. The polynomial under the second square root has not coinciding roots on $[0, 2]$ for $|\varepsilon| < \frac{1}{2}$ and never has the root $x = 0$. The only possibility is that it has the root $x = 2$. This happens when $t = \varepsilon + \frac{1}{4\varepsilon}$. □

Remark 3.2 The formula in (31) (and its consequences below) is pretty specific for the planar case. In Pinzari (2019, Equation 49), we proposed a general formula (holding for the planar and the spatial case), which, unfortunately, does not seem equally exploitable.

4 Set Out and Analytic Tools

For definiteness, we refer to perihelion librations about $(0, 0)$, the case $(0, \pi)$ being specular.

Using the identity in (30), the assumption A_3 in the introduction, which gives

$$\|\mathbf{x}'_{\mathcal{K}_0} \times \mathbf{y}'_{\mathcal{K}_0}\| = \|\mathbf{x}_{\mathcal{K}_0} \times \mathbf{y}_{\mathcal{K}_0}\| = G, \tag{35}$$

and the relation

$$\|\mathbf{y}'_{\mathcal{K}_0}\|^2 = \frac{|\mathbf{y}'_{\mathcal{K}_0} \cdot \mathbf{x}'_{\mathcal{K}_0}|^2}{\|\mathbf{x}'_{\mathcal{K}_0}\|^2} + \frac{\|\mathbf{y}'_{\mathcal{K}_0} \times \mathbf{x}'_{\mathcal{K}_0}\|^2}{\|\mathbf{x}'_{\mathcal{K}_0}\|^2} = R^2 + \frac{G^2}{r^2}, \tag{36}$$

we rewrite the Hamiltonians (8) as

$$\widehat{H}_1(R, G, r, g) = \frac{R^2}{2m_0} + \frac{G^2}{2m_0 r^2} - \frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{m_0^2}{r} \widehat{F}_{\beta\varepsilon(r)}\left(\widehat{E}_{\beta\varepsilon(r)}(G, g)\right)$$

$$- \frac{\beta}{\beta + \overline{\beta}} \frac{m_0^2}{r} \widehat{F}_{-\overline{\beta}\varepsilon(r)}\left(\widehat{E}_{-\overline{\beta}\varepsilon(r)}(G, g)\right)$$

$$\widehat{H}_2(R, G, r, g) = \frac{R^2}{2m_0} + \frac{G^2}{2m_0 r^2} - \frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{m_0^2}{r} \widehat{F}_{(\beta+\overline{\beta})\varepsilon(r)}\left(\widehat{E}_{(\beta+\overline{\beta})\varepsilon(r)}(G, g)\right)$$

$$- \frac{\beta}{\beta + \overline{\beta}} \frac{m_0^2}{r} \tag{37}$$

having neglected to write (as well as we shall do below) the dependence on Λ.

We suddenly remark that $\widehat{H}_1$ and $\widehat{H}_2$ are both even with respect to G and g separately, because so are the functions $\widehat{E}_\varepsilon(G, g)$ and the[7] term $\frac{G^2}{2m_0 r^2}$. Then, the manifolds

$$\mathcal{M}_0 := \{(R, G, r, g) : (G, g) = (0, 0)\}, \quad \mathcal{M}_\pi := \{(R, G, r, g) : (G, g) = (0, \pi)\} \tag{38}$$

which, by the discussions of Sect. 2, are invariant for $\widehat{E}_\varepsilon(G, g)$, keep to be so also for $\widehat{H}_i$. We focus on $\mathcal{M}_0$. On $\mathcal{M}_0$, the motions of the coordinates (R, r) are governed by the Hamiltonians

$$h_i(R, r) = \frac{R^2}{2m_0} + V_i(r) \tag{39}$$

where (using (33) and dehomogenizing)

$$V_1(r) = -\frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{2m_0^2}{\sqrt{r - 2\beta a} \left(\sqrt{r} + \sqrt{r - 2\beta a}\right)} - \frac{\beta}{\beta + \overline{\beta}} \frac{2m_0^2}{\sqrt{r + 2\overline{\beta} a} \left(\sqrt{r} + \sqrt{r + 2\overline{\beta} a}\right)}$$

$$V_2(r) = -\frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{2m_0^2}{\sqrt{r - 2(\beta + \overline{\beta})a} \left(\sqrt{r} + \sqrt{r - 2(\beta + \overline{\beta})a}\right)} - \frac{\beta}{\beta + \overline{\beta}} \frac{m_0^2}{r}. \tag{40}$$

The "potentials" V_1 and V_2 are well defined and increasing from $-\infty$ to 0 for $r > 2\beta a$, $r > 2(\beta + \overline{\beta})a$, respectively, so action-angle coordinates do not exist. In other words, closely to $\mathcal{M}_0$, $\widehat{H}_i$ has not close to an integrable system in the sense of Liouville–Arnold (Arnold (1963a)) and hence standard perturbation theory does not apply. In the next section, we develop an analytic theory suited to this case. It will be used to "decouple" the Hamiltonians.

4.1 A Normal Form Theory Without Quasiperiodic Unperturbed Motions

In this section, we describe a procedure for eliminating the angles[8] $\boldsymbol{\varphi}$ at high orders, given Hamiltonian of the form

$$H(\mathbf{I}, \boldsymbol{\varphi}, \mathbf{p}, \mathbf{q}, y, x) = h(\mathbf{I}, \mathbf{J}(\mathbf{p}, \mathbf{q}), y) + f(\mathbf{I}, \boldsymbol{\varphi}, \mathbf{p}, \mathbf{q}, y, x) \tag{41}$$

which we assume to be holomorphic on the neighborhood

$$\mathbb{P}_{\rho,s,\delta,r,\xi} = \mathbb{I}_\rho \times \mathbb{T}_s^n \times \mathbb{B}_\delta \times \mathbb{Y}_r \times \mathbb{X}_\xi \supset \mathbb{P} = \mathbb{I} \times \mathbb{T}^n \times \mathbb{B} \times \mathbb{Y} \times \mathbb{X},$$

---

[7] Note that this circumstance would not hold without assuming A_3, since, in that case, instead of the term $\frac{G^2}{r^2}$, we would have $\frac{(C-G)^2}{r^2}$ or $\frac{(C+G)^2}{r^2}$ (depending on the verse of rotation of $\mathbf{x}'$), which would break the symmetry.

[8] Note that the procedure described in this section does not seem to be related to Arnold (2006, §6.4.4), for the lack of slow–fast couples.

and

$$\mathbf{J}(\mathbf{p}, \mathbf{q}) = (p_1 q_1, \ldots, p_m q_m).$$

Here, $\mathbb{I} \subset \mathbb{R}^n$, $\mathbb{B} \subset \mathbb{R}^{2m}$, $\mathbb{Y} \subset \mathbb{R}$, $\mathbb{X} \subset \mathbb{R}$ are open and connected; $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ is the standard torus.

We denote as $\mathcal{O}_{\rho,s,\delta,r,\xi}$ the set of complex holomorphic functions

$$\phi : \quad \mathbb{P}_{\hat{\rho},\hat{s},\hat{\delta},\hat{r},\hat{\xi}} \to \mathbb{C}$$

for some $\hat{\rho} > \rho$, $\hat{s} > s$, $\hat{\delta} > \delta$, $\hat{r} > r$, $\hat{\xi} > \xi$, equipped with the norm

$$\|\phi\|_{\rho,s,\delta,r,\xi} := \sum_{k,h,j} \|\phi_{khj}\|_{\rho,r,\xi} e^{s|k|} \delta^{h+j}$$

where $\phi_{khj}(\mathbf{I}, y, x)$ are the coefficients of the Taylor–Fourier expansion[9]

$$\phi = \sum_{k,h,j} \phi_{khj}(\mathbf{I}, y, x) e^{iks} \mathbf{p}^h \mathbf{q}^j, \quad \|\phi\|_{\rho,r,\xi} := \sup_{\mathbb{I}_\rho \times \mathbb{Y}_r \times \mathbb{X}_\xi} |\phi(\mathbf{I}, y, x)|.$$

If ϕ is independent of x, we simply write $\|\phi\|_{\rho,r}$ for $\|\phi\|_{\rho,r,\xi}$.

If $\phi \in \mathcal{O}_{\rho,s,\delta,r,\xi}$, we define its "off-average" $\widetilde{\phi}$ and "average" $\overline{\phi}$ as

$$\widetilde{\phi} := \sum_{\substack{k,h,j: \\ (k,h-j)\neq(0,0)}} \phi_{khj}(\mathbf{I}, y, x) e^{iks} \mathbf{p}^h \mathbf{q}^j$$

$$\overline{\phi} := \phi - \widetilde{\phi} = \frac{1}{2\pi^2} \int_{[0,2\pi]^n} \Pi_{\mathbf{pq}} \phi(\mathbf{I}, \boldsymbol{\varphi}, \mathbf{J}(\mathbf{p}, \mathbf{q}), y, x) d\boldsymbol{\varphi},$$

with

$$\Pi_{\mathbf{pq}} \phi(\mathbf{I}, \boldsymbol{\varphi}, \mathbf{J}(\mathbf{p}, \mathbf{q}), y, x) := \sum_{k,h} \phi_{khh}(\mathbf{I}, y, x) e^{iks} \mathbf{p}^h \mathbf{q}^h.$$

We decompose

$$\mathcal{O}_{\rho,s,\delta,r,\xi} = \mathcal{Z}_{\rho,s,\delta,r,\xi} \oplus \mathcal{N}_{\rho,s,\delta,r,\xi} ,$$

where $\mathcal{Z}_{\rho,s,\delta,r,\xi}, \mathcal{N}_{\rho,s,\delta,r,\xi}$ are the "zero-average" and the "normal" classes

$$\mathcal{Z}_{\rho,s,\delta,r,\xi} := \{\phi \in \mathcal{O}_{\rho,s,\delta,r,\xi} : \quad \phi = \widetilde{\phi}\} = \{\phi \in \mathcal{O}_{\rho,s,\delta,r,\xi} : \quad \overline{\phi} = 0\} \quad (42)$$
$$\mathcal{N}_{\rho,s,\delta,r,\xi} := \{\phi \in \mathcal{O}_{\rho,s,\delta,r,\xi} : \quad \phi = \overline{\phi}\} = \{\phi \in \mathcal{O}_{\rho,s,\delta,r,\xi} : \quad \widetilde{\phi} = 0\}, \quad (43)$$

respectively. We finally let $\omega_{y,\mathbf{I},\mathbf{J}} := \partial_{y,\mathbf{I},\mathbf{J}} h$.

