Uniqueness and Non-uniqueness of Limit Cycles for Piecewise Linear Differential Systems with Three Zones and No Symmetry

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Abstract Some techniques for proving the existence and uniqueness of limit cycles for smooth differential systems are extended to continuous piecewise linear differential systems with two and three zones and no symmetry. For planar systems with three linearity zones, the existence of two limit cycles surrounding the only equilibrium point at the origin is rigorously shown for the first time. The usefulness of the achieved analytical results is illustrated by considering non-symmetric memristor-based electronic oscillators.

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1 Introduction and Statement of the Main Results

One of the most interesting problems in the qualitative theory of planar differential systems is the study of their limit cycles, see, for instance, the books Yan-Qian et al. (1986), Zhang Zhifen et al. (1992). This problem restricted to the planar polynomial differential equations is the famous second part of the 16th Hilbert problem (Hilbert 1900). Since Hilbert problem has up to now been intractable (see Ilyashenko 2002; Li 2003), Smale in (1998) proposed to study it restricted to the polynomial Liénard differential systems.

For smooth Liénard systems, there are many results on the nonexistence, existence and uniqueness of limit cycles, see, for instance, Carletti and Villari (2005), Dumortier and Li (1996), Dumortier and Rousseau (1990), Gasull et al. (2009), Khibnik et al. (1998), Llibre et al. (2009), Llibre and Valls (2013), Xiao and Zhang (2003), Zhang Zhifen et al. (1992). Going beyond the smooth case, a natural step is to allow non-smoothness while keeping the continuity, as has been done in some recent work (Freire et al. 2002; Hogan 2003; van Horssen 2005; Llibre et al. 2013). In a further step, other authors have considered a line of discontinuity in the vector field defining the planar system, see Giannakopoulos and Pliete (2001), Llibre and Ponce (2012), Zou et al. (2006).

Planar piecewise linear differential systems appear naturally as the most accurate mathematical models for a big amount of engineering devices exhibiting nonlinear dynamics. The description of all possible nonlinear responses for this family of systems and their rigorous mathematical justification are, however, tasks only partially undertaken. As a matter of fact, the lack of smoothness precludes the application of the results coming from standard differentiable dynamics, and new specific results are still needed even in the seemingly simpler cases. In particular, while the majority of known results for piecewise linear differential systems deal with two zones or three zones with symmetry, in this paper we focus the attention on non-symmetric systems. We review some results for systems with only two linear zones, extending some previous results, and give also new results for system with three zones. We also show that combining such results, it can be proved the existence of cases with two limit cycles surrounding the only equilibrium point, an outstanding result in the field.

In this work, we will study the limit cycles of the Liénard piecewise linear differential systems

$$\frac{dx}{d\tau} = x' = F(x) - y, \quad \frac{dy}{d\tau} = y' = g(x), \tag{1}$$

where

$$F(x) = \begin{cases} T_L(x+u) - T_C u & \text{if } x \leq -u, \\ T_C x & \text{if } -u \leq x \leq v, \\ T_R(x-v) + T_C v & \text{if } x \geq v, \end{cases} \tag{2}$$

and

$$g(x) = \begin{cases} \ell(x + u) - u & \text{if } x \leq -u, \\ x & \text{if } -u \leq x \leq v, \\ r(x - v) + v & \text{if } x \geq v, \end{cases} \tag{3}$$

where the constants u and v are positive, so that the straight lines $x = -u$ and $x = v$ split the phase plane in three linearity regions. In the case that these systems are symmetric with respect the origin of coordinates, i.e.,

$$u = v, \quad T_L = T_R, \quad \ell = r.$$

the study of their limit cycles is done, see Carmona et al. (2002), Freire et al. (2002), Freire et al. (1999), Freire et al. (1997), Llibre and Sotomayor (1996), and for a complete analysis the book Llibre and Teruel (2013).

Remark 1 Systems (1)–(3) constitute an important family of piecewise linear systems since under generic assumptions a big amount of systems in direct control and other areas can be written in such a form, see, for instance, Llibre and Sotomayor (1996) and Carmona et al. (2002). In fact, a piecewise linear *characteristic function* ϕ, given by

$$\phi(x) = \begin{cases} m_L(x + u) - u & \text{for } x \leq -u, \\ x & \text{for } -u \leq x \leq v, \\ m_R(x - v) + v & \text{for } x \geq v, \end{cases}$$

is very common in control systems of the form

$$\frac{d\mathbf{x}}{ds} = \dot{\mathbf{x}} = A\mathbf{x} + \phi(\mathbf{k} \cdot \mathbf{x})\mathbf{b}, \tag{4}$$

see, for instance, Bernardo et al. (2008), Lefschetz (1965), Narendra and Taylor (1973) and also Sect. 2, where A is a 2×2 real matrix and $\mathbf{x}$, $\mathbf{k}$ and $\mathbf{b}$ are in $\mathbb{R}^2$ and $\mathbf{k} \cdot \mathbf{x}$ denotes the usual inner product between the vectors $\mathbf{k}$ and $\mathbf{x}$. Remarkably enough, the only required hypothesis to pass from (4) to the form (1)–(3) is $D > 0$, where $D = \det B$, and $B = A + \mathbf{b}\mathbf{k}^t$.

In this paper, we shall study the limit cycles of systems (1) in the non-symmetric case with respect to the origin of coordinates. Of course, as a particular case, we shall get the symmetric case. The first main result in this direction is the following theorem that characterizes the number of limit cycles when there are only two linearity zones. We need the following definitions. Let

$$Y_{\pm}^r = \pm v \exp(\pm\gamma\theta_{\pm})\sqrt{(\sigma - \mu_{\pm})^2 + \omega^2},$$

where provided that $T_C \in (-2, 2)$ and $r < 0$, we take

$$\gamma = \frac{T_C}{\sqrt{4 - T_C^2}}, \quad \mu_\pm = \frac{T_R \pm \sqrt{T_R^2 - 4r}}{2r}, \quad \sigma = \frac{T_C}{2}, \quad \omega = \frac{\sqrt{4 - T_C^2}}{2},$$

and

$$\theta_\pm = 2 \arctan\left(\frac{\omega}{\sqrt{(\sigma - \mu_\pm)^2 + \omega^2} \pm (\sigma - \mu_\pm)}\right).$$

Moreover, let

$$Y_\pm^\ell = \pm u \exp(\pm \gamma \psi_\pm) \sqrt{(\sigma - \delta_\pm)^2 + \omega^2},$$

where it is assumed $\ell < 0$, and we take

$$\delta_\pm = \frac{T_L \pm \sqrt{T_L^2 - 4\ell}}{2\ell} \quad \text{and} \quad \psi_\pm = 2 \arctan\left(\frac{\omega}{\sqrt{(\sigma - \delta_\pm)^2 + \omega^2} \pm (\sigma - \delta_\pm)}\right).$$

Theorem 2 *Consider the differential systems* (1) *with only two linearity zones, where* $T_L = T_C$ *and* $\ell = 1$. *Assume* $T_C \neq 0$. *Then, the following statements hold.*

(a) *Two necessary conditions for the existence of periodic orbits are* $|T_C| < 2$ *and* $T_C T_R < 0$.

(b) *Assume that* $|T_C| < 2$ *and* $T_C T_R < 0$. *Then, system* (1) *has a periodic orbit*
   (b.1) when $T_C > 0$ *and* $r > 0$ *if and only if* $T_C + T_R/\sqrt{r} < 0$;
   (b.2) when $T_C < 0$ *and* $r > 0$ *if and only if* $T_C + T_R/\sqrt{r} > 0$;
   (b.3) when $T_C > 0$ *and* $r < 0$ *if and only if* $e^{\pi \gamma} Y_+^r + Y_-^r < 0$;
   (b.4) when $T_C < 0$ *and* $r < 0$ *if and only if* $e^{\pi \gamma} Y_+^r + Y_-^r > 0$.
   Moreover, in all cases that the origin is surrounded by a limit cycle this is unique, stable if $T_C > 0$ *and unstable if* $T_C < 0$.

   The proof of Theorem 2 is given in Sect. 5. Its dual result, whose proof is similar and we will not provide it, is the following.

Theorem 3 *Consider the differential systems* (1) *with only two linearity zones, where* $T_R = T_C$ *and* $r = 1$. *Assume* $T_C \neq 0$. *Then, the following statements hold.*

(a) *Two necessary conditions for the existence of periodic orbits are* $|T_C| < 2$ *and* $T_L T_C < 0$.

