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# The Properties of Solutions for a Generalized *b*-Family Equation with Peakons

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**Abstract** This paper deals with the Cauchy problem for a shallow water equation with high-order nonlinearities, $y_t + u^{m+1}y_x + bu^m u_x y = 0$, where $b$ is a constant, $m \in \mathbb{N}$, and we have the notation $y := (1 - \partial_x^2)u$ , which includes the famous Camassa–Holm equation, the Degasperis–Procesi equation, and the Novikov equation as special cases. The local well-posedness of strong solutions for the equation in each of the Sobolev spaces $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ is obtained, and persistence properties of the strong solutions are studied. Furthermore, although the $H^1(\mathbb{R})$-norm of the solution to the nonlinear model does not remain constant, the existence of its weak solutions in each of the low order Sobolev spaces $H^s(\mathbb{R})$ with $1 < s < \frac{3}{2}$ is established, under the assumption $u_0(x) \in H^s(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. Finally, the global weak solution and peakon solution for the equation are also given.

**Keywords** Persistence properties · Local well-posedness · Weak solution

**Mathematics Subject Classification** 35G25 · 35L05 · 35Q50

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## 1 Introduction

In this paper, we consider the Cauchy problem for the following shallow water equation with high-order nonlinearities:

$$\begin{cases} y_t + u^{m+1} y_x + b u^m u_x y = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $b$ is a constant and $m \in \mathbb{N}$, the notation $y := (1 - \partial_x^2)u$. It is easy to see that model (1.1) contains the three kinds of famous shallow water equation, that is, the Camassa–Holm equation, the Degasperis–Procesi equation, and the Novikov equation.

Obviously, if $m = 0$, $b \in \mathbb{R}$, Eq. (1.1) becomes a $b$-equation:

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.2}$$

which can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation. For the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated (see Dullin et al. Dullin et al. 2001, 2003, 2004). Equation (1.2) belongs to the following family of nonlinear dispersive partial differential equations:

$$u_t - \gamma u_{xxx} - \alpha^2 u_{xxt} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x,$$

where $\gamma, \alpha, c_1, c_2$ and $c_3$ are real constants. Up to rescaling there are only three equations that are asymptotically integrable within this family: the KdV equation, the Camassa–Holm (Eq. (1.2) with $b = 2$) equation and the Degasperis–Procesi equation (Eq. (1.2) with $b = 3$). In fact, the Camassa–Holm and Degasperis–Procesi equations are the only members of the $b$-equation family with a bi-Hamiltonian structure (Ivanov 2007), and these two kinds of shallow water equation have been studied extensively recently (see Bressan and Constantin 2007; Camassa and Holm 1993; Constantin and Lannes 2009; Constantin and Escher 2011; Constantin and Strauss 2000; Degasperis et al. 2002, 2003; Degasperis and Procesi 1999; Escher et al. 2006; Liu and Yin 2006; Xin and Zhang 2000; Yin 2004 and references therein). The solutions of the $b$-equation were studied numerically for various values of $b$ in Holm and Staley (2003a, 2003b), where $b$ was taken as a bifurcation parameter. The necessary conditions for integrability of the $b$-equation were investigated in Mikhailov and Novikov (2002). In Gilson and Pickering (1995), Hone (2009), Painlevé analysis is applied to these sorts of equation. The $b$-equation also admits peakon solutions for any $b \in \mathbb{R}$ (see Degasperis et al. 2003; Holm and Staley 2003a, 2003b). The well-posedness, blow-up phenomena, and global solutions for the $b$-equation were shown in Escher and Yin (2008), Mu et al. (2011).

On the other hand, taking $m = 1, b = 3$ in (1.1) we found the Novikov equation,

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.3}$$

which was recently discovered by Novikov in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity (Novikov 2009). Since then the Novikov equation has been studied by some researchers (Hone and Wang 2008; Hone et al. 2009; Ni and Zhou 2011; Jiang and Ni 2012). The Novikov equation possesses a matrix Lax pair, infinitely many conserved densities, a bi-Hamiltonian structure as well as peakon solutions (Hone and Wang 2008). These apparently exotic waves replicate a feature that is characteristic of the waves of great height waves of largest amplitude that are exact solutions of the governing equations for water waves (Constantin 2006; Constantin and Escher 2007, 2011; Toland 1996). The Novikov equation possesses the explicit formulas for multi-peakon solutions (Hone et al. 2009). It has been shown that the Cauchy problem for the Novikov equation is locally well-posed in Besov spaces and in Sobolev spaces and possesses persistence properties (Ni and Zhou 2011; Yan et al. 2012). Analogous to the Camassa–Holm equation, the Novikov equation displays the blow-up phenomenon (Jiang and Ni 2012) and global weak solutions (Wu and Yin 2011).

In fact, many different types of solution for various shallow water equations have been investigated. Wazwaz (2006, 2007) studied the solitary wave solutions for generalized $b$-family equation

$$u_t - u_{xxt} + (1+b)u^m u_x = b u_x u_{xx} + u u_{xxx} \tag{1.4}$$

for $m = 2$. Since then Eq. (1.4) has attracted a lot of researchers. When $m = 2$, peakon wave solutions of (1.4) with $b = 2$ were studied in Liu and Qian (2001), Tian and Song (2004),and the periodic blow-up solutions and limit forms for (1.3) were obtained in Liu and Guo (2008). Peakon wave solutions for $b = 3$ was also discussed in Liu and Ouyang (2007). Especially, when $m = 2$ and $b > -2$ is arbitrary, Liu (2010) gave several new types of the explicit nonlinear traveling wave solution of (1.4). For any positive integer $m$, Shen and Xu (2005) considered the bifurcations of the smooth and non-smooth traveling waves of (1.4) for $b = 2$, Zhang et al. (2007) analyzed (1.4) for $b = 3$. Recently, Deng et al. (2011) investigated the traveling wave solutions for Eq. (1.4). The local and global existence and blow-up phenomenon of solutions for Eq. (1.1) with $b = m + 2$ are considered by Li et al. (2012), Mi and Mu C. L. (2013).

Recently, applying the method of pseudoparabolic regularization, Hakkaev and Kirchev (2005) investigated the local well-posedness for generalized Camassa–Holm equation with high-order nonlinearities

$$u_t + \big(a(u)\big)_x = \left(b'(u)\frac{u_x^2}{2} + b(u)u_{xx}\right)_x, \tag{1.5}$$

where $b(u) = u^p$ and $a(u) = 2ku + \frac{p+2}{2}u^{p+1}$. The stability of peakons and orbital stability of solitary wave solution are also obtained in Hakkaev and Kirchev (2005).

Motivated by the results mentioned above, the goal of this paper is to establish the well-posedness and persistence property of strong solutions, and weak solutions and peakon solutions for problem (1.1). Most of our results can be extended to the periodic case. First, we use Kato's Theorem to obtain the existence and uniqueness of strong solutions for Eq. (1.1).

**Theorem 1.1** *Let $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Then there exist a maximal $T = T(\|u_0\|_{H^s(\mathbb{R})})$, and a unique solution $u(x,t)$ to the problem (1.1) such that*

$$u = u(\cdot, u_0) \in C\big([0,T); H^s(\mathbb{R})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{R})\big).$$

*Moreover, the solution depends continuously on the initial data, i.e. the mapping*

$$u_0 \to u(\cdot, u_0) : H^s(\mathbb{R}) \to C\big([0,T); H^s(\mathbb{R})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{R})\big)$$

*is continuous.*

In Himonas et al. (2007), Ni and Zhou (2011, 2012), Henry (2009), the spatial decay rate for the strong solution to the Camassa–Holm, $b$-equation, Novikov equation were established provided that the corresponding initial datum decays at infinity. This kind of property is the so-called persistence property. Similarly, for Eq. (1.1), we also have the following persistence properties for the strong solution. However, the hard question is that there are high nonlinearity in (1.1), which makes the proof of several required nonlinear estimates very difficult.

**Theorem 1.2** *Assume that* $u_0 \in C([0,T); H^s(\mathbb{R}))$ *with* $s > 3/2$ *satisfies*

$$\begin{aligned} &\big|u_0(x)\big|,\ \big|u_{0x}(x)\big| \sim O\big(\mathrm{e}^{-\theta x}\big) \quad \text{as } x \uparrow \infty, \\ &\Big(\text{respectively, } \big|u_0(x)\big|, \big|u_{0x}(x)\big| \sim O\big((1+x)^{-\alpha}\big) \text{ as } x \uparrow \infty\Big) \end{aligned}$$

*for some* $\theta \in (0,1)$*(respectively,* $\alpha \geq \frac{1}{m+1}$*), then the corresponding strong solution* $u \in C([0,T); H^s(\mathbb{R}))$ *to Eq.* (1.1) *satisfies, for some* $T > 0$,

$$\begin{aligned} &\big|u(x,t)\big|,\ \big|u_x(x,t)\big| \sim O\big(\mathrm{e}^{-\theta x}\big) \quad \text{as } x \uparrow \infty, \\ &\Big(\text{respectively, } \big|u(x)\big| \sim O\big((1+x)^{-\alpha}\big) \text{ as } x \uparrow \infty\Big) \end{aligned}$$

*uniformly in the time interval* $[0,T]$.

Since the "peakon" solution $u(t,x) = c^{\frac{1}{m+1}}\mathrm{e}^{-|x-ct|}, c > 0$ does not satisfy the asymptotic behavior in Theorem 1.2. The following result establishes the optimality of Theorem 1.2 and tells us that a strong non-trivial solution of (1.1) corresponding to data with fast decay at infinity will immediately behave asymptotically, in the $x$-variable at infinity, as the "peakon" solution

$$u(t,x) = c^{\frac{1}{m+1}}\mathrm{e}^{-|x-ct|}, \quad c > 0.$$

**Theorem 1.3** *Assume that* $u_0 \in C([0,T); H^s(\mathbb{R}))$ *with* $s > 3/2$ *satisfies*

$$\begin{aligned} &\big|u_0(x)\big| \sim O\big(\mathrm{e}^{-x}\big), \qquad \big|u_{0x}(x)\big| \sim O\big(\mathrm{e}^{-\theta x}\big) \quad \text{as } x \uparrow \infty, \\ &\Big(\text{respectively, } \big|u_0(x)\big| \sim O\big((1+x)^{-\alpha}\big), \big|u_{0x}(x)\big| \sim O\big((1+x)^{-\beta}\big) \text{ as } x \uparrow \infty\Big) \end{aligned}$$

*for some* $\theta \in (\frac{1}{m+1}, 1)$*(respectively,* $\alpha \geq \frac{1}{m+1}, \beta \in (\frac{\alpha}{m+1}, \alpha)$*), then the corresponding strong solution* $u \in C([0,T); H^s(\mathbb{R}))$ *to Eq.* (1.1) *satisfies for some* $T > 0$

$$\begin{aligned} &\big|u(x,t)\big| \sim O\big(\mathrm{e}^{-x}\big) \quad \text{as } x \uparrow \infty, \\ &\Big(\text{respectively, } \big|u(x)\big| \sim O\big((1+x)^{-\alpha}\big) \text{ as } x \uparrow \infty\Big) \end{aligned}$$

*uniformly in the time interval* $[0,T]$.

