Journal of Nonlinear Science (2020) 30:1503–1517
https://doi.org/10.1007/s00332-020-09615-y

On Liouville Type Theorem for Stationary Non-Newtonian Fluid Equations

Dongho Chae[1,2] · **Jörg Wolf**[1]

Received: 24 April 2019 / Accepted: 29 January 2020 / Published online: 8 February 2020

Abstract

In this paper, we prove a Liouville type theorem for non-Newtonian fluid equations in $\mathbb{R}^3$, having the diffusion term $\boldsymbol{A}_p(u) = \nabla \cdot (|\boldsymbol{D}(u)|^{p-2}\boldsymbol{D}(u))$ with $\boldsymbol{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$, $3/2 < p < 3$. In the case $3/2 < p \le 9/5$, we show that a suitable weak solution $u \in W^{1,p}(\mathbb{R}^3)$ satisfying $\liminf_{R\to\infty} |u_{B(R)}| = 0$ is trivial, i.e., $u \equiv 0$. On the other hand, for $9/5 < p < 3$ we prove the following Liouville type theorem: if there exists a matrix valued function $\boldsymbol{V} = \{V_{ij}\}$ such that $\partial_j V_{ij} = u_i$(summation convention), whose $L^{\frac{3p}{2p-3}}$ mean oscillation has the following growth condition at infinity,

$$\fint_{B(r)} |\boldsymbol{V} - \boldsymbol{V}_{B(r)}|^{\frac{3p}{2p-3}} \mathrm{d}x \le C r^{\frac{9-4p}{2p-3}} \quad \forall 1 < r < +\infty,$$

then $u \equiv 0$.

Keywords Non-Newtonian fluid equations · Liouville type theorem

Mathematics Subject Classification 35Q30 · 76D05 · 76D03

Communicated by Anthony Bloch.

✉ Dongho Chae
dchae@cau.ac.kr

Jörg Wolf
jwolf2603@cau.ac.kr

[1] Department of Mathematics, Chung-Ang University, Dongjak-gu, Heukseok-ro 84, Seoul 06974, Republic of Korea

[2] School of Mathematics, Korea Institute for Advanced Study, Dongdaemun-gu, Hoegi-ro 85, Seoul 02455, Republic of Korea

1 Introduction

We consider the following stationary non-Newtonian fluid equations in $\mathbb{R}^3$

$$-\boldsymbol{A}_p(u) + (u \cdot \nabla)u = -\nabla\pi \quad \text{in} \quad \mathbb{R}^3, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

where $u = (u_1, u_2, u_3) = u(x)$ is the velocity field, $\pi = \pi(x)$ is the scalar pressure and

$$\boldsymbol{A}_p(u) = \nabla \cdot (|\boldsymbol{D}(u)|^{p-2}\boldsymbol{D}(u)), \quad 1 < p < +\infty$$

with $\boldsymbol{D}(u) = \boldsymbol{D} = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ representing the symmetric gradient. Here, $|\boldsymbol{D}|^{p-2}\boldsymbol{D} = \boldsymbol{\sigma}(\boldsymbol{D})$ stands for the deviatoric stress tensor. System (1.1), (1.2) is popular among engineers, known as a power law model of non-Newtonian fluid, where the viscosity depends on the shear rate $|\boldsymbol{D}(u)|$. For $p = 2$, it reduces to the usual stationary Navier-Stokes equations, as pioneered by Leray (1933). For $1 < p < 2$, the fluid is called shear thinning, while in case $2 < p < +\infty$ the fluid is called shear thickening. For more details on the continuum mechanical background of the above equations, we refer to Wilkinson (1960). Concerning the existence and regularity of solutions to (1.1), we refer to Pokorný (1996), Frehse et al. (2003).

The Liouville type problem for the Navier–Stokes equations, as stated in Galdi's book (Galdi 2011, Remark X. 9.4, pp. 729), is a challenging open problem in the mathematical fluid mechanics. We refer Chae (2014), Chae and Yoneda (2013), Chae and Wolf (2016), Chamorro et al. (2019), Gilbarg and Weinberger (1978), Koch et al. (2009), Korobkov et al. (2015), Kozono et al. (2017), Seregin (2016, 2018), Seregin and Wang (2019) and the references therein for partial progresses for the problem. In those literatures, authors provided sufficient conditions for velocities to guarantee the triviality of solutions. Our aim is to study Liouville type theorems for more general equations than the Navier–Stokes equation, namely equations for non-Newtonian fluids modeling such as power law fluids.

We mention that similarly to the case of the Navier–Stokes equations our main theorem has no implications for further regularity properties of weak solutions of equations modeling power law fluids.

Let $u \in L^1_{loc}(\mathbb{R}^n)$ be a vector function, and let $\boldsymbol{V} = \{V_{ij}\} \in L^1_{loc}(\mathbb{R}^n; \mathbb{R}^{n\times n})$ be defined such that $\partial_j V_{ij} = u_i$, where the derivative is in the sense of distribution. Clearly, such $\boldsymbol{V}$ exists, although it is not unique. For instance, we may set $\{V_{ij}\}$

$$\begin{cases} V_{ii}(x) = \displaystyle\int_0^{x_i} u_i(x_1, \ldots, \xi_i, \ldots, x_n)\mathrm{d}\xi_i & \text{if} \quad 1 \le i \le n \\ V_{ij}(x) \equiv 0 \quad \text{if} \quad i \neq j. \end{cases}$$

In Seregin (2016, 2018), Seregin proved Liouville type theorem for the Navier–Stokes equations under hypothesis on the function $\boldsymbol{V}$ with restriction $V_{ij} = -\epsilon_{ijk}\psi_k$

with ϵ_{ijk} being the standard alternating tensor. In particular in Seregin (2018), it is shown that if $\boldsymbol{V} \in BMO(\mathbb{R}^3)$, then $u = 0$. In this paper, we would like to generalize this result for system (1.1), (1.2).