(b) *Assume that* $|T_C| < 2$ *and* $T_L T_C < 0$. *Then, system* (1) *has a periodic orbit*
   (b.1) when $T_C > 0$ *and* $\ell > 0$ *if and only if* $T_C + T_L/\sqrt{\ell} < 0$;
   (b.2) when $T_C < 0$ *and* $\ell > 0$ *if and only if* $T_C + T_L/\sqrt{\ell} > 0$;
   (b.3) when $T_C > 0$, *and* $\ell < 0$ *if and only if* $e^{\pi \gamma} Y_+^\ell + Y_-^\ell < 0$;

(b.4) *when $T_C < 0$, and $\ell < 0$ if and only if $e^{\pi\gamma}Y_+^\ell + Y_-^\ell > 0$.*
Moreover, in all cases that the origin is surrounded by a limit cycle, this is unique, stable if $T_C > 0$ and unstable if $T_C < 0$.

Theorems 2 and 3 improve and extend cases studied in Freire et al. (1998). Here, we give a shorter and clear proof using the techniques developed in Llibre et al. (2013). In Llibre et al. (2013), the authors proved Theorem 2 with the additional hypothesis that systems (1) have only one equilibrium point at the origin and with $T_C > 0$. This is equivalent to state that $r \geq 0$ in the case $T_L = T_C$ and $\ell = 1$ and that $\ell \geq 0$ in the case $T_R = T_C$ and $r = 1$. Thus, we extend here Theorem 2 of Llibre et al. (2013) to cover all cases.

Remark 4 We remark that if we consider, for instance, the case of Theorem 2 (b.1) and we allow T_C to vanish, then we have a center at the origin, its period annulus is bounded by the line $x = v$. If we perturb this degenerate situation by taking $T_C > 0$ and small, then the necessary and sufficient condition of statement (b.1) is automatically fulfilled, so that we conclude that such a perturbation implies a bifurcation of a limit cycle from the period annulus. This bifurcation and the analogous for the other statements of Theorems 2 and 3 are quantitatively studied in a more general setting in Ponce et al. (2013).

In the case of three linearity zones, our main result is the following.

Theorem 5 *Consider the differential systems* (1) *and assume $T_C > 0$. Then, the following statements hold.*

(a) *A necessary condition for the existence of periodic orbits is that the traces T_L, T_C, T_R have not the same sign.*

(b) *If $T_R, T_L < 0$, and $r, \ell \geq 0$, then the origin is surrounded by a limit cycle, which is unique and stable.*

(c) *If $T_R, T_L \leq 0$, then the origin is surrounded by at most one limit cycle, that if it exists is stable.*

(d) *If $T_L \leq 0$, $T_R > 0$, and $T_R - T_C r \leq 0$, then the origin is surrounded by at most one limit cycle, that if it exists is stable.*

(e) *If $T_R \leq 0$, $T_L > 0$, and $T_L - T_C \ell \leq 0$, then the origin is surrounded by at most one limit cycle, that if it exists is stable.*

Theorem 5 is proved in Sect. 6. Its immediate dual result, which is given without proof, is the following.

Theorem 6 *Consider the differential systems* (1) *and assume $T_C < 0$. Then, the following statements hold.*

(a) *A necessary condition for the existence of periodic orbits is that the traces T_L, T_C, T_R have not the same sign.*

(b) *If $T_R, T_L > 0$, and $r, \ell \geq 0$, then the origin is surrounded by a limit cycle, which is unique and unstable.*

(c) *If T_R, $T_L \geq 0$, then the origin is surrounded by at most one limit cycle, that if it exists is unstable.*

(d) *If $T_L \geq 0$, $T_R < 0$, and $T_R - T_C r \geq 0$, then the origin is surrounded by at most one limit cycle, that if it exists is unstable.*

(e) *If $T_R \geq 0$, $T_L < 0$, and $T_L - T_C \ell \geq 0$, then the origin is surrounded by at most one limit cycle, that if it exists is unstable.*

Statement (b) of Theorem 5 is Theorem 1 of Llibre et al. (2013), where the case of equal signs of T_R and T_L and different from the sign of T_C, with the origin as the unique equilibrium point of the system, i.e., $r \geq 0$ and $\ell \geq 0$, was previously studied. The remaining statements are new and represent a relevant contribution in the study of limit cycles of piecewise linear systems.

In Sect. 2, we show the usefulness of this kind of results in a relevant problem of nonlinear electronics.

Finally, for planar systems with three linearity zones, we give for the first time rigorous results assuring the existence of two limit cycles surrounding the same equilibrium point under particular assumptions. The characterization of all possible cases with two limit cycles is far from being completely solved, and it is left for future work. Here, we exploit the fact that there are situations where, by moving only the parameter T_C, it is possible to pass from a system with two zones to a system with three zones. Using this simple idea and combining adequately the previous results, we are able to prove the existence of at least two limit cycles surrounding the equilibrium at the origin. This is the aim of our last results.

Theorem 7 *Consider the differential systems* (1) *and assume* $0 < v < u$, $\ell > 0$, $r = 1$, $0 < T_R < 2$ *and* $T_L < 0$ *and satisfying* $T_R + T_L/\sqrt{\ell} < 0$. *Then, the following statements hold.*

(a) *If $0 < T_C \leq T_R$, then the origin is surrounded by a limit cycle, which is unique and stable.*

(b) *If $T_C = 0$, then the origin is surrounded by a bounded period annulus whose most external periodic orbit, which is tangent to the line $x = v$, is unstable. There exists also a stable limit cycle surrounding such period annulus.*

(c) *There exist $\varepsilon > 0$ such that if $-\varepsilon < T_C < 0$, then the origin is surrounded by at least two limit cycles, the smaller being unstable and the bigger stable.*

Particular phase portraits illustrating the three statements of Theorem 7 are drawn in Fig. 1 for $1 = v < u = 1.5$, $T_L = 1.75$, $T_R = 0.5$ and $\ell = r = 1$. Regarding the figure, in the left panel, we have $T_C = 0.5$, so that the system has only two linear zones and, according to Theorem 2, one stable limit cycle. For $T_C = 0$, we draw in the central panel the new stable limit cycle and the bounded center configuration within the old, dashed limit cycle. Finally, in the right panel we magnify the region bounded by the previous stable limit cycle, showing the two limit cycles that exist for $T_C = -0.05$. In any case, we also draw the piecewise linear graph of $y = F(x)$.

For a proof of Theorem 7, see Sect. 7. After changes of variables, and also reversing the time in some cases, we can write the following dual results for Theorem 7, whose proofs are direct consequences of the one of Theorem 7 and they will not be provided.
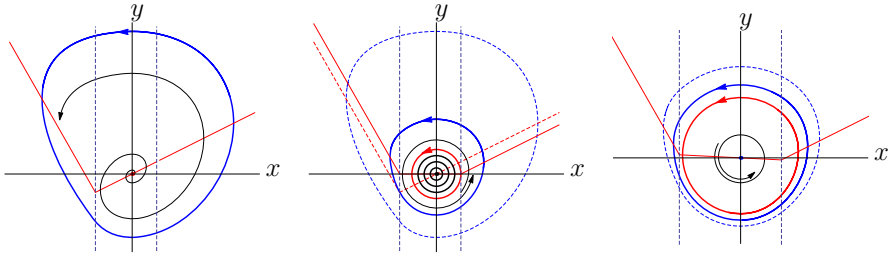
Fig. 1 Phase portraits corresponding to the three statements of Theorem 7 for $1 = v < u = 1.5$, $T_L = 1.75$, $T_R = 0.5$ and $\ell = r = 1$

Theorem 8 *Consider the differential systems* (1) *and assume* $0 < u < v$, $r > 0$, $\ell = 1$, $0 < T_L < 2$ *and* $T_R < 0$ *and satisfying* $T_L + T_R/\sqrt{r} < 0$. *Then, the following statements hold*

(a) *If* $0 < T_C \leq T_L$, *then the origin is surrounded by a limit cycle, which is unique and stable.*

(b) *If* $T_C = 0$, *then the origin is surrounded by a bounded period annulus whose most external periodic orbit, which is tangent to the line* $x = u$, *is unstable. There exists also a stable limit cycle surrounding such period annulus.*