---

[9] We denote as $\mathbf{x}^h := x_1^{h_1} \ldots x_n^{h_n}$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $h = (h_1, \ldots, h_n) \in \mathbb{N}^n$.

We shall prove the following result. Its peculiarity is that it does not need any non-resonance condition on the frequencies $\omega_{\mathbf{I}}$, which, as a matter of fact, might also be zero.

Theorem 4.1 *For any n, m, there exists a number $c_{n,m} \geq 1$ such that, for any $N \in \mathbb{N}$ such that, the following inequalities are satisfied:*

$$4N\mathcal{X} \left\| \Im \frac{\omega_{\mathbf{I}}}{\omega_y} \right\|_{\rho,r} < s, \quad 4N\mathcal{X} \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\rho,r} < 1, \quad \widetilde{c}_{n,m} N \frac{\mathcal{X}}{\mathrm{d}} \|f\|_{\rho,s,\delta,r,\xi} \left\| \frac{1}{\omega_y} \right\|_{\rho,s,\delta,r,\xi} < 1 \tag{44}$$

with $\mathrm{d} := \min\left\{ \rho s, r\xi, \delta^2 \right\}$, $\mathcal{X} := \sup\left\{ |x| : x \in \mathbb{X}_\xi \right\}$, one can find an operator

$$\Psi_* : \quad \mathcal{O}_{\rho,s,\delta,r,\xi} \to \mathcal{O}_{1/3(\rho,s,\delta,r,\xi)} \tag{45}$$

which carries H to

$$\mathrm{H}_* = \mathrm{h} + g_* + f_*$$

where $g_* \in \mathcal{N}_{1/3(\rho,s,\delta,r,\xi)}$, $f_* \in \mathcal{O}_{1/3(\rho,s,\delta,r,\xi)}$ and, moreover, the following inequalities hold:

$$\|g_* - \overline{f}\|_{1/3(\rho,s,\delta,r,\xi)} \leq 162\widetilde{c}_{n,m} \frac{\mathcal{X}}{\mathrm{d}} \left\| \frac{\widetilde{f}}{\omega_y} \right\|_{\rho,s,\delta,r,\xi} \|f\|_{\rho,s,\delta,r,\xi}$$

$$\|f_*\|_{1/3(\rho,s,\delta,r,\xi)} \leq \frac{1}{2^{N+1}} \|f\|_{\rho,s,\delta,r,\xi}. \tag{46}$$

The transformation Ψ_* can be obtained as a composition of time-one Hamiltonian flows and satisfies the following. If

$$(\mathbf{I}, \boldsymbol{\varphi}, \mathbf{p}, \mathbf{q}, y, x) := \Psi_*(\mathbf{I}_*, \boldsymbol{\varphi}_*, \mathbf{p}_*, \mathbf{q}_*, R_*, r_*),$$

the following uniform bounds hold:

$$\mathrm{d} \max\left\{ \frac{|\mathbf{I} - \mathbf{I}_*|}{\rho}, \frac{|\boldsymbol{\varphi} - \boldsymbol{\varphi}_*|}{s}, \frac{|\mathbf{p} - \mathbf{p}_*|}{\delta}, \frac{|\mathbf{q} - \mathbf{q}_*|}{\delta}, \frac{|y - y_*|}{r}, \frac{|x - x_*|}{\xi} \right\}$$

$$\leq \max\left\{ s|\mathbf{I} - \mathbf{I}_*|, \rho|\boldsymbol{\varphi} - \boldsymbol{\varphi}_*|, \delta|\mathbf{p} - \mathbf{p}_*|, \delta|\mathbf{q} - \mathbf{q}_*|, \xi|y - y_*|, r|x - x_*| \right\}$$

$$\leq 19\mathcal{X} \left\| \frac{f}{\omega_y} \right\|_{\rho,s,\delta,r,\xi}. \tag{47}$$

Remark 4.1 (Extensions)

(i) There is an obvious extension to the case that $\mathbb{I}_\rho$, $\mathbb{T}_s^n$ are replaced with $(\mathbb{I}_1)_{\rho_1} \times \cdots \times (\mathbb{I}_n)_{\rho_n}$, $\mathbb{T}_{s_1} \times \cdots \times \mathbb{T}_{s_n}$. In this case, the number s in the former equation in (44) is to be replaced with $\min_i\{s_i\}$. Moreover, the product ρs in the definition

of d is to be replaced with $\min_i\{\rho_i\,s_i\}$. Finally, the bound in (47) is to be changed taking into account the different sizes.

(ii) If f does not depend on some angle $\varphi_1, \ldots, \varphi_p$, the vector $\omega_{\mathbf{I}}$ in (44) is to be replaced with $\widehat{\omega}_{\mathbf{I}} := (\omega_{\mathbf{I}_{p+1}}, \ldots, \omega_{\mathbf{I}_n})$.

4.2 Outline of the Proof

The complete proof of Theorem 4.1 is provided in Appendix A, but here we aim to spend some word, so as to highlight the main ideas.

We proceed by recursion. We assume that, at a certain step, we have a system of the form

$$\mathrm{H}(\mathbf{I}, \boldsymbol{\varphi}, \mathbf{J}(\mathbf{p}, \mathbf{q}), y) = \mathrm{h}(\mathbf{I}, \mathbf{J}(\mathbf{p}, \mathbf{q}), y) + g(\mathbf{I}, \mathbf{J}(\mathbf{p}, \mathbf{q}), y, x) + f(\mathbf{I}, \boldsymbol{\varphi}, \mathbf{J}(\mathbf{p}, \mathbf{q}), y, x)$$

$$(48)$$

where $f \in \mathcal{O}_{\rho,s,\delta,r,\xi}$, $g \in \mathcal{N}_{\rho,s,\delta,r,\xi}$. At the first step, just take $g \equiv 0$.

After splitting f on its Taylor–Fourier basis

$$f = \sum_{k,h,j} f_{khj}(\mathbf{I}, y, x)e^{ik\cdot\boldsymbol{\varphi}}\mathbf{p}^h\mathbf{q}^j,$$

one looks for a time-one map

$$\Phi = e^{\mathcal{L}_\phi} = \sum_{k=0}^{\infty} \frac{\mathcal{L}_\phi^k}{k!} \quad \mathcal{L}_\phi(f) := \{\phi,\ f\}$$

generated by a small Hamiltonian ϕ which will be taken in the class $\mathcal{Z}_{\rho,s,\delta,r,\xi}$ in (42). Here,

$$\{\phi,\ f\} := \sum_{i=1}^{n}(\partial_{\mathbf{I}_i}\phi\,\partial_{\boldsymbol{\varphi}_i}f - \partial_{\mathbf{I}_i}f\,\partial_{\boldsymbol{\varphi}_i}\phi)$$

$$+ \sum_{i=1}^{m}(\partial_{\mathbf{p}_i}\phi\,\partial_{\mathbf{q}_i}f - \partial_{\mathbf{p}_i}f\,\partial_{\mathbf{q}_i}\phi) + (\partial_y\phi\partial_x f - \partial_y f\partial_x\phi)$$

denotes the Poisson parentheses of ϕ and f. One lets

$$\phi = \sum_{\substack{(k,h,j):\\(k,h-j)\neq(0,0)}} \phi_{khj}(\mathbf{I}, y, x)e^{ik\cdot\boldsymbol{\varphi}}\mathbf{p}^h\mathbf{q}^j. \qquad (49)$$

The operation

$$\phi \to \{\phi, \mathrm{h}\}$$

acts diagonally on the monomials in the expansion (49), carrying

$$\phi_{khj} \rightarrow -\left(\omega_y \partial_x \phi_{khj} + \lambda_{khj} \phi_{khj}\right), \quad \text{with} \quad \lambda_{khj} := (h - j) \cdot \omega_{\mathbf{J}} + ik \cdot \omega_{\mathbf{I}}. \quad (50)$$

Therefore, one defines

$$\{\phi, h\} =: -D_\omega \phi.$$

The formal application of $\Phi = e^{\mathcal{L}_\phi}$ yields:

$$e^{\mathcal{L}_\phi} H = e^{\mathcal{L}_\phi}(h + g + f) = h + g - D_\omega \phi + f + \Phi_2(h) + \Phi_1(g) + \Phi_1(f) \quad (51)$$

where the Φ_h's are the tails of $e^{\mathcal{L}_\phi}$, defined in Appendix A.

Next, one requires that the residual term $-D_\omega \phi + f$ lies in the class $\mathcal{N}_{\rho,s,\delta,r,\xi}$ in (43)

$$(-D_\omega \phi + f) \in \mathcal{N}_{\rho,s,\delta,r,\xi} \quad (52)$$

for ϕ.

Since we have chosen $\phi \in \mathcal{Z}_{\rho,s,\delta,r,\xi}$, by (50), we have that also $D_\omega \phi \in \mathcal{Z}_{\rho,s,\delta,r,\xi}$. So, Eq. (52) becomes

$$-D_\omega \phi + \widetilde{f} = 0.$$

In terms of the Taylor–Fourier modes, the equation becomes

$$\omega_y \partial_x \phi_{khj} + \lambda_{khj} \phi_{khj} = f_{khj} \quad \forall (k, h, j) : (k, h - j) \neq (0, 0). \quad (53)$$

In the standard situation, one typically proceeds to solve such equation via Fourier series:

$$f_{khj}(\mathbf{I}, y, x) = \sum_\ell f_{khj\ell}(\mathbf{I}, y) e^{i\ell x}, \quad \phi_{khj}(\mathbf{I}, y, x) = \sum_\ell \phi_{khj\ell}(\mathbf{I}, y) e^{i\ell x}$$

so as to find $\phi_{khj\ell} = \dfrac{f_{khj\ell}}{\mu_{khj\ell}}$ with the usual denominators $\mu_{khj\ell} := \lambda_{khj} + i\ell\omega_y$ which one requires not to vanish via, e.g., a "diophantine inequality" to be held for all (k, h, j, ℓ) with $(k, h - j) \neq (0, 0)$. In this standard case, there is not much freedom in the choice of ϕ. In fact, such solution is determined up to solutions of the homogenous equation

$$D_\omega \phi_0 = 0 \quad (54)$$

which, in view of the Diophantine condition, has the only trivial solution $\phi_0 \equiv 0$. *The situation is different if f is not periodic in x, or ϕ is not needed so. In such a case, it*

is possible to find a solution of (53), corresponding to a non-trivial solution of (54), where small divisors do not appear. This is

$$
\phi_{khj}(\mathbf{I}, y, x) = \begin{cases} \dfrac{1}{\omega_y} \displaystyle\int_0^x f_{khj}(\mathbf{I}, y, \tau) e^{\frac{\lambda_{khj}}{\omega_y}(\tau - x)} \mathrm{d}\tau & \text{if } (k, h - j) \neq (0, 0) \\[4mm] 0 & \text{otherwise.} \end{cases} \tag{55}
$$

Multiplying by $e^{ik\varphi}$ and summing over k, h and j, we obtain

$$
\phi(\mathbf{I}, \boldsymbol{\varphi}, p, q, y, x) = \frac{1}{\omega_y} \int_0^x \widetilde{f}\left(\mathbf{I}, \boldsymbol{\varphi} + \frac{\omega_{\mathbf{I}}}{\omega_y}(\tau - x), \, pe^{\frac{\omega_{\mathbf{I}}}{\omega_y}(\tau - x)}, \, qe^{-\frac{\omega_{\mathbf{I}}}{\omega_y}(\tau - x)}, \, y, \tau\right) \mathrm{d}\tau. \tag{56}
$$

In Appendix A, we shall prove that, under the assumptions (44), this function can be used to obtain a convergent time-one map and that the construction can be iterated so as to provide the proof of Theorem 4.1. The construction of the iterations and the proof of its convergence are obtained adapting the techniques of Pöschel (1993) to the present case.