*Remark 1.1* The notation means that

$$\left|f(x)\right| \sim O\left(\mathrm{e}^{-\theta x}\right) \quad \text{as } x \uparrow \infty \quad \text{if } \lim_{x\to\infty} \frac{f(x)}{\mathrm{e}^{-\theta x}} = L,$$

where $L$ is a constant (allowed to be zero).

Next, we apply the method of pseudoparabolic regularization to deal with the weak solution of Eq. (1.1). To this goal, we need rewrite Eq. (1.1). For a real number $s$ with $s > 0$, suppose that the function $u_0(x)$ is in $H^s(\mathbb{R})$, and let $u_{\epsilon 0}$ be the convolution $u_{\epsilon 0} = \phi_\epsilon * u_0$ of the function $\phi_\epsilon(x) = \epsilon^{-\frac{1}{4}}\phi(\epsilon^{-\frac{1}{4}}x)$ with $u_0$, where the function $\phi$ is such that the Fourier transform $\widehat{\phi}$ of $\phi$ satisfies $\widehat{\phi} \in C_0^\infty$, $\widehat{\phi(\xi)} \geq 0$ and $\widehat{\phi(\xi)} = 1$ for any $\xi \in (-1, 1)$. Thus we have $u_{\epsilon 0}(x) \in C^\infty$. It follows from Theorem 1.1 that for each $\epsilon$ satisfying $0 < \epsilon < \frac{1}{4}$, the Cauchy problem

$$\begin{cases} u_t - u_{xxt} + (m+3)u^m u_x = u^{m+1}u_{xxx} + (m+2)u^m u_x u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_{\epsilon 0}(x), & x \in \mathbb{R}, \end{cases} \tag{1.6}$$

has a unique solution $u_\epsilon \in C^\infty([0, T_\epsilon), H^\infty(\mathbb{R}))$, in which $T_\epsilon$ may depend on $\epsilon$. However, we shall show that under certain assumptions, there exist two constants $c$ and $T > 0$, both independent of $\epsilon$, such that the solution of problem (1.6) satisfies $\|u_\epsilon^m u_{\epsilon x}\|_{L^\infty(\mathbb{R})} \leq c$ for any $t \in [0, T)$ and exists a weak solution $u(x,t) \in L^2([0,T], H^s(\mathbb{R}))$ for problem (1.6). These results are summarized in the following two theorems.

**Theorem 1.4** *If $u_0(x) \in H^s(\mathbb{R})$ with $s \in [1, \frac{3}{2}]$ such that $\|u_0^m u_{0x}\|_{L^\infty(\mathbb{R})} < \infty$. Let $u_{\epsilon 0}$ be defined as in system (1.6). Then there exist two constants $c$ and $T > 0$, which are independent of $\epsilon$, such that $u_\epsilon$ of problem (1.6) satisfies $\|u_\epsilon^m u_{\epsilon x}\|_{L^\infty(\mathbb{R})} \leq c$ for any $t \in [0, T)$.*

Past the limit $\epsilon \to 0$ in Theorem 1.4, we can obtain the existence of weak solution in the space $L^2([0,T], H^s(\mathbb{R}))$ with $1 < s \leq \frac{3}{2}$ for Eq. (1.1).

**Theorem 1.5** *Suppose that $u_0(x) \in H^s(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ with $1 < s \leq \frac{3}{2}$. Then there exists a life span $T > 0$ such that problem (1.1) has a weak solution $u(x,t) \in L^2([0,T], H^s(\mathbb{R}))$ in the sense of distribution and $u^m u_x \in L^\infty([0,T] \times \mathbb{R})$.*

Finally, we consider global weak solution and peakon solution for problem (1.1).

**Theorem 1.6** *The single peakon takes the form $u(t,x) = c^{\frac{1}{m+1}}\mathrm{e}^{-|x-ct-x_0|}$, $c > 0$. Moveover, this peakon solitary wave is a global weak solution to Eq. (1.1).*

Moreover, we discuss the existence of multi-peakon solutions to Eq. (1.1).

**Theorem 1.7** *Equation (1.1) has peakon solutions of the form*

$$u(t,x)=\sum_{i=1}^{N}p_i(t)\mathrm{e}^{-|x-q_i(t)|}, \tag{1.7}$$

*whose positions $q_t(t)$ and amplitudes $p_j(t)$ are in accordance to the dynamical system*

$$\begin{aligned} p_j' &= \left(\sum_{i=1}^{N} p_i \mathrm{e}^{-|q_j-q_i(t)|}\right)^{m+1}, \\ q_j' &= (b-m-1)q_j\left(\sum_{i=1}^{N} p_i \mathrm{e}^{-|q_j-q_i|}\right)^{m}\left(\sum_{i=1}^{N} p_i \operatorname{sgn}(q_j-q_i)\mathrm{e}^{-|q_j-q_i|}\right). \end{aligned} \tag{1.8}$$

This paper is organized as follows. In the next section, the local well-posedness and persistence properties of strong solutions for the problem (1.1) are established, and Theorems 1.1–1.3 are proved. The existence of weak solutions for the problem (1.1) is proved in Sect. 3. In Sect. 4, we consider the global weak solution and peakon solutions for the problem (1.1), and prove Theorems 1.6–1.7.

## 2 Well-Posedness and Persistence Properties of Strong Solutions

*Notation* The space of all infinitely differentiable functions $f(x,t)$ with compact support in $\mathbb{R}\times[0,+\infty)$ is denoted by $C_0^\infty$. Let $p$ be any constant with $1\le p<\infty$ and denote $L^p=L^p(\mathbb{R})$ the space of all measurable functions $f$ such that $\|f\|_{L^p}^p=\int_{\mathbb{R}}|f(x)|^p\,\mathrm{d}x<\infty$. The space $L^\infty=L^\infty(\mathbb{R})$ with the standard norm $\|f\|_{L^\infty}=\inf_{m(\mathrm{e})=0}\sup_{x\in\mathbb{R}/\mathrm{e}}|f(x)|$. For any real number $s$, let $H^s=H^s(\mathbb{R})$ denote the Sobolev space with the norm defined by

$$\|f\|_{H^s}=\left(\int_{\mathbb{R}}\left(1+|\xi|^2\right)^s\left|\widehat{f}(\xi,t)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}<\infty,$$

where $\widehat{f}(\xi,t)=\int_{\mathbb{R}}\mathrm{e}^{-ix\xi}f(x,t)\,\mathrm{d}x$. Let $C([0,T];H^s(\mathbb{R}))$ denote the class of continuous functions from $[0,T]$ to $H^s$.

*Proof of Theorem 1.1* To prove well-posedness we apply Kato's semigroup approach (Kato 1975). For this, we rewrite the Cauchy problem of Eq. (1.1) as the following transport equation:

$$\begin{cases} u_t+u^{m+1}u_x+F(u)=0, \\ u(x,0)=u_0(x), \end{cases} \tag{2.1}$$

where $F(u):=P*E(u)$. $E(u)=\frac{m(b-m-1)}{2}u^{m-1}(\partial_x u)^3+\partial_x(\frac{b}{m+2}u^{m+2}+\frac{3m+3-b}{2}\times u^m u_x^2)$ and $P(x)=\frac{1}{2}\mathrm{e}^{-|x|}$. Similar to Constantin and Escher (1998), we can choose the notation $A(u) = u^{m+1}\partial_x$, $Y = H^s$, $X = H^{s-1}$ and $Q = \Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. Following closely the considerations made in Constantin and Escher (1998), Mu et al. (2011), Lai and Wu (2010), we obtain the statement of Theorem 1.1. □

*Proof of Theorem 1.2* We introduce the notation $M = \sup_{t\in[0,T]} \|u(t)\|_{H^s}$. For the first step we will give estimates on $\|u(x,t)\|_{L^\infty}$. Integrating both sides with respect to $x$ by multiplying the first equation of (2.1) by $u^{2p-1}$ with $p \in \mathbb{Z}^+$, we can get

$$\int_{\mathbb{R}} u^{2p-1} u_t \,\mathrm{d}x + \int_{\mathbb{R}} u^{2p-1}\big(u^{m+1}u_x\big)\,\mathrm{d}x + \int_{\mathbb{R}} u^{2p-1}\big(P * E(u)\big)\,\mathrm{d}x = 0. \tag{2.2}$$

Note that the estimates

$$\int_{\mathbb{R}} u^{2p-1} u_t \,\mathrm{d}x = \frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\big\|u(x,t)\big\|_{L^{2p}}^{2p} = \big\|u(x,t)\big\|_{L^{2p}}^{2p-1}\frac{\mathrm{d}}{\mathrm{d}t}\big\|u(x,t)\big\|_{L^{2p}},$$

and

$$\left|\int_{\mathbb{R}} u^{2p-1}\big(u^{m+1}u_x\big)\,\mathrm{d}x\right| \le \big\|u^m u_x(x,t)\big\|_{L^\infty}\big\|u(x,t)\big\|_{L^{2p}}^{2p}$$

are true. Moreover, we use Hölder's inequality

$$\left|\int_{\mathbb{R}} u^{2p-1}\big(P * E(u)\big)\,\mathrm{d}x\right| \le \big\|u(x,t)\big\|_{L^{2p}}^{2p-1}\big\|P * E(u)\big\|_{L^{2p}}.$$

From (2.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|u(x,t)\big\|_{L^{2p}} \le \big\|u^m u_x(x,t)\big\|_{L^\infty}\big\|u(x,t)\big\|_{L^{2p}} + \big\|P * E(u)\big\|_{L^{2p}}.$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$. Form the above inequality we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|u(x,t)\big\|_{L^\infty} \le M^{m+1}\big\|u(x,t)\big\|_{L^\infty} + \big\|P * E(u)\big\|_{L^\infty},$$

where we are using

$$\big\|u_x(x,t)\big\|_{L^\infty}\big\|u(x,t)\big\|_{L^\infty}^m \le \big\|u_x(x,t)\big\|_{H^{\frac{1}{2}+}}\big\|u(x,t)\big\|_{H^{\frac{1}{2}+}}^m \le \big\|u(x,t)\big\|_{H^s}^{m+1} \le M^{m+1}.$$