(c) *There exist* $\varepsilon > 0$ *such that if* $-\varepsilon < T_C < 0$, *then the origin is surrounded by at least two limit cycles, the smaller being unstable and the bigger stable.*

Theorem 9 *Consider the differential systems* (1) *and assume* $0 < v < u$, $\ell > 0$, $r = 1$, $-2 < T_R < 0$ *and* $T_L > 0$ *and satisfying* $T_R + T_L/\sqrt{\ell} > 0$. *Then, the following statements hold.*

(a) *If* $T_R \leq T_C < 0$, *then the origin is surrounded by a limit cycle, which is unique and unstable.*

(b) *If* $T_C = 0$, *then the origin is surrounded by a bounded period annulus whose most external periodic orbit, which is tangent to the line* $x = v$, *is stable. There exists also an unstable limit cycle surrounding such period annulus.*

(c) *There exist* $\varepsilon > 0$ *such that if* $0 < T_C < \varepsilon$, *then the origin is surrounded by at least two limit cycles, the smaller being stable and the bigger unstable.*

Theorem 10 *Consider the differential systems* (1) *and assume* $0 < u < v$, $r > 0$, $\ell = 1$, $-2 < T_L < 0$ *and* $T_R > 0$ *and satisfying* $T_L + T_R/\sqrt{r} > 0$. *Then, the following statements hold.*

(a) *If* $T_L \leq T_C < 0$, *then the origin is surrounded by a limit cycle, which is unique and unstable.*

(b) *If* $T_C = 0$, *then the origin is surrounded by a bounded period annulus whose most external periodic orbit, which is tangent to the line* $x = u$, *is stable. There exists also an unstable limit cycle surrounding such period annulus.*

(c) *There exist* $\varepsilon > 0$ *such that if* $0 < T_C < \varepsilon$, *then the origin is surrounded by at least two limit cycles, the smaller being stable and the bigger unstable.*

A different approach leading also to the existence of two nested limit cycles, using boundary equilibrium bifurcations, is given in Ponce et al. (2015), where the only equilibrium point surrounded by the limit cycles is no longer in the central zone.

2 Application to the Study of a Simple Oscillator with One Memristor

Following Corinto et al. (2011) memristors are two-terminal electronic passive devices for which a nonlinear relationship links charge and flux Chua (1971). They seem to be at the basis of future generation oscillatory associative and dynamic memories; as another important feature, nanoscale memristors have potential to reproduce the behavior of biological synapses. Here, we apply our previous results to the analysis of an elementary oscillator endowed with one flux-controlled memristor, see Fig. 2 and Itoh and Chua (2008). For the importance of memristors, see, for instance, Strukov et al. (2008).

In the shown circuit, the values of L and C for the impedance and capacitance are positive constants, while the resistor R has a negative value. This negative resistor is typically realized by an auxiliary active device, responsible for the energy supplied to the circuit, see Corinto et al. (2011). From Kirchoff's laws, we see that

$$i_R(\tau) - i_L(\tau) = 0,$$
$$i_L(\tau) - i_C(\tau) - i_M(\tau) = 0,$$
$$v_R(\tau) + v_L(\tau) + v_C(\tau) = 0,$$
$$v_C(\tau) - v_M(\tau) = 0,$$

where v, i stand for the voltage and current, respectively, across the corresponding element of the circuit.

Integrating with respect to time the above equations, and assuming as in Corinto et al. (2011) that all the initial conditions are zero, we get

$$q_R(\tau) - q_L(\tau) = 0, \tag{5}$$
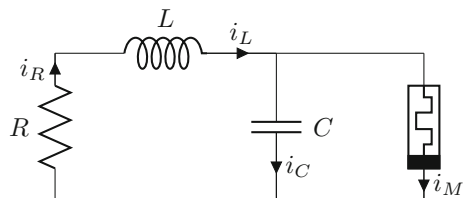$$q_L(\tau) - q_C(\tau) - q_M(\tau) = 0, \tag{6}$$
$$\varphi_R(\tau) + \varphi_L(\tau) + \varphi_C(\tau) = 0, \tag{7}$$
$$\varphi_C(\tau) - \varphi_M(\tau) = 0, \tag{8}$$

where q and φ stand, respectively, for the charge and flux associated with each element.

Fig. 2 The simple oscillator with one memristor analyzed in this section. Note that the negative value considered for R makes it the only active element in the circuit

We denote by f_M the flux–charge characteristic of the memristor and, after recalling the constitutive equations of the involved bipoles, namely

$$\varphi_R(\tau) = Rq_R(\tau),$$
$$\varphi_L(\tau) = L\frac{d}{d\tau}q_L(\tau),$$
$$q_C(\tau) = C\frac{d}{d\tau}\varphi_C(\tau),$$
$$q_M(\tau) = f_M(\varphi_M(\tau)),$$

we arrive at the equations

$$\frac{d}{d\tau}\varphi_C(\tau) = \frac{1}{C}q_C(\tau) = \frac{1}{C}\left[q_L(\tau) - q_M(\tau)\right],$$
$$\frac{d}{d\tau}q_L(\tau) = \frac{1}{L}\varphi_L(\tau) = \frac{1}{L}\left[-\varphi_C(\tau) - \varphi_R(\tau)\right].$$

We denote the state variables by $x_1 = \varphi_C(\tau)$ and $x_2 = q_L(\tau)$, and using (5) and (8), to write $\varphi_R(\tau) = Rq_R(\tau) = Rq_L(\tau)$ and $q_M(\tau) = f_M(\varphi_M(\tau)) = f_M(\varphi_C(\tau))$, we have the differential system

$$\frac{dx_1}{d\tau} = \frac{1}{C}\left[-f_M(x_1) + x_2\right], \tag{9}$$
$$\frac{dx_2}{d\tau} = \frac{1}{L}\left[-x_1 - Rx_2\right]. \tag{10}$$

Instead of the symmetric piecewise linear function considered in Itoh and Chua (2008) and Corinto et al. (2011), we adopt here a more general model for the nonlinear flux–charge characteristic of the memristor, namely

$$f_M(x) = \begin{cases} b_L(x + u) - au & \text{for } x \leq -u, \\ ax & \text{for } -u \leq x \leq v, \\ b_R(x - v) + av & \text{for } x \geq v. \end{cases}$$

So we consider here both symmetric and non-symmetric oscillators. A rescaling of the time by putting $\tau = Cs$ and the use of the positive parameters

$$G = -\frac{1}{R} > 0, \quad v = -\frac{RC}{L} = \frac{C}{GL} > 0.$$

brings the system to

$$\frac{dx_1}{ds} = -f_M(x_1) + x_2,$$
$$\frac{dx_2}{ds} = -v(Gx_1 - x_2),$$

and by doing the change of variables $x = x_1$, $y = vx_1 - x_2$, we get

$$\frac{dx}{ds} = vx - f_M(x) - y,$$
$$\frac{dy}{ds} = v[Gx - f_M(x)].$$

A natural assumption is to take that the determinant in the central region is to be positive, which requires $a < G$. We assume such condition in the sequel, and after introducing for convenience a positive constant ω, we have

$$D = v(G - a) = \omega^2 > 0.$$

Therefore, under the above assumption, using the change $(x, y, s) \mapsto (X, Y, \tau)$ given by

$$X = x, \quad Y = \frac{y}{\sqrt{D}} = \frac{y}{\omega} \quad \text{and} \quad \tau = \omega s,$$

we get that system (9)–(10) can be written as (1), satisfying (2)–(3) with

$$T_L = \frac{v - b_L}{\omega}, \quad T_C = \frac{v - a}{\omega}, \quad T_R = \frac{v - b_R}{\omega},$$

and

$$\ell = \frac{Gv - b_L v}{Gv - av} = \frac{G - b_L}{G - a}, \quad r = \frac{Gv - b_R v}{Gv - av} = \frac{G - b_R}{G - a}.$$

We are now able to apply in a convenient way our previous results; we do not try to illustrate all the statements but only the most significant cases. We consider first the case of non-symmetric memristors with three linear zones in the situation corresponding to $T_C > 0$, T_L, $T_R < 0$ with $\ell, r \geq 0$, a case not considered in Corinto et al. (2011). Such a situation leads, under our previous assumption $a < G$, to the inequalities $a < v < b_L \leq G$ and $a < v < b_R \leq G$, what implies in particular the necessity of the inequality $v < G$, that is the design condition $R^2C < L$ (we recall that $v = -RC/L$ and $G = -1/R > 0$). A direct application of Theorem 5(b) gives the following result.