5 Proof of Theorem 1.1

The purpose of this section is state and prove a more precise version of Theorem 1.1 (Theorem 5.1). We shall obtain it as an application of Theorem 4.1 to the Hamiltonians $\widehat{H}_i$. Therefore, we need to introduce a change in new coordinates $\mathcal{C}$ which put the $\widehat{H}_i$'s in the suited form (41). Since the potentials V_i in (40) are, for large r, asymptotic to $-\frac{m_0^2}{r}$, it is convenient to rewrite the functions $\widehat{H}_i$ in (37) as

$$
\widehat{H}_1(R, G, r, g) = \left(\frac{R^2}{2m_0} - \frac{m_0^2}{r}\right) + \frac{G^2}{2m_0 r^2} - \frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{m_0^2}{r} \left(\widehat{F}_{-\overline{\beta}\varepsilon(r)}\left(\widehat{E}_{\beta\varepsilon(r)}(G, g)\right) - 1\right)
$$
$$
\qquad\qquad - \frac{\beta}{\beta + \overline{\beta}} \frac{m_0^2}{r} \left(\widehat{F}_{-\overline{\beta}\varepsilon(r)}\left(\widehat{E}_{-\overline{\beta}\varepsilon(r)}(G, g)\right) - 1\right)
$$
$$
\widehat{H}_2(R, G, r, g) = \left(\frac{R^2}{2m_0} - \frac{m_0^2}{r}\right) + \frac{G^2}{2m_0 r^2} - \frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{m_0^2}{r} \left(\widehat{F}_{(\beta + \overline{\beta})\varepsilon(r)}\left(\widehat{E}_{-\overline{\beta}\varepsilon(r)}(G, g)\right) - 1\right)
$$

and take $\mathcal{C}$ as the composition of two independent and canonical changes

$$
\mathcal{C}_1 : (\mathcal{G}, \gamma) \rightarrow (G, g) , \quad \mathcal{C}_2 : (y, x) \rightarrow (R, r)
$$

where $\mathcal{C}_1$ is defined as in (26), with $k = 0$, while $\mathcal{C}_2$ is defined via the formulae

$$
\begin{cases}
R(y, x) = \dfrac{m_0^3}{y} \sqrt{\dfrac{\cos \xi'(x) + 1}{1 - \cos \xi'(x)}} \\[2em]
r(y, x) = \dfrac{y^2}{m_0^3}(1 - \cos \xi'(x))
\end{cases}
\tag{57}
$$

where $\xi'(x)$ solves

$$
\xi' - \sin \xi' = x.
\tag{58}
$$

$\mathcal{C}_2$ has been chosen so that

$$
H_\omega \circ \mathcal{C}_1 = \left(\frac{R^2}{2m_0} - \frac{m_0^2}{r} \right) \circ \mathcal{C}_1 = -\frac{m_0^5}{2y^2}.
$$

Using the new coordinates, we have

$$
\begin{aligned}
\widehat{H}_1 &= -\frac{m_0^5}{2y^2} + \frac{m_0^2}{r(y, x)} \left(\varepsilon(y, x) \frac{(\Lambda^2 - \mathcal{G}^2)}{2\Lambda^2} \cos^2 \gamma - \frac{\overline{\beta}}{\beta + \overline{\beta}} \left(\widehat{F}_{\beta \varepsilon(y,x)} \left(\widehat{E}_{\beta \varepsilon(y,x)}(\mathcal{G}, \gamma) \right) - 1 \right) \right. \\
&\quad \left. - \frac{\beta}{\beta + \overline{\beta}} \left(\widehat{F}_{-\overline{\beta}\varepsilon(y,x)} \left(\widehat{E}_{-\overline{\beta}\varepsilon(y,x)}(\mathcal{G}, \gamma) \right) - 1 \right) \right) \\
\widehat{H}_2 &= -\frac{m_0^5}{2y^2} + \frac{m_0^2}{r(y, x)} \left(\varepsilon(y, x) \frac{(\Lambda^2 - \mathcal{G}^2)}{2\Lambda^2} \cos^2 \gamma \right. \\
&\quad \left. - \frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{m_0^2}{r(y, x)} \left(\widehat{F}_{(\beta + \overline{\beta})\varepsilon(y,x)} \left(\widehat{E}_{(\beta + \overline{\beta})\varepsilon(y,x)}(\mathcal{G}, \gamma) \right) - 1 \right) \right)
\end{aligned}
$$

having abusively denoted as $\varepsilon(y, x)$ the function $\varepsilon(r(y, x))$, and using the formulae in (27)–(29)

$$
\widehat{E}_\varepsilon(\mathcal{G}, \gamma) := \frac{\mathcal{G}}{\Lambda} + \varepsilon \left(1 - \frac{\mathcal{G}^2}{\Lambda^2} \right) \cos^2 \gamma.
\tag{59}
$$

We now determine a domain of holomorphy of $\widehat{H}_i$. Recall that we use, as mass parameters, the numbers β_*, β^* in (10). For the coordinates (y, x), we choose the complex domains $\mathbb{Y}_{\sqrt{m_0^3 \alpha_-}}$, $\mathbb{X}_{\sqrt{\varepsilon_0}}$, where

$$
\mathbb{Y} := \left\{ y \in \mathbb{R} : 2\sqrt{m_0^3 \alpha_-} < y < \sqrt{m_0^3 \alpha_+} \right\}, \quad \mathbb{X} := \left\{ x \in \mathbb{R} : |x - \pi| \le \pi - 2\sqrt{\varepsilon_0} \right\}
\tag{60}
$$

with $0 < \varepsilon_0 < 1$, $0 < \alpha_- < \alpha_+/4$ verifying

$$\alpha_- \varepsilon_0 > \frac{4\beta^* a}{c_0} \quad \text{with} \tag{61}$$

with β^* as in (10) and c_0 being such that for any $0 < \varepsilon_0 < 1$ and for any $x \in \mathcal{X}_{\sqrt{\varepsilon_0}}$, Eq. (58) has a unique solution $\xi'(x)$ which depends analytically on x and verifies

$$|1 - \cos \xi'(x)| \geq c_0 \varepsilon_0. \tag{62}$$

(The existence of such a number c_0 is well known.) For the coordinates $\mathcal{G}, \gamma$, we choose the domains $\mathbb{G}_\delta$, $\mathbb{T}_{s_0}$, with $0 < \delta < \Lambda$, s_0 fixed, and $\mathbb{G} := \left\{ \mathcal{G} \in \mathbb{R} : \Lambda - \delta < \mathcal{G} < \Lambda \right\}$. Remark that, since, in $\widehat{\mathrm{H}}_i$, there is no dependence of G, but only on G^2, $\mathcal{G} = \Lambda$ is a regular point for $\widehat{\mathrm{H}}_i$. Then, we let

$$\mathbb{D} := \mathbb{G} \times \mathbb{T} \times \mathbb{Y} \times \mathbb{X} \subset \mathbb{R}^4$$

and

$$\mathbb{D}_{\delta, s_0, \sqrt{m_0^3 \alpha_-}, \sqrt{\varepsilon_0}} := \mathbb{G}_\delta \times \mathbb{T}_{s_0} \times \mathbb{Y}_{\sqrt{m_0^3 \alpha_-}} \times \mathbb{X}_{\sqrt{\varepsilon_0}} \subset \mathbb{C}^4.$$

We now check that, under the further assumptions

$$0 < \delta \leq \frac{\Lambda}{4}, \quad C^*(s_0) \frac{\delta}{\Lambda} < 1 \quad C^*(s_0) := 16 \left(\sup_{\mathbb{T}_{s_0}} |\sin \gamma| \right)^2 \tag{63}$$

$\widehat{\mathrm{H}}_i$ are holomorphic functions on the domain

$$\mathbb{D}_{\delta, s_0, \sqrt{m_0^3 \alpha_-}, \sqrt{\varepsilon_0}} := \mathbb{Y}_{\sqrt{m_0^3 \alpha_-}} \times \mathbb{X}_{\sqrt{\varepsilon_0}} \times \mathbb{G}_\delta \times \mathbb{T}_{s_0}. \tag{64}$$

By (62), the first equation in (60) and the expression of $\mathrm{r}(y, x)$ in (57), we have

$$|\mathrm{r}(y, x)| \geq c_0 \alpha_- \varepsilon_0 \tag{65}$$

and hence, because of (61),

$$|\beta^* \varepsilon(y, x)| = \left| \frac{\beta^* a}{\mathrm{r}(y, x)} \right| \leq \frac{\beta^* a}{c_0 \alpha_- \varepsilon_0} < \frac{1}{4}. \tag{66}$$

