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Optimal Harvesting of a Stochastic Logistic Model with Time Delay

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Abstract This note is concerned with the optimal harvesting of a stochastic logistic model with time delay. The classical optimal harvesting question of this type of model is difficult because it is very difficult to obtain the explicit solution of the corresponding delay Fokker–Planck equation. The main aim of this note was to find a new approach to overcome this problem. In this note, using the ergodic method, sufficient and necessary criteria for the existence of optimal harvesting policy of our model are obtained. At the same time, the optimal harvesting effort and the maximum of harvesting yield are given. This method provides a new approach to study the optimal harvesting problem of stochastic population models, which can be also applied to investigate stochastic multi-species models.

Keywords Optimal harvesting · Stochastic perturbations · Ergodic method

Mathematics Subject Classification 60H30 · 60H10 · 92D25

1 Introduction

In recent years, optimal harvesting in managing natural resources has received much attention. Because the growth of species in the natural world is inevitably affected by environmental noises, many scholars have considered the optimal harvesting of stochastic population systems. By solving the corresponding Fokker–Planck equation,

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Beddington and May (1977) established the optimal harvesting policy for a stochastic logistic model. Using the same method, Li and Wang (2010) obtained the optimal harvesting policy for a stochastic Gilpin–Ayala model. The optimal harvesting of stochastic population models was also examined in Alvarez and Shepp (1998), Braumann (2002), Lande et al. (1995), Liu and Bai (2014), Ludwig and Varah (1979), Lungu and Øksendal (1997), Song et al. (2011) and Zou and Wang (2014).

On the other hand, all species in the natural world should exhibit time delay, for example, they need time to mature (see e.g., Gopalsamy (1992)). Therefore, it is important to consider the optimal harvesting of stochastic population models with time delay. However, as far as we know, no result of this aspect has been reported. One possible reason is that it is very difficult to get the explicit solution of the corresponding delay Fokker–Planck equation.

In this note, we use the ergodic method to study this problem. One advantage of this method is that it is unnecessary to solve the corresponding Fokker–Planck equation. We obtain the sufficient and necessary conditions for the existence of optimal harvesting policy. At the same time, the optimal harvest effort and the maximum of expectation of sustainable yield are obtained. At the end of this note, it is shown that this method can be also applied to investigate stochastic multi-species systems, and as an example, we establish the optimal harvest effort and the maximum of expectation of sustainable yield of a two-species stochastic delayed competitive system with harvesting.

2 Main Results

Consider the following stochastic delayed logistic model with harvesting

$$\mathrm{d}N(t) = N(t)\Big(r - h - aN(t) - bN(t-\tau)\Big)\mathrm{d}t + \sigma N(t)\mathrm{d}W(t) \tag{1}$$

with initial condition $N(\theta) = \phi(\theta) \in C([-\tau, 0]; R_+)$, where $N(t)$ is the population size at time t; r, h, a, and b are positive constants, h is the harvesting effort, $\tau > 0$ is the delay; $W(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\in R_+}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\in R_+}$, σ^2 is the intensity of the environmental noise, $C([-\tau, 0]; R_+)$ represents the family of continuous functions from $[-\tau, 0]$ to $R_+ := (0, +\infty)$.

Our aim was to obtain the optimal harvesting effort h^* such that the expectation of sustained yield $Y(h) = \lim_{t\to+\infty} \mathbb{E}(hN(t))$ is maximum. To this end, let us prepare some lemmas.

Lemma 1 *For any initial data $\phi(\theta) \in C([-\tau, 0]; R_+)$, Eq. (1) has a unique global positive solution $N(t)$ almost surely (a.s.). Moreover, for any $p > 0$, there is $K(p) > 0$ such that*

$$\limsup_{t\to+\infty} \mathbb{E}(N(t))^p \le K(p). \tag{2}$$

Proof The proof is rather standard and hence is omitted (see e.g., Bahar and Mao (2004)). □

Lemma 2 (Liu and Wang (2013)) *Let* $z(t) \in C(\Omega \times [0, +\infty); R_+)$.