Proposition 11 *Consider the memristor-based oscillator under study with the assumption of vanishing flux and charge initial conditions, as modeled by* (9)–(10). *Under the design condition $R^2C < L$, if the additional hypotheses $a < v$, $v < b_L \leq G$ and $v < b_R \leq G$ are fulfilled, then the circuit exhibits robust oscillations, corresponding to a stable limit cycle in the phase plane state variables.*

Let us now consider rather non-symmetric memristors, namely those with a characteristic with only two zones, in the situation, for instance, where $0 < T_L = T_C < 2$, $T_R < 0$, $\ell = 1$ and $r > 0$. From Theorem 2(b.1), we can deduce the following

result. Note that this case is also relevant when the system has indeed three zones, but the lack of symmetry allows to discard one zone whenever we can reasonably assume that the periodic orbit will only use two zones. Thus, we take $b_L = a$ so that $T_L = T_C$ and $\ell = 1$, and we require that $0 < T_C < 2$ (unstable focus at the origin) and $T_R < 0$, so to fulfill the necessary conditions for oscillation. The condition $T_C < 2$ translates after some elementary algebra to $(v + a)^2 < 4vG$, and the necessary and sufficient condition $T_C + T_R/\sqrt{r} < 0$, written in terms of the parameters of the circuit, completes the statement.

Proposition 12 *Consider the memristor-based oscillator under study with the assumption of vanishing flux and charge initial conditions, as modeled by* (9)-(10). *Under the design condition $R^2C < L$, if the additional hypotheses $b_L = a < v$, $(v + a)^2 < 4vG$ and $v < b_R < G$ are fulfilled, then the circuit exhibits stable oscillations if and only if $(2v - a - b_R)G < v^2 - ab_R$.*

These results complement the analysis done in Corinto et al. (2011), where the symmetric situation $b_L = b_R$ with $u = v = 1$ was considered, and only the case $\ell, r < 0$ was analyzed.

3 Preliminary Results

Considering the more general case of three zones, which includes the particular case of two zones, the piecewise linear functions $F(x)$ and $g(x)$ induce a partition of $\mathbb{R}^2$ into three open strips separated by two straight lines, as follows

$$\mathcal{S}_L = \{(x, y) : x < -u\}, \quad \mathcal{S}_C = \{(x, y) : -u < x < v\}, \quad \mathcal{S}_R = \{(x, y) : x > v\},$$

and the straight lines are $\Gamma_L = \{(x, y) : x = -u\}$, and $\Gamma_R = \{(x, y) : x = v\}$. So system (1) is a piecewise linear differential system with three different linearity regions separated by two parallel straight lines.

We note that systems (1) are analytic in $\mathbb{R}^2 \setminus (\Gamma_L \cup \Gamma_R)$ but only of class $\mathcal{C}^0$ on $\mathbb{R}^2$. Since they satisfy a Lipschitz condition in the whole $\mathbb{R}^2$, we can apply to systems (4) the classical theorems on existence, uniqueness and continuity on initial conditions and parameters. Note that their solutions are $\mathcal{C}^1$, but not $\mathcal{C}^2$.

Now, we classify the equilibria of system (1).

Proposition 13 *The following statements hold for the piecewise linear differential system* (1).

(a) *If $\ell \geq 0$ and $r \geq 0$, then the origin is the unique equilibrium, and it is hyperbolic.*
(b) *If $\ell \geq 0$ and $r < 0$, then there are two equilibria, the origin and $e_R = (x_R, y_R) = ((r - 1)v/r, (T_Cr - T_R)v/r)$ in $\mathcal{S}_R$, the origin is hyperbolic and e_R is a saddle.*
(c) *If $\ell < 0$ and $r \geq 0$, then there are two equilibria: the origin and $e_L = (x_L, y_L) = ((1 - \ell)u/\ell, (T_L - T_C\ell)u/\ell)$ in $\mathcal{S}_L$, the origin is hyperbolic and e_L is a saddle.*
(d) *If $\ell < 0$ and $r < 0$, then there are three equilibria: the origin, e_L in $\mathcal{S}_L$ and e_R in $\mathcal{S}_R$, the origin is hyperbolic, and e_L and e_R are saddles.*

Proof The proof is straightforward, see, for instance, Theorem 2.15, or the hyperbolic singular point theorem of Dumortier et al. (2006). □

The next result will be useful when system (1) has more than one equilibrium point.

Lemma 14 *Consider the Liénard differential system* (1) *with* $r < 0$ *or* $\ell < 0$. *If it has a periodic orbit, then it is contained in the strip* $x_2 < x < x_1$ *where*

$$x_1 = \begin{cases} x_R & if\ r < 0, \\ +\infty & if\ r \geq 0 \end{cases} \quad and \quad x_2 = \begin{cases} x_L & if\ \ell < 0, \\ -\infty & if\ \ell \geq 0. \end{cases}$$

Proof We first prove the case $r < 0$. Taking into account the behavior of the vector field on the line $x = x_R$, it follows that the periodic orbit is contained in the region $x < x_R$. Otherwise, since on the straight line $x = x_R$, we have $\dot{x} > 0$ if $y > y_R$ and $\dot{x} < 0$ if $y < y_R$, and the point e_R of Proposition 13 is in the interior of the bounded region V limited by the periodic orbit, we have a contradiction because the sum of the indices of the equilibrium points contained in V is not equal to 1, see, for instance, page 148 of Zhang Zhifen et al. (1992).

The proof for the case $\ell < 0$ follows in an analogous way. □

We note that system (1) is invariant under the following symmetry:

$$(x, y, t, T_L, T_R, \ell, r, u, v) \mapsto (-x, -y, t, T_R, T_L, r, \ell, v, u). \tag{11}$$

This symmetry will be useful to split the analysis of the system with three zones into different subcases with only two zones.

It is easy to check that the traces in the regions $\mathcal{S}_L$, $\mathcal{S}_C$ and $\mathcal{S}_R$ are given by T_L, T_C and T_R, respectively. By the Bendixson theorem, see, for instance, Theorem 7.10 in Dumortier et al. (2006), these traces cannot have the same sign when limit cycles exist.

Without loss of generality, we can also assume that $T_C > 0$. Clearly we can change the sign of T_C doing the change of variables $(x, y, \tau) \mapsto (x, -y, -\tau)$. *From now on in the rest of the paper, we will assume that* $T_C > 0$. We remark that following the proofs it becomes clear that when $T_C < 0$ the stability of the limit cycle (if exists) is unstable, because when $T_C > 0$ it is stable.

We say that a vector field has the *nonnegative rotation property* whenever along any half-ray starting from the origin the angle of the vector field measured with respect to the positive direction of the x-axis does not decrease as long as one moves far from the origin.

We will use the Massera's method for uniqueness of limit cycles. To this end, we recall that a *period annulus* is a region in the plane completely filled by non-isolated periodic orbits. For a periodic orbit surrounding the origin, we say that it is *star-like with respect to the origin* when any segment joining the origin and a point of the periodic orbit has no other points in common with the periodic orbit, and consequently such segments are in the interior of the periodic orbit.

Theorem 15 (Massera's method). *Consider a Liénard differential system $x' = F(x) - y$ and $y' = g(x)$ defined in the open strip*

$$S = \left\{(x, y) \in \mathbb{R}^2 : x_2 < x < x_1\right\}$$

for some $x_2 < 0 < x_1$. Assume that $xg(x) > 0$ for $x \in S$, and that $F(0) = 0$, so that the only equilibrium point in S is the origin. Assume also that the system in S has the nonnegative rotation property and has no period annulus surrounding the origin. If the system has a periodic orbit, then it is star-like with respect to the origin and it is a limit cycle, which is unique and stable.

Proof Assume first that $x_1 = \infty$, $x_2 = -\infty$, i.e., system $x' = F(x) - y$ and $y' = g(x)$ has the origin as the unique equilibrium point. Then, Theorem 15 is just Lemma 1 of Llibre et al. (2013).