By inequalities (65)–(66) and Proposition 3.3, we only need to check that, if $\varepsilon_* \in \{\beta\varepsilon, -\bar{\beta}\varepsilon\}$ for $i = 1$ and $\varepsilon_* = (\beta + \bar{\beta})\varepsilon$ for $i = 2$, then Eq. (34) with $\varepsilon = \varepsilon_*$ and $t = \widehat{\mathrm{E}}_{\varepsilon_*}(\mathcal{G}, \gamma)$ has not solutions in $\mathbb{D}_{\delta, s_0, \sqrt{m_0^3 \alpha_-}, \sqrt{\varepsilon_0}}$. We prove that any such solution

would verify $|\varepsilon_*| \geq \frac{1}{4}$, which would contradict (66), as $|\beta^*\varepsilon| \geq |\varepsilon_*|$. Using (59), Eq. (34) with $\varepsilon = \varepsilon_*$ and $t = \widehat{\mathrm{E}}_{\varepsilon_*}(\mathcal{G}, \gamma)$ is

$$4\varepsilon_*^2 \left(1 - \left(1 - \frac{\mathcal{G}^2}{\Lambda^2}\right)\cos^2\gamma\right) - 4\frac{\mathcal{G}}{\Lambda}\varepsilon_* + 1 = 0 .$$

We solve for ε_*:

$$\varepsilon_* = \frac{1}{2}\left(\frac{\mathcal{G}}{\Lambda} + \sin\gamma\sqrt{\frac{\mathcal{G}^2}{\Lambda^2} - 1}\right)$$

with the double determination of the square root. Since $\left|\frac{\mathcal{G}}{\Lambda}\right| \geq 1 - \frac{\delta}{\Lambda} \geq \frac{3}{4}$ and, if $c_*(s_0) := \sup_{\mathbb{T}_{s_0}} |\sin\gamma|$, $\left|\sin\gamma\sqrt{\frac{\mathcal{G}^2}{\Lambda^2} - 1}\right| \leq c^*(s_0)\sqrt{\frac{\delta}{\Lambda}} \leq \frac{1}{4}$, we have $|\varepsilon_*| \geq \frac{1}{2}\left(\frac{3}{4} - \frac{1}{4}\right) = \frac{1}{4}$, as claimed.

We are now ready to state the result. Observe that the motions

$$\mathcal{G} = \mathcal{G}_0 = \text{constant}, \quad |\gamma(T) - \gamma(0)| = 2\pi$$

correspond, using the coordinates (G, g), to librations about (0, 0) if $0 < \mathcal{G}_0 < \Lambda$; about $(0, \pi)$ if $-\Lambda < \mathcal{G}_0 < 0$. We shall prove the existence, in $\widehat{H}_i$'s, of motions close to these ones.

Theorem 5.1 *Let α_-, α_+, β, $\overline{\beta}$, δ, ε_0 and s_0 be fixed; β_*, β^* as in (10). There exist two numbers $C^* > C_* > 1$, both independent of α_-, α_+, β, $\overline{\beta}$, δ, ε_0, with C^* possibly depending on s_0, while C_* independent of s_0, such that, if the following inequalities are satisfied*

$$0 < \varepsilon_0 < 1, \quad 0 < \delta \leq \frac{\Lambda}{4}, \quad \frac{4\beta^* a}{c_0\alpha_-\varepsilon_0} < 1, \quad \frac{C^*\delta}{\beta_*\Lambda} \leq 1$$

$$\frac{1}{N_0} := C_* \max\left\{\frac{\beta_*\Lambda}{c_0^2\varepsilon_0^2\delta s_0}\sqrt{\frac{a}{\alpha_-}} , \frac{\beta_*}{c_0^2\varepsilon_0^{5/2}}\frac{a}{\alpha_-}\right\}\frac{\alpha_+^{3/2}}{\alpha_-^{3/2}} < \frac{c_0^2\varepsilon_0^2\alpha_-^2}{2\alpha_+^2} \quad (67)$$

it is possible to find new coordinates $(\mathcal{G}_*, \gamma_*, y_*, x_*)$ and a time T such that any solution $\Gamma_*(t) = (\mathcal{G}_*(t), \gamma_*(t), y_*(t), x_*(t))$ of $\widehat{H}_i$ with initial datum $\Gamma_*(0) = (\mathcal{G}_*(0), \gamma_*(0), y_*(0), x_*(0)) \in \mathbb{D}$ such that

$$|\mathcal{G}(0) - \Lambda| \leq \frac{\delta}{2}, \quad 2\sqrt{m_0^3\alpha_-} \leq |y_*(0)| \leq \frac{\sqrt{m_0^3\alpha_-} + \sqrt{m_0^3\alpha_+}}{2}, \quad x_*(0) = \pi \quad (68)$$

stays in $\mathbb{D}$ for all $0 \leq t \leq T$ and $\mathcal{G}_*$ varies a little in the course of such time:

$$|\mathcal{G}_*(t) - \mathcal{G}_*(0)| \leq C_* \frac{2^{-N_0}}{s_0}\frac{m_0^2 a\beta_*}{c_0^2\varepsilon_0^2\alpha_-^2}t \quad \text{for all } 0 \leq t \leq T.$$

If, in addition,

$$\eta := C_* \max \left\{ \frac{\alpha_+^2}{\beta_* \sqrt{\alpha_-^3 a}} \ , \ \frac{\alpha_+^2}{c_0^2 \varepsilon_0^{5/2} \alpha_-^2} \sqrt{\frac{a}{\alpha_-}} \ , \ \frac{\alpha_+^2}{c_0^2 \varepsilon_0^2 \alpha_-^2} \frac{\Lambda}{s_0 \delta} 2^{-N_0} \right\} < 1, \quad (69)$$

then motions close to librations occur, in the sense that, also

$$|\gamma_*(T) - \gamma_*(0)| \geq 3\pi.$$

The time T can be taken to be

$$T = \frac{\Lambda \alpha_+^3}{\beta_* m_0^2 a} \frac{3\pi}{\eta}. \quad (70)$$

Finally, the change

$$(\mathcal{G}_*, \gamma_*, y_*, x_*) \rightarrow (\mathcal{G}, \gamma, y, x)$$

is real-analytic and close to the identity, in the sense that

$$|\mathcal{G} - \mathcal{G}_*| \leq \frac{\Lambda}{N_0}, \quad |\gamma - \gamma_*| \leq \frac{s_0}{N_0}, \quad |y - y_*| \leq \frac{\sqrt{m_0^3 \alpha_-}}{N_0}, \quad |x - x_*| \leq \frac{\sqrt{\varepsilon_0}}{N_0}.$$

Remark 5.1 (Proof of Theorem 1.1) Inequalities (67), (68) and (69) are simultaneously satisfied provided that the following holds. Fix $0 < \varepsilon_0 < 1$ and $0 < \delta \leq \frac{\Lambda}{4}$ once forever. Then, identify δ as the size of U_0 and 2^{-N_0} as the size of V_0. Take

$$s_0 \vartriangleright \frac{\Lambda}{\delta \varepsilon_0^4}, \quad \frac{\alpha_+}{a} = 2^8 \frac{\alpha_-}{a} \vartriangleright \frac{C^*(s_0)^2}{\varepsilon_0^8}$$

$$\min \left\{ \varepsilon_0^4 \frac{\delta}{\Lambda} s_0 \sqrt{\frac{\alpha_-}{a}}, \ \varepsilon_0^{9/2} \frac{\alpha_-}{a} \right\} \vartriangleright \beta^* \geq \beta_* \vartriangleright \max \left\{ C^* \frac{\delta}{\Lambda}, \ \sqrt{\frac{\alpha_-}{a}} \right\}.$$

For short, we have written "$a \vartriangleright b$" if there exist $c > 1$, independent of δ, Λ, α_-, α_+, a and s_0 such that $a > cb$. Note that here it is essential that α_-, s_0, β_* and β^* can be chosen arbitrarily large.

Proof During the proof, we shall make extensive use of Cauchy[10] inequalities.

---

[10] Observe that the assumptions (67) include, in particular, (61) and (63), so $\widehat{H}_i$'s are holomorphic on the domain (64).

We aim to apply Theorem 4.1, with $\mathbf{I} = \mathcal{G}$, $\boldsymbol{\varphi} = \gamma$, (y, x) as in (57), $\mathrm{h}(y) = -\frac{m_0^5}{2y^2}$ and, finally

$$
f(\mathcal{G}, \gamma, y, x) =
\begin{cases}
\frac{m_0^2}{\mathrm{r}(y,x)} \left(\varepsilon(y, x) \frac{(\Lambda^2 - \mathcal{G}^2)}{2\Lambda^2} \cos^2 \gamma - \frac{\overline{\beta}}{\beta + \overline{\beta}} \left(\widehat{\mathrm{F}}_{\beta\varepsilon(y,x)} \left(\widehat{\mathrm{E}}_{\beta\varepsilon(y,x)}(\mathcal{G}, \gamma) \right) - 1 \right) \right. \\
\left. - \frac{\beta}{\beta + \overline{\beta}} \left(\widehat{\mathrm{F}}_{-\overline{\beta}\varepsilon(y,x)} \left(\widehat{\mathrm{E}}_{-\overline{\beta}\varepsilon(y,x)}(\mathcal{G}, \gamma) \right) - 1 \right) \right) \quad i = 1 \\[2mm]
\frac{m_0^2}{\mathrm{r}(y,x)} \left(\varepsilon(y, x) \frac{(\Lambda^2 - \mathcal{G}^2)}{2\Lambda^2} \cos^2 \gamma \right. \\
\left. - \frac{\overline{\beta}}{\beta + \overline{\beta}} \frac{m_0^2}{\mathrm{r}(y,x)} \left(\widehat{\mathrm{F}}_{(\beta+\overline{\beta})\varepsilon(y,x)} \left(\widehat{\mathrm{E}}_{(\beta+\overline{\beta})\varepsilon(y,x)}(\mathcal{G}, \gamma) \right) - 1 \right) \right) \\
\quad i = 2
\end{cases}
\tag{71}
$$

In our case, $\mathbf{p}$, $\mathbf{q}$ do not exist and the unperturbed term h does not depend on $\mathbf{I} = \mathcal{G}$. Therefore, we have only to verify the last condition in (44). We have

$$
\omega_y = \frac{m_0^5}{y^3}, \quad \mathrm{d} = \min\{\delta s_0, \sqrt{m_0^3 \alpha_- \varepsilon_0}\}, \quad \mathcal{X} = \sqrt{4\pi^2 + \varepsilon_0} \leq 3\pi
$$

