



# On the Extension of Onsager's Conjecture for General Conservation Laws

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Received: 23 June 2018 / Accepted: 4 September 2018 / Published online: 21 September 2018  
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## Abstract

The aim of this work is to extend and prove the Onsager conjecture for a class of conservation laws that possess generalized entropy. One of the main findings of this work is the “universality” of the Onsager exponent,  $\alpha > 1/3$ , concerning the regularity of the solutions, say in  $C^{0,\alpha}$ , that guarantees the conservation of the generalized entropy, regardless of the structure of the genuine nonlinearity in the underlying system.

**Keywords** Onsager's conjecture · Conservation laws · Conservation of entropy

**Mathematics Subject Classification** 35Q31

## 1 Introduction

In this work, we aim at extending and proving the Onsager conjecture for a class of conservation laws that admit a generalized entropy. Roughly speaking, the Onsager conjecture (Onsager 1949) states that weak solutions of the three-dimensional Euler equations of inviscid incompressible flows conserve energy if the velocity field  $u \in C^{0,\alpha}$ , for  $\alpha > \frac{1}{3}$ , and that the critical exponent  $\alpha = \frac{1}{3}$  is sharp. This conjecture has been the subject of intensive investigation for the last two decades. The sufficient condition direction was proved by Eyink (1994) for the case when  $\alpha > \frac{1}{2}$ . Later, a complete proof was established by Constantin and Titi (1994) (see also Cheskidov et al. 2008) under slightly weaker regularity assumptions on the solution which involve a similar exponent  $\alpha > \frac{1}{3}$ . Duchon and Robert (2000) have shown, under similar sufficient conditions to those in Constantin and Titi (1994), a local version of the conservation of energy. It is worth mentioning that the above results are established

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Communicated by Paul Newton.

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in the absence of physical boundaries, i.e., periodic boundary conditions or the whole space. However, due to the well-recognized dominant role of the boundary in the generation of turbulence (cf. Bardos and Titi (2013) and references therein) it seems very reasonable to investigate the analogue of the Onsager conjecture in bounded domains. Indeed, for the three-dimensional Euler equations in a smooth bounded domain  $\Omega$ , subject to no-normal flow (slip) boundary conditions, it has been shown in Bardos and Titi (2018) that a weak solution conserves the energy provided the velocity field  $u \in C^{0,\alpha}(\overline{\Omega})$ , for  $\alpha > \frac{1}{3}$ , (see also Robinson et al. (2017) for the case of the upper-half space under stronger conditions on the pressure term). A local version, analogue to that of Duchon and Robert (2000), was established recently in Bardos et al. under slightly weaker conditions to those in Bardos and Titi (2018), but at the expense of additional sufficient conditions concerning the vanishing behavior of the energy flux near the boundary.

Showing the sharpness of the exponent  $\alpha = \frac{1}{3}$  in Onsager's conjecture turns out to be much more subtle. This direction has been underlined by a series of contributions (cf. Isett 2016; Buckmaster et al. 2018 and references therein) where weak solutions,  $u \in C^{0,\alpha}$ , with  $\alpha < \frac{1}{3}$ , that dissipate energy were constructed using the convex integration machinery. Notice, however, that there exists a family of weak solutions to the three-dimensional Euler equations, that are not more regular than  $L^2$ , and which conserve the energy, cf. Bardos and Titi (2010).

It is most natural to ask whether the analogue of the Onsager conjecture is valid for other systems of conservation laws. Indeed, there has been some intensive recent work extending the Onsager conjecture for other physical systems, in the absence of physical boundaries, see, e.g., Akramov and Wiedemann (2018), Drivas and Eyink (2017), Feireisl et al. (2017), Gwiazda et al. (2018), Yu (2017) and references therein. In this paper, we consider systems of conservation laws with physical boundaries. We use the approach of Duchon and Robert (2000), as it has been outlined and extended in Bardos et al. in the presence of physical boundaries, to establish the local conservation of “generalized entropies” (conserved quantities, which are not necessarily convex) for systems of conservation laws which possess such generalized entropies. This is accomplished provided the underlying weak solutions are locally in  $C^{0,\alpha}$ , for  $\alpha > \frac{1}{3}$ . One of the primary findings of this work is the universality of the Onsager critical exponent  $\alpha = \frac{1}{3}$ , regardless of the structure of the genuine nonlinearity in the underlying system. Notably, in a forthcoming paper (Bardos et al. 2018) we will show the extension of these results and provide additional explicit physical examples, using more delicate harmonic analysis tools and function spaces. In particular, we will show similar results under slightly weaker regularity assumptions of the solutions which are required to belong to some “exotic” function spaces with exponent  $\alpha = \frac{1}{3}$ .