Because of Gronwall's inequality, we get

$$\big\|u(x,t)\big\|_{L^\infty} \le \exp\big(M^{m+1}t\big)\left(\big\|u_0(x)\big\|_{L^\infty} + \int_0^t \big\|\big(P * E(u)\big)(x,\tau)\big\|_{L^\infty}\,\mathrm{d}\tau\right).$$

Next, we will give estimates on $\|u_x(x,t)\|_{L^\infty}$. Differentiating (2.1) with respect to the $x$-variable produces the equation

$$u_{xt} + u^{m+1}u_{xx} + (m+1)u^m u_x^2 + \partial_x\big(P * E(u)\big) = 0. \tag{2.3}$$

Multiplying this equation by $(u_x)^{2p-1}$ with $p \in \mathbb{Z}^+$, integrating the result in the $x$-variable, and using integration by parts:

$$\begin{aligned}
&\int_{\mathbb{R}} (u_x)^{2p-1} u_{xt}\,\mathrm{d}x = \frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\big\|u_x(x,t)\big\|_{L^{2p}}^{2p} = \big\|u_x(x,t)\big\|_{L^{2p}}^{2p-1}\frac{\mathrm{d}}{\mathrm{d}t}\big\|u_x(x,t)\big\|_{L^{2p}},\\
&\left|\int_{\mathbb{R}} (u_x)^{2p-1}\big(u^m u_x^2\big)\,\mathrm{d}x\right| \le \big\|u(x,t)\big\|_{L^\infty}^m \big\|u_x(x,t)\big\|_{L^\infty}\big\|u_x(x,t)\big\|_{L^{2p}}^{2p},\\
&\left|\int_{\mathbb{R}} (u_x)^{2p-1}\big(u^{m+1} u_{xx}\big)\,\mathrm{d}x\right| = \left|\frac{m+1}{2p}\int_{\mathbb{R}} u^m u_x^{2p+1}\,\mathrm{d}x\right|\\
&\qquad\qquad \le \frac{m+1}{2p}\big\|u(x,t)\big\|_{L^\infty}^m \big\|u_x(x,t)\big\|_{L^\infty}\big\|u_x(x,t)\big\|_{L^{2p}}^{2p}.
\end{aligned}$$

From the above inequalities, we also get the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|u_x(x,t)\big\|_{L^{2p}} \le \left(m+1+\frac{m+1}{2p}\right)M^{m+1}\big\|u_x(x,t)\big\|_{L^{2p}} + \big\|\partial_x\big(P * E(u)\big)\big\|_{L^{2p}},$$

where we are using $\|u_x(x,t)\|_{L^\infty}\|u(t)\|_{L^\infty}^m \le M^{m+1}$. Then passing to the limit in this inequality and using Gronwall's inequality one obtains

$$\begin{aligned}
&\big\|u_x(x,t)\big\|_{L^\infty}\\
&\quad\le \exp\big((m+1)M^{m+1}t\big)\left(\big\|u_{0x}(x)\big\|_{L^\infty} + \int_0^t \big\|\partial_x\big(P * E(u)\big)(x,\tau)\big\|_{L^\infty}\,\mathrm{d}\tau\right).
\end{aligned}$$

We shall now repeat the arguments using the weight

$$\varphi_N(x) = \begin{cases} 1, & x \le 0,\\ \mathrm{e}^{\theta x}, & 0 < x < N,\\ \mathrm{e}^{\theta N}, & x \ge N,\end{cases}$$

where $N \in \mathbb{Z}^+$ and $\theta \in (0,1)$. Observe that for all $N$ we have

$$0 \le \varphi_N'(x) \le \varphi_N(x), \quad \text{for all } x \in \mathbb{R}. \tag{2.4}$$

Using the notation $E(u)$, from (2.1) we get

$$\partial_t(u\varphi_N) + \big(u^{m+1}\varphi_N\big)u_x + \varphi_N\big(P * E(u)\big) = 0,$$

and from (2.3), we also obtain

$$\partial_t(\varphi_N\partial_x u) + u^{m+1}\varphi_N\partial_x^2 u + (m+1)u^m(\varphi_N\partial_x u)\partial_x u + \varphi_N\partial_x\big(P * E(u)\big) = 0.$$

We need to eliminate the second derivatives in the second term in the above equality. Thus, combining integration by parts and (2.4) we find

$$\begin{aligned}
&\left|\int_{\mathbb{R}} u^{m+1}\varphi_N \partial_x^2 u(\partial_x u\varphi_N)^{2p-1}\right| \\
&\quad = \left|\int_{\mathbb{R}} u^{m+1}(\partial_x u\varphi_N)^{2p-1}\big(\partial_x(\varphi_N\partial_x u) - \partial_x u\varphi_N'\big)\,\mathrm{d}x\right| \\
&\quad = \left|\int_{\mathbb{R}} \frac{1}{2p}u^{m+1}\partial_x\big((\partial_x u\varphi_N)^{2p}\big) - u^{m+1}(\partial_x u\varphi_N)^{2p-1}\partial_x u\varphi_N'\,\mathrm{d}x\right| \\
&\quad \le \big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\|u\|_{L^\infty}^m\|\partial_x u\varphi_N\|_{L^{2p}}^{2p}.
\end{aligned}$$

Hence, as in the weightless case, we have

$$\begin{aligned}
&\|u\varphi_N\|_{L^\infty} + \|\partial_x u\varphi_N\|_{L^\infty} \\
&\quad \le \exp\big((m+1)M^{m+1}t\big)\big(\big\|u_0(x)\varphi_N\big\|_{L^\infty} + \big\|u_{0x}(x)\varphi_N\big\|_{L^\infty}\big) \\
&\qquad + \exp\big((m+1)M^{m+1}t\big)\int_0^t \big(\big\|\varphi_N\partial_x\big(E(u)\big)\big\|_{L^\infty} + \big\|\varphi_N\big(E(u)\big)\big\|_{L^\infty}\big)\,\mathrm{d}\tau.
\end{aligned}$$

A simple calculation shows that there exists $C>0$, depending only on $\theta\in(0,1)$, such that for any $N\in\mathbb{Z}^+$,

$$\varphi_N\int_{\mathbb{R}}\frac{1}{\varphi_N(y)}\,\mathrm{d}y \le C = \frac{4}{1-\theta}.$$

Thus, we have

$$\begin{aligned}
\big|\varphi_N\big(1-\partial_x^2\big)^{-1}\big(u^{m-1}u_x^3\big)\big| &= \frac{1}{2}\left|\varphi_N\int_{\mathbb{R}} \mathrm{e}^{-|x-y|}\big(u^{m-1}u_x^3\big)(y)\,\mathrm{d}y\right| \\
&= \frac{1}{2}\left|\varphi_N\int_{\mathbb{R}} \mathrm{e}^{-|x-y|}\frac{1}{\varphi_N(y)}(\varphi_N u_x)\big(u^{m-1}u_x^2\big)(y)\,\mathrm{d}y\right| \\
&\le \frac{1}{2}\left(\varphi_N\int_{\mathbb{R}} \mathrm{e}^{-|x-y|}\frac{1}{\varphi_N(y)}\,\mathrm{d}y\right)\|\varphi_N u_x\|_{L^\infty}\big\|u^{m-1}u_x^2\big\|_{L^\infty} \\
&\le c\|\varphi_N u_x\|_{L^\infty}\big\|u^{m-1}u_x^2\big\|_{L^\infty},
\end{aligned}$$

and

$$\begin{aligned}
\big|\varphi_N\big(1-\partial_x^2\big)^{-1}\partial_x\big(u^{m-1}u_x^3\big)\big| &= \frac{1}{2}\left|\varphi_N\int_{\mathbb{R}} \operatorname{sgn}(x-y)\mathrm{e}^{-|x-y|}\big(u^{m-1}u_x^3\big)(y)\,\mathrm{d}y\right| \\
&\le c\|\varphi_N u_x\|_{L^\infty}\big\|u^{m-1}u_x^2\big\|_{L^\infty}.
\end{aligned}$$

Using the same method, we can estimate the other terms:

$$\begin{aligned}
\big|\varphi_N\big(1-\partial_x^2\big)^{-1}\big(u^{m+1}u_x\big)\big| &\le c\|u\|_{L^\infty}^{m+1}\|\varphi_N u_x\|_{L^\infty}, \\
\big|\varphi_N\big(1-\partial_x^2\big)^{-1}\partial_x\big(u^{m+1}u_x\big)\big| &= \big|\varphi_N\big(1-\partial_x^2\big)^{-1}\partial_x^2\big(u^{m+2}\big)\big|
\end{aligned}$$

$$\leq \left|\varphi_N\left(u^{m+2}\right)\right| + \left|\varphi_N\left(1-\partial_x^2\right)^{-1}\left(u^{m+2}\right)\right|$$
$$\leq c\|u\|_{L^\infty}^{m+1}\|\varphi_N u\|_{L^\infty},$$

and

$$\left|\varphi_N\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^m u_x^2\right)\right| \leq c\|\varphi_N u\|_{L^\infty}\left\|u^{m-1}u_x^2\right\|_{L^\infty},$$
$$\left|\varphi_N\left(1-\partial_x^2\right)^{-1}\partial_x^2\left(u^m u_x^2\right)\right| \leq \left|\varphi_N u^m u_x^2\right| + \left|\varphi_N\left(1-\partial_x^2\right)^{-1}\left(u^m u_x^2\right)\right|$$
$$\leq c\|\varphi_N u\|_{L^\infty}\left\|u^{m-1}u_x^2\right\|_{L^\infty},$$

Thus, it follows that there exists a constant $C > 0$ which depends only on $M, m$ and $T$, such that

$$\begin{aligned}
&\|u\varphi_N\|_{L^\infty} + \|\partial_x u\varphi_N\|_{L^\infty} \\
&\quad\leq C\left(\|u_0\varphi_N\|_{L^\infty} + \|u_{0x}\varphi_N\|_{L^\infty}\right) \\
&\qquad + C\int_0^t \left(\left(\|u\|_{L^\infty}^{m+1} + \left\|u^{m-1}u_x^2\right\|_{L^\infty}\right)\left(\|\varphi_N\partial_x u\|_{L^\infty} + \|\varphi_N u\|_{L^\infty}\right)\right)\mathrm{d}\tau \\
&\quad\leq C\left(\|u_0\varphi_N\|_{L^\infty} + \|u_{0x}\varphi_N\|_{L^\infty}\right) + C\int_0^t \left(\|\varphi_N\partial_x u\|_{L^\infty} + \|\varphi_N u\|_{L^\infty}\right)\mathrm{d}\tau.
\end{aligned}$$