(having used $\varepsilon_0 < 1$) and

$$
\left\| \frac{1}{\omega_y} \right\|_{\sqrt{m_0^3 \alpha_-}, \sqrt{\varepsilon_0}, \delta, s_0} \leq 2\sqrt{\frac{\alpha_+^3}{m_0}}
$$

$$
\| f \|_{\sqrt{m_0^3 \alpha_-}, \sqrt{\varepsilon_0}, \delta, s_0} \leq \frac{m_0^2 a}{c_0^2 \varepsilon_0^2 \alpha_-^2} \left(C_1 \frac{\delta}{\Lambda} + C_2 \beta_* \right) \leq C_* \frac{m_0^2 a \beta_*}{c_0^2 \varepsilon_0^2 \alpha_-^2} =: \Delta \tag{72}
$$

with[11] C_1, C_2, C_* independent of α_-, α_+, δ, β_*, β^* but C_1 possibly depending on s_0, while C_2, C_* independent of s_0. We have chosen the number C^* in (67) larger than or equal to $2C_1/C_2$ and the number C_* larger than or equal to $3C_2/2$, so that $\left(C_1 \frac{\delta}{\Lambda} + C_2 \beta_* \right) \leq \frac{3}{2} C_2 \beta_* \leq C_* \beta_*$. We have

$$
\widetilde{c}_{1,0} \frac{\mathcal{X}}{\mathrm{d}} \| f \|_{\sqrt{m_0^3 \alpha_-}, \sqrt{\varepsilon_0}, \delta, s_0} \left\| \frac{1}{\omega_y} \right\|_{\sqrt{m_0^3 \alpha_-}, \sqrt{\varepsilon_0}, \delta, s_0}
$$

$$
\leq C_* \max \left\{ \frac{\beta_* \Lambda}{c_0^2 \varepsilon_0^2 \delta s_0} \sqrt{\frac{a}{\alpha_-}}, \frac{\beta_* \Lambda}{c_0^2 \varepsilon_0^2 \sqrt{m_0^3 \alpha_- \varepsilon_0}} \sqrt{\frac{a}{\alpha_-}} \right\} \frac{\alpha_+^{3/2}}{\alpha_-^{3/2}}
$$

$$
= C_* \max \left\{ \frac{\beta_* \Lambda}{c_0^2 \varepsilon_0^2 \delta s_0} \sqrt{\frac{a}{\alpha_-}}, \frac{\beta_*}{c_0^2 \varepsilon_0^{\frac{5}{2}}} \frac{a}{\alpha_-} \right\} \frac{\alpha_+^{3/2}}{\alpha_-^{3/2}}
$$

---

[11] Split f in (71) as $f = f_1 + f_2$, where $f_1 = \frac{m_0^2}{\mathrm{r}(y,x)} \varepsilon(y, x) \frac{(\Lambda^2 - \mathcal{G}^2)}{2\Lambda^2} \cos^2 \gamma$. The term proportional to C_i corresponds to be a upper bound of $\left\| \frac{f_i}{\omega_y} \right\|$. C_2 can be chosen to be independent of s_0 because $\| f_2 \|$ goes to zero as $s_0 \to \infty$.

$$\leq \frac{1}{N_0}.$$

Therefore, the last condition in (44) is immediately implied by $N < N_0$, with N_0 as in (67). We then find a real-analytic transformation

$$\phi_* : \quad (\mathcal{G}_*, \gamma_*, y_*, x_*) \in \mathbb{D}_{\sqrt{m_0^3 \alpha_-}/3, \sqrt{\varepsilon_0}/3, \delta/3, s_0/3} \quad \to \quad (\mathcal{G}, \gamma, y, x) \in \mathbb{D}_{\delta, s_0, \sqrt{m_0^3 \alpha_-}, \sqrt{\varepsilon_0}}$$

which leads $\widehat{H}_i$ to

$$\widehat{H}_* = h(y_*) + g_*(y_*, x_*, \mathcal{G}_*) + f_*(\mathcal{G}_*, \gamma_*, y_*, x_*) \tag{73}$$

where g_* and f_* satisfy the following bounds:

$$\|g_* - \overline{f}\| \leq 2\Delta, \quad \|g_*\| \leq 2^{-N}\Delta$$

with $\overline{f}(y_*, x_*, \mathcal{G}_*)$ the γ_*-average of $f(y_*, x_*, \mathcal{G}_*, \gamma_*)$ and Δ as in (72). Let now $\Gamma_*(t) = (\mathcal{G}_*(t), \gamma_*(t), y_*(t), x_*(t))$ be a solution of $\widehat{H}_i$ with initial datum $\Gamma_*(0) = (\mathcal{G}_*(0), \gamma_*(0), y_*(0), x_*(0)) \in \mathbb{D}$ and verifying (68). We look for a time $T > 0$ such that $\Gamma_*(t) \in \mathbb{D}$ for all $0 \leq t \leq T$. We show that we can take T as in (70), which, for convenience, we rewrite as

$$T = \min \left\{ \sqrt{\frac{\alpha_-^3}{m_0}}, \frac{\sqrt{m_0^3 \alpha_- \varepsilon_0}}{\Delta}, 2^{N_0} \frac{s_0 \delta}{\Delta} \right\} \tag{74}$$

where Δ is as in (72). Equation (73) implies

$$|y_*(t) - y_*(0)| \leq \frac{\Delta t}{\sqrt{\varepsilon_0}}.$$

So, for $t \leq \frac{\sqrt{m_0^3 \alpha_- \varepsilon_0}}{\Delta}$, we have

$$t \leq \frac{|y_*(0)| - \sqrt{m_0^3 \alpha_-}}{\Delta} \sqrt{\varepsilon_0} \quad \Longrightarrow \quad |y_*(t) - y_*(0)| \leq |y_*(0)| - \sqrt{m_0^3 \alpha_-}$$

namely, $y_*(t) \in \mathbb{Y}$ for $0 \leq t \leq T$. Since $|y| \geq \sqrt{m_0^3 \alpha_-}$ for all this time, we also have

$$|x_*(t) - x_*(0)| \leq \left(\sqrt{\frac{m_0}{\alpha_-^3}} + \frac{\Delta}{\sqrt{m_0^3 \alpha_-}} \right) t.$$

Inequalities $0 < \varepsilon_0 < 1$ and $t \leq \min\left\{ \sqrt{\frac{\alpha_-^3}{m_0}}, \frac{\sqrt{m_0^3 \alpha_-}}{\Delta} \right\}$ imply

$$t \leq \frac{2}{2\max\left\{\sqrt{\frac{m_0}{\alpha_-^3}}, \frac{\Delta}{\sqrt{m_0^3\alpha_-}}\right\}} \leq \frac{\pi - \sqrt{\varepsilon_0}}{\sqrt{\frac{m_0}{\alpha_-^3}} + \frac{\Delta}{\sqrt{m_0^3\alpha_-}}} \implies |x_*(t) - x_*(0)| \leq \pi - \sqrt{\varepsilon_0}$$

and hence $x_*(t) \in \mathbb{X}$ for $0 \leq t \leq T$. Since $0 \leq t \leq 2^N \frac{s_0 \delta}{\Delta}$, we have

$$|\mathcal{G}_*(t) - \mathcal{G}_*(0)| \leq \frac{2^{-(N+1)}\Delta t}{s_0} \leq \frac{\delta}{2}$$

and hence $\mathcal{G}_*(t) \in \mathbb{G}$. Let us now evaluate the variation of γ_* during the time T. We have

$$|\gamma_*(T) - \gamma_*(0)| \geq \inf |\partial_{\mathcal{G}_*}(g_* + f_*)|t \geq \left(\inf |\partial_{\mathcal{G}_*}\overline{f}| - \sup |\partial_{\mathcal{G}_*}(|g_* - \overline{f}| + |f_*|)|\right)T$$
$$\geq \left(\inf |\partial_{\mathcal{G}_*}\overline{f}| - \frac{\Delta}{\Lambda}N_0^{-1}\right)T.$$

Proceeding as in (72) and using Cauchy inequalities, one sees that $\inf |\partial_{\mathcal{G}_*}\overline{f}| \geq c^*\beta_* m_0^2 \frac{a}{\Lambda \alpha_+^2}$. So, using (72) and $N_0^{-1} < \frac{c_0^2 \varepsilon_0^2 \alpha_-^2}{2\alpha_+^2}$,

$$|\gamma_*(T) - \gamma_*(0)| \geq c^*\beta_* m_0^2 \frac{a}{\Lambda \alpha_+^2}\left(1 - \frac{\alpha_+^2}{c_0^2 \varepsilon_0^2 \alpha_-^2 N_0}\right)T \geq \frac{c^*}{2}\beta_* m_0^2 \frac{a}{\Lambda \alpha_+^2}T$$
$$= \frac{c^*}{2}m_0^2 \beta_* \frac{a}{\Lambda \alpha_+^2}\min\left\{\sqrt{\frac{\alpha_-^3}{m_0}}, \frac{\sqrt{m_0^3 \alpha_- \varepsilon_0}}{\Delta}, 2^N\frac{s_0 \delta}{\Delta}\right\}$$
$$= c^\circ \min\left\{\beta_*\sqrt{\frac{\alpha_-^3 a}{\alpha_+^4}}, \frac{c_0^2 \varepsilon_0^{5/2}\alpha_-^2}{\alpha_+^2}\sqrt{\frac{\alpha_-}{a}}, \frac{c_0^2 \varepsilon_0^2 \alpha_-^2}{\alpha_+^2}2^{N_0}s_0\frac{\delta}{\Lambda}\right\} =: \frac{3\pi}{\eta}$$

with c^*, c° independent of $\alpha_-, \alpha_+, \beta, \overline{\beta}, \delta, \varepsilon_0$ and s_0. We then see that $|\gamma_*(T) - \gamma_*(0)|$ is lower-bounded by 3π as soon as the condition in (69) is satisfied. $\qquad\square$

A Proof of Theorem 4.1

In this section, we use the definitions and the notations introduced in Sect. 4.1, plus the following further ones.

Definition A.1 Given h and f as in (41), we call[12] the function (56) NQP-*primitive of f relatively to* h or, simply, NQP-*primitive of f*.

Example A.1 Let $n = 1$ and $f(I, \varphi) = a(I) \cos \varphi$. The NQP-primitive of f is

$$
\phi(I, \varphi, y, x) = \frac{1}{\omega_y} \int_0^x a(I) \cos \left(\varphi + \frac{\omega_I}{\omega_y} (\tau - x) \right) d\tau = \frac{a(I)}{\omega_I} \int_{-\frac{\omega_I}{\omega_y} x}^0 \cos(\varphi + \psi) d\psi
$$

$$
= \frac{a(I)}{\omega_I} \left(\sin \varphi - \sin \left(\varphi - \frac{\omega_I}{\omega_y} x \right) \right)
$$

having changed variable $\tau \to \psi = \frac{\omega_I}{\omega_y}(\tau - x)$.