(a) If there exist constants $T > 0$, $\lambda \geq 0$, *and* $\lambda_0 > 0$ *such that*

$$\ln z(t) \leq \lambda t - \lambda_0 \int_0^t z(s)\mathrm{d}s + \sigma W(t)$$

for all $t \geq T$, *then* $\limsup_{t\to+\infty} t^{-1} \int_0^t z(s)\mathrm{d}s \leq \lambda/\lambda_0 \quad a.s.$

(b) If there exist three positive constants T, λ, *and* λ_0 *such that*

$$\ln z(t) \geq \lambda t - \lambda_0 \int_0^t z(s)\mathrm{d}s + \sigma W(t)$$

for all $t \geq T$, *then* $\liminf_{t\to+\infty} t^{-1} \int_0^t z(s)\mathrm{d}s \geq \lambda/\lambda_0 \quad a.s.$

Lemma 3 *For Eq. (1),*

(i) If $h > r - 0.5\sigma^2$, *then* $\lim_{t\to+\infty} N(t) = 0,\ a.s.;$

(ii) If $h = r - 0.5\sigma^2$, *then* $\lim_{t\to+\infty} t^{-1} \int_0^t N(s)\mathrm{d}s = 0,\ a.s.;$

(iii) If $h < r - 0.5\sigma^2$, *then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t N(s)\mathrm{d}s = \frac{r - h - 0.5\sigma^2}{a + b},\ a.s. \tag{3}$$

Proof The proof is similar to that of Liu and Wang (2013) but for the completeness of the paper we will give it briefly. Consider the following stochastic logistic equation:

$$\mathrm{d}X(t) = X(t)\Big(r - h - aX(t)\Big)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t),\ X(\theta) = \phi(\theta). \tag{4}$$

By the work of Liu and Wang (2013),

$$\begin{cases} \lim_{t\to+\infty} t^{-1} \int_0^t X(s)\mathrm{d}s = \frac{r-h-0.5\sigma^2}{a}\ a.s., & \text{if} \quad h \leq r - 0.5\sigma^2; \\ \lim_{t\to+\infty} X(t) = 0 \quad a.s., & \text{if} \quad h > r - 0.5\sigma^2. \end{cases}$$

According to the comparison theorem for stochastic differential delay equations (Bao and Yuan 2011),

$$N(t) \leq X(t)\ a.s., t \geq -\tau. \tag{5}$$

This completes the proof of (i) and (ii).

Now let us prove (iii). If $h < r - 0.5\sigma^2$, then

$$\lim_{t\to+\infty} t^{-1}\int_{t-\tau}^{t} X(s)\mathrm{d}s = \lim_{t\to+\infty}\left(t^{-1}\int_0^t X(s)\mathrm{d}s - t^{-1}\int_0^{t-\tau} X(s)\mathrm{d}s\right) = 0.$$

This, together with (5), means

$$\lim_{t\to+\infty} t^{-1}\int_{t-\tau}^{t} N(s)\mathrm{d}s = 0. \tag{6}$$

Applying Itô's formula to (1) leads to

$$\begin{aligned}\ln N(t) - \ln N(0) &= (r - 0.5\sigma^2 - h)t - a\int_0^t N(s)\mathrm{d}s - b\int_0^t N(s-\tau)\mathrm{d}s + \sigma W(t)\\ &= (r - 0.5\sigma^2 - h)t - (a+b)\textstyle\int_0^t N(s)\mathrm{d}s\\ &\quad + b\left[\int_{t-\tau}^{t} N(s)\mathrm{d}s - \int_{-\tau}^{0} N(s)\mathrm{d}s\right] + \sigma W(t).\end{aligned} \tag{7}$$

In view of (6), for arbitrary $\varepsilon > 0$, there is $T > 0$ such that for $t \geq T$,

$$-\varepsilon \leq t^{-1}\ln N(0) + bt^{-1}\left[\int_{t-\tau}^{t} N(s)\mathrm{d}s - \int_{-\tau}^{0} N(s)\mathrm{d}s\right] \leq \varepsilon.$$

Substituting the above inequalities into (7) yields

$$\ln N(t) \leq (r - 0.5\sigma^2 - h + \varepsilon)t - (a+b)\int_0^t N(s)\mathrm{d}s + \sigma W(t), \tag{8}$$

$$\ln N(t) \geq (r - 0.5\sigma^2 - h - \varepsilon)t - (a+b)\int_0^t N(s)\mathrm{d}s + \sigma W(t). \tag{9}$$

Since $r - 0.5\sigma^2 > h$, we can let ε be sufficiently small such that $r - 0.5\sigma^2 - h - \varepsilon > 0$. Applying (a) and (b) in Lemma 2 to (8) and (9), respectively, one can observe that

$$\begin{aligned}\frac{r - 0.5\sigma^2 - h - \varepsilon}{a+b} &\leq \liminf_{t\to+\infty} t^{-1}\int_0^t N(s)\mathrm{d}s \leq \limsup_{t\to+\infty} t^{-1}\\ &\int_0^t N(s)\mathrm{d}s \leq \frac{r - 0.5\sigma^2 - h + \varepsilon}{a+b}.\end{aligned}$$

Then the desired assertion follows from the arbitrariness of ε. □

Lemma 4 *If $a > b$, then Eq. (1) is asymptotically stable in distribution, i.e., there is a probability measure $\nu(\cdot)$ such that the transition probability $p(t, \phi, \cdot)$ of $N(t)$ converges weakly to $\nu(\cdot)$ as $t \to +\infty$ for every $\phi \in C([-\tau, 0]; R_+)$.*

Proof The proof will be divided into three steps.