For a measurable set $\Omega \subset \mathbb{R}^n$, we denote by $|\Omega|$ the n-dimensional Lebesgue measure of Ω, and for $f \in L^1(\Omega)$ we use the notation

$$f_\Omega := \fint_\Omega f \mathrm{d}x := \frac{1}{|\Omega|} \int_\Omega f \mathrm{d}x.$$

In contrast to the case $p = 2$, it is still open whether any weak solution to system (1.1), (1.2) is regular or not. Therefore, in the present paper we only work with weak solutions satisfying the local energy inequality the solution of which are called suitable weak solution.

Definition 1.1 Let $\frac{3}{2} \leq p < +\infty$.

1. We say $u \in W^{1,p}_{loc}(\mathbb{R}^3)$ is a weak solution to (1.1), (1.2) if the following identity is fulfilled

$$\int\limits_{\mathbb{R}^3} \Big(|\boldsymbol{D}(u)|^{p-2} \boldsymbol{D}(u) - u \otimes u \Big) : \boldsymbol{D}(\varphi) \mathrm{d}x = 0 \tag{1.3}$$

for all vector fields $\varphi \in C^\infty_c(\mathbb{R}^3)$ with $\nabla \cdot \varphi = 0$.

2. A pair $(u, \pi) \in W^{1,p}_{loc}(\mathbb{R}^3) \times L^{\frac{3}{2}}_{loc}(\mathbb{R}^3)$ is called a suitable weak solution to (1.1), (1.2) if besides (1.3) the following local energy inequality holds

$$\begin{aligned} &\int\limits_{\mathbb{R}^3} |\boldsymbol{D}(u)|^p \phi \mathrm{d}x \\ &\quad \leq - \int_{\mathbb{R}^3} |\boldsymbol{D}(u)|^{p-2} \boldsymbol{D}(u) : u \otimes \nabla\phi \mathrm{d}x + \int\limits_{\mathbb{R}^3} \Big(\frac{1}{2}|u|^2 + \pi \Big) u \cdot \nabla\phi \mathrm{d}x \end{aligned} \tag{1.4}$$

for all nonnegative $\phi \in C^\infty_c(\mathbb{R}^3)$.

Remark 1.2 In case $\frac{9}{5} \leq p < +\infty$ any weak solution to (1.1), (1.2) is a suitable weak solution. Indeed, by Sobolev's embedding theorem we have $u \in L^{\frac{9}{2}}_{loc}(\mathbb{R}^3)$, which yields $|u|^2|\nabla u| \in L^1_{loc}(\mathbb{R}^3)$. In addition, as we will see below in Sect. 2 from (1.3) we get $\pi \in L^{\frac{9}{4}}_{loc}(\mathbb{R}^3)$ such that for all $\varphi \in W^{1,\frac{9}{5}}(\mathbb{R}^3)$ with compact support

$$\int\limits_{\mathbb{R}^3} \Big(|\boldsymbol{D}(u)|^{p-2} \boldsymbol{D}(u) : \boldsymbol{D}(\varphi) + u \otimes u : \boldsymbol{D}(\varphi) \Big) \mathrm{d}x = \int\limits_{\mathbb{R}^3} \pi \nabla \cdot \varphi \mathrm{d}x. \tag{1.5}$$

Thus, inserting $\varphi = u\phi$ into (1.5), where $\phi \in C^\infty_c(\mathbb{R}^3)$, and applying integration by parts, we get (1.4) where the inequality is replaced by equality.

Our aim in this paper is to prove the following.

Theorem 1.3

(i) *Let* $\frac{3}{2} \le p \le \frac{9}{5}$*. We suppose* $(u, \pi) \in W^{1,p}_{loc}(\mathbb{R}^3) \times L^{\frac{3}{2}}_{loc}(\mathbb{R}^3)$ *is a suitable weak solution of* (1.1)*,* (1.2)*. If*

$$\int\limits_{\mathbb{R}^3} |\nabla u|^p \mathrm{d}x < +\infty, \quad \liminf_{R\to\infty} |u_{B(R)}| = 0 \tag{1.6}$$

then $u \equiv 0$.

(ii) *Let* $\frac{9}{5} < p < 3$*. We suppose* $(u, \pi) \in W^{1,p}_{loc}(\mathbb{R}^3) \times L^{\frac{3}{2}}_{loc}(\mathbb{R}^3)$ *is a weak solution of* (1.1)*,* (1.2)*. Assume there exists* $\boldsymbol{V} = \{V_{ij}\} \in W^{2,p}_{loc}(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ *such that* $\partial_j V_{ij} = u_i$*, and*

$$\fint_{B(r)} |\boldsymbol{V} - \boldsymbol{V}_{B(r)}|^{\frac{3p}{2p-3}} \mathrm{d}x \le C r^{\frac{9-4p}{2p-3}} \quad \forall 1 < r < +\infty. \tag{1.7}$$

Then, $u \equiv 0$.

Remark 1.4 In the case $\frac{6}{5} < p < \frac{3}{2}$, our method does not work, since it requires $u \in L^3_{loc}(\mathbb{R}^3)$ in order to satisfy the local energy inequality. Indeed, by Sobolev's embedding theorem it follows $W^{1,p}_{loc}(\mathbb{R}^3) \hookrightarrow L^{\frac{3p}{3-p}}_{loc}(\mathbb{R}^3)$ and $\frac{3p}{3-p} \ge 3$ if and only if $p \ge \frac{3}{2}$.

Remark 1.5 Obviously $\boldsymbol{V} \in BMO(\mathbb{R}^3)$ implies condition (1.7). In fact, (1.7) is guaranteed by $\boldsymbol{V} \in C^{0,\alpha}(\mathbb{R}^3)$ wih $\alpha = \frac{9-4p}{3p} > 0$ thanks to the Campanato theorem (Giaquinta 1983).
Choosing $p = 2$ in (ii) of the above theorem, we immediate obtain the following corollary on the Navier–Stokes equations, coinciding with a special case of Chae and Wolf (2019).