Definition A.2 (*Time-one flow*) Let $\mathcal{L}_\phi(\cdot) := \{\phi, \cdot\}$. For a given $\phi \in \mathcal{O}_{\rho,s,\delta,r,\xi}$ and $h \in \mathbb{N}$, we denote as Φ_h the formal series

$$
\Phi_h := \sum_{j \geq h} \frac{\mathcal{L}_\phi^j}{j!} . \tag{75}
$$

We call $\Psi := \Phi_0 := e^{\mathcal{L}_\phi}$ *time-one flow* generated by ϕ.

Definition A.3 We call NQP-*homological transformation* the time-one flow $\Psi := \Phi_0 = e^{\mathcal{L}_\phi}$ generated by ϕ in (49), with ϕ_{khj} as in (55).

Below, we prove that the NQP-homological transformation is well defined. Later, we shall prove that the composition of $N + 1$ NQP-homological transformations $\Psi_1 = e^{\mathcal{L}_{\phi_1}}, \ldots, \Psi_{N+1} = e^{\mathcal{L}_{\phi_{N+1}}}$, where

$$
\Psi_j : \quad H_{j-1} \to H_j
$$

with $H_0 = H$ provides the transformation Ψ_* in (45).

Lemma A.1 (Pöschel 1993) *There exists an integer number $\bar{c}_{n,m}$ such that, for any $\phi \in \mathcal{O}_{\rho,s,\delta,r,\xi}$ and any $r' < r$, $s' < s$, $\rho' < \rho$, $\xi' < \xi$, $\delta' < \delta$ such that*

$$
\frac{\bar{c}_{n,m} \|\phi\|_{\rho,s,\delta,r,\xi}}{d} < 1 \quad d := \min \left\{ \rho'\sigma', r'\xi', \delta'^2 \right\}
$$

then the series in (75) converge uniformly so as to define the family $\{\Phi_h\}_{h=0,1,\ldots}$ of operators

$$
\Phi_h : \quad \mathcal{O}_{\rho,s,\delta,r,\xi} \to \mathcal{O}_{\rho-\rho',s-s',\delta-\delta',r-r',\xi-\xi'} .
$$

---

[12] NQP stands for "non-quasiperiodic."

Moreover, the following bound holds (showing, in particular, uniform convergence):

$$\|\mathcal{L}_\phi^j[g]\|_{\rho-\rho',s-s',\delta-\delta',r-r',\xi-\xi'} \leq j!\left(\frac{\overline{c}_{n,m}\|\phi\|_{\rho,s,\delta,r,\xi}}{d}\right)^j \|g\|_{\rho,s,\delta,r,\xi} \qquad (76)$$

for all $g \in \mathcal{O}_{\rho,s,\delta,r,\xi}$.

Remark A.1 (Pöschel 1993) The bound (76) immediately implies

$$\|\Phi_h g\|_{\rho-\rho',s-s',\delta-\delta',r-r',\xi-\xi'} \leq \frac{\left(\frac{\overline{c}_{n,m}\|\phi\|_{\rho,s,\delta,r,\xi}}{d}\right)^h}{1 - \frac{\overline{c}_{n,m}\|\phi\|_{\rho,s,\delta,r,\xi}}{d}} \|g\|_{\rho,s,\delta,r,\xi} \quad \forall g \in \mathcal{O}_{\rho,s,\delta,r,\xi}. \tag{77}$$

Lemma A.2 (Iterative lemma) *There exists a number* $\widetilde{c}_{n,m} > 1$ *such that the following holds. For any choice of positive numbers* r', ρ', s', ξ', δ' *satisfying*

$$2\rho' < \rho, \quad 2r' < r, \quad 2\xi' < \xi \tag{78}$$

$$2s' < s, \quad 2\delta' < \delta, \quad \mathcal{X}\left\|\Im\frac{\omega_{\mathbf{I}}}{\omega_y}\right\|_{\rho,r} < s - 2s', \quad \mathcal{X}\left\|\frac{\omega_{\mathbf{J}}}{\omega_y}\right\|_{\rho,r} < \log\frac{\delta}{2\delta'} \tag{79}$$

and provided that the following inequality holds true

$$\widetilde{c}_{n,m}\frac{\mathcal{X}}{d}\left\|\frac{\widetilde{f}}{\omega_y}\right\|_{\rho,s,\delta,r,\xi} < 1 \quad d := \min\left\{\rho'\sigma', r'\xi', \delta'^2\right\}, \tag{80}$$

one can find an operator

$$\Phi : \quad \mathcal{O}_{\rho,s,\delta,r,\xi} \to \mathcal{O}_{\rho_+,s_+,\delta_+,r_+,\xi_+}$$

with

$$r_+ := r - 2r', \quad \rho_+ := \rho - 2\rho', \quad \xi_+ := \xi - 2\xi', \quad s_+ := s - 2s' - \mathcal{X}\left\|\Im\frac{\omega_{\mathbf{I}}}{\omega_y}\right\|_{\rho,r}$$

$$\delta_+ := \delta e^{-\mathcal{X}\left\|\frac{\omega_{\mathbf{J}}}{\omega_y}\right\|_{\rho,r}} - 2\delta'$$

which carries the Hamiltonian H *in* (48) *to*

$$\mathrm{H}_+ := \Phi[\mathrm{H}] = h + g + \overline{f} + f_+$$

where

$$\|f_+\|_{r_+,\rho_+,\xi_+,s_+,\delta_+}$$
$$\leq \widetilde{c}_{n,m}\left(\frac{\mathcal{X}}{d}\left\|\frac{\widetilde{f}}{\omega_y}\right\|_{\rho,s,\delta,r,\xi}\|f\|_{\rho,s,\delta,r,\xi} + \|\{\phi,g\}\|_{\rho_1-\rho',s_1-s',\delta_1-\delta',r_1-r',\xi_1-\xi'}\right) \tag{81}$$

with

$$\rho_1 := \rho, \quad s_1 := s - \mathcal{X} \left\| \mathfrak{I} \frac{\omega_{\mathbf{I}}}{\omega_y} \right\|_{\rho,r}, \quad \delta_1 := \delta e^{-\mathcal{X} \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\rho,r}}, \quad r_1 := r, \quad \xi_1 := \xi$$

for a suitable $\phi \in \mathcal{O}_{\rho_1,s_1,\delta_1,r_1,\xi_1}$ verifying

$$\| \phi \|_{\rho_1,s_1,\delta_1,r_1,\xi_1} \leq \mathcal{X} \left\| \frac{\widetilde{f}}{\omega_y} \right\|_{\rho,s,\delta,r,\xi}. \tag{82}$$

Furthermore, if

$$(\mathbf{I}_+, \boldsymbol{\varphi}_+, \mathbf{p}_+, \mathbf{q}_+, y_+, x_+) := \Phi(\mathbf{I}, \boldsymbol{\varphi}, \mathbf{p}, \mathbf{q}, y, x)$$

the following uniform bounds hold:

$$\max \left\{ s'|\mathbf{I} - \mathbf{I}_+|, \ \rho'|\boldsymbol{\varphi} - \boldsymbol{\varphi}_+|, \ \delta'|\mathbf{p} - \mathbf{p}_+|, \ \delta|\mathbf{q} - \mathbf{q}_+|, \ \xi'|y - y_+|, \ r'|x - x_+| \right\}$$
$$\leq 2 \mathcal{X} \left\| \frac{\widetilde{f}}{\omega_y} \right\|_{\rho,s,\delta,r,\xi}. \tag{83}$$

Remark A.2 The right-hand side of (81) benefits of the absence of small divisors, as no "ultraviolet" (i.e., with size $\sim e^{-Ns} \| f \|$) term appears.

Proof Let $\overline{c}_{n,m}$ be as in Lemma A.1. We shall choose $\widetilde{c}_{n,m}$ suitably large with respect to $\overline{c}_{n,m}$.

Let ϕ_{khj} as in (55). Let us fix

$$0 < \overline{r} \leq r, \quad 0 < \overline{\rho} \leq \rho, \quad 0 < \overline{\xi} \leq \xi, \quad 0 < \overline{s} < s, \quad 0 < \overline{\delta} < \delta \tag{84}$$

and assume that

$$\mathcal{X} \left\| \mathfrak{I} \frac{\omega_{\mathbf{I}}}{\omega_y} \right\|_{\rho,r} \leq s - \overline{s}, \quad \mathcal{X} \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\rho,r} \leq \log \frac{\delta}{\overline{\delta}}. \tag{85}$$

Then, we have

$$\| \phi_{khj} \|_{\overline{\rho},\overline{r},\overline{\xi}} \leq \left\| \frac{f_{khj}}{\omega_y} \right\|_{\overline{\rho},\overline{r},\overline{\xi}} \left\| \int_0^x |e^{-\frac{\lambda_{khj}}{\omega_y}\tau}| \right\|_{\overline{\rho},\overline{r},\overline{\xi}} \, d\tau \leq \mathcal{X} \left\| \frac{f_{khj}}{\omega_y} \right\|_{\overline{\rho},\overline{r},\overline{\xi}} e^{\mathcal{X} \left\| \frac{\lambda_{khj}}{\omega_y} \right\|_{\overline{\rho},\overline{r}}}.$$

Since

$$\left\| \frac{\lambda_{khj}}{\omega_y} \right\|_{\overline{\rho},\overline{r}} \leq (h + j) \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\overline{\rho},\overline{r}} + |k| \left\| \mathfrak{I} \frac{\omega_{\mathbf{I}}}{\omega_y} \right\|_{\overline{\rho},\overline{r}},$$

we have

$$\|\phi_{khj}\|_{\overline{\rho},\overline{r},\overline{\xi}} \le \mathcal{X} \left\| \frac{\widetilde{f}_{khj}}{\omega_y} \right\|_{\overline{\rho},\overline{r},\overline{\xi}} e^{(h+j)\mathcal{X}\left\|\frac{\omega_{\mathbf{J}}}{\omega_y}\right\|_{\overline{\rho},\overline{r}}+|k|\mathcal{X}\left\|\Im\frac{\omega_{\mathbf{I}}}{\omega_y}\right\|_{\overline{\rho},\overline{r}}}$$

which yields (after multiplying by $e^{|k|\overline{s}}(\overline{\delta})^{j+h}$ and summing over k, j, h with $(k, h - k) \ne (0, 0)$) to

$$\|\phi\|_{\overline{\rho},\overline{r},\overline{\xi},\overline{s},\overline{\delta}} \le \mathcal{X} \left\| \frac{\widetilde{f}}{\omega_y} \right\|_{\overline{r},\overline{\rho},\overline{\xi},\overline{s}+\mathcal{X}\left\|\Im\frac{\omega_{\mathbf{I}}}{\omega_y}\right\|_{\overline{\rho},\overline{r}},\overline{\delta}e^{\mathcal{X}\left\|\frac{\omega_{\mathbf{J}}}{\omega_y}\right\|_{\overline{\rho},\overline{r}}}}.$$