Hence, for any $n \in \mathbb{Z}$ and any $t \in [0, T]$ we have

$$\begin{aligned}
\|u\varphi_N\|_{L^\infty} + \|\partial_x u\varphi_N\|_{L^\infty} &\leq C\left(\|u_0\varphi_N\|_{L^\infty} + \|u_{0x}\varphi_N\|_{L^\infty}\right) \\
&\leq C\left(\left\|u_0\max\left\{1, \mathrm{e}^{\theta x}\right\}\right\|_{L^\infty} + \left\|u_{0x}\max\left\{1, \mathrm{e}^{\theta x}\right\}\right\|_{L^\infty}\right).
\end{aligned}$$

Finally, taking the limit as $N$ goes to infinity, we find that for any $t \in [0, T]$,

$$\left\|u\mathrm{e}^{\theta x}\right\|_{L^\infty} + \left\|\partial_x u\mathrm{e}^{\theta x}\right\|_{L^\infty} \leq C\left(\left\|u_0\max\left\{1, \mathrm{e}^{\theta x}\right\}\right\|_{L^\infty} + \left\|u_{0x}\max\left\{1, \mathrm{e}^{\theta x}\right\}\right\|_{L^\infty}\right).$$

By an argument similar to the one used above and the proof of Theorem 1.1 in Ni and Zhou (2012), we get

$$\begin{aligned}
&\left\|u(1+x)^\alpha\right\|_{L^\infty} + \left\|\partial_x u(1+x)^\alpha\right\|_{L^\infty} \\
&\quad\leq C\left(\left\|u_0\max\left\{1, (1+x)^\alpha\right\}\right\|_{L^\infty} + \left\|u_{0x}\max\left\{1, (1+x)^\alpha\right\}\right\|_{L^\infty}\right),
\end{aligned}$$

which completes the proof of Theorem 1.2. □

Next, we give a simple proof for Theorem 1.3.

*Proof of Theorem 1.3* We use Theorem 1.2 to prove this theorem.

For any $t_1 \in [0, T]$, integrating Eq. (2.1) over the time interval $[0, t_1]$ we get

$$u(x, t_1) - u(x, 0) + \int_0^{t_1}\left(u^{m+1}u_x\right)(x, \tau)\,\mathrm{d}\tau + \int_0^{t_1}\left(P * E(u)\right)(x, \tau)\,\mathrm{d}\tau = 0. \quad (2.5)$$

From Theorem 1.2 it follows that

$$\int_0^{t_1} \big(u^{m+1}u_x\big)(x,\tau)\,\mathrm{d}\tau \sim O\big(\mathrm{e}^{-(m+2)\theta x}\big) \quad \text{as } x \uparrow \infty,$$

and so

$$\int_0^{t_1} \big(u^{m+1}u_x\big)(x,\tau)\,\mathrm{d}\tau \sim O\big(\mathrm{e}^{-x}\big) \quad \text{as } x \uparrow \infty.$$

We shall show that the last term in (2.5) is $O(\mathrm{e}^{-x})$; thus we have

$$\int_0^{t_1} \big(P * E(u)\big)(x,\tau)\,\mathrm{d}\tau = P(x) * \int_0^{t_1} \big(E(u)\big)(x,\tau)\,\mathrm{d}\tau \doteq P(x) * \rho(x).$$

From the given condition and Theorem 1.2. we know $\rho(x) \sim O(\mathrm{e}^{-x})$ as $x \uparrow \infty$. Since

$$P(x) * \rho(x) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-|x-y|}\rho(y)\,\mathrm{d}y = \frac{1}{2}\mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{y}\rho(y)\,\mathrm{d}y + \frac{1}{2}\mathrm{e}^{x}\int_{x}^{\infty} \mathrm{e}^{-y}\rho(y)\,\mathrm{d}y,$$

we have

$$\mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{y}\rho(y)\,\mathrm{d}y = O(1)\mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{2y}\,\mathrm{d}y \sim O(1)\mathrm{e}^{-x} \sim O\big(\mathrm{e}^{-x}\big) \quad \text{as } x \uparrow \infty,$$

$$\mathrm{e}^{x}\int_{x}^{\infty} \mathrm{e}^{-y}\rho(y)\,\mathrm{d}y = O(1)\mathrm{e}^{x}\int_{x}^{\infty} \mathrm{e}^{-2y}\,\mathrm{d}y \sim O(1)\mathrm{e}^{-x} \sim O\big(\mathrm{e}^{-x}\big) \quad \text{as } x \uparrow \infty.$$

Thus

$$\int_0^{t_1} \big(P * E(u)\big)(x,\tau)\,\mathrm{d}\tau \sim O\big(\mathrm{e}^{-x}\big) \quad \text{as } x \uparrow \infty.$$

From (2.5) and $|u_0(x)| \sim O(\mathrm{e}^{-x})$ as $x \uparrow \infty$, we know

$$\big|u(x,t_1)\big| \sim O\big(\mathrm{e}^{-x}\big) \quad \text{as } x \uparrow \infty.$$

By the arbitrariness of $t_1 \in [0,T]$, we get

$$\big|u(x,t)\big| \sim O\big(\mathrm{e}^{-x}\big) \quad \text{as } x \uparrow \infty$$

uniformly in the time interval $[0,T]$.

By an argument similar to the one used above and the proof of Theorem 1.2 in Ni and Zhou (2012), we get

$$\big|u(x,t)\big| \sim O\big((1+x)^{-\alpha}\big) \quad \text{as } x \uparrow \infty$$

uniformly in the time interval $[0,T]$. This completes the proof of Theorem 1.3. □

## 3 Existence of Weak Solutions

In order to establish the proofs of Theorems 1.4 and 1.5, we give several lemmas.

**Lemma 3.1** (See Kato 1975) *If $r > 0$, then $H^r \cap L^\infty$ is an algebra, and*

$$\|fg\|_{H^r} \le c\big(\|f\|_{L^\infty}\|g\|_{H^r} + \|g\|_{L^\infty}\|h\|_{H^r}\big),$$

*where c is a constant depending only on r.*

**Lemma 3.2** (See Kato 1975) *If $r > 0$, then*

$$\big\|\big[\Lambda^r, f\big]g\big\|_{L^2} \le c\big(\|\partial_x f\|_{L^\infty}\|\Lambda^{r-1}g\|_{L^2} + \big\|\Lambda^r f\big\|_{L^2}\|g\|_{L^\infty}\big),$$

*where $[\Lambda^r, f]g = \Lambda^r(fg) - f\Lambda^r g$ with $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$, and c is a constant depending only on r.*

**Lemma 3.3** *For $s \ge 1$ and $f(x) \in H^s$ and letting $k_1 > 0$ be an integer such that $k_1 \le s-1$, then $f, f', \ldots, f^{k_1}$ are bounded uniformly continuous functions which converge to 0 at $x = \pm\infty$.*

*Proof* This proof was stated by Bona and Smith (1975, p. 559). □

Now for $s \ge 2$, multiplying Eq. (1.1) by $u$, we have

$$uu_t - uu_{txx} = -(b+1)u^{m+2}u_x + bu^{m+1}u_x u_{xx} + u^{m+2}u_{xxx}. \tag{3.1}$$

Integrating by parts on $\mathbb{R}$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big(u^2+u_x^2\big)\,\mathrm{d}x &= (b-m+2)\int_{\mathbb{R}}\big(u^m u_x u_{xx}\big)\,\mathrm{d}x \\ &= \frac{(b-m+2)(m+1)}{2}\int_{\mathbb{R}} u^m u_x^3\,\mathrm{d}x,\end{aligned}$$

from which we have

$$\int_{\mathbb{R}}\big(u^2+u_x^2\big)\,\mathrm{d}x = (b-m+2)(m+1)\int_0^t\bigg[\int_{\mathbb{R}} u^m u_x^3\,\mathrm{d}x\bigg]\mathrm{d}\tau + \int_{\mathbb{R}}\big(u_0^2+u_{0x}^2\big)\,\mathrm{d}x. \tag{3.2}$$

**Lemma 3.4** *Let $s \ge 4$ and let the function $u(x,t)$ be a solution of the problem (1.1) and the initial data $u_0(x) \in H^s$, then we have*

$$2\pi\|u\|_{H^1}^2 \le \int_{\mathbb{R}}\big(u_0^2+u_{0x}^2\big)\,\mathrm{d}x + (m+1)|m+2-b|\int_0^t\big\|u^m u_x\big\|_{L^\infty}\|u\|_{H^1}^2\,\mathrm{d}\tau. \tag{3.3}$$

*For* $q\in(0,s-1]$, *there is a constant* $c$ *depending only on* $q$ *such that*

$$\int_{\mathbb{R}}\left(\Lambda^{q+1}u\right)^2\mathrm{d}x\le\int_{\mathbb{R}}\left(\Lambda^{q+1}u_0\right)^2\mathrm{d}x+c\int_0^t\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\left(\|u\|_{H^q}^2+\|u\|_{H^{q+1}}^2\right)+\|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}^3\|u\|_{H^q}^2\,\mathrm{d}\tau. \tag{3.4}$$

*If* $q\in[0,s-1]$, *there is a constant* $c$ *depending only on* $q$ *such that*

$$\|u_t\|_{H^q}\le c\|u\|_{H^1}^{m+1}\|u\|_{H^{q+1}}. \tag{3.5}$$

*Proof* Using $2\pi\|u\|_{H^1}^2\le\int_{\mathbb{R}}(u^2+u_x^2)\,\mathrm{d}x$ and (3.2), we deduce (3.3).