Corollary 1.6 *Let* (u, π) *be a smooth solution of the stationary Navier–Stokes equations on* $\mathbb{R}^3$*. Suppose there exists* $\boldsymbol{V} \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ *such that* $\nabla \cdot \boldsymbol{V} = u$*, and*

$$\fint_{B(r)} |\boldsymbol{V} - \boldsymbol{V}_{B(r)}|^6 \mathrm{d}x \le C r \quad \forall 1 < r < +\infty. \tag{1.8}$$

Then, $u \equiv 0$.

Remark 1.7 (1) A similar result as Corollary 1.6 has been obtained by Seregin in Seregin (2018). Instead of (1.8), he imposed

$$\left(\fint_{B(r)} |\boldsymbol{V} - \boldsymbol{V}_{B(r)}|^s \mathrm{d}x \right)^{\frac{1}{s}} \le C r^{\alpha(s)} \quad \forall 1 < r < +\infty \tag{1.9}$$

with $0 < \alpha(s) < \frac{s-3}{6(s-1)}$ for $s > 3$. In case $s = 6$, our result improves Seregin's result.

(2) We believe that condition (1.7) can be generalized by replacing the exponent $\frac{2p}{2p-3}$ by $s_0 \le s < \frac{2p}{2p-3}$ for suitable $s_0 > 1$. In order to keep the paper less technical, however, we restrict ourselves to the case $s = \frac{2p}{2p-3}$ here.

2 Proof of Theorem 1.3

We start our discussion of estimating the pressure for both of the cases (i) and (ii). First note that by the hypothesis $u \in W^{1,p}_{loc}(\mathbb{R}^3)$ and due to Sobolev's embedding theorem it holds $u \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$. This yields

$$|\boldsymbol{D}(u)|^{p-2}\boldsymbol{D}(u) - u \otimes u \in L^q_{loc}(\mathbb{R}^3), \quad q = \min\left\{\frac{3p}{6-2p}, \frac{p}{p-1}\right\}.$$

Given $0 < R < +\infty$, and noting that $q \ge \frac{3}{2}$ for $p \ge \frac{3}{2}$, we may define the functional $F \in W^{-1,s}(B(R))$, $\frac{3}{2} \le s \le q$, by means of

$$\langle F, \varphi\rangle = \int\limits_{B(R)} (|\boldsymbol{D}(u)|^{p-2}\boldsymbol{D}(u) - u \otimes u) : \boldsymbol{D}(\varphi)\mathrm{d}x, \quad \varphi \in W^{1,s'}_0(B(R)),$$

where we set $s' = \frac{s}{s-1}$. Since u is a weak solution to (1.1), (1.2) in view of (Sohr 2001, Lemma 2.1.1) there exists a unique $\pi_R \in L^q(B(R))$ with $\int\limits_{B(R)} \pi_R \mathrm{d}x = 0$ such that

$$\langle F, \varphi\rangle = \int\limits_{B(R)} \pi_R \nabla \cdot \varphi \mathrm{d}x \quad \forall \varphi \in W^{1,s'}_0(B(R)).$$

Furthermore, we get for all $\frac{3}{2} \le s \le q$

$$\int\limits_{B(R)} |\pi_R|^s \mathrm{d}x \le c\|F\|^s_{W^{-1,s}(B(R))} \le c\||\boldsymbol{D}(u)|^{p-2}\boldsymbol{D}(u) - u \otimes u\|^s_{L^s(B(R))}, \tag{2.1}$$

with a constant $c > 0$, depending only on p but independent of $0 < R < +\infty$. Let $1 < \rho < R < +\infty$. We set $\widetilde{\pi}_R = \pi_R - (\pi_R)_{B(1)}$. From the definition of the pressure π_R, it follows that

$$\int\limits_{B(\rho)} (\widetilde{\pi}_R - \widetilde{\pi}_\rho)\nabla \cdot \varphi \mathrm{d}x = 0 \quad \forall \varphi \in W^{1,s'}_0(B(\rho)).$$

This shows that $\widetilde{\pi}_R - \widetilde{\pi}_\rho$ is constant in $B(\rho)$. Since $(\widetilde{\pi}_R - \widetilde{\pi}_\rho)_{B(1)} = 0$ it follows that $\widetilde{\pi}_\rho = \widetilde{\pi}_R$ in $B(\rho)$. This allows us to define $\pi \in L^q_{loc}(\mathbb{R}^3)$ by setting $\pi = \widetilde{\pi}_R$ in

$B(R)$. In particular, $\pi - \pi_{B(R)} = \pi_R$. Thus, thanks to (2.1) we estimate by Hölder's inequality

$$\int\limits_{B(R)} |\pi - \pi_{B(R)}|^s \mathrm{d}x \leq c \| |\boldsymbol{D}(u)|^{p-2}\boldsymbol{D}(u) - u \otimes u \|^s_{L^s(B(R))}$$
$$\leq cR^{\frac{s(3-p)}{p}} \Bigg(\int\limits_{B(R)} |\boldsymbol{D}(u)|^p \mathrm{d}x \Bigg)^{s(p-1)} + c \int\limits_{B(R)} |u|^{2s} \mathrm{d}x.$$

Hence,

$$\|\pi - \pi_{B(R)}\|_{L^s(B(R))} \leq cR^{\frac{3-p}{p}} \|\boldsymbol{D}(u)\|^{p-1}_{L^p(B(R))} + c\|u\|^2_{L^{2s}(B(R))}. \tag{2.2}$$

Note that $q = \frac{9}{4}$ whenever $\frac{9}{5} \leq p < +\infty$. This yields the existence of the pressure $\pi \in L^{\frac{9}{4}}_{loc}(\mathbb{R}^3)$.