## 2 Local and Global Generalized Entropies Conservation

In this section, we state and prove our main result. In the first subsection, we establish the local entropy conservation for any weak solution that belongs to the Hölder space  $C^{0,\alpha}$  with  $\alpha > 1/3$ . In the second subsection, we state a fundamental Lemma

concerning nonlinear commutator estimate of Friedrichs mollifier, and in the last subsection we state additional sufficient conditions for establishing the global entropy conservation.

### 2.1 Local Entropy Conservation

Let  $Q \subset \mathbb{R}^{d+1}$  be an open set, and consider in  $Q$  the following system of conservation laws:

$$\sum_{0 \leq i \leq d} \partial_{x_i} A_i(u) = 0, \tag{2.1}$$

where  $x \mapsto u(x)$  is the unknown vector field defined in  $Q$  with values in an open convex set  $\mathcal{M} \subset \subset \mathbb{R}^k$ , while the vectors  $A_i, i = 0, 1, \dots, d$ , are  $C^2$  vector-valued functions defined in  $\mathcal{M}$  with values in  $\mathbb{R}^l$ , where  $A_i^j$ , for  $j = 1, \dots, l$ , denotes the  $j$ -th component of  $A_i$ . Below  $\mathcal{L}(\mathbb{R}^l; \mathbb{R})$  stands for the set of linear operators from  $\mathbb{R}^l$  to  $\mathbb{R}$ .

**Theorem 2.1** *Suppose that  $B : \mathcal{M} \mapsto \mathcal{L}(\mathbb{R}^l; \mathbb{R})$  is a  $C^1$  map, represented by a row vector  $B(u) = (b_1(u), b_2(u), \dots, b_l(u))$ , defines a generalized entropy, i.e., for every  $i = 0, 1, \dots, l$  there exists a  $C^2$  flux  $q_i : \mathcal{M} \mapsto \mathbb{R}$  such that one has:*

$$B(u) \cdot \nabla_u A_i(u) = \nabla_u q_i(u), \text{ for } i = 0, 1, \dots, d. \tag{2.2}$$

*Suppose that  $u$  is a weak solution of (2.1). Moreover, supposed that for every  $\mathcal{K} \subset \subset Q$  there exists  $\alpha > \frac{1}{3}$ , which might depend on  $\mathcal{K}$ , such that  $u \in C^{0,\alpha}(\mathcal{K})$ . Then the following equation holds in  $\mathcal{D}'(Q)$*

$$\sum_{0 \leq i \leq d} \partial_{x_i} q_i(u) = 0. \tag{2.3}$$

**Remark 2.2** (i) Observe that since  $\mathcal{M}$  is assumed to be an open convex, and hence simply-connected, set then the generalized entropy condition (2.2) is equivalent to the relation:

$$\sum_{1 \leq j \leq l} \frac{\partial b_j}{\partial u_\beta} \frac{\partial A_i^j}{\partial u_\alpha} = \sum_{1 \leq j \leq l} \frac{\partial b_j}{\partial u_\alpha} \frac{\partial A_i^j}{\partial u_\beta}, \text{ for all } \alpha, \beta = 1, 2, \dots, l, \text{ and } i = 1, 2, \dots, d. \tag{2.4}$$

For more details about entropy cf. Dafermos (2000), Chapters 3 and 5.

(ii) Notice that equation (2.3) is, in a sense, the analogue of the local conservation of energy for the three-dimensional incompressible Euler equations as presented in Duchon and Robert (2000) (see also Bardos et al.).