Step 1. By (2), there is a $T > 0$ such that for all $t \geq T$, $\mathbb{E}(N(t))^p \leq 1.5K(p)$. On the other hand, it follows from the continuity of $\mathbb{E}(N(t))^p$ that we can find a $L_1(p) > 0$ such that $\mathbb{E}(N(t))^p \leq L_1(p)$ for $t \leq T$. Let $L_2(p) = \max\{1.5K(p), L_1(p)\}$, then

$$\mathbb{E}(N(t))^p \leq L_2(p), t \geq -\tau. \tag{10}$$

Eq. (1) is equivalent to the following integral equation

$$N(t)=N(0)+\int_0^t N(s)\Big[r-h-aN(s)-bN(s-\tau)\Big]\mathrm{d}s+\int_0^t \sigma N(s)\mathrm{d}W(s). \tag{11}$$

Clearly,

$$\begin{aligned}
&\mathbb{E}\Big|N(t)\Big(r-h-aN(t)-bN(t-\tau)\Big)\Big|^p\\
&=\mathbb{E}\Big[N^p(t)\Big|r-h-aN(t)-bN(t-\tau)\Big|^p\Big]\\
&\leq 0.5\mathbb{E}(N(t))^{2p}+0.5\mathbb{E}\Big|r-h-aN(t)-bN(t-\tau)\Big|^{2p}\\
&\leq 0.5L_2(2p)+4^{2p}\Big((r+h)^{2p}+a^{2p}\mathbb{E}(N(t))^{2p}+b^{2p}\mathbb{E}(N(t-\tau))^{2p}\Big)\\
&=0.5L_2(2p)+4^{2p}\Big((r+h)^{2p}+(a^{2p}+b^{2p})L_2(2p)\Big)=:L_3(p).
\end{aligned}\tag{12}$$

At the same time, in view of the moment inequality for stochastic integrals (see e.g., Mao and Yuan (2006), p. 69), one can see that for $0 \leq t_1 \leq t_2$ and $p > 2$,

$$\begin{aligned}
\mathbb{E}\Big|\int_{t_1}^{t_2}\sigma N(s)\mathrm{d}W(s)\Big|^p &\leq \sigma^{2p}\Big[\frac{p(p-1)}{2}\Big]^{p/2}(t_2-t_1)^{(p-2)/2}\int_{t_1}^{t_2}\mathbb{E}(N(s))^p\mathrm{d}s\\
&\leq \sigma^{2p}\Big[\frac{p(p-1)}{2}\Big]^{p/2}(t_2-t_1)^{p/2}L_2(p).
\end{aligned}$$

Substituting the above inequality and (12) into (11), we can obtain that for $0 < t_1 < t_2 < \infty,\ t_2 - t_1 \leq 1$,

$$\begin{aligned}
&\mathbb{E}\Big(|N(t_2)-N(t_1)|^p\Big)\\
&=\mathbb{E}\Big|\int_{t_1}^{t_2}N(s)\Big[r-h-aN(s)-bN(s-\tau)\Big]\mathrm{d}s+\int_{t_1}^{t_2}\sigma N(s)\mathrm{d}W(s)\Big|^p\\
&\leq 2^{p-1}\mathbb{E}\Big|\int_{t_1}^{t_2}N(s)\Big[r-h-aN(s)-bN(s-\tau)\Big]\mathrm{d}s\Big|^p+2^{p-1}\mathbb{E}\Big|\int_{t_1}^{t_2}\sigma N(s)\mathrm{d}W(s)\Big|^p\\
&\leq 2^{p-1}(t_2-t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\Big|N(s)\Big[r-h-aN(s)-bN(s-\tau)\Big]\Big|^p\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+2^{p-1}\sigma^{2p}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{p/2}L_2(p)\\
&=2^{p-1}(t_2-t_1)^{p}L_3(p)+2^{p-1}\sigma^{2p}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{p/2}L_2(p)\\
&\le 2^{p-1}(t_2-t_1)^{p/2}\left[(t_2-t_1)^{p/2}+\left(\frac{p(p-1)}{2}\right)^{p/2}\right]L_4(p)\\
&\le 2^{p-1}(t_2-t_1)^{p/2}\left[1+\left(\frac{p(p-1)}{2}\right)^{p/2}\right]L_4(p),
\end{aligned}
$$