Note that the right-hand side is well defined because of (85). In the case of the choice

$$\overline{r} = r =: r_1, \quad \overline{\rho} = \rho =: \rho_1, \quad \overline{\xi} = \xi =: \xi_1, \quad \overline{s} = s - \mathcal{X} \left\| \Im \frac{\omega_{\mathbf{I}}}{\omega_y} \right\|_{\rho,r} =: s_1$$

$$\overline{\delta} = \delta e^{-\mathcal{X}\left\|\frac{\omega_{\mathbf{J}}}{\omega_y}\right\|_{\rho,r}} =: \delta_1$$

(which, in view of the two latter inequalities in (79), satisfies (84)–(85)) the inequality becomes (82). An application of Lemma A.1, with r, ρ, ξ, s, δ replaced by $r_1 - r'$, $\rho_1 - \rho'$, $\xi_1 - \xi'$, $s_1 - s'$, $\delta_1 - \delta'$, concludes with a suitable choice of $\widetilde{c}_{n,m} > \overline{c}_{n,m}$ and (by (51))

$$f_+ := \Phi_2(\mathrm{h}) + \Phi_1(g) + \Phi_1(f).$$

Observe that the bound (81) follows from Eqs. (77) and (76) and the identities

$$\Phi_2[\mathrm{h}] = \sum_{j=2}^{\infty} \frac{\mathcal{L}_\phi^j(\mathrm{h})}{j!} = \sum_{j=1}^{\infty} \frac{\mathcal{L}_\phi^{j+1}(\mathrm{h})}{(j+1)!} = -\sum_{j=1}^{\infty} \frac{\mathcal{L}_\phi^j(\widetilde{f})}{(j+1)!}$$

$$\Phi_1[g] = \sum_{j=1}^{\infty} \frac{\mathcal{L}_\phi^j(g)}{j!} = \sum_{j=0}^{\infty} \frac{\mathcal{L}_\phi^{j+1}(g)}{(j+1)!} = \sum_{j=0}^{\infty} \frac{\mathcal{L}_\phi^j(g_1)}{(j+1)!}$$

with $g_1 := \mathcal{L}_\phi(g) = \{\phi, g\}$. The bounds in (83) are a consequence of equalities of the kind

$$\mathbf{I}_+ - \mathbf{I} = \sum_{j=0}^{\infty} \frac{\mathcal{L}_\phi^{j+1}(\mathbf{I})}{(j+1)!} = \sum_{j=0}^{\infty} \frac{\mathcal{L}_\phi^j(-\partial_\varphi\phi)}{(j+1)!}$$

(and similar). □

The proof of the Theorem 4.1 goes through iterate applications of Lemma A.2. We premise the following

Remark A.3 Replacing conditions in (79) with the stronger ones[13]

$$3s' < s, \quad 3\delta' < \delta, \quad \mathcal{X} \left\| \Im \frac{\omega_{\mathbf{I}}}{\omega_y} \right\|_{\rho,r} < s', \quad \mathcal{X} \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\rho,r} < \frac{\delta'}{\delta} \tag{86}$$

[and keeping (78), (80) unvaried] one can take, for $s_+, \delta_+, s_1, \delta_1$ the simpler expressions

$$s_{+\text{new}} = s - 3s', \quad \delta_{+\text{new}} = \delta - 3\delta', \quad s_{1\text{new}} := s - s', \quad \delta_{1\text{new}} = \delta - \delta'$$

(while keeping $r_+, \rho_+, \xi_+, r_1, \rho_1, \xi_1$ unvaried). Indeed, since $1 - e^{-x} \le x$ for all x,

$$\delta_1 = \delta e^{-\mathcal{X} \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\rho,r}} = \delta - \delta(1 - e^{-\mathcal{X} \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\rho,r}}) \ge \delta - \delta \mathcal{X} \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\rho,r} \ge \delta - \delta' = \delta_{1\text{new}}.$$

This also implies $\xi_+ = \delta_1 - \delta' \ge \delta - 2\delta' = \xi_{+\text{new}}$. That $s_+ \ge s_{+\text{new}}, s_1 \ge s_{1\text{new}}$ is even more immediate.

Now, we can proceed with the

Proof of Theorem 4.1 Let $\widetilde{c}_{n,m}$ be as in Lemma A.2. We shall choose $\overline{c}_{n,m}$ suitably large with respect to $\widetilde{c}_{n,m}$.

We apply Lemma A.2 with

$$2\rho' = \frac{\rho}{3}, \quad 3s' = \frac{s}{3}, \quad 3\delta' = \frac{\delta}{3}, \quad 2r' = \frac{r}{3}, \quad 2\xi' = \frac{\xi}{3}, \quad g \equiv 0.$$

We make use of the stronger formulation described in Remark A.3. Conditions in (78) and the two former conditions in (86) are trivially true. The two latter inequalities in (86) reduce to

$$\mathcal{X} \left\| \Im \frac{\omega_{\mathbf{I}}}{\omega_y} \right\|_{\rho,r} < \frac{s}{9}, \quad \mathcal{X} \left\| \frac{\omega_{\mathbf{J}}}{\omega_y} \right\|_{\rho,r} < \frac{1}{9},$$

and they are certainly satisfied by assumption (44), for $N > 2$. Since

$$d = \min\{\rho' s', r' \xi', \delta'^2\} = \min\{\rho s/36, r\xi/54, \delta^2/81\} \ge \frac{1}{81} \min\{\rho s, r\xi, \delta^2\} = \frac{d}{81},$$

we have that condition (80) is certainly implied by the last inequality in (44), once one chooses $c_{n,m} > 81\widetilde{c}_{n,m}$. By Lemma A.2, it is then possible to conjugate H to

$$\mathrm{H}_1 = \mathrm{h}_0 + \overline{f} + f_1$$

---

[13] The three first inequalities in (86) are immediately seen to be stronger than the corresponding three first inequalities in (79). On the other hand, rewriting the second inequality in (86) as $\frac{\delta'}{\delta} < 1 - \frac{2\delta'}{\delta}$ and using the inequality (which holds for all $x \ge 1$) $\log x \ge 1 - \frac{1}{x}$ with $x = \frac{\delta}{2\delta'}$, we have also $\frac{\delta'}{\delta} < \log \frac{\delta}{2\delta'}$.

with $f_1 \in \mathcal{O}_{\rho^{(1)}, s^{(1)}, \delta^{(1)}, r^{(1)}, \xi^{(1)},}$, where $(\rho^{(1)}, s^{(1)}, \delta^{(1)}, r^{(1)}, \xi^{(1)},) := 2/3 (\rho, s, \delta, r, \xi)$ and

$$\|f_1\|_{\rho^{(1)}, s^{(1)}, \delta^{(1)}, r^{(1)}, \xi^{(1)},} \leq 81 \widetilde{c}_{n,m} \frac{\mathcal{X}}{d} \left\| \frac{\widetilde{f}}{\omega_y} \right\|_{\rho, s, \delta, r, \xi} \|f\|_{\rho, s, \delta, r, \xi} \leq \frac{\|f\|_{\rho, s, \delta, r, \xi}}{2} \tag{87}$$

since $c_{n,m} \geq 162 \widetilde{c}_{n,m}$ and $N \geq 1$. Now, we aim to apply Lemma A.2 N times, again as described in Remark A.3, each time with parameters

$$\rho'_j = \frac{\rho}{6N}, \quad s'_j = \frac{s}{9N}, \quad \delta'_j = \frac{\delta}{9N}, \quad r'_j = \frac{r}{6N}, \quad \xi'_j = \frac{\xi}{6N}. \tag{88}$$

Therefore, we find, at each step

$$\rho^{(j+1)} := \rho^{(1)} - j \frac{\rho}{3N}, \quad s^{(j+1)} := s^{(1)} - j \frac{s}{3N}, \quad \delta^{(j+1)} := \delta^{(1)} - j \frac{\delta}{3N}$$

$$r^{(j+1)} := r^{(1)} - j \frac{r}{3N}, \quad \xi^{(j+1)} := \xi^{(1)} - j \frac{\xi}{3N}$$

$$\rho^{(j)}_1 := \rho^{(j)}, \quad s^{(j)}_1 := s^{(j)} - \frac{s}{9N}, \quad \delta^{(j)}_1 := \delta^{(j)} - \frac{\delta}{9N}$$

$$r^{(j)}_1 := r^{(j)}, \quad \xi^{(j)}_1 := \xi^{(j)}, \quad \mathcal{X}_j := \sup\{|x| : \ x \in \mathbb{X}_{\xi_j}\} \tag{89}$$

with $1 \leq j \leq N$.

We assume that for a certain $1 \leq i \leq N$ and all $1 \leq j \leq i$, we have $H_j \in \mathcal{O}_{\rho^{(j)}, s^{(j)}, \delta^{(j)}, r^{(j)}, \xi^{(j)},}$ of the form

$$H_j = h_0 + g_{j-1} + f_j, \quad g_{j-1} \in \mathcal{N}_{\rho^{(j)}, s^{(j)}, \delta^{(j)}, r^{(j)}, \xi^{(j)},}, \quad g_{j-1} - g_{j-2} = \overline{f}_{j-1} \tag{90}$$

$$\|f_j\|_{\rho^{(j)}, s^{(j)}, \delta^{(j)}, r^{(j)}, \xi^{(j)},} \leq \frac{\|f_1\|_{\rho^{(1)}, s^{(1)}, \delta^{(1)}, r^{(1)}, \xi^{(1)},}}{2^{j-1}} \tag{91}$$

with $g_{-1} \equiv 0$, $g_0 = \overline{f}$. We want to prove that, if $i < N$, Lemma A.2 can be applied once again, so as to conjugate H_i to a suitable H_{i+1} such that (90)–(91) are true with $j = i + 1$. To this end, according to the discussion in Remark A.3, we check the stronger inequalities

$$2\rho'_i < \rho^{(i)}, \quad 2r'_i < r^{(i)}, \quad 2\xi'_i < \xi^{(i)} \tag{92}$$

$$\mathcal{X}_i \left\| \Im \frac{\omega_\mathbf{I}}{\omega_y} \right\|_{\rho_i, r_i} < s'_i, \quad \mathcal{X}_i \left\| \frac{\omega_\mathbf{J}}{\omega_y} \right\|_{\rho_i, r_i} < \frac{\delta'_i}{\delta_i} \tag{93}$$

$$\widetilde{c}_{n,m} \frac{\mathcal{X}_i}{d_i} \left\| \frac{f_i}{\omega_y} \right\|_{\rho_i, s_i, \delta_i, r_i, \xi_i} < 1. \tag{94}$$

where $d_i := \min\{\rho_i' s_i', r_i' \xi_i', \delta_i'^2\}$. Conditions (92) and (93) are certainly verified, since in fact they are implied by the definitions above (using also $\delta_i \leq \frac{2}{3}\delta$, $\mathcal{X}_i \leq \mathcal{X}$) and the two former inequalities in (44). To check the validity of (94), we firstly observe that

$$d_i = \min\{\rho'^{(i)} s'^{(i)}, \ (\delta'^{(i)})^2, \ r'^{(i)} \xi_j', \} \geq \frac{d}{81N^2}. \tag{95}$$