The proof of Theorem 2.1 is an extension of the ideas introduced in Bardos et al. What has to be proven is that for any  $\phi \in \mathcal{D}(Q)$  one has:

$$\sum_{0 \leq i \leq d} \langle \partial_{x_i} \phi, q_i(u) \rangle_x = 0. \tag{2.5}$$

The support of  $\phi$  being given one introduces three open sets  $Q_i$  such that  $\text{supp } \phi \subset\subset Q_1 \subset\subset Q_2 \subset\subset Q_3 \subset\subset Q$  and such for  $i = 1, 2$ , one has

$$\sup_{x \in Q_i, y \in \mathbb{R}^{d+1} \setminus Q_{i+1}} |x - y| > \epsilon_0,$$

for some  $\epsilon_0 > 0$  that depends on the support of  $\phi$  and  $Q$ .

Then one introduces a  $C^\infty$  cutoff function  $I : \mathbb{R}^{d+1} \mapsto [0, 1]$ , which is zero outside  $Q_3$  and is equal to 1 on  $Q_2$ . For any distribution  $T \in \mathcal{D}'(Q)$  one denotes by  $\bar{T} \in \mathcal{D}'(\mathbb{R}^{d+1})$  the distribution defined for every  $\psi \in \mathcal{D}(\mathbb{R}^{d+1})$  by the formula

$$\langle \psi, \bar{T} \rangle_x = \langle I\psi, T \rangle_x. \tag{2.6}$$

Eventually, we use standard  $C^\infty(\mathbb{R}^{d+1})$  radially symmetric compactly supported Friedrichs mollifier  $x \mapsto \rho_\epsilon(x)$ , with support inside the ball of radius  $\epsilon > 0$ . For any distribution  $T \in \mathcal{D}'(\mathbb{R}^{d+1})$ , we define the distribution  $T^\epsilon := T * \rho_\epsilon$ . Next, we fix  $\epsilon \in (0, \frac{\epsilon_0}{2})$ , which we will eventually let it tend to zero.

Observe that from (2.1) one infers that

$$\sum_{0 \leq i \leq d} \partial_{x_i} A_i(\bar{u}) = 0, \text{ in } \mathcal{D}'(Q_2). \tag{2.7}$$

Notice that for any  $\Psi \in C_c^2(Q_1; \mathcal{L}(\mathbb{R}^l, \mathbb{R}))$  one has that  $\Psi^\epsilon \in \mathcal{D}(Q_2; \mathcal{L}(\mathbb{R}^l, \mathbb{R}))$ . Therefore, as a result of (2.7) one has

$$\begin{aligned} 0 &= \left\langle \Psi^\epsilon, \sum_{0 \leq i \leq d} \partial_{x_i} A_i(\bar{u}) \right\rangle_x = - \sum_{0 \leq i \leq d} \langle (\partial_{x_i} \Psi)^\epsilon, A_i(\bar{u}) \rangle_x \\ &= \sum_{0 \leq i \leq d} \int_{\mathbb{R}^{d+1}} \partial_{x_i} \Psi(x) \cdot (A_i(\bar{u}))^\epsilon(x) \, dx \\ &= - \sum_{0 \leq i \leq d} \int_{\mathbb{R}^{d+1}} \partial_{x_i} \Psi(x) \cdot A_i((\bar{u})^\epsilon)(x) \, dx \\ &\quad - \sum_{0 \leq i \leq d} \int_{\mathbb{R}^{d+1}} \partial_{x_i} \Psi(x) \cdot \left( (A_i(\bar{u}))^\epsilon(x) - A_i((\bar{u})^\epsilon)(x) \right) \, dx. \end{aligned} \tag{2.8}$$

Now, we replace  $\Psi$ , in (2.8), by  $\phi B((\bar{u})^\epsilon) \in C_c^2(Q_1; \mathcal{L}(\mathbb{R}^l, \mathbb{R}))$ . Thus, the right-hand side of (2.8) is the sum of two terms:

$$J_\epsilon = - \sum_{0 \leq i \leq d} \int_{\mathbb{R}_x^{d+1}} \partial_{x_i} (\phi B((\bar{u})^\epsilon))(x) \cdot A_i((\bar{u})^\epsilon)(x) \, dx, \tag{2.9}$$

and

$$K_\epsilon = \sum_{0 \leq i \leq d} \int_{\mathbb{R}_x^{d+1}} \partial_{x_i} (\phi B((\bar{u})^\epsilon))(x) \cdot \left( A_i((\bar{u})^\epsilon)(x) - (A_i(\bar{u}))^\epsilon(x) \right) \, dx. \tag{2.10}$$