where $L_4(p)=\max\{L_3(p),\sigma_1^{2p}L_2(p)\}$. Then it follows from the Kolmogorov continuity criterion (see e.g., Karatzas and Shreve (1991)) that almost every sample path of $N(t)$ is uniformly continuous on $t\ge -\tau$.

Step 2. let $N^{\phi}(t)$ and $N^{\varphi}(t)$ be two solutions of Eq. (1) with initial conditions $\phi(\theta)\in C([-\tau,0];R_+)$ and $\varphi(\theta)\in C([-\tau,0];R_+)$, respectively. Define $V_1(t)=|\ln N^{\phi}(t)-\ln N^{\varphi}(t)|$. By Itô's formula, we have

$$
\begin{aligned}
\mathrm{d}^+V_1(t)&=\mathrm{sgn}\Big(N^{\phi}(t)-N^{\varphi}(t)\Big)\mathrm{d}\Big(\ln N^{\phi}(t)-\ln N^{\varphi}(t)\Big)\\
&=\mathrm{sgn}\Big(N^{\phi}(t)-N^{\varphi}(t)\Big)\Big[-a\Big(N^{\phi}(t)-N^{\varphi}(t)\Big)\\
&\qquad -b\Big(N^{\phi}(t-\tau)-N^{\varphi}(t-\tau)\Big)\Big]\mathrm{d}t\\
&\le -a|N^{\phi}(t)-N^{\varphi}(t)|\mathrm{d}t+b\Big|N^{\phi}(t-\tau)-N^{\varphi}(t-\tau)\Big|\mathrm{d}t.
\end{aligned}
$$

Define $V_2(t)=b\int_{t-\tau}^{t}\Big|N^{\phi}(s)-N^{\varphi}(s)\Big|\mathrm{d}s$. Then

$$
\begin{aligned}
\mathrm{d}^+(V_1(t)+V_2(t))&\le -a\Big|N^{\phi}(t)-N^{\varphi}(t)\Big|\mathrm{d}t+b\Big|N^{\phi}(t-\tau)-N^{\varphi}(t-\tau)\Big|\mathrm{d}t\\
&\quad +b\Big|N^{\phi}(t)-N^{\varphi}(t)\Big|\mathrm{d}t-b\Big|N^{\phi}(t-\tau)-N^{\varphi}(t-\tau)\Big|\mathrm{d}t\\
&=-(a-b)\Big|N^{\phi}(t)-N^{\varphi}(t)\Big|\mathrm{d}t.
\end{aligned}
$$

Therefore,

$$
V_1(t)+V_2(t)\le V_1(0)+V_2(0)-(a-b)\int_0^t|N^{\phi}(s)-N^{\varphi}(s)|\mathrm{d}s.
$$

That is to say

$$
(a-b)\int_0^t|N^{\phi}(s)-N^{\varphi}(s)|ds\le V_1(0)+V_2(0)<\infty.
$$

Therefore, $|N^{\phi}(t) - N^{\varphi}(t)| \in L^1[0, \infty)$. It then follows from the uniform continuity of $N(t)$ and Barbalat's conclusion (Barbalat 1959) that

$$\lim_{t \to +\infty} |N^{\phi}(t) - N^{\varphi}(t)| = 0, \ a.s. \tag{13}$$

Step 3. This step of proof is a modification of that in Bao et al. (2009) and Mao and Yuan (2006), and hence we only give the outline. Let $p(t, \phi, \mathrm{d}y)$ be the transition probability of the process $N(t)$, and let $P(t, \phi, A)$ stand for the probability of event $N(t) \in A$ given the initial data $N(\theta) = \phi(\theta)$. It follows from (2) and Chebyshev's inequality that the family of transition probability $p(t, \phi, \mathrm{d}y)$ is tight. Now let $\mathscr{P}(C([-\tau, 0]; R_+))$ be all the probability measures defined on $C([-\tau, 0]; R_+)$. For any two measures $P_1, \ P_2 \in \mathscr{P}$ define the metric

$$\mathrm{d}_L(P_1, P_2) = \sup_{f \in L} \left| \int_{R_+} f(x) P_1(\mathrm{d}x) - \int_{R_+} f(x) P_2(\mathrm{d}x) \right|,$$

where

$$L = \left\{ f : C([-\tau, 0]; R_+) \to R \middle| |f(x) - f(y)| \le ||x - y||, |f(\cdot)| \le 1 \right\}.$$