We write Eq. (1.1) in the equivalent form

$$u_t-u_{xxt}=-\frac{b+1}{m+2}\left(u^{m+2}\right)_x+\frac{1}{m+2}\partial_x^3u^{m+2}+\frac{b-3(m+1)}{2}\partial_x\left(u^mu_x^2\right)-\frac{m[b-(m+1)]}{2}u^{m-1}u_x^3. \tag{3.6}$$

Since $\partial_x^2=-\Lambda^2+1$, the Parseval equality gives rise to

$$\int_{\mathbb{R}}\left(\Lambda^qu\right)\Lambda^q\partial_x^3u^{m+2}\,\mathrm{d}x=-(m+2)\int_{\mathbb{R}}\left(\Lambda^{q+1}u\right)\Lambda^{q+1}\left(u^{m+1}u_x\right)\mathrm{d}x+(m+2)\int_{\mathbb{R}}\left(\Lambda^qu\right)\Lambda^q\left(u^{m+1}u_x\right)\mathrm{d}x.$$

For any $q\in(0,s-1]$, applying $(\Lambda^qu)\Lambda^q$ to both sides for Eq. (3.6), respectively, and integrating with respect to $x$ again,using integration by parts, one obtains

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\left(\Lambda^qu\right)^2+\left(\Lambda^qu_x\right)^2\right)\mathrm{d}x
&=-b\int_{\mathbb{R}}\left(\Lambda^qu\right)\Lambda^q\left(u^{m+1}u_x\right)\mathrm{d}x-\int_{\mathbb{R}}\left(\Lambda^{q+1}u\right)\Lambda^{q+1}\left(u^{m+1}u_x\right)\mathrm{d}x\\
&\quad-\frac{b-3(m+1)}{2}\int_{\mathbb{R}}\left(\Lambda^qu_x\right)\Lambda^q\left(u^mu_x^2\right)\mathrm{d}x\\
&\quad-\frac{m[b-(m+1)]}{2}\int_{\mathbb{R}}\left(\Lambda^qu\right)\Lambda^q\left(u^{m-1}u_x^3\right)\mathrm{d}x.\end{aligned} \tag{3.7}$$

We will estimate the terms on the right-hand side of (3.7) separately. For the first term, by using the Cauchy–Schwartz inequality and Lemmas 3.1 and 3.2; we have

$$\begin{aligned}&\int_{\mathbb{R}}\left(\Lambda^qu\right)\Lambda^q\left(u^{m+1}u_x\right)\mathrm{d}x\\
&\quad=\int_{\mathbb{R}}\left(\Lambda^qu\right)\left[\Lambda^q\left(u^{m+1}u_x\right)-u^{m+1}\Lambda^qu_x\right]\mathrm{d}x+\int_{\mathbb{R}}\left(\Lambda^qu\right)u^{m+1}\Lambda^qu_x\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}&\le c\|u\|_{H^q}\big((m+1)\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\|u\|_{H^q}+\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\|u\|_{H^q}\big)\\&\quad+\frac{m+1}{2}\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\big\|\Lambda^q u\big\|_{L^2}^2\\&\le c\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\|u\|_{H^q}^2.\end{aligned}$$

Using the above estimate or the second term on the right-hand side of (3.7) yields

$$\int_{\mathbb{R}}\big(\Lambda^{q+1}u\big)\Lambda^{q+1}\big(u^{m+1}u_x\big)\,\mathrm{d}x=c\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\|u\|_{H^{q+1}}^2.$$

For the third term on the right-hand side of (3.7), using the Cauchy–Schwartz inequality, and Lemma 3.1, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\big(\Lambda^q u_x\big)\Lambda^q\big(u^m u_x^2\big)\,\mathrm{d}x&\le\big\|\Lambda^q u_x\big\|_{L^2}\big\|\Lambda^q\big(u^m u_x^2\big)\big\|_{L^2}\\&\le c\|u\|_{H^{q+1}}\big(\big\|u^m u_x\big\|_{L^\infty}\|u_x\|_{H^q}+\|u_x\|_{L^\infty}\big\|u^m u_x\big\|_{H^q}\big)\\&\le c\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\|u\|_{H^{q+1}}^2.\end{aligned}$$

For the last term on the right-hand side of (3.7), using Lemma 3.1 repeatedly results in

$$\begin{aligned}&\int_{\mathbb{R}}\big(\Lambda^q u\big)\Lambda^q\big(u^{m-1}u_x^3\big)\,\mathrm{d}x\\&\quad=\int_{\mathbb{R}}\big(\Lambda^q u\big)\big[\Lambda^q\big(\big(u^{m-1}u_x^3\big)\big)-u^{m-1}\Lambda^q u_x^3\big]\,\mathrm{d}x+\int_{\mathbb{R}}\big(\Lambda^q u\big)u^{m-1}\Lambda^q u_x^3\,\mathrm{d}x\\&\quad\le c\|u\|_{H^q}\big((m-1)\|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}\|u\|_{H^q}^3+\|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}^3\|u\|_{H^q}\big)\\&\qquad+\frac{m-1}{2}\|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}^3\big\|\Lambda^q u\big\|_{L^2}^2\\&\quad\le c\|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}^3\|u\|_{H^q}^2.\end{aligned}$$

By the above inequalities, it follows from (3.7) that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}\big(\big(\Lambda^q u\big)^2+\big(\Lambda^q u_x\big)^2\big)\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}}\big(\big(\Lambda^q u_0\big)^2+\big(\Lambda^q u_{0x}\big)^2\big)\,\mathrm{d}x\\&\quad\le c\int_0^t\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\big(\|u\|_{H^q}^2+\|u\|_{H^{q+1}}^2\big)+\|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}^3\|u\|_{H^q}^2\,\mathrm{d}\tau.\end{aligned}\tag{3.8}$$

Thus, we get (3.4).

Applying the operator $(1-\partial_x^2)^{-1}$ by multiplying both sides of (3.6) yields the equation

$$\begin{aligned}u_t+u^{m+1}u_x=&-\frac{m(b-m-1)}{2}\big(1-\partial_x^2\big)^{-1}u^{m-1}(\partial_x u)^3\\&-\big(1-\partial_x^2\big)^{-1}\partial_x\bigg(\frac{b}{m+2}u^{m+2}+\frac{3m+3-b}{2}u^m(\partial_x u)^2\bigg).\end{aligned}\tag{3.9}$$

Multiplying both sides of (3.9) by $(\Lambda^q u_t)\Lambda^q$ for $q \in [0, s-1]$ and integrating the resultant equation by parts give rise to

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\Lambda^q u_t\right)^2 \mathrm{d}x + \int_{\mathbb{R}} \left(\Lambda^q u_t\right)\Lambda^q \left(u^{m+1} u_x\right) \mathrm{d}x \\
&\quad = -\frac{m(b-m-1)}{2} \int_{\mathbb{R}} \left(\Lambda^q u_t\right)\left(1-\partial_x^2\right)^{-1} \Lambda^q \left(u^{m-1}(\partial_x u)^3\right) \mathrm{d}x \\
&\qquad - \int_{\mathbb{R}} \left(\Lambda^q u_t\right)\left(1-\partial_x^2\right)^{-1} \Lambda^q \partial_x \left(\frac{b}{m+2} u^{m+2} + \frac{3m+3-b}{2} u^m (\partial_x u)^2\right). \qquad (3.10)
\end{aligned}$$

Using $\|u^{m+1}u_x\|_{H^q} \le c\|u^{m+2}\|_{H^{q+1}} \le c\|u\|_{L^\infty}^{m+1}\|u\|_{H^{q+1}} \le c\|u\|_{H^1}^{m+1}\|u\|_{H^{q+1}}$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \left(\Lambda^q u_t\right)\Lambda^q \left(u^{m+1} u_x\right) \mathrm{d}x &\le \left\|\Lambda^q u_t\right\|_{L^2} \left\|\Lambda^q \left(u^{m+1} u_x\right)\right\|_{L^2} \le c\|u_t\|_{H^q} \left\|u^{m+1} u_x\right\|_{H^q} \\
&\le c\|u_t\|_{H^q}\|u\|_{H^1}^{m+1}\|u\|_{H^{q+1}}.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\Lambda^q u_t\right)\left(1-\partial_x^2\right)^{-1} \Lambda^q \left(u^{m-1}(\partial_x u)^3\right) \mathrm{d}x \\
&\quad \le \left\|\Lambda^q u_t\right\|_{L^2} \left(\int_{\mathbb{R}} \left(1+\xi^2\right)^q \left(\int_{\mathbb{R}} \widehat{u^{m-1}u_x}(\xi-\eta)\widehat{u_x}^2(\eta)\,\mathrm{d}\eta\right)^2\right)^{\frac{1}{2}},
\end{aligned}$$

it follows from Young's inequality ($\|f \star g\|_r \le \|f\|_p\|g\|_q$, $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$) and from the inequality $(1+\xi^2)^l \le c[(1+(\xi-\eta)^2)^l + (1+\eta^2)^l]$, $l>0$, that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\Lambda^q u_t\right)\left(1-\partial_x^2\right)^{-1} \Lambda^q \left(u^{m-1}(\partial_x u)^3\right) \mathrm{d}x \\
&\quad \le \left\|\Lambda^q u_t\right\|_{L^2} \left(\int_{\mathbb{R}} c\left(\int_{\mathbb{R}} \left[\left(1+(\xi-\eta)^2\right)^{\frac{q}{2}} + \left(1+\eta^2\right)^{\frac{q}{2}}\right] \right.\right. \\
&\qquad \left.\left. \times \widehat{u^{m-1}u_x}(\xi-\eta)\widehat{u_x}^2(\eta)\,\mathrm{d}\eta\right)^2\right)^{\frac{1}{2}} \\
&\quad \le c\left\|\Lambda^q u_t\right\|_{L^2}\left(\left\|\Lambda^q \widehat{u^{m-1}} u_x \star \widehat{u_x}^2\right\|_{L^2} + \left\|\widehat{u^{m-1}u_x} \star \widehat{\Lambda^{q-1}u_x^2}\right\|_{L^2}\right) \\
&\quad \le c\left\|\Lambda^q u_t\right\|_{L^2}\left(\left\|\Lambda^q \widehat{u^{m-1}} u_x\right\|_{L^2}\left\|\widehat{u_x}^2\right\|_{L^1} + \left\|\widehat{u^{m-1}u_x}\right\|_{L^2}\left\|\widehat{\Lambda^{q-1}u_x^2}\right\|_{L^1}\right) \\
&\quad \le c\left\|\Lambda^q u_t\right\|_{L^2}\left(\left\|u^{m-1}u_x\right\|_{H^q}\|u_x\|_{L^2} + \left\|u^{m-1}u_x\right\|_{L^2}\|u_x\|_{H^{q-1}}\right) \\
&\quad \le c\|u_t\|_{H^q}\|u\|_{L^\infty}^{m-1}\|u\|_{H^1}^2\|u\|_{H^{q+1}}.
\end{aligned}$$

On the right-hand side of (3.10), we have

$$\begin{aligned}
&\int_{\mathbb{R}} (\Lambda^q u_t)(1-\partial_x^2)^{-1}\Lambda^q\partial_x\left(\frac{b}{m+2}u^{m+2}+\frac{3m+3-b}{2}u^m(\partial_x u)^2\right)\\
&\quad\le c\|u_t\|_{H^q}\left(\int_{\mathbb{R}}(1+\xi^2)^{q-1}\,\mathrm{d}\xi\right.\\
&\qquad\left.\times\left(\int_{\mathbb{R}}\left[\widehat{u^{m+1}}(\xi-\eta)\widehat{u}(\eta)+\widehat{u^m u_x}(\xi-\eta)u_x(\eta)\right]\mathrm{d}\eta\right)^2\right)^{\frac{1}{2}}\\
&\quad\le c\|u_t\|_{H^q}\|u\|_{L^\infty}^m\|u\|_{H^1}\|u\|_{H^{q+1}}.
\end{aligned}$$

Applying the Sobolev inequality $\|u\|_{L^\infty}\le\|u\|_{H^1}$ to the above three estimates, then, from (3.10) we find the inequality

$$\|u_t\|_{H^q}\le c\|u\|_{H^1}^{m+1}\|u\|_{H^{q+1}}$$

for a constant $c>0$. This completes the proof of Lemma 3.4. □

For an arbitrary positive Sobolev exponent $s>0$, we give the following lemma.