Thanks to (2.2) one has for  $J_\epsilon$ :

$$\begin{aligned} J_\epsilon &= - \sum_{0 \leq i \leq d} \int_{\mathbb{R}_x^{d+1}} \partial_{x_i} (\phi B((\bar{u})^\epsilon))(x) \cdot A_i((\bar{u})^\epsilon)(x) \, dx \\ &= \sum_{0 \leq i \leq d} \int_{\mathbb{R}_x^{d+1}} (\phi B((\bar{u})^\epsilon))(x) \cdot \partial_{x_i} A_i((\bar{u})^\epsilon)(x) \, dx \\ &= \sum_{0 \leq i \leq d} \int_{\mathbb{R}_x^{d+1}} \left[ \phi(B(\eta(\bar{u})^\epsilon)) \cdot (\nabla_u A_i((\bar{u})^\epsilon)) \cdot \partial_{x_i} (\bar{u})^\epsilon \right] \, dx \\ &= \sum_{0 \leq i \leq d} \int_{\mathbb{R}_x^{d+1}} \phi(x) \partial_{x_i} q_i((\bar{u})^\epsilon)(x) \, dx = - \sum_{0 \leq i \leq d} \int_{\mathbb{R}_x^{d+1}} \partial_{x_i} \phi(x) q_i((\bar{u})^\epsilon)(x) \, dx. \end{aligned} \tag{2.11}$$

Since  $u \in C^{0,\alpha}(Q_2)$ ,  $q_i \in C^2(\mathcal{M})$  and  $\phi \in \mathcal{D}(Q_1)$ , then by virtue of the Lebesgue Dominant Convergence theorem this last term in (2.11) converges, when  $\epsilon \rightarrow 0$ , to

$$- \sum_{0 \leq i \leq d} \int_Q \partial_{x_i} \phi(x) q_i(u) \, dx = \sum_{0 \leq i \leq d} \langle \phi, \partial_{x_i} q_i(u) \rangle_x.$$

To complete the proof, one has to show that for  $\alpha > \frac{1}{3}$  the term  $K_\epsilon$  converges to 0, as  $\epsilon \rightarrow 0$ . Obviously, one has:

$$\begin{aligned} &\| \partial_{x_i} (\phi B((\bar{u})^\epsilon)) \|_{L^\infty(Q_1)} \\ &\leq \| \phi \|_{C^1(Q_1)} \| B \|_{C^0(\mathcal{M})} + C \| \phi \|_{L^\infty(Q_1)} \| B \|_{C^2(\mathcal{M})} \| u \|_{C^{0,\alpha}(Q_2)} \epsilon^{\alpha-1}. \end{aligned} \tag{2.12}$$

Therefore, the proof is completed by virtue of the following estimate.

$$\| (A_i((\bar{u})^\epsilon) - (A_i(\bar{u}))^\epsilon) \|_{L^\infty(Q_1)} \leq C \| A_i \|_{C^2(\mathcal{M})} \| u \|_{C^{0,\alpha}(Q_2)}^2 \epsilon^{2\alpha}, \tag{2.13}$$

which we will establish in the next section.

### 2.2 Nonlinear Commutator Estimate for Friedrichs Mollifier

Since  $\bar{u}$  in (2.13) belongs to  $C^{0,\alpha}(\mathbb{R}^{d+1})$ , with the Hölder exponent  $\alpha$  that corresponds to the compact set  $\bar{Q}_3$ , and is compactly supported in  $Q_3$ , estimate (2.13) follows from the following more “general lemma” concerning nonlinear commutator estimate for the Friedrichs mollifier (see also Gwiazda et al. 2018). Notably, the estimate below, which generalizes those established for quadratic nonlinearities in Constantin and Titi (1994) and Bardos and Titi (2018), shows that regardless of the structure of the nonlinearity one always obtains the same exponent for  $\epsilon$ , i.e.,  $2\alpha$  in (2.13). Combining (2.13) with (2.12) implies the “universality” of the Onsager exponent, i.e.,  $\alpha > 1/3$ , for conservation laws.