Since the family of transition probability $p(t, \phi, \mathrm{d}y)$ is tight, then by (13) and a standard procedure in Bao et al. (2009) (Lemmas 3.4 and 3.5) or Mao and Yuan (2006) (pp. 213–215), we can prove that sequence $\{p(t, \phi, \cdot) : t \ge 0\}$ is Cauchy in metric space $\mathscr{P}(C([-\tau, 0]; R_+))$, and for any compact subset $\mathcal{K} \in R_+$, $\lim_{t \to +\infty} d_L(p_1(t, \phi, \cdot), p_2(t, \varphi, \cdot)) = 0$ uniformly in $\phi, \varphi \in \mathcal{K}$. To complete the proof, it suffices to prove that there is a probability measure $\nu(\cdot) \in \mathscr{P}(C([-\tau, 0]; R_+))$ such that $p(t, \phi, \cdot)$ converges weakly to $\nu(\cdot)$. The proof is similar to that of Theorem 3.1 in Bao et al. (2009) and hence is omitted. □

Now we are in the position to give our main result.

Theorem 1 *For model (1), let $a > b$.*

(I) If $h > r - 0.5\sigma^2$, then $\lim_{t \to +\infty} N(t) = 0, \ a.s.$;
(II) If $h = r - 0.5\sigma^2$, then $\lim_{t \to +\infty} t^{-1} \int_0^t N(s)ds = 0, \ a.s.$;
(III) If $h < r - 0.5\sigma^2$, then the optimal harvesting effort h^* is

$$h^* = 0.5(r - 0.5\sigma^2),$$

the maximum of expectation of sustainable yield is

$$Y^* = \frac{(r - 0.5\sigma^2)^2}{4(a + b)}.$$

Proof From Lemma 3, it suffices to show (III). By Lemma 4, model (1) is asymptotically stable in distribution, then it has a unique invariant probability measure $\nu(\cdot)$ by Kolmogorov–Chapman equation (see e.g., Mao and Yuan (2006)), and this invariant measure is ergodic (Prato and Zabczyk (1996), Theorem 3.2.6). That is to say, if g is an integrable function with respect to ν, then

$$\lim_{t\to+\infty} t^{-1}\int_0^t \mathbb{E}[g(N(s))]\mathrm{d}s = \int_0^{+\infty} g(N)\mathrm{d}\nu(N). \tag{14}$$

On the other hand, since Eq. (1) is asymptotically stable in distribution, then measure $\nu(\cdot)$ is strong mixing (Prato and Zabczyk (1996), Corollary 3.4.3), i.e., the solution $N(t,\omega)$ of (1) converges to a random variable $Z(\omega)$ in distribution, and $\nu(\cdot)$ is the measure induced by $Z(\omega)$. In other words,

$$\lim_{t\to+\infty} \mathbb{E}[g(N(t))] = \mathbb{E}[g(Z)] = \int_\Omega g(Z(\omega))\mathrm{d}P(\omega) = \int_0^{+\infty} g(N)\mathrm{d}\nu(N). \tag{15}$$

We are now in the position to show that $g(N) = N$ is integrable with respect to measure $\nu(\cdot)$. In fact, for any integer $L > 0$, $N \wedge L$ is integrable with respect to measure $\nu(\cdot)$. In view of (2) and the dominated convergence theorem as well as the ergodic property of $\nu(\cdot)$,

$$\begin{aligned} K(1)\wedge L &= \lim_{t\to+\infty}\frac{1}{t}\int_0^t (K(1)\wedge L)\mathrm{d}s \geq \lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}[N(s)\wedge L]\mathrm{d}s \\ &= \int_0^{+\infty}[N\wedge L]\nu(\mathrm{d}N). \end{aligned}$$

Letting $L \to +\infty$, we can obtain that $g(N) = N$ is integrable with respect to measure $\nu(\cdot)$. This, together with (15), (14), and (3), implies

$$\begin{aligned} Y(h) &= \lim_{t\to+\infty}\mathbb{E}(hN(t)) = \lim_{t\to+\infty} h\mathbb{E}(N(t)) \\ &= h\int_0^{+\infty} N d\nu(N) = h\lim_{t\to+\infty} t^{-1}\int_0^t \mathbb{E}[N(s)]ds \\ &= \frac{h(r-h-\sigma^2/2)}{a+b}. \end{aligned}$$