**Lemma 3.5** *For $u_0\in H^s$ with $s>0$ and $u_{\epsilon 0}=\phi_\epsilon\star u_0$, the following estimates hold for any $\epsilon$ with $0<\epsilon<\frac{1}{4}$,*

$$\|u_{\epsilon 0x}\|_{L^\infty}\le c\|u_{0x}\|_{L^\infty}\quad \text{and}\quad \|u_{\epsilon 0}\|_{H^q}\le c,\quad \text{if } q\le s, \tag{3.11}$$

$$\|u_{\epsilon 0}\|_{H^q}\le c\epsilon^{\frac{s-q}{4}},\quad \text{if } q>s, \tag{3.12}$$

$$\|u_{\epsilon 0}-u_0\|_{H^q}\le c\epsilon^{\frac{s-q}{4}},\quad \text{if } q\le s, \tag{3.13}$$

$$\|u_{\epsilon 0}-u_0\|_{H^s}=o(1), \tag{3.14}$$

*where $c$ is a constant independent of $\epsilon$.*

*Proof* This proof is similar to that of Lemma 5 in Bona and Smith (1975) and Lemma 4.5 in Lai and Wu (2011), so we omit it. □

**Lemma 3.6** *For $s\ge 1$ and $u_0\in H^s$, there exists a constant $c$ independent of $\epsilon$, such that the solution $u_\epsilon$ of problem (1.6) satisfies*

$$\|u_\epsilon\|_{H^1}\le c\exp\left\{(m+1)|m+2-b|\int_0^t\left\|u_\epsilon^m u_{\epsilon x}\right\|_{L^\infty}\mathrm{d}\tau\right\}\quad \text{for } t\in[0,T_\epsilon). \tag{3.15}$$

*Proof* Using $u_0\in H^s$, we know the $u_{\epsilon 0}\in C^\infty$. It follows from Theorem 1.1 that $u_\epsilon(x,t)\in C^\infty([0,T_\epsilon),H^\infty)$. Thus, all the assumptions in Lemma 3.4 are valid. From

(3.3) and (3.11), we get

$$\begin{aligned}
\|u_\epsilon\|_{H^1}^2 &\le \int_{\mathbb{R}} \left(u_\epsilon^2 + u_{\epsilon x}^2\right) \mathrm{d}x \\
&= \int_{\mathbb{R}} \left(u_{\epsilon 0}^2 + u_{\epsilon 0x}^2\right) \mathrm{d}x + (m+1)|m+2-b| \int_0^t \left\|u_\epsilon^m u_{\epsilon x}\right\|_{L^\infty} \|u_\epsilon\|_{H^1}^2 \,\mathrm{d}\tau \\
&\le \|u_{\epsilon 0}\|_{H^1}^2 + (m+1)|m+2-b| \int_0^t \left\|u_\epsilon^m u_{\epsilon x}\right\|_{L^\infty} \|u_\epsilon\|_{H^1}^2 \,\mathrm{d}\tau \\
&\le c + (m+1)|m+2-b| \int_0^t \left\|u_\epsilon^m u_{\epsilon x}\right\|_{L^\infty} \|u_\epsilon\|_{H^1}^2 \,\mathrm{d}\tau.
\end{aligned}$$

Using Gronwall's inequality, we can obtain the inequality (3.15), which finishes the proof of Lemma 3.6. □

*Proof of Theorem 1.4* Using the notation $u = u_\epsilon$ and differentiating (3.9) with respect to $x$ give rise to

$$u_{xt} - \frac{b}{m+2} u^{m+2} - \frac{3m+1-b}{2} u^m (\partial_x u)^2 + u^{m+1} u_{xx} = G \tag{3.16}$$

with $G = -\frac{m(b-m-1)}{2} \Lambda^{-2} \partial_x (u^{m-1} u_x^3) - \Lambda^{-2}(\frac{b}{m+2} u^{m+2} + \frac{3m+3-b}{2} u^m u_x^2)$.

Letting $p > 0$ be an integer and multiplying (3.16) by $(u^m u_x)^{2p+1}$, then integrating the resulting equation with respect to $x$, and using

$$\int_{\mathbb{R}} u^{m+1} u_{xx} \left(u^m u_x\right)^{2p+1} \mathrm{d}x = -\frac{m+1+m(2p+1)}{2p+2} \int_{\mathbb{R}} u^{m(2p+2)} (u_x)^{2p+3} \,\mathrm{d}x,$$

yield the equality

$$\begin{aligned}
&\frac{1}{2p+2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \left(u^m u_x\right)^{2p+2} \mathrm{d}x - \frac{b}{m+2} \int_{\mathbb{R}} u^{m+2} \left(u^m u_x\right)^{2p+1} \mathrm{d}x \\
&\quad = \int_{\mathbb{R}} G \cdot \left(u^m u_x\right)^{2p+1} \mathrm{d}x \\
&\qquad + \left(\frac{m+1+m(2p+1)}{2p+2} + \frac{3m+1-b}{2}\right) \int_{\mathbb{R}} u^{m(2p+2)} (u_x)^{2p+3} \,\mathrm{d}x.
\end{aligned}$$

Applying the Hölder inequality, we get

$$\begin{aligned}
&\frac{1}{2p+2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \left(u^m u_x\right)^{2p+2} \mathrm{d}x \\
&\quad \le \left| \frac{m+1}{2p+2} + \frac{3m+1-b}{2} \right| \left\|u^m u_x\right\|_{L^\infty} \int_{\mathbb{R}} \left|u^m u_x\right|^{2p+2} \mathrm{d}x
\end{aligned}$$

$$+\left\{\frac{b}{m+2}\left(\int_{\mathbb{R}}\left|u^{m+2}\right|^{2p+2}\mathrm{d}x\right)^{\frac{1}{2p+2}}+\left(\int_{\mathbb{R}}|G|^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}}\right\}$$

$$\times\left(\int_{\mathbb{R}}\left|u^m u_x\right|^{2p+2}\mathrm{d}x\right)^{\frac{2p+1}{2p+2}},$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}}(u^m u_x)^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}}$$

$$\le\left\{\frac{b}{m+2}\left(\int_{\mathbb{R}}\left|u^{m+2}\right|^{2p+2}\mathrm{d}x\right)^{\frac{1}{2p+2}}+\left(\int_{\mathbb{R}}|G|^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}}\right\}$$

$$+\left|\frac{m+1}{2p+2}+\frac{3m+1-b}{2}\right|\left\|u^m u_x\right\|_{L^\infty}\left(\int_{\mathbb{R}}\left|u^m u_x\right|^{2p+2}\mathrm{d}x\right)^{\frac{1}{2p+2}}.$$

Since $\|f\|_{L^p}\to\|f\|_{L^\infty}$ as $p\to\infty$ for any $f\in L^\infty\cap L^2$, integrating the above inequality with respect to $t$ and taking the limit as $p\to\infty$ result in the estimate

$$\left\|u^m u_x\right\|_{L^\infty}\le\left\|u_0^m u_{0x}\right\|_{L^\infty}+c\int_0^t\left(\left\|u^{m+2}\right\|_{L^\infty}+\|G\|_{L^\infty}+\left\|u^m u_x\right\|_{L^\infty}^2\right)\mathrm{d}\tau. \quad (3.17)$$

Using the algebraic property of $H^s$ with $s>\frac{1}{2}$ and Lemma 3.6 leads to

$$\left\|u^{m+2}\right\|_{L^\infty}\le c\left\|u^{m+2}\right\|_{H^{\frac{1}{2}+}}\le c\|u\|_{H^1}^{m+2}$$

$$\le c\exp\left\{(m+2)(m+1)|m+2-b|\int_0^t\left\|u^m u_x\right\|_{L^\infty}\mathrm{d}\tau\right\}, \quad (3.18)$$

and

$$\|G\|_{L^\infty}=\left\|\frac{m(b-m-1)}{2}\Lambda^{-2}\partial_x\left(u^{m-1}u_x^3\right)\right.$$

$$\left.+\Lambda^{-2}\left(\frac{b}{m+2}u^{m+2}+\frac{3m+3-b}{2}u^m u_x^2\right)\right\|_{L^\infty}$$

$$\le c\left(\left\|\Lambda^{-2}\partial_x\left(u^{m-1}u_x^3\right)\right\|_{H^{\frac{1}{2}+}}+\left\|\Lambda^{-2}u^{m+2}\right\|_{H^{\frac{1}{2}+}}+\left\|\Lambda^{-2}u^m u_x^2\right\|_{H^{\frac{1}{2}+}}\right)$$

$$\le c\left(\left\|u^{m-1}u_x^3\right\|_{H^0}+\left\|u^{m+2}\right\|_{H^0}+\left\|u^m u_x^2\right\|_{H^0}\right)$$

$$\le c\left(\left\|u^{m-1}u_x^2\right\|_{L^\infty}\|u\|_{H^1}+\|u\|_{H^1}^{m+2}+\left\|u^m u_x\right\|_{L^\infty}\|u\|_{H^1}\right).$$