Now assume either $x_1 = x_R$ or $x_2 = x_L$. It follows from Lemma 14 that any periodic orbit must be contained in the strip $x_2 < x < x_1$. Now, we can apply the arguments of the proof of Lemma 1 of Llibre et al. (2013) to the strip S and the theorem follows.   □

4 Computation of the Points $Y_\pm^r$ and $Y_\pm^\ell$

In this section, we consider the system with two linearity zones, i.e., with either $T_L = T_C$ and $\ell = 1$ (obtained by suppressing the left zone and extending the central zone to the left), or $T_R = T_C$ and $r = 1$ (obtained by suppressing the right zone and extending the central zone to the right). Without loss of generality, we can consider the case $T_L = T_C$ and $\ell = 1$, because the other case can be studied in a similar way. Hence, in this section, we will work with the system

$$x' = F(x) - y, \quad \text{and} \quad y' = g(x), \tag{12}$$

where

$$F(x) = \begin{cases} T_C x & \text{if } x < v, \\ T_R(x - v) + T_C v & \text{if } x \geq v, \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if } x < v, \\ r(x - v) + v & \text{if } x \geq v. \end{cases}$$

Moreover, we will consider the focus-saddle case, i.e., the case in which $T_C \in (-2, 2)$ and $r < 0$. To alleviate expressions, we can do a homogeneous rescaling by a factor of v, which is equivalent to assume $v = 1$. At the end, it will suffice to multiply both coordinates by v to undo the rescaling.

We suppose for the focus at the origin the eigenvalues $\sigma \pm i\omega$, where $\sigma^2 + \omega^2 = 1$, and define the ratio γ between the real and imaginary parts of such eigenvalues, so that

$$2\sigma = T_C, \quad \omega = \sqrt{1 - \sigma^2}, \quad \gamma = \frac{\sigma}{\omega} = \frac{T_C}{\sqrt{4 - T_C^2}}.$$

With this notation, if (x_i, y_i) is the starting point for an orbit using the focus dynamics and we want to know the final point (x_f, y_f) along the orbit after a time t, then we can write the computations in terms of the phase angle $\theta = \omega t$ as follows,

$$\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \exp(\gamma\theta) \begin{pmatrix} \cos\theta + \gamma\sin\theta & -\dfrac{\sin\theta}{\omega} \\ \dfrac{\sin\theta}{\omega} & \cos\theta - \gamma\sin\theta \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}. \tag{13}$$

We remark that the phase angle θ does not coincide in general with the geometrical angle of the orbit as seen from the focus at the origin, with the exception of the cases when $\theta = n\pi$ with $n \in \mathbb{Z}$.

Regarding the saddle point, their eigenvalues are denoted by $\lambda_- < 0 < \lambda_+$ where

$$\lambda_\pm = \frac{T_R \pm \sqrt{T_R^2 - 4r}}{2},$$

so that

$$T_R = \lambda_+ + \lambda_-, \quad r = \lambda_+ \cdot \lambda_-,$$

and we also introduce for convenience the notation

$$\mu_\pm = \frac{\lambda_\pm}{r} = \frac{1}{\lambda_\mp} = \frac{T_R \pm \sqrt{T_R^2 - 4r}}{2r}.$$

Note that the saddle point, originally at $(x_R, y_R) = (v - v/r, T_C v - T_R v/r)$, after the introduced rescaling, becomes the point $(1 - 1/r, T_C - T_R/r) = (1 - \mu_+\mu_-, 2\sigma - \mu_+ - \mu_-)$. The $\lambda_\pm$-eigenvectors of the matrix

$$\begin{pmatrix} T_R & -1 \\ r & 0 \end{pmatrix} = \begin{pmatrix} \lambda_+ + \lambda_- & -1 \\ \lambda_+\lambda_- & 0 \end{pmatrix}$$

are $(1, \lambda_\mp)^T$. The linear $\lambda_\pm$-invariant manifolds that emanate from the saddle following the eigenvectors $(1, \lambda_\mp)^T$ intersect the line $x = 1$ at the points $(1, y_\pm)$ when

$$\begin{pmatrix} 1 - \mu_+\mu_- \\ 2\sigma - \mu_+ - \mu_- \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ \lambda_\mp \end{pmatrix} = \begin{pmatrix} 1 \\ y_\pm \end{pmatrix},$$

Fig. 3 Remarkable points for $r < 0$, $0 < T_C < 2$ and $T_R < 0$. The *graphs* of functions $F(x)$ and $g(x)$ are drawn in *solid* and *dashed*, respectively. The *vertical lines* $x = v$ and $x = x_R$ are also drawn
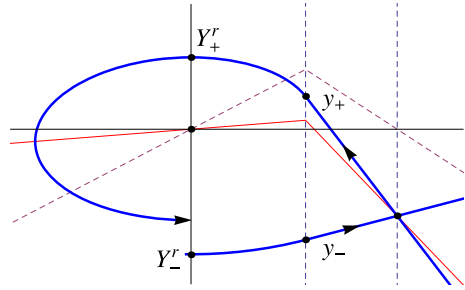
that is, for $\alpha = \mu_+\mu_- = 1/r$. Thus

$$y_\pm = 2\sigma - \mu_+ - \mu_- + \mu_+\mu_-\lambda_\mp = 2\sigma - \mu_+ - \mu_- + \mu_\mp = 2\sigma - \mu_\pm.$$

To compute now the values $Y_\pm^r$, it suffices to solve the equation

$$\begin{pmatrix} 0 \\ Y_\pm^r \end{pmatrix} = v \exp(\gamma\theta) \begin{pmatrix} \cos\theta + \gamma\sin\theta & -\dfrac{\sin\theta}{\omega} \\ \dfrac{\sin\theta}{\omega} & \cos\theta - \gamma\sin\theta \end{pmatrix} \begin{pmatrix} 1 \\ 2\sigma - \mu_\pm \end{pmatrix}, \quad (14)$$

where we have added the factor v to undo the rescaling, and the phase angle must satisfy $0 < \pm\theta < \pi$, since we must integrate forward (backward) in time to get Y_+^r (Y_-^r), see Fig. 3.

Assume first we want to compute Y_+^r. From the first coordinate, we get

$$\cos\theta + \gamma\sin\theta - (2\sigma - \mu_+)\frac{\sin\theta}{\omega} = \cos\theta - \gamma\sin\theta + \mu_+\frac{\sin\theta}{\omega} = 0. \quad (15)$$

Equivalently, we write

$$\omega\cos\theta = (\sigma - \mu_+)\sin\theta,$$

and so, from

$$\tan\theta = \frac{\omega}{\sigma - \mu_+},$$

we obtain

$$\cos\theta = \text{sign}(\sigma - \mu_+)\frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{\sigma - \mu_+}{\sqrt{(\sigma - \mu_+)^2 + \omega^2}} = \frac{\sigma - \mu_+}{\sqrt{1 - 2\sigma\mu_+ + \mu_+^2}},$$

and then

$$\sin\theta = \frac{\omega}{\sqrt{1 - 2\sigma\mu_+ + \mu_+^2}}.$$

To avoid problems with the determination of the arctan function, since it is not difficult to build examples with $\theta > \pi/2$, we use the trigonometric identity

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta},$$

leading to

$$\theta = 2 \arctan \left(\frac{\omega}{\sqrt{1 - 2\sigma \mu_+ + \mu_+^2} + \sigma - \mu_+} \right). \tag{16}$$

Taking into account that, from (15), we know that

$$\cos \theta - \gamma \sin \theta = -\mu_+ \frac{\sin \theta}{\omega},$$

we finally obtain from the second coordinate of (14) that

$$Y_+^r = v \exp(\gamma \theta) \frac{1 - \mu_+(2\sigma - \mu_+)}{\sqrt{1 - 2\sigma \mu_+ + \mu_+^2}} = v \exp(\gamma \theta) \sqrt{1 - 2\sigma \mu_+ + \mu_+^2},$$

where θ is given in (16).