Clearly, $Y(h)$ has a unique extreme value at $h^* = (r-\sigma^2/2)/2$, and the maximum of expectation of sustainable yield is $Y^* = \frac{(r-\sigma^2/2)^2}{4(a+b)}$. □

Remark 5 Consider the following stochastic logistic population model with harvesting:

$$\mathrm{d}N(t) = N(t)(r-h-aN(t))\mathrm{d}t + \sigma N\mathrm{d}W(t). \tag{16}$$

It follows from Theorem 1 that the optimal harvesting policy exists if and only if $h < r - \sigma^2/2$, and $h^* = (r - \sigma^2/2)/2$, $Y^* = \frac{(r-\sigma^2/2)^2}{4a}$. These results coincide with the classical conclusions established by Beddington and May (1977).

Remark 6 In Theorem 1, the delay τ does not appear in the optimal criterion because there is a requirement that the parameters satisfy $a > b$. This result is similar to the Hopf bifurcation criterion of the corresponding deterministic differential delay model. Setting $\sigma = 0$, Eq. (1) reduces to the deterministic differential delay model

$$\mathrm{d}N(t) = N(t)\Big(r - h - aN(t) - bN(t-\tau)\Big)\mathrm{d}t, \quad r > h,$$

for which it is well known (Ruan (2006), Theorem 2) that

(i) if $a \geq b$, then the steady state $N^* = \frac{r-h}{a+b}$ is asymptotically stable for all delay $\tau \geq 0$;
(ii) If $a < b$, then there is a critical value τ_0 given by

$$\tau_0 = \frac{a+b}{(r-h)\sqrt{b^2-a^2}} \arcsin \frac{\sqrt{b^2-a^2}}{b}$$

such that N^* is stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. A Hopf bifurcation occurs at N^* when τ passes through τ_0.

Similar result for stochastic delay differential equation was established by Maekey and Neehaeva (1994). Consider the following linear stochastic delay differential equation

$$\mathrm{d}x(t) = [-Ax(t) - Bx(t-\tau)]\mathrm{d}t + \sigma \mathrm{d}W(t), \ A > 0, \ B > 0$$

for which it has been shown (Maekey and Neehaeva (1994)) that if $A > B$, then its trivial solution is stochastically stable for all delay $\tau \geq 0$.

Remark 7 In Theorem 1, we used $Y(h) = \lim_{t\to+\infty} \mathbb{E}(hN(t))$ as the yield function and obtained its maximum. However, if one chooses other functions as the yield functions, then the maximal yields may be different. For example, let $\rho(x)$ be the stationary probability density of (1), and we choose $\tilde{Y}(h) = h \max_x \rho(x)$ as the yield function. In general,

$$\max_h \left\{ h \max_x \rho(x) \right\} \neq \max_h \left\{ \int_0^{+\infty} hx\rho(x)\mathrm{d}x \right\} = \max_h \left\{ \lim_{t\to+\infty} \mathbb{E}(hN(t)) \right\}.$$

That is to say, $\max_h \{Y(h)\} \neq \max_h \{\tilde{Y}(h)\}$. At the same time, our ergodic method cannot be applied to investigate the maximum of $\tilde{Y}(h)$.