Let $1 < r < +\infty$ be arbitrarily chosen, and $r \leq \rho < R \leq 2r$. We set $\overline{R} = \frac{R+\rho}{2}$. Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a cut off function, which is radially non-increasing with $\zeta = 1$ on $B(\rho)$ and $\zeta = 0$ on $\mathbb{R}^3 \backslash B(\overline{R})$ satisfying $|\nabla \zeta| \leq c(R-\rho)^{-1}$. From (1.4) with $\phi = \zeta^p$ we get

$$\int_{B(\bar{R})} |\boldsymbol{D}(u)|^p \zeta^p \mathrm{d}x \leq - \int\limits_{B(\overline{R})} |\boldsymbol{D}(u)|^{p-2} \nabla \zeta^p \cdot \boldsymbol{D}(u) \cdot u \mathrm{d}x +$$
$$+ \frac{1}{2} \int\limits_{B(\overline{R})} |u|^p u \cdot \nabla \zeta^p + \int\limits_{B(\overline{R})} (\pi - \pi_{B(\overline{R})}) u \cdot \nabla \zeta^p \mathrm{d}x.$$

Applying Hölder's and Young's inequality, we get from above

$$\int\limits_{B(\rho)} |\boldsymbol{D}(u)|^p \zeta^p \mathrm{d}x \leq c(R-\rho)^{-p} \int\limits_{B(\overline{R}) \backslash B(\rho)} |u|^p \mathrm{d}x + c(R-\rho)^{-1} \int\limits_{B(\overline{R}) \backslash B(\rho)} |u|^3 \mathrm{d}x$$
$$+ c(R-\rho)^{-1} \int\limits_{B(\overline{R}) \backslash B(\rho)} |\pi - \pi_{B(\overline{R})}||u| \mathrm{d}x$$
$$= I + II + III. \tag{2.3}$$

The case $\frac{3}{2} \leq p \leq \frac{9}{5}$: Observing (1.8) and applying Sobolev's embedding theorem, we get

$$u \in L^{\frac{3p}{3-p}}(\mathbb{R}^3). \tag{2.4}$$

In (2.3) we take $\rho = \frac{R}{2}$. Applying Hölder's inequality, we easily get

$$I + II \le c\Bigg(\int\limits_{\mathbb{R}^3\setminus B(\frac{R}{2})} |u|^{\frac{3p}{3-p}}\mathrm{d}x\Bigg)^{\frac{3-p}{3}} + cR^{\frac{5p-9}{p}}\Bigg(\int\limits_{\mathbb{R}^3\setminus B(\frac{R}{2})} |u|^{\frac{3p}{3-p}}\mathrm{d}x\Bigg)^{\frac{3-p}{p}}.$$

Using (2.4) and recalling that $p \le \frac{9}{5}$, we see that $I + II = o(R)$ as $R \to +\infty$.

Applying Hölder's inequality along with (2.2) with $s = \frac{3}{2}$, we estimate

$$\begin{aligned} III &\le cR^{-1}\Big(R^{\frac{3-p}{p}}\|\boldsymbol{D}(u)\|^{p-1}_{L^p(B(\overline{R}))} + c\|u\|^2_{L^3(B(\overline{R}))}\Big)\Bigg(\int\limits_{\mathbb{R}^3\setminus B(\frac{R}{2})} |u|^3\mathrm{d}x\Bigg)^{\frac{1}{3}} \\ &\le c\|\nabla u\|^{p-1}_{L^p}\Bigg(\int\limits_{\mathbb{R}^3\setminus B(\frac{R}{2})} |u|^{\frac{3p}{3-p}}\mathrm{d}x\Bigg)^{\frac{3-p}{3p}} \\ &\quad + cR^{\frac{5p-9}{p}}\|u\|^2_{L^{\frac{3p}{3-p}}}\Bigg(\int\limits_{\mathbb{R}^3\setminus B(\frac{R}{2})} |u|^{\frac{3p}{3-p}}\mathrm{d}x\Bigg)^{\frac{3-p}{3p}}. \end{aligned}$$

Observing (2.4) along with $p \le \frac{9}{5}$, we find $III = o(R)$ as $R \to +\infty$. Inserting the above estimates into the right-hand side of (2.3), we deduce that $\boldsymbol{D}(u) \equiv 0$, which implies that $u = u(x)$ is a linear function x. Taking into account the condition (1.6), we obtain $u \equiv 0$.

The case $\frac{9}{5} < p < 3$: In order to estimate I and II, we choose another cut off function $\psi \in C^\infty(\mathbb{R}^3)$, which is radially non-increasing with $\psi = 1$ on $B(\overline{R})$ and $\psi = 0$ on $\mathbb{R}^3\setminus B(R)$ satisfying $|\nabla\psi| \le c(R-\rho)^{-1}$. Recalling that $u = \nabla\cdot\boldsymbol{V}$, applying integration by parts and applying the Hölder inequality, we find

$$\begin{aligned} &\int\limits_{B(R)} |u|^p\psi^p\mathrm{d}x \\ &\quad = \int\limits_{B(R)} \partial_i(V_{ij} - (V_{ij})_{B(R)})u_j|u|^{p-2}\psi^p\mathrm{d}x \\ &\quad = -\int\limits_{B(R)} (V_{ij} - (V_{ij})_{B(R)})\Big(\partial_i u_j|u|^{p-2} + (p-2)u_j u_k\partial_i u_k|u|^{p-4}\Big)\psi^p\mathrm{d}x \\ &\qquad - \int\limits_{B(R)} (V_{ij} - (V_{ij})_{B(R)})u_j|u|^{p-2}\partial_i\psi^p\mathrm{d}x \end{aligned}$$

$$\leq c\Bigg(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^p \mathrm{d}x\Bigg)^{\frac{1}{p}} \Bigg(\int\limits_{B(R)} |\nabla u|^p \mathrm{d}x\Bigg)^{\frac{1}{p}} \Bigg(\int\limits_{B(R)} |u|^p \psi^p \mathrm{d}x\Bigg)^{\frac{p-2}{p}}$$
$$+ c(R-\rho)^{-1}\Bigg(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^p \mathrm{d}x\Bigg)^{\frac{1}{p}} \Bigg(\int\limits_{B(R)} |u|^p \psi^p \mathrm{d}x\Bigg)^{\frac{p-1}{p}}.$$