The computations for Y_-^r are identical if we change θ by $-\theta$ and μ_+ by μ_-. To eliminate any ambiguity, however, and to use also a positive angle θ in $(0, \pi)$, we start from

$$\begin{pmatrix} 0 \\ Y_-^r \end{pmatrix} = v \exp(-\gamma \theta) \begin{pmatrix} \cos \theta - \gamma \sin \theta & \dfrac{\sin \theta}{\omega} \\ -\dfrac{\sin \theta}{\omega} & \cos \theta + \gamma \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 2\sigma - \mu_- \end{pmatrix}, \tag{17}$$

solving for

$$\cos \theta - \gamma \sin \theta + \frac{\sin \theta}{\omega}(2\sigma - \mu_-) = 0,$$

leading now to

$$\omega \cos \theta = -(\sigma - \mu_-) \sin \theta, \quad \tan \theta = -\frac{\omega}{\sigma - \mu_-},$$

so that

$$\cos \theta = -\frac{\sigma - \mu_-}{\sqrt{1 - 2\sigma \mu_- + \mu_-^2}},$$

and

$$\sin \theta = \frac{\omega}{\sqrt{1 - 2\sigma \mu_- + \mu_-^2}}.$$

Using now that

$$\cos \theta + \gamma \sin \theta = \mu_- \frac{\sin \theta}{\omega},$$

we see that

$$Y_-^r = -v \exp(-\gamma \theta) \frac{1 - \mu_-(2\sigma - \mu_-)}{\sqrt{1 - 2\sigma \mu_- + \mu_-^2}} = -v \exp(-\gamma \theta)\sqrt{1 - 2\sigma \mu_- + \mu_-^2},$$

where θ is given by

$$\theta = 2 \arctan \left(\frac{\omega}{\sqrt{1 - 2\sigma \mu_- + \mu_-^2} - (\sigma - \mu_-)} \right). \tag{18}$$

Therefore, the common expression for both ordinates is

$$Y_\pm^r = \pm v \exp(\pm \gamma \theta_\pm)\sqrt{1 - 2\sigma \mu_\pm + \mu_\pm^2},$$

with

$$\theta_\pm = 2 \arctan \left(\frac{\omega}{\sqrt{1 - 2\sigma \mu_\pm + \mu_\pm^2} \pm (\sigma - \mu_\pm)} \right).$$

In a similar way, we obtain $Y_\pm^\ell$.

5 Proof of Theorem 2

In this section, we consider system (12). Then, Theorem 2 can be stated as follows.

Theorem 16 *Consider the differential system* (12) *with* $T_C \neq 0$. *Then, the following statements hold.*

(a) *Two necessary conditions for the existence of periodic orbits are* $|T_C| < 2$ *and* $T_R T_C < 0$.
(b) *If* $|T_C| < 2$ *and* $T_R T_C < 0$, *then the system has periodic orbits*
   (b.1) *when* $T_C > 0$ *and* $r > 0$ *if and only if* $T_C + T_R/\sqrt{r} < 0$;
   (b.2) *when* $T_C < 0$ *and* $r > 0$ *if and only if* $T_C + T_R/\sqrt{r} > 0$;

(b.3) *when $T_C > 0$ and $r < 0$ if and only if $e^{\pi T_C / \sqrt{4-T_C^2}} Y_+^r + Y_-^r < 0$;*

(b.4) *when $T_C < 0$ and $r < 0$ if and only if $e^{\pi T_C / \sqrt{4-T_C^2}} Y_+^r + Y_-^r > 0$.*
In all cases, the origin is surrounded by a limit cycle, which is unique, stable if $T_C > 0$ and unstable if $T_C < 0$.

Now, we shall prove Theorem 16 when $T_C > 0$. The case $T_C < 0$ can be proved in a similar way. Note that the traces in the regions $\mathcal{S}_C, \mathcal{S}_R$ are given by T_C and T_R, respectively. By the Bendixson theorem, see, for instance, Theorem 7.10 in Dumortier et al. (2006), they cannot have the same sign for the existence of limit cycles and thus $T_R T_C < 0$. First we show that system (12) can be transformed in another system with the nonnegative rotation property. This is the statement of the following proposition, where we prove more things that we will use later on.

Proposition 17 *Consider system (12) with $T_C > 0$ for all x when $r \geq 0$, or its restriction to the region $x < x_R$ when $r < 0$. We assume that $T_R \leq T_C$. Then, the system can be transformed in another system having the nonnegative rotation property if $T_R - T_C r \leq 0$, or if $T_R - T_C r > 0$, $r < 0$ and $T_R \leq 0$.*

Proof To show the nonnegative rotation property, we will compute the slope of the vector field along half-rays of the form $y = \lambda x$. Since it will appear $F(x) - \lambda x$ in some denominators, we can disregard the points of vertical slope in which $F(x) = \lambda x$.

Now, we transform the system by introducing a new first variable $z = z(x)$, namely

$$z = \text{sgn}(x)\sqrt{2G(x)} \text{ where } G(x) = \int_0^x g(s)\,ds. \tag{19}$$

Note that $z = x$ for $x \leq v$ and that this change of variable is injective for $x > v$ as long as $g(x) > 0$, that is, for all x when $r \geq 0$, or for $x < x_R$ when $r < 0$. Note that $z^2(x) = 2G(x)$ and thus we have $z(x)z'(x) = g(x)$ for all x, and so

$$\frac{dz}{dx} = \frac{g(x)}{z}. \tag{20}$$

Therefore,

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = \frac{g(x)}{z}(F(x) - y),$$

and in the new variables, the system is

$$\frac{dy}{dz} = \frac{dy/dt}{dz/dt} = \frac{z}{F(x(z)) - y},$$

which is equivalent to the system $\dot{z} = F(x(z)) - y$, $\dot{y} = z$, where the dot now indicates the derivative with respect to an implicitly defined, different time parameterization of the orbits. Now, we study the slope of the vector field along the half-rays of the form $y = \lambda z$.

Then, we can write

$$m_\lambda(z) = \left.\frac{dy}{dz}\right|_{y=\lambda z} = \frac{z}{F(x(z)) - \lambda z},$$

and to analyze the monotone character of this slope along the half-rays, we compute its derivative with respect to z, namely

$$\frac{dm_\lambda(z)}{dz} = \frac{F(x(z)) - \lambda z - z\left[F'(x(z))x'(z) - \lambda\right]}{[F(x(z)) - \lambda z]^2},$$

not to be considered at $z = v$, where this derivative has a jump discontinuity.

In the region $x < v$, since $z = x$, we have

$$\frac{dm_\lambda(z)}{dz} = \frac{F(x(z)) - zF'(x(z))x'(z)}{[F(x(z)) - \lambda z]^2} = \frac{T_C z - T_C z}{[F(x(z)) - \lambda z]^2} = 0.$$

So the slope of the vector field is constant along the half-rays.

For the region $x > v$, we get

$$\frac{dm_\lambda(z)}{dz} = \frac{(T_C - T_R)v - T_R\left[zx'(z) - x(z)\right]}{[F(x(z)) - \lambda z]^2}. \tag{21}$$

Now, we study the value of $zx'(z) - x(z)$ for $x > v$. From (20) and the equality $z^2 = 2G(x) = r(x - v)^2 + 2vx - v^2$ for $x > v$, we have

$$\begin{aligned}
zx'(z) - x(z) &= \frac{z^2 - xg(x)}{g(x)} = \frac{r(x-v)^2 + 2vx - v^2 - rx(x-v) - vx}{g(x)} \\
&= \frac{(x-v)(rx - rv + v - rx)}{g(x)} = \frac{v(1-r)(x-v)}{g(x)}.
\end{aligned}$$

Then, (21) can be rewritten as

$$\frac{dm_\lambda(z)}{dz} = v\frac{(T_C - T_R)g(x) - T_R(1-r)(x-v)}{g(x)\left[F(x(z)) - \lambda z\right]^2}, \tag{22}$$

and the sign of the above expression, as long as $g(x) > 0$, is controlled by the sign of the numerator, namely

$$\begin{aligned}
(T_C - T_R)v &+ \left[(T_C - T_R)r - T_R(1-r)\right](x-v) \\
&= (T_C - T_R)v - (T_R - T_C r)(x-v),
\end{aligned}$$

an expression which is affine in $x - v$.

Since our hypothesis implies $T_C - T_R \geq 0$, we see that the first constant term is always nonnegative. If $T_R - T_C r \leq 0$, then the second term is also nonnegative for

$x > v$ and we are done. The case $T_R - T_C r > 0$ leads always to a change in the sign of the expression for z big enough, so that we cannot guarantee the monotone increasing character of the slope along half-rays in the whole plane. When $r < 0$, however, we only need to study what happens for $x < x_R$, that is, for $x - v < -v/r$. Substituting now $x = x_R$ in the above expression, we get

$$(T_C - T_R)v + (T_R - T_C r)\frac{v}{r} = -T_R v + T_R \frac{v}{r} = -T_R v \left(1 - \frac{1}{r}\right),$$

which for $T_R \leq 0$ is nonnegative. Thus, we can also assure in such a case the required monotonicity for $x < x_R$, and the proposition follows.									□

Noticing that for $T_R \leq 0$ the hypothesis $T_C - T_R \geq 0$ is strictly satisfied, we see immediately that we can have $T_R - T_C r \leq 0$ (and then Proposition 17 applies) or $T_R - T_C r > 0$, that is, $r < T_R/T_C \leq 0$, but then since $T_R \leq 0$ Proposition 17 also applies. In short, we get the following result.