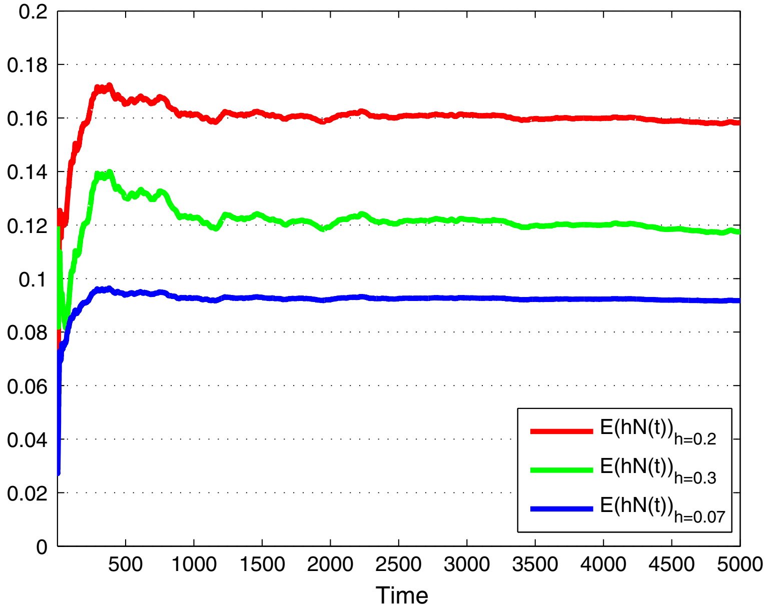

Fig. 1 Expectation of sustainable yield of (1) for $r = 0.5$, $a = 0.2 > b = 0.05$, $\tau = 5$, $\sigma^2 = 0.2$, $\phi(\theta) = 0.2 + 0.1 \sin\theta$, $\theta \in [-5, 0]$, step size $\Delta t = 0.1$. *Red line* is with $h = h^* = 0.2$, *green line* is with $h = 0.3$, and *blue line* is with $h = 0.07$ (Color figure online)

3 Numerical Simulations

Now let us use the Monte Carlo method (see e.g., Bruti-Liberati and Platen (2010)) to illustrate the theoretical results. In Fig. 1, we let $r = 0.5$, $a = 0.2 > b = 0.05$, $\tau = 5$, $\sigma^2 = 0.2$, initial condition $\phi(\theta) = 0.2 + 0.1 \sin\theta$, $\theta \in [-5, 0]$. The only difference between conditions of red line, green line, and blue line is that the value of h is different. Red line is with $h = h^* = (r - \sigma^2/2)/2 = 0.2$, green line is with $h = 0.3$ ($Y(h) = 0.12$), and blue line is with $h = 0.07$ ($Y(h) = 0.0924$). Figure 1 indicates that if $h = h^* = 0.2$ (red line), then the expectation of sustainable yield is maximum and $Y^* = \frac{(r-\sigma^2/2)^2}{4(a+b)} = 0.16$. To see these more clearly, we let the mean asymptotic value $Y(h) = \lim_{t\to+\infty} \mathbb{E}(hN(t))$ take an approximate value $\mathbb{E}(hN(20000))$ and plot it as a function of h in Fig. 2. The values of parameters in Fig. 2 are the same with that in Fig. 1. Figure 2 also shows that $Y(h)$ takes the maximum value at 0.2 and $Y^* = 0.16$.

4 Concluding Remarks

This note considers the optimal harvesting of a stochastic delay logistic population model. Using the ergodic method, sufficient and necessary conditions for the existence of optimal harvesting strategy are established, and the optimal harvesting effort and the maximum of expectation of sustainable yield are also obtained. The results reveal that time delay has no impact on the optimal harvesting strategy in some cases. As far

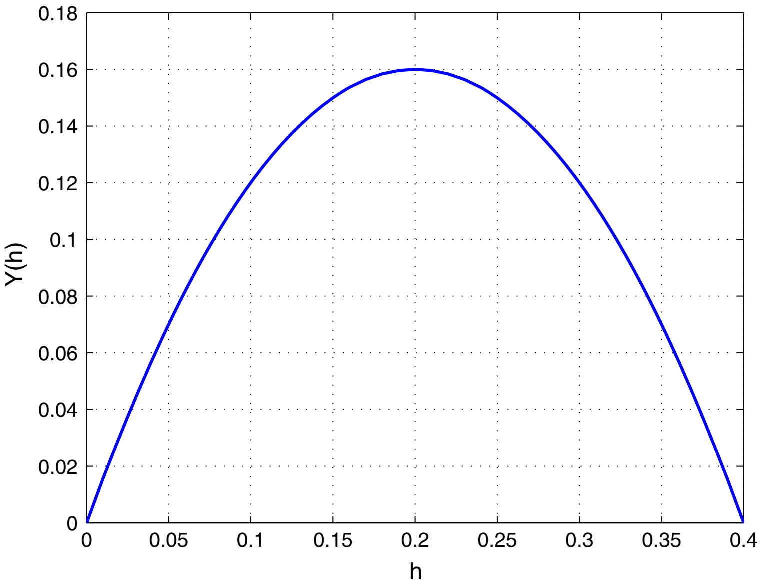

Fig. 2 Mean asymptotic value $Y(h)$ of (1) for $r = 0.5$, $a = 0.2$, $b = 0.05$, $\tau = 5$, $\sigma^2 = 0.2$, $\phi(\theta) = 0.2 + 0.1\sin\theta$, $\theta \in [-5, 0]$, step size $\Delta h = 0.005$

as we know, this note is the first attempt to investigate the optimal harvesting problem of stochastic population models with delay. The traditional method is difficult to use, because it is difficult to obtain the explicit solution of the corresponding delay Fokker–Planck equation. One advantage of our method is that it is not necessary to solve the corresponding Fokker–Planck equation.