**Lemma 2.3** *For any  $F \in C^2(\mathcal{M}; \mathbb{R}^l)$  and for any compactly supported function  $v \in C_c^{0,\alpha}(\mathbb{R}^{d+1}; \mathcal{M})$  one has:*

$$\|(F(v))^\epsilon - F(v^\epsilon)\|_{L^\infty} \leq C(\|F\|_{C^2(\mathcal{M})})\|v\|_{C^{0,\alpha}(\mathbb{R}^{d+1})}^2 \epsilon^{2\alpha}. \tag{2.14}$$

**Proof** First observe that if  $F$  is an affine map one has:

$$(F(v))^\epsilon - F(v^\epsilon) = 0. \tag{2.15}$$

Therefore, combining (2.15) with the Taylor formula applied to  $F(v(x - y))$ , viewed as a function of  $y$ , about  $v_\epsilon(x)$  gives the following estimates:

$$\begin{aligned} |(F(v))^\epsilon(x) - F(v^\epsilon(x))| &= \left| \left( \int_{\mathbb{R}^{d+1}} F(v(x - y)) \rho_\epsilon(y) dy \right) - F(v^\epsilon(x)) \right| \\ &= \left| \int_{\mathbb{R}^{d+1}} (F(v(x - y)) - F(v^\epsilon(x))) \rho_\epsilon(y) dy \right| \\ &= \left| \int_{\mathbb{R}^{d+1}} \left( \int_0^1 (\nabla_v^2 F(sv(x - y) + (1 - s)v^\epsilon(x))) (1 - s) ds \right) (v(x - y) - v^\epsilon(x))^2 \rho_\epsilon(y) dy \right| \tag{2.16} \\ &\leq \|F\|_{C^2(\mathcal{M})} \int_{\mathbb{R}^{d+1}} |v(x - y) - v^\epsilon(x)|^2 \rho_\epsilon(y) dy \\ &= \|F\|_{C^2(\mathcal{M})} \int_{\mathbb{R}^{d+1}} |v(x - y) - \int_{\mathbb{R}^{d+1}} v(x - z) \rho_\epsilon(z) dz|^2 \rho_\epsilon(y) dy \\ &= \|F\|_{C^2(\mathcal{M})} \int_{\mathbb{R}^{d+1}} \left| \int_{\mathbb{R}^{d+1}} (v(x - y) - v(x - z)) \rho_\epsilon(z) dz \right|^2 \rho_\epsilon(y) dy. \end{aligned}$$

Since  $x \mapsto \rho_\epsilon(x)$  is equal to zero for  $|x| > \epsilon$  then in the last term of (2.16) one has to restrict oneself to the values when  $|y| < \epsilon$  and  $|z| < \epsilon$ . This in turn implies that  $|(x - y) - (x - z)| \leq 2\epsilon$ . Consequently one has

$$\begin{aligned} \|(F(v))^\epsilon - F(v^\epsilon)\|_{L^\infty} &\leq \|F\|_{C^2(\mathcal{M})} \int_{\mathbb{R}^{d+1}} \left| \int_{\mathbb{R}^{d+1}} (2\epsilon)^\alpha \|v\|_{C^{0,\alpha}(\mathbb{R}^{d+1})} \rho_\epsilon(z) dz \right|^2 \rho_\epsilon(y) dy \\ &= \epsilon^{2\alpha} \|F\|_{C^2(\mathcal{M})} \|v\|_{C^{0,\alpha}(\mathbb{R}^{d+1})}^2. \end{aligned}$$

□

### 2.3 Sufficient Conditions for Global Entropy Conservation

Theorem 2.1 can obviously be applied to the case of conservation of energy. Consider as above a weak solution  $u$  of the following conservation law:

$$\partial_t A_0(u) + \sum_{1 \leq i \leq d} \partial_{x_i} (A_i(u)) = 0, \tag{2.17}$$

and assume that this equation has an extra conservation law  $u \mapsto \eta(u)$  (or entropy as usually called) with corresponding fluxes  $q_0(u) = \eta(u)$  and  $q_j(u)$ , for  $j = 1, 2, \dots, d$ , satisfying

$$\begin{aligned} \nabla_u \eta(u) \cdot \nabla_u A_0(u) &= \nabla_u \eta(u), \quad \text{and} \\ \nabla_u \eta(u) \cdot \nabla_u A_j(u) &= \nabla_u q_j(u), \quad \text{for } j = 1, 2, \dots, d. \end{aligned} \tag{2.18}$$