Using Hölder's inequality, Young's inequality and observing (1.7), we obtain

$$\begin{aligned}
\int\limits_{B(R)} |u|^p \psi^p \mathrm{d}x &\leq c\Bigg(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^p \mathrm{d}x\Bigg)^{\frac{1}{2}} \Bigg(\int\limits_{B(R)} |\nabla u|^p \mathrm{d}x\Bigg)^{\frac{1}{2}} \\
&\quad + c(R-\rho)^{-p} \int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^p \mathrm{d}x \\
&\leq cR^{3-p}\Bigg(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^{\frac{3p}{2p-3}} \mathrm{d}x\Bigg)^{\frac{2p-3}{6}} \Bigg(\int\limits_{B(R)} |\nabla u|^p \mathrm{d}x\Bigg)^{\frac{1}{2}} \\
&\quad + c(R-\rho)^{-p} R^{6-2p}\Bigg(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^{\frac{3p}{2p-3}} \mathrm{d}x\Bigg)^{\frac{2p-3}{3}} \\
&\leq cR^{\frac{9-2p}{3}}\Bigg(\int\limits_{B(R)} |\nabla u|^p \mathrm{d}x\Bigg)^{\frac{1}{2}} + c(R-\rho)^{-p} R^{\frac{18-4p}{3}}.
\end{aligned}$$

Since $R \geq 1$, and $p > 9/5$ we have $R^{\frac{9-2p}{3}} \leq R^p$ and $R^{\frac{18-4p}{3}} \leq R^{2p}$, and therefore

$$I \leq c(R-\rho)^{-p} R^p \Bigg(\int\limits_{B(R)} |\nabla u|^p \mathrm{d}x\Bigg)^{\frac{1}{2}} + (R-\rho)^{-2p} R^{2p}.$$

To estimate II, we proceed similar. We first estimate the L^3 norm of u as follows:

$$\begin{aligned}
&\int\limits_{B(R)} |u|^3 \psi^3 \mathrm{d}x \\
&\quad = \int\limits_{B(R)} \partial_i (V_{ij} - (V_{ij})_{B(R)}) u_j |u| \psi^3 \mathrm{d}x \\
&\quad = -\int\limits_{B(R)} (V_{ij} - (V_{ij})_{B(R)}) \partial_i (u_j |u|) \psi^3 \mathrm{d}x - \int\limits_{B(R)} (V_{ij} - (V_{ij})_{B(R)}) u_j |u| \partial_i \psi^3 \mathrm{d}x
\end{aligned}$$

$$\leq c\Big(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^{\frac{3p}{2p-3}}\mathrm{d}x\Big)^{\frac{2p-3}{3p}}\Big(\int\limits_{B(R)} |u|^3\psi^3\mathrm{d}x\Big)^{\frac{1}{3}}\Big(\int\limits_{B(R)} |\nabla u|^p\mathrm{d}x\Big)^{\frac{1}{p}}$$
$$+ c(R-\rho)^{-1}\Big(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^3\mathrm{d}x\Big)^{\frac{1}{3}}\Big(\int\limits_{B(R)} |u|^3\psi^3\mathrm{d}x\Big)^{\frac{2}{3}}.$$

Using Young's inequality, we get

$$\begin{aligned}\int\limits_{B(R)} |u|^3\psi^3\mathrm{d}x &\leq c\Big(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^{\frac{3p}{2p-3}}\mathrm{d}x\Big)^{\frac{2p-3}{2p}}\Big(\int\limits_{B(R)} |\nabla u|^p\mathrm{d}x\Big)^{\frac{3}{2p}}\\ &\quad + c(R-\rho)^{-3}\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^3\mathrm{d}x\\ &\leq c\Big(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^{\frac{3p}{2p-3}}\mathrm{d}x\Big)^{\frac{2p-3}{2p}}\Big(\int\limits_{B(R)} |\nabla u|^p\mathrm{d}x\Big)^{\frac{3}{2p}}\\ &\quad + c(R-\rho)^{-3}R^{\frac{3(3-p)}{p}}\Big(\int\limits_{B(R)} |\boldsymbol{V} - \boldsymbol{V}_{B(R)}|^{\frac{3p}{2p-3}}\mathrm{d}x\Big)^{\frac{2p-3}{p}}. \qquad (2.5)\end{aligned}$$

Once more appealing to (1.7), and recalling $R \geq 1$, $p > 9/5$, and thus $R^{\frac{9-p}{p}} \leq R^4$, we arrive at

$$\begin{aligned} II &\leq c(R-\rho)^{-1}R\Big(\int\limits_{B(R)} |\nabla u|^p\mathrm{d}x\Big)^{\frac{3}{2p}} + c(R-\rho)^{-4}R^{\frac{9-p}{p}}\\ &\leq c(R-\rho)^{-1}R\Big(\int\limits_{B(R)} |\nabla u|^p\mathrm{d}x\Big)^{\frac{3}{2p}} + c(R-\rho)^{-4}R^4. \qquad (2.6)\end{aligned}$$

It remains to estimate III. Using Hölder's inequality and Young's inequality, we infer

$$III \leq c(R-\rho)^{-1}\int\limits_{B(\overline{R})} |\pi - \pi_{B(\overline{R})}|^{\frac{3}{2}}\mathrm{d}x + c(R-\rho)^{-1}\int\limits_{B(\overline{R})} |u|^3\mathrm{d}x. \qquad (2.7)$$

Combining (2.7), (2.6) and (2.2), we obtain

$$III \leq cR^{\frac{3(3-p)}{2p}}(R-\rho)^{-1}\Big(\int\limits_{B(\overline{R})} |\nabla u|^p\mathrm{d}x\Big)^{\frac{3(p-1)}{2p}} + c(R-\rho)^{-1}\int\limits_{B(\overline{R})} |u|^3\mathrm{d}x.$$

The second term on the right-hand side can be absorbed into II. We also observe here, $R^{\frac{3(3-p)}{2p}} < R$ thanks to $R \geq 1$ and $p > 9/5$.