Corollary 18 *Consider system* (12) *with $T_C > 0$ for all x when $r \geq 0$, or its restriction to the region $x < x_R$ when $r < 0$. If $T_R \leq 0$, then the system can be transformed in another system having the nonnegative rotation property independently on the value of r.*

Now, we show that if system (12) has a periodic orbit, then it is unique.

Proposition 19 *Assume that system* (12) *with $T_C > 0$ has a periodic orbit. Then, the periodic orbit surrounds the origin and it is a limit cycle, which is unique and stable.*

Proof By Proposition 17, the system can be transformed into one, which has the nonnegative rotation property. Therefore, by Theorem 15 such a periodic orbit is a limit cycle which is unique. Applying Theorem 15, we get that the limit cycle of system (12) is stable because we are assuming that $T_C > 0$.									□

To prove statement (b), we need to study the existence of periodic orbits surrounding the origin. To do this, we will use the positive and negative parts of the y-axis as domain and range for defining two different half-return maps, namely a right half-return map P_R and a left half-return map P_L.

We start by studying the left half-return map P_L defined in the positive y-axis, by taking the orbit starting at the point $(0, y)$ with $y > 0$, and coming back to the negative y-axis at the point $(0, -P_L(y))$. Since the system becomes purely linear, it is easy to see, see, for instance, Freire et al. (2012) that P_L is a linear function given by $P_L(y) = e^{\pi T_C/\sqrt{4-T_C^2}} y$.

Now, we study the qualitative properties of the right half-return map P_R defined in the negative y-axis, by taking the orbit starting at the point $(0, -y)$ with $y > 0$, and coming back to the positive y-axis at the point $(0, P_R(y))$. The following lemma is Lemma 3 of Llibre et al. (2013).

Lemma 20 *Consider a Liénard differential system with a continuous vector field given by $\dot{x} = F(x) - y$ and $\dot{y} = g(x)$. Assume that $F(x)$ is positive and increasing*

for small positive values of x, it has a positive zero only at $x = \bar{x} > 0$, and it is decreasing to $-\infty$ as $x \to \infty$ monotonically for $x > \bar{x}$. Assuming also that $g(0) = 0$ and $g(x) > 0$ for all $x > 0$, the following statements hold. The orbits starting at the point $(0, -y)$ with $y > 0$, enter the half-plane $x > 0$ and go around the origin in a counterclockwise path, coming back to the y-axis at the point $(0, P_R(y))$ with $P_R(y)) > 0$. The difference $P_R(y) - y$ is positive for small values of y, but eventually becomes negative and tends to $-\infty$ when $y \to \infty$.

The map P_R when $r < 0$ is not defined for all positive values of y. We shall compute now its domain of definition. Two separatrices of the saddle (x_R, y_R) intersect the line $x = v$, the stable at the point (v, y_-) and the unstable at the point (v, y_+), where

$$y_{\pm} = \frac{\left(2T_C r - T_R \mp \sqrt{T_R^2 - 4r}\right) v}{2r},$$

see Sect. 4 and Fig. 3. The flow of system (12) in $x < v$ starting at the point (v, y_-) in backward time intersects the negative y-axis at the point $(0, Y_-^r)$.

The flow of system (12) in $x < v$ starting at the point (v, y_+) in forward time intersects the positive y-axis at the point $(0, P_R(-Y_-^r))$, where it is assumed

$$P_R(-Y_-^r) = \lim_{y \nearrow (-Y_-^r)} P_R(y) = Y_+^r.$$

Hence, the map $P_R : (0, -Y_-^r) \to (0, Y_+^r)$.

Corollary 21 *Assume that system (12) has $T_C > 0$. Then, the orbits starting at the point $(0, -y)$ with $y > 0$ enter the half-plane $x > 0$ and go around the origin in a counterclockwise path, coming back to the y-axis at the point $(0, P_R(y))$ with $P_R(y)) > 0$. The difference $P_R(y) - y$ is positive for small values of y, and*

(a) *it eventually becomes negative and tends to $-\infty$ when $y \to \infty$, if $r > 0$;*
(b) *tends to $Y_+^r + Y_-^r$ when $y \to -Y_-^r$, if $r < 0$.*

Proof Since we are assuming that $T_C > 0$ and $T_R < 0$, we get that $F(x)$ is positive and increasing for small positive values of x, and it has a positive zero only at $x = v(1 - T_C/T_R) > 0$. It is decreasing to $-\infty$ as $x \to \infty$ monotonically for $x > v(1 - T_C/T_R)$. Note that $g(0) = 0$ and, when $r > 0$, $g(x) > 0$ for all $x > 0$. Now we are under the assumptions of Lemma 20 and the statement (a) of the corollary follows from it.

The proof of statement (b) follows directly from the domain of definition of the map P_R and its image.   □

Now, we continue with the proof of Theorem 16 and we need to show that in fact, we have a periodic orbit. Note that for system (12) the origin is a node when $|T_C| \geq 2$, and a focus when $|T_C| < 2$. Moreover, it cannot be a node because some of its invariant manifolds are straight lines that should extend to the infinity, precluding the existence of periodic orbits. Then, it must be a focus, and thus, $|T_C| < 2$. This concludes the proof of statement (a).

Now, we prove statements (b.1) and (b.3). Statement (b.1) can be proved with exactly the same proof as in the proof of Theorem 2 in Llibre et al. (2013). In this case, i.e., when $r > 0$, we can show that a necessary and sufficient condition for the existence of periodic orbits in this case is that

$$T_C + \frac{T_R}{\sqrt{r}} < 0.$$

Now, we prove statement (b.3), i.e., we need to study the existence of periodic orbits when $r < 0$. Note that as explained above, we have that

$$P_L(y) = e^{\pi T_C/\sqrt{4-T_C^2}} y, \tag{23}$$

and in particular

$$P_L(Y_+^r) = e^{\pi T_C/\sqrt{4-T_C^2}} Y_+^r. \tag{24}$$

If $P_L(Y_+^r) < -Y_-^r$, as shown in Fig. 3, then it is easy to conclude the existence of a trapping region that, since the focus at the origin is unstable, assures the existence of a periodic orbit. Its uniqueness and stability come from Proposition 19.

Suppose that $P_L(Y_+^r) = -Y_-^r$, then we have a homoclinic connection. We claim that inside the region limited by the homoclinic loop there are no periodic solutions. For proving the claim, we shall use the next result, see Theorem 1 in page 364 of Perko (1991).

Theorem 22 *Let p be a topological saddle of an analytic differential system in the plane and let γ be a homoclinic loop at p. Then, the orbits near γ contained in the region limited by γ tend to γ in forward (respectively backward) time if and only the trace of the linear part of the system at p is negative (respectively positive).*

Corollary 23 *Let e_R be a saddle of the Liénard piecewise linear differential system (1) and assume that this saddle has a homoclinic loop γ. Then, the orbits near γ contained in the region limited by γ tend to γ in forward (respectively backward) time if and only the trace of the linear part of the system at p is negative (respectively positive).*

Proof Since the Liénard piecewise linear differential system (1) can be a limit of analytic differential systems in the plane having homoclinic loops tending to the homoclinic loop γ of system (1), by Theorem 22 the corollary follows.   □

Now, we prove the claim. Since $T_R < 0$, the trace of the saddle e_R is negative. By Corollary 23, the homoclinic loop surrounding the focus is stable. By Proposition 19 inside the region limited by the loop, there is at most one periodic solution surrounding the focus and it is stable. But this is in contradiction with the fact that the homoclinic loop is stable. Hence, the claim is proved.

If finally $P_L(Y_+^r) > -Y_-^r$, then we have a trapping region in backward time. By considering that the focus is stable when the time is reversed, the only possibilities for

periodic orbits are a semi-stable periodic orbit, or two or more periodic orbits, again in contradiction with Proposition 19. Statement (b.3) is shown, and this concludes the proof of Theorem 16.

6 Proof of Theorem 5

To prove Theorem 5, we will state and prove several auxiliary results.