Consequently, the above gives formally the extra conservation law:

$$\partial_t \eta(u) + \sum_{1 \leq i \leq d} \partial_{x_i} q_i(u) = 0. \tag{2.19}$$

Then, applying Theorem 2.1 with,  $x_0 = t$  and  $(B(u) = \nabla_u \eta(u))$  one has the following:

**Theorem 2.4** *Suppose that  $u \in L^\infty(Q)$  is defined in a time cylindrical domain  $Q = (T_1, T_2) \times \Omega$ . Suppose also that  $\Omega$  is a bounded open set with a Lipschitz boundary  $\partial\Omega$ . This implies, in particular, the existence of  $\delta_0 > 0$  such that for every  $x \in \Omega$  with  $d(x, \partial\Omega) < \delta_0$  the function  $x \mapsto d(x, \partial\Omega)$  is Lipschitz and that there exists a unique point  $\hat{x} \in \partial\Omega$ , depending on  $x$ , such that*

$$d(x, \partial\Omega) = |x - \hat{x}| \quad \text{and} \quad \nabla_x d(x, \partial\Omega) = -\vec{n}(\hat{x}). \tag{2.20}$$

*Suppose that  $u$  is a weak solution of (2.17) with the following properties:*

1. *For any  $\tilde{Q} \subset\subset Q$  there exists  $\alpha > \frac{1}{3}$ , which might depend on  $\tilde{Q}$ , such that  $u \in C^{0,\alpha}(\tilde{Q})$ .*

2. *Let  $\delta \in (0, \frac{\delta_0}{2})$ , denote by  $\Omega_\delta = \Omega \cap \{d(x, \partial\Omega) < \frac{\delta}{2}\}$  and by  $Q_\delta = (T_1, T_2) \times \Omega_\delta$ . Then*

$$\lim_{\delta \rightarrow 0} \sup_{(t,x) \in Q_\delta} \left| \sum_{1 \leq i \leq d} q_i(u(t, x)) \vec{n}_i(\hat{x}) \right| = 0. \tag{2.21}$$

*Then the solution  $u$  conserves the total entropy  $\eta(u)$ , i.e., for almost every  $(t_1, t_2)$  satisfying  $T_1 < t_1 \leq t_2 < T_2$*

$$\int_\Omega \eta(u(t_1, x)) dx = \int_\Omega \eta(u(t_2, x)) dx. \tag{2.22}$$

**Proof** Theorem 2.1 implies that  $u$  satisfies the entropy relation (2.3) in the sense of distribution. Hence one considers a test function in  $\mathcal{D}(Q)$  of the following form  $\phi(t, x) =$

$\theta(t) \times \chi\left(\frac{d(x, \partial\Omega)}{\delta}\right)$ . Here  $\chi \in C^\infty([0, \infty))$  is a cutoff function  $\chi : [0, \infty) \rightarrow [0, 1]$  satisfying  $\chi(s) = 0$  for  $s \in [0, \frac{1}{4}]$  and  $\chi(s) = 1$  for  $s \in [\frac{1}{2}, \infty)$ ; and  $\theta \in \mathcal{D}((T_1, T_2))$ . Parameter  $\delta$  is chosen to be small enough such that  $\text{supp } \theta \subset (T_1 + \delta, T_2 - \delta)$ , and  $\delta \in (0, \frac{\delta_0}{2})$ .

Then one has:

$$0 = \left\langle \phi, \partial_t \eta(u) + \sum_{1 \leq i \leq d} \partial_{x_i} q_i(u) \right\rangle_{t,x} = - \int_Q \eta(u(t, x)) \chi\left(\frac{d(x, \partial\Omega)}{\delta}\right) \frac{d}{dt} \theta(t) dx dt$$

$$- \int_{Q_\delta} \theta(t) \left( \sum_{1 \leq i \leq d} q_i(u(t, x)) \vec{n}_i(\hat{x}) \frac{1}{\delta} \chi'\left(\frac{d(x, \partial\Omega)}{\delta}\right) \right) dx dt. \quad (2.23)$$

Letting  $\delta \rightarrow 0$ , then by the Lebesgue Dominant Convergence theorem one obtains first the trivial relation

$$\lim_{\delta \rightarrow 0} \int_Q \eta(u(t, x)) \chi\left(\frac{d(x, \partial\Omega)}{\delta}\right) \frac{d}{dt} \theta(t) dx dt = \int_{T_1}^{T_2} \left( \frac{d}{dt} \theta(t) \int_\Omega \eta(u(t, x)) dx \right) dt. \quad (2.24)$$