Thus, inserting the estimate of II, and once more using $R \geq 1$, we find

$$III \leq cR(R-\rho)^{-1}\Bigg(\int\limits_{B(\overline{R})} |\nabla u|^p \mathrm{d}x\Bigg)^{\frac{3(p-1)}{2p}} + cR(R-\rho)^{-1}\Bigg(\int\limits_{B(R)} |\nabla u|^p \mathrm{d}x\Bigg)^{\frac{3}{2p}} \\ + cR^4(R-\rho)^{-4}.$$

Inserting the estimates of I, II and III into the right-hand side of (2.3), and applying Young's inequality, we are led to

$$\begin{aligned} \int\limits_{B(R)} |\boldsymbol{D}(u)|^p \zeta^p \mathrm{d}x &\leq \frac{1}{2}\int\limits_{B(R)} |\nabla u|^p \mathrm{d}x + cR^4(R-\rho)^{-4} + cR^{2p}(R-\rho)^{-2p} \\ &\quad + cR^{\frac{2p}{2p-3}}(R-\rho)^{-\frac{2p}{2p-3}} + cR^{\frac{2p}{3-p}}(R-\rho)^{-\frac{2p}{3-p}} \\ &\leq \frac{1}{2}\int\limits_{B(R)} |\nabla u|^p \mathrm{d}x + cR^m(R-\rho)^{-m}, \end{aligned} \tag{2.8}$$

where we set

$$m = \max\left\{4, 2p, \frac{2p}{2p-3}, \frac{2p}{3-p}\right\},$$

and used the fact that $R^\alpha(R-\rho)^{-\alpha} \leq R^\beta(R-\rho)^{-\beta}$ for $\alpha \leq \beta$. Furthermore, applying Calderón–Zygmund's inequality, we infer

$$\begin{aligned} \int\limits_{B(\rho)} |\nabla u|^p \mathrm{d}x &\leq \int\limits_{\mathbb{R}^3} |\nabla(u\zeta)|^p \mathrm{d}x \\ &\leq \int\limits_{B(R)} |\boldsymbol{D}(u)|^p \zeta^p \mathrm{d}x + c(R-\rho)^{-p}\int\limits_{B(R)} |u|^p \mathrm{d}x. \end{aligned} \tag{2.9}$$

Estimating the left-hand side of (2.8) from below by (2.9), and applying the iteration Lemma in (Giaquinta 1983, V. Lemma 3.1), we deduce that

$$\int\limits_{B(\rho)} |\nabla u|^p \mathrm{d}x \leq cR^m(R-\rho)^{-m} \tag{2.10}$$

for all $r \leq \rho < R \leq 2r$. Choosing $R = 2r$ and $\rho = r$ in (2.10), and passing $r \to +\infty$, we find

$$\int_{\mathbb{R}^3} |\nabla u|^p \mathrm{d}x < +\infty. \tag{2.11}$$

Similarly, from (2.6) and (2.11), we get the estimate

$$r^{-1}\int\limits_{B(r)}|u|^3\mathrm{d}x \le c \quad \forall 1<r<+\infty. \tag{2.12}$$

Next, we claim that

$$r^{-1}\int\limits_{B(3r)\setminus B(2r)}|u|^3\mathrm{d}x = o(1) \quad \text{as} \quad r\to+\infty. \tag{2.13}$$

Let $\psi\in C^\infty(\mathbb{R}^3)$ be a cut off function for the annulus $B(3r)\setminus B(2r)$ in $B(4r)\setminus B(r)$, i.e., $0\le\psi\le 1$ in $\mathbb{R}^3$, $\psi=0$ in $\mathbb{R}^3\setminus(B(4r)\setminus B(r))$, $\psi=1$ on $B(3r)\setminus B(2r)$ and $|\nabla\psi|\le cr^{-1}$. Recalling that $u_i=\partial_j V_{ij}$, and applying integration by parts, using Hölder's inequality along with (1.7) we calculate

$$\begin{aligned}
&\int\limits_{B(4r)\setminus B(r)}|u|^3\psi^3\mathrm{d}x\\
&\quad=\int\limits_{B(4r)\setminus B(r)}\partial_j(V_{ij}-(V_{ij})_{B(4r)})u_i|u|\psi^3\mathrm{d}x\\
&\quad=-\int\limits_{B(4r)\setminus B(r)}(V_{ij}-(V_{ij})_{B(4r)})\partial_j(u_i|u|)\psi^3\mathrm{d}x\\
&\qquad-\int\limits_{B(4r)\setminus B(r)}(V_{ij}-(V_{ij})_{B(4r)})(u_i|u|)\partial_j\psi^3\mathrm{d}x\\
&\quad\le c\Bigg(\int\limits_{B(4r)}|\boldsymbol{V}-\boldsymbol{V}_{B(4r)}|^{\frac{3p}{2p-3}}\mathrm{d}x\Bigg)^{\frac{2p-3}{3p}}\Bigg(\int\limits_{B(4r)\setminus B(r)}|u|^3\psi^3\mathrm{d}x\Bigg)^{\frac13}\\
&\qquad\Bigg(\int\limits_{B(4r)\setminus B(r)}|\nabla u|^p\mathrm{d}x\Bigg)^{\frac1p}\\
&\qquad+cr^{-1}\Bigg(\int\limits_{B(4r)}|\boldsymbol{V}-\boldsymbol{V}_{B(4r)}|^{\frac{3p}{2p-3}}\mathrm{d}x\Bigg)^{\frac{2p-3}{3p}}\Bigg(\int\limits_{B(4r)\setminus B(r)}|u|^3\psi^3\mathrm{d}x\Bigg)^{\frac13}\\
&\qquad\Bigg(\int\limits_{B(4r)\setminus B(r)}|u|^p\mathrm{d}x\Bigg)^{\frac1p}\\
&\quad\le cr^{\frac23}\Bigg(\int\limits_{B(4r)\setminus B(r)}|u|^3\psi^3\mathrm{d}x\Bigg)^{\frac13}\Bigg(\int\limits_{B(4r)\setminus B(r)}|\nabla u|^p\mathrm{d}x\Bigg)^{\frac1p}\\
&\qquad+cr^{-\frac13}\Bigg(\int\limits_{B(4r)\setminus B(r)}|u|^3\psi^3\mathrm{d}x\Bigg)^{\frac13}\Bigg(\int\limits_{B(4r)\setminus B(r)}|u|^p\mathrm{d}x\Bigg)^{\frac1p}.
\end{aligned}\tag{2.14}$$