The next auxiliary result proves the uniqueness of the limit cycle (if it exists) in Theorem 5.

Proposition 24 *Assume that system* (1) *with* $T_C > 0$ *has three linearity zones and satisfies one of the following conditions.*

(a) *The sign of* T_R *and* T_L *are equal and different from the sign of* T_C, *that is,* $T_R, T_L \leq 0$;
(b) $T_L \leq 0$, $T_R > 0$ *and* $T_R - T_C r \leq 0$;
(c) $T_R \leq 0$, $T_L > 0$ *and* $T_L - T_C \ell \leq 0$.

If system (1) *has a periodic orbit, then it surrounds the origin, and it is a limit cycle, which is unique and stable.*

Proof The proof of statement (c) is very similar to the proof of statement (b), so we do not do it. First we consider system (1) restricted to $x \geq 0$ that it can be considered with only two linearity zones as system (12) restricted to $x \geq 0$. We shall prove simultaneously statements (a) and (b).

From Proposition 17 and Corollary 18, system (12) can be transformed into another having the nonnegative rotation property for all the half-rays contained in $x \geq 0$. By using the symmetry given in (11), and applying again Proposition 17 and Corollary 18 , we can deduce that such systems can also be transformed into an equivalent system having the nonnegative rotation property for all the half-rays contained in $x \leq 0$. Note that the change of variables (19) is the same in the whole plane, i.e., in $x \geq 0$ and in $x \leq 0$. In other words, the change of variables (19) produce that simultaneously in $x \leq 0$ and $x \geq 0$ the nonnegative rotation property holds. In short, system (1) with three linearity zones can be transformed into a system with the nonnegative rotation property for all the half-rays contained in S. Consequently, from Theorem 15, we conclude that for a such system, if there exists a periodic orbit, then it surrounds the origin and it is a limit cycle that is unique and stable. □

To prove statement (b) of Theorem 5, we need to prove the existence of such a periodic orbit surrounding the origin. To do this, again we will use the positive and negative parts of the y-axis as domain and range for defining two different half-return maps, namely a right half-return map P_R and a left half-return map P_L.

We start by studying the qualitative properties of the right half-return map P_R defined in the negative y-axis, by taking the orbit starting at the point $(0, -y)$ with $y > 0$, and coming back to the positive y-axis at the point $(0, P_R(y))$. We have the following result whose proof is exactly the same as Corollary 21.

Corollary 25 *Assume that system* (1) *has three linearity zones and that the signs of* T_R *and* T_L *are equal and different from the sign of* $T_C > 0$. *Then, the orbits starting at the point* $(0, -y)$ *with* $y > 0$ *enter the half-plane* $x > 0$ *and go around the origin in a counterclockwise path, coming back to the y-axis at the point* $(0, P_R(y))$ *with* $P_R(y)) > 0$. *The difference* $P_R(y) - y$ *is positive for small values of y, eventually becomes negative and*

(a) *tends to* $-\infty$ *when* $y \to \infty$ *if* $r > 0$,
(b) *tends to* $Y_+^r + Y_-^r$ *when* $y \nearrow (-Y_-^r)$, *if* $r < 0$.

Now, we can do a similar study for the left half-return map P_L defined in the positive y-axis, by taking the orbit starting at the point $(0, y)$, with $y > 0$ and coming back to the negative y-axis at the point $(0, -P_L(y))$. More precisely, we have the following result.

Corollary 26 *Assume that system* (1) *has three linearity zones and that the signs of* T_R *and* T_L *are equal and different from the sign of* $T_C > 0$. *Then, the orbits starting at the point* $(0, y)$, *with* $y > 0$ *enter the half-plane* $x < 0$ *and go around the origin in a counterclockwise path, coming back to the y-axis at the point* $(0, -P_L(y))$ *with* $P_L(y) > 0$. *The difference* $P_L(y) - y$ *is positive for small values of y, eventually becomes negative and*

(a) *tends to* $-\infty$ *when* $y \to \infty$, *if* $\ell > 0$,
(b) *tends to* $Y_+^\ell + Y_-^\ell$ *when* $y \nearrow Y_+^\ell$, *if* $\ell < 0$.

With these results, we can prove Theorem 5.

Proof of Theorem 5 Note that the traces in the regions $\mathcal{S}_L$, $\mathcal{S}_C$ and $\mathcal{S}_R$ are given by T_L, T_C and T_R, respectively. By the Bendixson theorem, see, for instance, Theorem 7.10 in Dumortier et al. (2006), they cannot have the same sign for the existence of limit cycles. This proves statement (a) of the theorem.

Statements (c), (d) and (e) are immediate consequences of Proposition 24.

To prove statement (b), we first look for the existence of periodic orbits by showing that system (1) has at least one periodic orbit. The existence of periodic orbits is equivalent to the existence of two positive values y_L and y_R such that

$$P_R(y_R) = y_L, \quad y_R = P_L(y_L).$$

Adding and subtracting the above equations, we get an equivalent system of sufficient and necessary conditions for the existence of periodic orbits, namely

$$\begin{aligned} P_R(y_R) + y_R &= y_L + P_L(y_L), \\ P_R(y_R) - y_R &= y_L - P_L(y_L). \end{aligned} \tag{25}$$

Furthermore, conditions (25) for the existence of periodic orbits translate now to the existence of a value $Y > 0$ being solution of the single equation $\widehat{P}_R(Y) = -\widehat{P}_L(Y)$, that is, of

$$\widehat{P}_R(Y) + \widehat{P}_L(Y) = 0,$$

where

$$\widehat{P}_R(Y) = P_R(y_R) - y_R, \quad \widehat{P}_L(Y) = P_L(y_L) - y_L,$$

being y_R the unique solution of $Y = P_R(y_R) + y_R$ and y_L the unique solution of $Y = P_L(y_L) + y_L$ (we recall that P_R and P_L are monotone functions).

The function $\widehat{P}_R(Y) + \widehat{P}_L(Y)$ is positive for small values $Y > 0$ and, when $r > 0$ and $\ell > 0$, such function is negative when $Y \to \infty$ as a consequence of statement (a) of Corollaries 25 and 26.

Now, we apply the mean value theorem to conclude the existence of at least a solution, and so a periodic orbit of the system. The uniqueness and the stability of the periodic orbit come from Proposition 24. This concludes the proof of statement (b) and so Theorem 5 is completely shown.   □

7 Proof of Theorem 7

Proof of Theorem 7 To show statement (a), we begin by considering the case $T_C = T_R$. Since $r = 1$, then we have a system with only two linearity zones, and we can apply Theorem 3 (b.1) to conclude the existence of one stable limit cycle $\mathcal{L}$ surrounding the origin. Clearly, the limit cycle $\mathcal{L}$ cannot be contained in the band $x > -u$ where the system is purely linear and so have points in the band $x < -u$. Inspired by Figueiredo (1960), we take the closed curve defined by $\mathcal{L}$ as the boundary of a region in the plane, and we claim that such a region remains positive invariant for the flow of the system for every value of $T_C \in (0, T_R)$. Effectively, moving the value of T_C in such a range, we do not alter the value of $g(x)$ but only the value of $F(x)$. Furthermore, from (2), it is easy to conclude that

$$F(x)|_{T_C \in (0, T_R)} < F(x)|_{T_C = T_R} < 0$$

for all $x > 0$, while we have the opposite inequality for all $x < 0$. Thus, at any point of the curve defined by $\mathcal{L}$, for all $T_C \in (0, T_R)$, the y-component of the vector field is equal to the one when $T_C = T_R$ and the x-component is modified in such a way the new vector field points inward the interior of $\mathcal{L}$, and our claim is shown. Since the origin is unstable, the stable limit cycle persists and statement (a) is shown.

When $T_C = 0$, recalling Remark 4 and since $v < u$, we have a circular period annulus tangent to the line $x = v$ and totally contained in the band $x > -u$. Since $T_R > 0$, the nearby orbits with points in $x > v$ tend to go farther and the outermost periodic orbit of the annulus is unstable. However, the reasoning of statement (a) is also valid and the curve $\mathcal{L}$ still defines a trapping region, so that there exists at least one stable limit cycle using the three linearity zones and surrounding the period annulus of the origin. Statement (b) is shown.

For the proof of statement (c), it suffices to use Remark 4 again and the fact that the stable limit cycle will persist under small perturbations in the value of T_C.   □

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