While for the term

$$- \int_{Q_\delta} \theta(t) \left( \sum_{1 \leq i \leq d} q_i(u(t, x)) \vec{n}_i(\hat{x}) \frac{1}{\delta} \chi'\left(\frac{d(x, \partial\Omega)}{\delta}\right) \right) dx dt, \quad (2.25)$$

one uses the estimate:

$$\left| \int_{Q_\delta} \theta(t) \left( \sum_{1 \leq i \leq d} q_i(u(t, x)) \vec{n}_i(\hat{x}) \frac{1}{\delta} \chi'\left(\frac{d(x, \partial\Omega)}{\delta}\right) \right) dx dt \right|$$

$$\leq \sup_{(t,x) \in Q_\delta} \left| \sum_{1 \leq i \leq d} q_i(u(t, x)) \vec{n}_i(\hat{x}) \right| \left( \int_{\Omega_\delta} \frac{1}{\delta} \left| \chi'\left(\frac{d(x, \partial\Omega)}{\delta}\right) \right| dx \right)$$

$$\times \int_{T_1}^{T_2} |\theta(t)| dt$$

$$\leq C \sup_{(t,x) \in Q_\delta} \left| \sum_{1 \leq i \leq d} q_i(u(t, x)) \vec{n}_i(\hat{x}) \right|. \quad (2.26)$$

Thanks to (2.21) the right-hand side in the inequality above tends to zero, as  $\delta \rightarrow 0$ . Hence, from all the above one infers that for every  $\theta \in \mathcal{D}(T_1, T_2)$ ,

$$\int_{T_1}^{T_2} \left( \frac{d}{dt} \theta(t) \int_\Omega \eta(u(t, x)) dx \right) dt = 0$$

$$\text{or } \frac{d}{dt} \int_\Omega \eta(u(t, x)) dx = 0 \text{ in } \mathcal{D}'(T_1, T_2). \quad (2.27)$$



The above implies that  $\int_{\Omega} \eta(u(t, x)) dx = \text{const.}$ , for almost every  $t \in (T_1, T_2)$ , since  $\int_{\Omega} \eta(u(t, x)) dx$  is a continuous function for all  $t \in (T_1, T_2)$ . This concludes our proof.  $\square$

**Remark 2.5** Observe that the above theorem can be applied to the incompressible Euler equations

$$\begin{aligned} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p &= 0, \\ \nabla \cdot v &= 0. \end{aligned} \quad (2.28)$$

where  $u$  in the above theorem is the column vector  $u = \begin{pmatrix} v \\ p \end{pmatrix}$ ,  $\eta(u) = \frac{|v|^2}{2}$ ,  $A_0(u) = \begin{pmatrix} v \\ 0 \end{pmatrix}$ , and  $A_i(u) = \begin{pmatrix} v_i v + p \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{pmatrix} \\ v_i \end{pmatrix}$ , for  $i = 1, 2, 3$ .

In the forthcoming paper (Bardos et al. 2018), we will also provide some additional physical examples for which Theorem 2.1 can be applied.

**Remark 2.6** We remark that Lemma 2.3 could be generalized in the spirit of Lemma 2.1 of Conti et al. (2012) to give

$$\|F(v^\epsilon) - (F(v))^\epsilon\|_{C^r} \leq C \|F\|_{C^{r+2}} \|v\|_{C^{r+\alpha}}^2 \epsilon^{2\alpha} \quad (2.29)$$

**Acknowledgements** EST would like to thank the École Polytechnique for its kind hospitality, where this work was completed, and the École Polytechnique Foundation for its partial financial support through the 2017–2018 “Gaspard Monge Visiting Professor” Program. EST was also supported in part by the ONR Grant N00014-15-1-2333, and by the Einstein Stiftung/Foundation, Berlin, through the Einstein Visiting Fellow Program. This work was partially supported by the Simons-Foundation Grant 346300 and the Polish Government MNiSW 2015–2019 matching fund. PG and AŚG received support from the National Science Centre (Poland), 2015/18/MST1/00075.

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