Using Lemma 3.6 again

$$\begin{aligned}
&\int_0^t \|G\|_{L^\infty}\,\mathrm{d}\tau \\
&\quad \le c\int_0^t \big(\|u^m u_x\|^2_{L^\infty}\|u\|_{H^1} + \|u\|^{m+2}_{H^1} + \|u^m u_x\|_{L^\infty}\|u\|_{H^1}\big)\,\mathrm{d}\tau \\
&\quad \le c\int_0^t \left\{\big(\|u^m u_x\|^2_{L^\infty} + 1 + \|u^m u_x\|_{L^\infty}\big)\exp\left(c\int_0^\tau \|u^m u_x\|_{L^\infty}\,\mathrm{d}\xi\right)\right\}\mathrm{d}\tau, \quad (3.19)
\end{aligned}$$

where $c$ is independent of $\epsilon$. Applying (3.11), (3.17)–(3.19) and writing out the subscript $\epsilon$ of $u$, we obtain

$$\begin{aligned}
\|u_\epsilon^m u_{\epsilon x}\|_{L^\infty} \le \|u_0^m u_{0x}\|_{L^\infty} & \\
&+ c\int_0^t \Big\{\big(\|u_\epsilon^m u_{\epsilon x}\|^2_{L^\infty} + 2 + \|u_\epsilon^m u_{\epsilon x}\|_{L^\infty}\big) \\
&\times \exp\left(c\int_0^\tau \|u_\epsilon^m u_{\epsilon x}\|_{L^\infty}\,\mathrm{d}\xi\right) + \|u_\epsilon^m u_{\epsilon x}\|^2_{L^\infty}\Big\}\,\mathrm{d}\tau.
\end{aligned}$$

It follows from the contraction mapping principle that there is a $T > 0$ such that the equation

$$\begin{aligned}
\|W\|_{L^\infty} &= \|u_0^m u_{0x}\|_{L^\infty} \\
&\quad + c\int_0^t \left\{\big(\|W\|^2_{L^\infty} + 2 + \|W\|_{L^\infty}\big)\exp\left(c\int_0^\tau \|W\|_{L^\infty}\,\mathrm{d}\xi\right) + \|W\|^2_{L^\infty}\right\}\mathrm{d}\tau
\end{aligned}$$

has a unique solution $W \in C[0, T]$, and from the above inequality, we know that the variable $T$ only depends on $c$ and $\|u_0^m u_{0x}\|_{L^\infty}$. Using the theorem on p. 51 in Li and Olver (2000) or Theorem II in Sect. 1.1 in Walter (1970) one derives that there are constants $T > 0$ and $c > 0$ independent of $\epsilon$ such that $\|u_\epsilon^m u_{\epsilon x}\|_{L^\infty} \le W(t)$ for arbitrary $t \in [0, T]$, which leads to the conclusion of Theorem 1.4. □

*Remark 3.1* Under the assumptions of Theorem 1.4, there exist two constants $T$ and $c$, both independent of $\epsilon$, such that the solution $u_\epsilon$ of problem (1.6) satisfies $u_\epsilon^m u_{\epsilon x} \le c$ for any $t \in [0, T]$. This states that, in Lemma 3.6, there exists a $T$ independent of $\epsilon$ such that (3.15) holds.

Using Theorem 1.4, Lemma 3.6, (3.4), (3.5), the notation $u_\epsilon = u$ and Gronwall's inequality results in the inequalities

$$\|u_\epsilon\|_{H^q} \le c\exp\left\{c\int_0^t \|u^m u_x\|_{L^\infty}\,\mathrm{d}\tau\right\} \le c,$$

and

$$\|u_{\epsilon t}\|_{H^r} \le \|u_\epsilon\|_{H^{r+1}}\|u_\epsilon\|^{m+1}_{H^1} \le c,$$

where $q \in (0, s], r \in (0, s-1]$ and $t \in [0, T)$. It follows from Aubin's compactness theorem that there is a subsequence of $\{u_\epsilon\}$, denoted by $\{u_{\epsilon_n}\}$, such that $\{u_{\epsilon_n}\}$ and their temporal derivatives $\{u_{\epsilon_n t}\}$ are weakly convergent to a function $u(x,t)$ and its derivative $u_t$ in $L^2([0,T], H^s)$ and $L^2([0,T], H^{s-1})$, respectively. Moreover, for any real number $R_1 > 0$, $\{u_{\epsilon_n}\}$ is strongly convergent to the function $u$ in the space $L^2([0,T], H^q(-R_1, R_1))$ for $q \in (0, s]$ and $\{u_{\epsilon_n t}\}$ strongly converges to $u_t$ in the space $L^2([0,T], H^r(-R_1, R_1))$ for $r \in (0, s-1]$.

*Proof of Theorem 1.5* From Theorem 1.4, we know that $\{u^m_{\epsilon_n} u_{\epsilon_n x}\}(\epsilon_n \to 0)$ is bounded in the space $L^\infty$. Thus, the sequences $u_{\epsilon_n}$, $u_{\epsilon_n x}$, $u^2_{\epsilon_n}$, and $u^3_{\epsilon_n}$ are weakly convergent to $u, u_x, u_x^2$, and $u_x^3$, respectively, in the space $L^2([0,T], H^r(-R_1, R_1))$ for any $r \in (0, s-1]$. Hence, $u$ satisfies the equation

$$
\begin{aligned}
&-\int_0^T \int_{\mathbb{R}} u(g_t - g_{xxt})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_0^T \int_{\mathbb{R}} \left[ \left( \frac{b+1}{m+2} u^{m+2} - \frac{b-3(m+1)}{2} u^m u_x^2 \right) g_x \right. \\
&\qquad \left. - \frac{1}{m+2} u^{m+2} g_{xxx} + \frac{m[b-(m+1)]}{2} u^{m-1} u_x^3 g \right] \mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

with $u(x,0) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0,T] \times \mathbb{R})$ is a separable Banach space and $u^m_{\epsilon_n} u_{\epsilon_n x}$ is a bounded sequence in the dual space $X^* = L^\infty([0,T] \times \mathbb{R})$ of $X$, there exists a subsequence of $u^m_{\epsilon_n} u_{\epsilon_n x}$, still denoted by $u^m_{\epsilon_n} u_{\epsilon_n x}$, weakly star convergent to a function $v$ in $L^\infty([0,T] \times \mathbb{R})$. As $u^m_{\epsilon_n} u_{\epsilon_n x}$ weakly converges to $u^m u_x$ in $L^2([0,T] \times \mathbb{R})$, we have the result that $u^m u_x = v$ almost everywhere. Thus, we obtain $u^m u_x \in L^\infty([0,T] \times \mathbb{R})$. □

## 4 Global Weak Solution and Peakon Solution

The main purpose of this section is to show that there exists a unique global weak solution to the problem (1.1) provided that the initial data $y_0$ satisfies certain sign conditions. In fact, the problem (1.1) can be rewritten as

$$
\begin{cases}
u_t + u^{m+1} u_x + F(u) = 0, & t > 0,\ x \in \mathbb{R}, \\
u(x,0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\tag{4.1}
$$

where

$$
\begin{aligned}
F(u) = {}& \frac{m(b-m-1)}{2} \left(1-\partial_x^2\right)^{-1} u^{m-1} (\partial_x u)^3 \\
&+ \left(1-\partial_x^2\right)^{-1} \partial_x \left( \frac{b}{m+2} u^{m+2} + \frac{3m+3-b}{2} u^m (\partial_x u)^2 \right).
\end{aligned}
\tag{4.2}
$$

**Definition 4.1** Let $u_0 \in H^1$. If $u$ belongs to $L^\infty_{\mathrm{loc}}([0,T);H^1)$ and satisfies the identity

$$\int_0^T \int_{\mathbb{R}} \big(u\psi_t + \big(u^{m+1}u_x + F(u)\big)\psi\big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0(x)\psi(0,x) = 0$$

for all $\psi \in C_0^\infty([0,T)\times\mathbb{R})$, then $u$ is called a weak solution to (4.1). If $u$ is a weak solution on $[0,T)$ for every $T>0$, then it is called a global weak solution to (4.1).

**Proposition 4.1**

(i) *Every strong solution is a weak solution.*
(ii) *If $u$ is a weak solution and $u \in C([0,T);H^s)\cap([0,T);H^{s-1})$ with $s>3/2$, then it is a strong solution.*

*Proof* The proof is similar to that of Proposition 4.1 in Constantin and Escher (1998), Wu and Yin (2011), so we omit it. □

Next, we prove that the peakon solitary wave $u(t,x) = c^{\frac{1}{m+1}}\mathrm{e}^{-|x-ct-x_0|}$, $c>0$, is a global weak solution to Eq. (1.1).

*Proof of Theorem 1.6* Without loss of generality, we set $x_0 = 0$. Note that $u_t = \operatorname{sgn}(x-ct)cu$, $u_x = -\operatorname{sgn}(x-ct)u$; then

$$-u_t + u^{m+1}u_x = -\big(cu + u^{m+2}\big)\operatorname{sgn}(x-ct). \tag{4.3}$$

On the other hand, by a simple computation, we get

$$\begin{aligned}
F(u) &= \int_{\mathbb{R}} \frac{m(b-m-1)}{4}\mathrm{e}^{-|x-y|}u^{m-1}(\partial_x u)^3(t,y)\,\mathrm{d}y \\
&\quad - \operatorname{sgn}(x-y)\int_{\mathbb{R}} \frac{1}{2}\mathrm{e}^{-|x-y|}\bigg(\frac{b}{m+2}u^{m+2}(t,y) \\
&\quad + \frac{3m+3-b}{2}u^m(\partial_x u)^2(t,y)\bigg)\,\mathrm{d}y \\
&= -\operatorname{sgn}(x-ct)\int_{\mathbb{R}} \frac{m(b-m-1)}{4}\mathrm{e}^{-|x-y|}u^{m+2}(t,y)\,\mathrm{d}y \\
&\quad - \operatorname{sgn}(x-y)\int_{\mathbb{R}} \frac{1}{2}\mathrm{e}^{-|x-y|}\bigg(\frac{b}{m+2}u^{m+2}(t,y) + \frac{3m+3-b}{2}u^{m+2}(t,y)\bigg)\,\mathrm{d}y \\
&= -\int_{-\infty}^{x} \frac{1}{2}\mathrm{e}^{y-x}\bigg(\operatorname{sgn}(y-ct)\frac{m(b-m-1)}{2} \\
&\quad + \frac{b}{m+2} + \frac{3m+3-b}{2}\bigg)u^{m+2}(t,y)\,\mathrm{d}y
\end{aligned}$$

$$+\int_x^{\infty}\frac{1}{2}e^{x-y}\biggl(-\operatorname{sgn}(y-ct)\frac{m(b-m-1)}{2}$$
$$+\frac{b}{m+2}+\frac{3m+3-b}{2}\biggr)u^{m+2}(t,y)\,dy.$$