It is useful to mention that the ergodic method can be applied to cover multi-species models with/without time delay. For instance, consider the following stochastic delayed competitive system with harvesting

$$\begin{cases} dN_1(t) = N_1(t)\Big[r_1 - h_1 - a_{11}N_1(t) - a_{12}N_2(t-\tau_1)\Big]dt + \sigma_1 N_1(t)dW_1(t), \\ dN_2(t) = N_2(t)\Big[r_2 - h_2 - a_{21}N_1(t-\tau_2) - a_{22}N_2(t)\Big]dt + \sigma_2 N_2(t)dW_2(t), \end{cases} \tag{17}$$

with initial conditions

$$N_i(\theta) = \phi_i(\theta) \in C([-\tau, 0]; R_+), \ \tau = \max\{\tau_1, \tau_2\}, \ i = 1, 2,$$

where $r_i > 0$, $a_{ij} > 0$, $\tau_i \geq 0$. The aim was to obtain the optimal harvesting effort $h^* = (h_1^*, h_2^*)$ such that

(i) the expectation of sustained yield $Y(h) = \lim_{t\to+\infty} \mathbb{E}(h_1N_1(t) + h_2N_2(t))$ is maximum;
(ii) Both N_1 and N_2 will not go to extinction.

Similar to the study of (1), we use the following algorithm:

Step 1. To show that for any $p > 0$, there is $K(p)$ such that

$$\limsup_{t\to+\infty} \mathbb{E}(N_i(t))^p \le K(p),\ i = 1, 2. \tag{18}$$

Step 2. Using (18) to prove

$$\lim_{t\to+\infty} |N_i^{\phi_i}(t) - N_i^{\varphi_i}(t)| = 0,\ a.s.,\ i = 1, 2, \tag{19}$$

where $(N_1^{\phi_1}(t), N_2^{\phi_2}(t))$ and $(N_1^{\varphi_1}(t), N_2^{\varphi_2}(t))$ are two solutions of Eq. (17) with initial conditions $\phi_i(\theta) \in C([-\tau, 0]; R_+)$ and $\varphi_i(\theta) \in C([-\tau, 0]; R_+)$, respectively, $i = 1, 2$.

Step 3. Applying (18) and (19) to show that Eq. (17) is asymptotically stable in distribution.

Step 4. Using Lemma 3 and the comparison theorem to prove that

$$\lim_{t\to+\infty} t^{-1} \int_0^t N_i(s)\mathrm{d}s = \text{a positive constant},\ a.s.,\ i = 1, 2,$$

Step 5. According to the conclusions in Steps 3 and 4, one can obtain the optimal harvest effort and the maximum of expectation of sustainable yield.

Making use of the steps given above, we have

Theorem 2 *For Eq. (17), set $b_i = r_i - 0.5\sigma_i^2$, $i = 1, 2$, let $a_{11} > a_{21}$ and $a_{22} > a_{12}$. If*

$$(b_1 - h_1)a_{22} > (b_2 - h_2)a_{12},\ (b_2 - h_2)a_{11} > (b_1 - h_1)a_{21},$$

then

$$\lim_{t\to+\infty} t^{-1} \int_0^t N_1(s)\mathrm{d}s = \frac{(b_1 - h_1)a_{22} - (b_2 - h_2)a_{12}}{a_{11}a_{22} - a_{12}a_{21}},$$

$$\lim_{t\to+\infty} t^{-1} \int_0^t N_2(s)\mathrm{d}s = \frac{(b_2 - h_2)a_{11} - (b_1 - h_1)a_{21}}{a_{11}a_{22} - a_{12}a_{21}},$$

and the optimal harvesting effort (h_1^*, h_2^*) is the solution of the equations

$$\begin{cases} (a_{12} + a_{21})h_2 - 2h_1a_{22} + b_1a_{22} - b_2a_{12} = 0, \\ (a_{12} + a_{21})h_1 - 2h_2a_{11} + b_2a_{11} - b_1a_{21} = 0, \end{cases}$$

moreover, the expectation of sustainable yield is

$$Y^* = h_1^*\left(\frac{(b_1 - h_1^*)a_{22} - (b_2 - h_2^*)a_{12}}{a_{11}a_{22} - a_{12}a_{21}}\right) + h_2^*\left(\frac{(b_2 - h_2^*)a_{11} - (b_1 - h_1^*)a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\right)$$

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