Using then $c_{n,m} > 162\widetilde{c}_{n,m}, \mathcal{X}_i < \mathcal{X}$, Eq. (87), the inequality in (91) with $j = i$ and the last inequality in (44), we easily conclude

$$\|f_i\|_{\rho^{(i)},s^{(i)},\delta^{(i)},r^{(i)},\xi^{(i)}} \leq \|f_1\|_{\rho^{(1)},s^{(1)},\delta^{(1)},r^{(1)},\xi^{(1)},} \leq 81\widetilde{c}_{n,m} \frac{\mathcal{X}}{d} \left\| \frac{\widetilde{f}}{\omega_y} \right\|_{\rho,s,\delta,r,\xi} \|f\|_{\rho,s,\delta,r,\xi}$$

$$\leq \frac{1}{\widetilde{c}_{n,m}} \frac{d}{81N^2} \frac{1}{\mathcal{X}} \left\| \frac{1}{\omega_y} \right\|_{\rho,r}^{-1} \leq \frac{1}{\widetilde{c}_{n,m}} \frac{d_i}{\mathcal{X}_i} \left\| \frac{1}{\omega_y} \right\|_{\rho^{(i)},r^{(i)}}^{-1}$$

which implies (94).

Then, the iterative lemma is applicable to H_i, which is conjugated to

$$H_{i+1} = h_0 + g_i + f_{i+1}, \quad g_i \in \mathcal{N}_{\rho^{(i+1)},s^{(i+1)},\delta^{(i+1)},r^{(i+1)},\xi^{(i+1)}}, \quad g_i - g_{i-1} = \overline{f}_i \tag{96}$$

with g_i, f_{i+1} satisfying (90) with $j = i + 1$. We prove that (91) holds with $j = i + 1$, so as to complete the inductive step. By the thesis of the iterative lemma,

$$\|f_{i+1}\|_{\rho^{(i+1)},s^{(i+1)},\delta^{(i+1)},r^{(i+1)},\xi^{(i+1)}} \leq \widetilde{c}_{n,m} \Big(\frac{\mathcal{X}_i}{d_i} \left\| \frac{\widetilde{f}_i}{\omega_y} \right\|_{\rho^{(i)},s^{(i)},\delta^{(i)},r^{(i)},\xi^{(i)}} \|f_i\|_{\rho^{(i)},s^{(i)},\delta^{(i)},r^{(i)},\xi^{(i)}}$$

$$+ \|\{\phi_i, g_{i-1}\}\|_{\rho_1^{(i)}-\rho'^{(i)},s_1^{(i)}-s'^{(i)},\delta_1^{(i)}-\delta'^{(i)},r_1^{(i)}-r'^{(i)},\xi_1^{(i)}-\xi'^{(i)}} \Big)$$

On the other hand, using (87), (95) and the last assumption in (44) and (91) with $j = i$, we obtain

$$\frac{\mathcal{X}_i}{d_i} \left\| \frac{\widetilde{f}_i}{\omega_y} \right\|_{\rho^{(i)},s^{(i)},\delta^{(i)},r^{(i)},\xi^{(i)}} \leq \frac{81N^2\chi_0}{d} \left\| \frac{1}{\omega_y} \right\| \|\widetilde{f}_i\| \leq \frac{81N^2\chi_0}{d} \left\| \frac{1}{\omega_y} \right\| \|f_i\|$$

$$\leq \frac{81N^2\chi_0}{d} \left\| \frac{1}{\omega_y} \right\| \|f_1\|$$

$$\leq \frac{(81)^2\widetilde{c}_{n,m}N^2\chi_0^2}{d^2} \left\| \frac{1}{\omega_y} \right\| \left\| \frac{\widetilde{f}}{\omega_y} \right\|_{\rho,s,\delta,r,\xi} \|f\|_{\rho,s,\delta,r,\xi}$$

$$\leq \frac{1}{6\widetilde{c}_{n,m}}.$$

Furthermore, using (76) with $j = 1$, (82) and

$$\|g_0\|_{\rho_1^{(i)}-\rho'^{(i)},s_1^{(i)}-s'^{(i)},\delta_1^{(i)}-\delta'^{(i)},r_1^{(i)}-r'^{(i)},\xi_1^{(i)}-\xi'^{(i)}} \leq \|f\|_{\rho_1^{(i)}-\rho'^{(i)},s_1^{(i)}-s'^{(i)},\delta_1^{(i)}-\delta'^{(i)},r_1^{(i)}-r'^{(i)},\xi_1^{(i)}-\xi'^{(i)}}$$

$$\leq \|f\|_{\rho^{(0)}, s^{(0)}, \delta^{(0)}, r^{(0)}, \xi^{(0)}}$$

and, with $j = 1, \ldots, i - 1$

$$
\begin{aligned}
\|g_j - g_{j-1}\|_{\rho_1^{(i)} - \rho'^{(i)}, s_1^{(i)} - s'^{(i)}, \delta_1^{(i)} - \delta'^{(i)}, r_1^{(i)} - r'^{(i)}, \xi_1^{(i)} - \xi'^{(i)}} &\leq \|f_j\|_{\rho_1^{(i)} - \rho'^{(i)}, s_1^{(i)} - s'^{(i)}, \delta_1^{(i)} - \delta'^{(i)}, r_1^{(i)} - r'^{(i)}, \xi_1^{(i)} - \xi'^{(i)}} \\
&\leq \|f_j\|_{\rho^{(j)}, s^{(j)}, \delta^{(j)}, r^{(j)}, \xi^{(j)},} \\
&\leq \frac{\|f_1\|_{\rho^{(1)}, s^{(1)}, \delta^{(1)}, r^{(1)}, \xi^{(1)},}}{2^{j-1}}
\end{aligned}
\tag{97}
$$

we obtain

$$
\begin{aligned}
&\|\{\phi_i, g_{i-1}\}\|_{\rho_1^{(i)} - \rho'^{(i)}, s_1^{(i)} - s'^{(i)}, \delta_1^{(i)} - \delta'^{(i)}, r_1^{(i)} - r'^{(i)}, \xi_1^{(i)} - \xi'^{(i)}} \\
&\leq \|\{\phi_i, g_0\}\|_{\rho_1^{(i)} - \rho'^{(i)}, s_1^{(i)} - s'^{(i)}, \delta_1^{(i)} - \delta'^{(i)}, r_1^{(i)} - r'^{(i)}, \xi_1^{(i)} - \xi'^{(i)}} \\
&\quad + \sum_{j=1}^{i-1} \|\{\phi_i, g_j - g_{j-1}\}\|_{\rho_1^{(i)} - \rho'^{(i)}, s_1^{(i)} - s'^{(i)}, \delta_1^{(i)} - \delta'^{(i)}, r_1^{(i)} - r'^{(i)}, \xi_1^{(i)} - \xi'^{(i)}} \\
&\leq 2 \frac{\overline{c}_{n,m} \chi_i}{d_i} \left\| \frac{\widetilde{f_i}}{\omega_y} \right\|_{\rho^{(i)}, s^{(i)}, \delta^{(i)}, r^{(i)}, \xi^{(i)}} \left(\frac{1}{N} \|g_0\|_{\rho_1^{(i)} - \rho'^{(i)}, s_1^{(i)} - s'^{(i)}, \delta_1^{(i)} - \delta'^{(i)}, r_1^{(i)} - r'^{(i)}, \xi_1^{(i)} - \xi'^{(i)}} \right. \\
&\quad \left. + \sum_{j=1}^{i-1} \|g_j - g_{j-1}\|_{\rho_1^{(i)} - \rho'^{(i)}, s_1^{(i)} - s'^{(i)}, \delta_1^{(i)} - \delta'^{(i)}, r_1^{(i)} - r'^{(i)}, \xi_1^{(i)} - \xi'^{(i)}} \right) \\
&\leq 162 N^2 \frac{\overline{c}_{n,m} \chi_i}{d} \left\| \frac{1}{\omega_y} \right\| \left(\frac{\|f\|_{\rho, s, \delta, r, \xi}}{N} + 2 \times 81 \widetilde{c}_{n,m} \frac{\chi}{d} \left\| \frac{\widetilde{f}}{\omega_y} \right\|_{\rho, s, \delta, r, \xi} \|f\|_{\rho, s, \delta, r, \xi} \right) \\
&\quad \times \|f_i\|_{\rho^{(i)}, s^{(i)}, \delta^{(i)}, r^{(i)}, \xi^{(i)}} \leq \frac{1}{3} \|f_i\|_{\rho^{(i)}, s^{(i)}, \delta^{(i)}, r^{(i)}, \xi^{(i)}}
\end{aligned}
$$

where the last summand is to be taken into account only when $i \geq 2$ and we have used that, when $j = 0$, d_i can be replaced with $N d_i$. Collecting the bounds above, we find (91) holds with $j = i + 1$. Then, the iterative lemma can be applied N times, and we get (46) as a direct consequence of (87) and (97) and of the validity of (91) for all $1 \leq i \leq N$. The inequalities in (47) follow from (83) and (88). For example,

$$|\mathbf{I}_* - \mathbf{I}_0| \leq \sum_{j=1}^{N} |\mathbf{I}^{(j)} - \mathbf{I}^{(j-1)}| \leq 2 \frac{\chi_0}{s'^{(0)}} \left\| \frac{f_0}{\omega_y} \right\| + 4 \frac{\chi_1}{s'^{(1)}} \left\| \frac{1}{\omega_y} \right\| \|f_1\| \leq 19 \frac{\chi_0}{s^{(0)}} \left\| \frac{f_0}{\omega_y} \right\|,$$

etc.　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　□

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