If $x < ct$, we deduce

$$\begin{aligned}
2F(u) &= a_1\int_{\infty}^{x} e^{y-x}u^{m+2}(t,y)\,dy + a_2\int_x^{ct} e^{x-y}u^{m+2}(t,y)\,dy \\
&\quad - a_1\int_{ct}^{\infty} e^{x-y}u^{m+2}(t,y)\,dy \\
&= c^{\frac{m+2}{m+1}}\biggl(a_1\int_{\infty}^{x} e^{y-x}e^{(m+2)(y-ct)}\,dy + a_2\int_x^{ct} e^{x-y}e^{(m+2)(y-ct)}\,dy \\
&\quad - a_1\int_{ct}^{\infty} e^{x-y}e^{(m+2)(-y+ct)}\,dy\biggr) \\
&= c^{\frac{m+2}{m+1}}\biggl(\frac{a_1}{m+3}e^{(m+3)y-x-ct(m+2)}\Big|_{\infty}^{x} + \frac{a_2}{m+1}e^{(m+1)y+x-ct(m+2)}\Big|_{x}^{ct} \\
&\quad + \frac{a_1}{(m+3)}e^{-(m+3)y+x+ct(m+2)}\Big|_{ct}^{\infty}\biggr) \\
&= c^{\frac{m+2}{m+1}}\biggl(\frac{a_1}{m+3}e^{(m+2)(x-ct)} + \frac{a_2}{m+1}\bigl(e^{x-ct}-e^{(m+2)(x-ct)}\bigr) \\
&\quad - \frac{a_1}{(m+3)}e^{x-ct}\biggr) \\
&= c^{\frac{m+2}{m+1}}\biggl[\biggl(\frac{a_1}{m+3}-\frac{a_2}{m+1}\biggr)e^{(m+2)(x-ct)} + \biggl(\frac{a_2}{m+1}-\frac{a_1}{m+3}\biggr)e^{x-ct}\biggr] \\
&= -2c^{\frac{m+2}{m+1}}\bigl[e^{(m+2)(x-ct)}+e^{x-ct}\bigr] \\
&= -2cu-2u^{m+2}
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{m(b-m-1)}{2}-\frac{b}{m+2}-\frac{3m+3-b}{2} \\
&= \frac{bm(m+3)-(m+1)(m+2)(m+3)}{2(m+2)}, \\
a_2 &= \frac{m(b-m-1)}{2}+\frac{b}{m+2}+\frac{3m+3-b}{2} \\
&= \frac{bm(m+1)-(m+1)(m+2)(m-3)}{2(m+2)},
\end{aligned}$$

and then

$$F(u) = -cu - u^{m+2} \quad \text{if } x < ct.$$

By a similar computation, we have

$$F(u) = cu + u^{m+2} \quad \text{if } x > ct,$$

therefore

$$F(u) = \big(cu + u^{m+2}\big)\,\mathrm{sgn}(x - ct). \tag{4.4}$$

Combining (4.3) with (4.4), we obtain

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \big(u\psi_t + \big(u^{m+1}u_x + F(u)\big)\psi\big)\,\mathrm{d}x\,\mathrm{d}t + \int_R u_0\psi(0,x)\,\mathrm{d}x \\ &\quad = \int_0^T \int_{\mathbb{R}} \big(-u_t + u^{m+1}u_x + F(u)\big)\varphi\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

Thus, by Definition 4.1, the peakon solitary wave $u(t,x) = c^{\frac{1}{m+1}}\mathrm{e}^{-|x-ct-x_0|}$ is a global weak solution to Eq. (1.1). □

*Proof of Theorem 1.7* We now derive the multi-peakon solutions of Eq. (1.1). We assume that Eq. (1.1) has an $N$-peakon solution of the form (1.7). It follows from Definition 4.1 that for any $\psi(t,x) \in C_c^\infty([0,\infty)\times\mathbb{R})$ the solution (4.1) satisfies

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}} \bigg[u_t + u^{m+1}u_x + \frac{m(b-m-1)}{2}\big(1-\partial_x^2\big)^{-1}u^{m-1}(\partial_x u)^3 \\ &\quad + \big(1-\partial_x^2\big)^{-1}\partial_x\bigg(\frac{b}{m+2}u^{m+2} + \frac{3m+3-b}{2}u^m(\partial_x u)^2\bigg)\bigg]\varphi(x)\,\mathrm{d}x\,\mathrm{d}t = 0,\end{aligned} \tag{4.5}$$

which is equivalent to the following equation:

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}} \bigg[u_t(\phi - \phi_{xx}) + \frac{1}{m+2}u^{m+2}\phi_{xxx} + \frac{m(b-m-1)}{2}u^{m-1}(\partial_x u)^3\phi \\ &\quad - \phi_x\bigg(\frac{b+1}{m+2}u^{m+2} + \frac{3m+3-b}{2}u^m(\partial_x u)^2\bigg)\bigg]\,\mathrm{d}x\,\mathrm{d}t = 0,\end{aligned} \tag{4.6}$$

where $\varphi = \phi - \phi_{xx}$, $\phi(t,x) \in C_c^\infty([0,\infty)\times\mathbb{R})$.

A straightforward computation gives

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} u_t(\phi - \phi_{xx})\,\mathrm{d}x\,\mathrm{d}t &= \sum_{i=1}^N \int_0^\infty \int_{-\infty}^{q_j(t)} \big(p_j' - p_j q_j'\big)\mathrm{e}^{x-q_j}(\phi - \phi_{xx})\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \sum_{i=1}^N \int_0^\infty \int_{q_j(t)}^\infty \big(p_j' + p_j q_j'\big)\mathrm{e}^{-(x-q_j)}(\phi - \phi_{xx})\,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^\infty \sum_{i=1}^N \big(p_j'\phi(q_j) + p_j q_j'\phi_x(q_j)\big)\,\mathrm{d}t,\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\frac{1}{m+2}\int_{\mathbb{R}} u^{m+2}\phi_{xxx}\,\mathrm{d}x \\
&\quad= -\left(\int_{-\infty}^{q_1} + \sum_{j=1}^{N-1}\int_{q_j}^{q_{j+1}} + \int_{q_N}^{\infty}\right) u^{m+1}u_x\phi_{xx}\,\mathrm{d}x \\
&\quad= -u^{m+1}u_x\phi_x\left(\Big|_{-\infty}^{q_1} + \sum_{j=1}^{N-1}\Big|_{q_j}^{q_{j+1}} + \Big|_{q_N}^{\infty}\right) + \int_{\mathbb{R}}\big((m+1)u^m u_x^2 + u^{m+1}u_{xx}\big)\phi_x\,\mathrm{d}x \\
&\quad= \big[-u^{m+1}u_x\phi_x + \big((m+1)u^m u_x^2 + u^{m+2}\big)\phi\big]\left(\Big|_{-\infty}^{q_1} + \sum_{j=1}^{N-1}\Big|_{q_j}^{q_{j+1}} + \Big|_{q_N}^{\infty}\right) \\
&\qquad - \int_{\mathbb{R}}\big(m(m+1)u^{m-1}u_x^3 + 2(m+1)u^{m+1}u_x + (m+2)u^{m+1}u_x\big)\phi\,\mathrm{d}x
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}}\left(\frac{b+1}{m+2}u^{m+2} + \frac{3m+3-b}{2}u^m(\partial_x u)^2\right)\phi_x\,\mathrm{d}x \\
&\quad= -\left[\left(\frac{b+1}{m+2}u^{m+2} + \frac{3m+3-b}{2}u^m u_x^2\right)\phi\right]\left(\Big|_{-\infty}^{q_1} + \sum_{j=1}^{N-1}\Big|_{q_j}^{q_{j+1}} + \Big|_{q_N}^{\infty}\right) \\
&\qquad + \int_{\mathbb{R}}\bigg((b+1)u^{m+1}u_x + (3m+3-b)u^{m+1}u_x \\
&\qquad + \frac{m(3m+3-b)}{2}u^{m-1}u_x^3\bigg)\,\mathrm{d}x.
\end{aligned} \tag{4.9}$$

Thus, combining (4.8) with (4.9), we get

$$\begin{aligned}
&\int_{\mathbb{R}}\bigg[\frac{1}{m+2}u^{m+2}\phi_{xxx} + \frac{m(b-m-1)}{2}u^{m-1}(\partial_x u)^3\phi \\
&\qquad - \phi_x\left(\frac{b+1}{m+2}u^{m+2} + \frac{3m+3-b}{2}u^m(\partial_x u)^2\right)\bigg]\,\mathrm{d}x \\
&\quad= \left(-u^{m+1}u_x\phi_x + \frac{b-m-1}{2}u^m u_x^2\phi\right)\left(\Big|_{-\infty}^{q_1} + \sum_{j=1}^{N-1}\Big|_{q_j}^{q_{j+1}} + \Big|_{q_N}^{\infty}\right) \\
&\quad= -2\sum_{j=1}^{N}\left[p_j\left(\sum_{i=1}^{N}p_i \mathrm{e}^{-|q_j-q_i|}\right)^{m+1}\phi_x(q_j)\right]
\end{aligned}$$

$$- 2(b-m-1)\sum_{j=1}^{N}\left[p_j\left(\sum_{i=1}^{N}p_i e^{-|q_j-q_i|}\right)^m \times\left(\sum_{i=1}^{N}p_i\,\mathrm{sgn}(q_j-q_i)e^{-|q_j-q_i|}\right)\phi(q_j)\right]. \tag{4.10}$$

Substituting (4.7), (4.10) into (4.6), we obtain the following system:

$$\begin{aligned} p_j' &= \left(\sum_{i=1}^{N}p_i e^{-|q_j-q_i(t)|}\right)^{m+1}, \\ q_j' &= (b-m-1)q_j\left(\sum_{i=1}^{N}p_i e^{-|q_j-q_i|}\right)^m\left(\sum_{i=1}^{N}p_i\,\mathrm{sgn}(q_j-q_i)e^{-|q_j-q_i|}\right) \end{aligned} \tag{4.11}$$

this leads to the conclusion of Theorem 1.7. □

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