Let us define $\widetilde{u}_{B(4r)\setminus B(r)} = \frac{1}{\int \psi \mathrm{d}x} \int\limits_{B(4r)\setminus B(r)} u\psi \mathrm{d}x$. Recalling that $u = \nabla \cdot (\boldsymbol{V} - \boldsymbol{V}_{B(2r)})$, using integration by parts, Hölder's inequality, together with (1.7) we get

$$
\begin{aligned}
|\widetilde{u}_{B(4r)\setminus B(r)}| &\le \frac{1}{\int \psi \mathrm{d}x}\Bigg| \int\limits_{B(4r)\setminus B(r)} (\boldsymbol{V} - \boldsymbol{V}_{B(4r)}) \cdot \nabla\psi \mathrm{d}x \Bigg| \\
&= cr^{-1} \fint_{B(4r)} |\boldsymbol{V} - \boldsymbol{V}_{B(4r)}| \mathrm{d}x \le cr^{-1} \left(\fint_{B(4r)} |\boldsymbol{V} - \boldsymbol{V}_{B(4r)}|^{\frac{3p}{2p-3}} \mathrm{d}x \right)^{\frac{2p-3}{3p}} \\
&\le cr^{\frac{9-7p}{3p}}.
\end{aligned}
\tag{2.15}
$$

By the triangular inequality, we have

$$
\begin{aligned}
\Bigg(\int\limits_{B(4r)\setminus B(r)} |u|^p \mathrm{d}x \Bigg)^{\frac{1}{p}} &\le \Bigg(\int\limits_{B(4r)\setminus B(r)} |u - u_{B(4r)\setminus B(r)}|^p \mathrm{d}x \Bigg)^{\frac{1}{p}} \\
&\quad + \Bigg(\int\limits_{B(4r)\setminus B(r)} |u_{B(4r)\setminus B(r)} - \widetilde{u}_{B(4r)\setminus B(r)}|^p \mathrm{d}x \Bigg)^{\frac{1}{p}} \\
&\quad + \Bigg(\int\limits_{B(4r)\setminus B(r)} |\widetilde{u}_{B(4r)\setminus B(r)}|^p \mathrm{d}x \Bigg)^{\frac{1}{p}} \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Using the Poincaré inequality and (2.15), we find

$$
I_1 + I_3 \le cr \Bigg(\int\limits_{B(4r)\setminus B(r)} |\nabla u|^p \mathrm{d}x \Bigg)^{\frac{1}{p}} + cr^{\frac{18-7p}{3p}}. \tag{2.16}
$$

For I_2, we use the Hölder inequality, and then, the Poincaré inequality to estimate

$$
\begin{aligned}
I_2 &= \Bigg(\int\limits_{B(4r)\setminus B(r)} \Bigg| \frac{1}{\int \psi \mathrm{d}x} \int\limits_{B(4r)\setminus B(r)} (u - u_{B(4r)\setminus B(r)})\psi \mathrm{d}x \Bigg|^p \mathrm{d}x \Bigg)^{\frac{1}{p}} \\
&\le c \Bigg(\int\limits_{B(4r)\setminus B(r)} |u - u_{B(4r)\setminus B(r)}|^p \mathrm{d}x \Bigg)^{\frac{1}{p}} \le cr \Bigg(\int\limits_{B(4r)\setminus B(r)} |\nabla u|^p \mathrm{d}x \Bigg)^{\frac{1}{p}}.
\end{aligned}
\tag{2.17}
$$

Combining (2.16) and (2.17), we get

$$
\Bigg(\int\limits_{B(4r)\setminus B(r)} |u|^p \mathrm{d}x \Bigg)^{\frac{1}{p}} \le cr^{\frac{18-7p}{3p}} + cr \Bigg(\int\limits_{B(4r)\setminus B(r)} |\nabla u|^p \mathrm{d}x \Bigg)^{\frac{1}{p}}. \tag{2.18}
$$

Inserting (2.18) into the last term of (2.14) and the dividing result by $\left(\int\limits_{B(4r)\setminus B(r)}|u|^3\psi^3\mathrm{d}x\right)^{\frac{1}{3}}$, we find

$$r^{-1}\int\limits_{B(4r)\setminus B(r)}|u|^3\psi^3\mathrm{d}x \le cr^{-\frac{1}{3}}\Bigg(\int\limits_{B(4r)\setminus B(r)}|\nabla u|^p\mathrm{d}x\Bigg)^{\frac{1}{p}} + cr^{\frac{18-11p}{3p}}.$$

Thus, observing (2.11) and $p > 9/5$, we obtain the claim (2.13).

Let $1 < r < +\infty$ be arbitrarily chosen. By $\zeta \in C^\infty(\mathbb{R}^n)$ we denote a cut off function, which is radially non-increasing with $\zeta = 1$ on $B(2r)$ and $\zeta = 0$ on $\mathbb{R}^3\setminus B(3r)$ such that $|\nabla\zeta| \le cr^{-1}$. We multiply (1.1) by $u\zeta$ integrate over $B(3r)$ and apply integration by parts. This yields

$$\begin{aligned}\int\limits_{B(3r)}|\nabla u|^p\zeta^2\mathrm{d}x &= \int\limits_{B(3r)}|\nabla u|^{p-2}\nabla\zeta^2\cdot\nabla u\cdot u\mathrm{d}x\\ &\quad+\frac{1}{2}\int\limits_{B(3r)}|u|^2u\cdot\nabla\zeta+\int\limits_{B(3r)}(\pi-\pi_{B(3r)})u\cdot\nabla\zeta\mathrm{d}x\\ &\le c\int_{B(3r)\setminus B(r)}|\nabla u|^p\mathrm{d}x + cr^{-p}\int_{B(3r)\setminus B(r)}|u|^p\mathrm{d}x\\ &\quad+cr^{-1}\int\limits_{B(3r)\setminus B(2r)}|u|^3\mathrm{d}x + cr^{-1}\int\limits_{B(3r)\setminus B(2r)}|\pi-\pi_{B(3r)}||u|\mathrm{d}x\\ &= I + II + III + IV. \end{aligned}\tag{2.19}$$

Using (2.12), we immediately get

$$I = o(1) \quad \text{as} \quad r \to +\infty.$$

From (2.18) and (2.11) it follows that

$$\begin{aligned} II &= c\left\{r^{-1}\left(\int_{B(3r)\setminus B(r)}|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right\}^p\\ &\le cr^{\frac{18-10p}{3}} + c\int_{B(3r)\setminus B(r)}|\nabla u|^p\mathrm{d}x = o(1) \quad \text{as} \quad r\to+\infty. \end{aligned}\tag{2.20}$$

From (2.13), we also find $III = o(1)$ as $r \to +\infty$. Finally, applying Hölder's inequality and using (2.13), we get

$$IV \le c\Bigg(r^{-1}\int\limits_{B(3r)}|\pi-\pi_{B(3r)}|^{\frac{3}{2}}\mathrm{d}x\Bigg)^{\frac{2}{3}}\Bigg(r^{-1}\int\limits_{B(3r)\setminus B(r)}|u|^3\mathrm{d}x\Bigg)^{\frac{1}{3}}$$

$$= c\left(r^{-1}\int\limits_{B(3r)} |\pi - \pi_{B(3r)}|^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}} o(1) \tag{2.21}$$

as $r \to +\infty$. Using estimate (2.2) with $B(3r)$ in place of $B(\overline{R})$, we obtain

$$r^{-1}\int\limits_{B(3r)} |\pi - \pi_{B(3r)}|^{\frac{3}{2}}\mathrm{d}x \le cr^{\frac{9-5p}{2p}}\left(\int\limits_{B(3r)} |\nabla u|^{p}\mathrm{d}x\right)^{\frac{3(p-1)}{2p}} + cr^{-1}\int\limits_{B(3r)} |u|^{3}\mathrm{d}x.$$

By virtue of (2.11) and (2.12), the right-hand side of the above inequality is bounded for $r \ge 1$. Therefore, (2.21) shows that $IV = o(1)$ as $r \to +\infty$. Inserting the above estimates of I, II, III and IV into the right-hand side of (2.19), we deduce that

$$\int\limits_{B(r)} |\nabla u|^{p}\mathrm{d}x = o(1) \quad \text{as} \quad r \to +\infty.$$

Accordingly, $u \equiv const$ and by means of (2.12) it follows $u \equiv 0$. ☐

Acknowledgements Chae was partially supported by NRF Grants 2016R1A2B3011647, while Wolf has been supported by NRF Grants 2017R1E1A1A01074536.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

Chae, D.: Liouville-type theorem for the forced Euler equations and the Navier–Stokes equations. Commun. Math. Phys. **326**, 37–48 (2014)

Chae, D., Wolf, J.: On Liouville type theorem for the stationary Navier-Stokes equations. Cal. Var. PDE **58**(3), 111 (2019)

Chae, D., Wolf, J.: On Liouville type theorems for the steady Navier–Stokes equations in $\mathbb{R}^3$. J. Differ. Equ. **261**, 5541–5560 (2016)

Chae, D., Yoneda, T.: On the Liouville theorem for the stationary Navier–Stokes equations in a critical space. J. Math. Anal. Appl. **405**(2), 706–710 (2013)

Chamorro, D., Jarrin, O., Lemarié-Rieusset, P.-G.: Some Liouville theorems for stationary Navier–Stokes equations in Lebesgue and Morrey spaces. C.R. Math. Acad. Sci. Paris. **357**(2), 175–179 (2019)

Frehse, J., Málek, J., Steinhauer, M.: On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. SIAM J. Math. Anal. **34**(5), 1064–1083 (2003)

Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier–Stokes Equations: Steady-State Problems. Springer Monographs in Mathematics, 2nd edn. Springer, New York (2011)

Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Annals of Mathematics Studies No. 105. Princeton University Press, Princeton (1983)

Gilbarg, D., Weinberger, H.F.: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral. Ann. Sc. Norm. Super. Pisa **4**(5), 381–404 (1978)

Koch, G., Nadirashvili, N., Seregin, G., Šverák, V.: Liouville theorems for the Navier–Stokes equations and applications. Acta Math. **203**, 83–105 (2009)

Korobkov, M., Pileckas, K., Russo, R.: The Liouville theorem for the steady-state Navier–Stokes problem for axially symmetric 3D solutions in absence of swirl. J. Math. Fluid Mech. **17**(2), 287–293 (2015)

Kozono, H., Terasawa, Y., Wakasugi, Y.: A remark on Liouville-type theorems for the stationary Navier–Stokes equations in three space dimensions. J. Funct. Anal. **272**, 804–818 (2017)

Leray, J.: Étude de diverses équations intégrales non linéaire et de quelques problémes que pose l'hydrodynamique. J. Math. Pures Appl. **12**, 1–82 (1933)

Pokorný, M.: Cauchy problem for the non-Newtonian viscous incompressible fluid. Appl. Math. **41**(3), 169–201 (1996)

Seregin, G.: Liouville type theorem for stationary Navier–Stokes equations. Nonlinearity **29**, 2191–2195 (2016)

Seregin, G.: Remarks on Liouville type theorems for steady-state Navier–Stokes equations. Algebra Anal. **30**(2), 238–248 (2018)

Seregin, G., Wang, W.: Sufficient conditions on Liouville type theorems for the 3D steady Navier-Stokes equations. Algebra i Analiz. **31**(2), 269–278 (2019)

Sohr, H.: The Navier–Stokes Equations. An Elementary Functional Analytic Approach. Birkhäuser, Basel (2001)

Wilkinson, W.L.: Non-Newtonian Fluids. Fluid Mechanics, Mixing and Heat Transfer. Pergamon Press, London (1960)