

## Stability of the $\mu$ -Camassa–Holm Peakons

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**Abstract** The  $\mu$ -Camassa–Holm ( $\mu$ CH) equation is a nonlinear integrable partial differential equation closely related to the Camassa–Holm equation. We prove that the periodic peaked traveling wave solutions (peakons) of the  $\mu$ CH equation are orbitally stable.

**Keywords** Water waves · Camassa–Holm equation · Peakons · Stability

**Mathematics Subject Classification (2000)** 35Q35 · 37K45

### 1 Introduction

The nonlinear partial differential equation

$$\mu(u_t) - u_{xxt} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in S^1 = \mathbb{R}/\mathbb{Z}, \quad (1.1)$$

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where  $u(x, t)$  is a real-valued spatially periodic function and  $\mu(u) = \int_{S^1} u(x, t) dx$  denotes its mean, was recently introduced in Khesin et al. (2008) as an integrable equation arising in the study of the diffeomorphism group of the circle. It describes the propagation of self-interacting, weakly nonlinear orientation waves in a massive nematic liquid crystal under the influence of an external magnetic field. The closest relatives of (1.1) are the Camassa–Holm equation (Camassa and Holm 1993; Fokas and Fuchssteiner 1981)

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (1.2)$$

and the Hunter–Saxton (Hunter and Saxton 1991) equation

$$-u_{txx} = 2u_x u_{xx} + uu_{xxx}. \quad (1.3)$$

In fact, each of Eqs. (1.1)–(1.3) can be written in the form

$$y_t + uy_x + 2u_x y = 0, \quad y = Au, \quad (1.4)$$

where the operator  $A$  is given by  $A = \mu - \partial_x^2$  in the case of (1.1),  $A = 1 - \partial_x^2$  in the case of (1.2), and  $A = -\partial_x^2$  in the case of (1.3). Following Lenells et al. (2010), we will refer to Eq. (1.1) as the  $\mu$ -Camassa–Holm ( $\mu$ CH) equation.

Equations (1.1)–(1.3) share many remarkable properties. (a) They are all completely integrable systems with a corresponding Lax pair formulation, a bi-Hamiltonian structure, and an infinite sequence of conservation laws; see Camassa and Holm (1993), Constantin and McKean (1999), Hunter and Zheng (1994), Khesin et al. (2008). (b) They all arise geometrically as equations for geodesic flow in the context of the diffeomorphism group of the circle  $\text{Diff}(S^1)$  endowed with a right-invariant metric (Khesin et al. 2008; Khesin and Misiólek 2003; Kouranbaeva 1999; Misiólek 1998; Shkoller 1998). (c) They are all models for wave breaking (each equation admits initially smooth solutions which break in finite time in such a way that the wave remains bounded while its slope becomes unbounded); see Camassa and Holm (1993), Constantin (1997), Constantin and Escher (1998, 2000) Constantin and McKean (1999), Hunter and Saxton (1991), Khesin et al. (2008), Misiólek (2002).

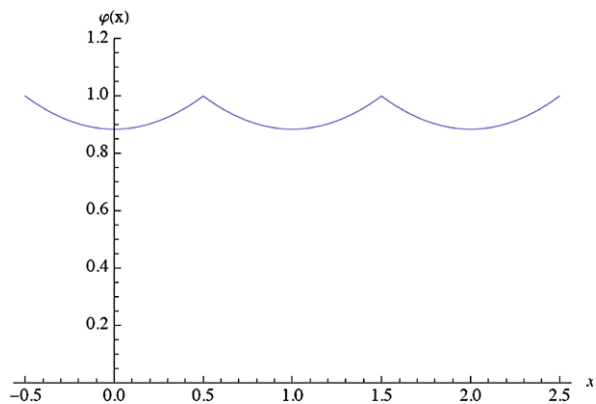
A particularly interesting feature of the Camassa–Holm equation is that it admits peaked soliton solutions (Camassa and Holm 1993). These solutions (called peakons) are traveling waves with a peak at their crest, and they occur in both the periodic and the nonperiodic setting. It was noted in Lenells et al. (2010) that the  $\mu$ CH equation also admits peakons: For any  $c \in \mathbb{R}$ , the peaked traveling wave  $u(x, t) = c\varphi(x - ct)$ , where (see Fig. 1)

$$\varphi(x) = \frac{1}{26}(12x^2 + 23) \quad \text{for } x \in [-1/2, 1/2] \quad (1.5)$$

and  $\varphi$  is extended periodically to the real line, is a solution of (1.1). Note that the height of the peakon  $c\varphi(x - ct)$  is proportional to its speed.

If waves such as the peakons are to be observable in nature, they must be stable under small perturbations; therefore, the stability of the peakons is of great interest. Since a small change in the height of a peakon yields another one traveling at a different speed, the correct notion of stability here is that of *orbital stability*: A periodic

**Fig. 1** The periodic peakon  $\varphi(x)$  of the  $\mu$ CH equation



wave with an initial profile close to a peakon remains close to some translate of it for all later times. That is, the shape of the wave remains approximately the same for all times.

The Camassa–Holm peakons are orbitally stable in the nonperiodic setting (Constantin and Strauss 2000) as well as in the periodic case (Lenells 2004a). In this paper, we show that the periodic  $\mu$ CH peakons given by (1.5) are also orbitally stable.

**Theorem 1.1** *The periodic peakons of Eq. (1.1) are orbitally stable in  $H^1(S^1)$ .*

An outline of the proof of Theorem 1.1 is given in Sect. 2, while a detailed proof is presented in Sect. 3. We conclude the paper with Sect. 4 where we discuss some results on the existence of solutions to (1.1).

## 2 Outline of Proof

There are two standard methods for studying the stability of a solution of a dispersive wave equation. The first method consists of linearizing the equation around the solution. In many cases, nonlinear stability is governed by the linearized equation. However, for the  $\mu$ CH and CH equations, the nonlinearity plays a dominant role rather than being a higher order perturbation of the linear terms. Thus, it is not clear how to prove nonlinear stability of the peakons using the linearized problem. Moreover, the peakons  $c\varphi(x - ct)$  are continuous but not differentiable, which makes it hard to analyze the spectrum of the operator linearized around  $c\varphi$ .

The second method is variational in nature. In this approach, the solution is realized as an energy minimizer under appropriate constraints. Stability follows if the uniqueness of the minimizer can be established (otherwise one only obtains the stability of the set of minima). A proof of the stability of the Camassa–Holm peakons using the variational approach is given in Constantin and Molinet (2001) for the case on the line and in Lenells (2004b) for the periodic case.

In this paper, we prove the stability of the peakon (1.5) using a method that is different from the two described above. Taking  $c = 1$  for simplicity, our approach can be

described as follows. To each function  $w : S^1 \rightarrow \mathbb{R}$ , we associate a real-valued function  $F_w(M, m)$  of two real variables  $(M, m)$  in such a way that the correspondence  $w \mapsto F_w$  has the following properties:

- If  $u(x, t)$  is a solution of (1.1) with maximal existence time  $T > 0$ , then

$$F_{u(t)}(M_{u(t)}, m_{u(t)}) \geq 0, \quad t \in [0, T), \tag{2.1}$$

where  $M_{u(t)} = \max_{x \in S^1} \{u(x, t)\}$  and  $m_{u(t)} = \min_{x \in S^1} \{u(x, t)\}$  denote the maximum and minimum of  $u$  at time  $t$ , respectively.

- For the peakon, we have  $F_\varphi \equiv F_{\varphi(\cdot)} = F_{\varphi(\cdot-t)}$  and  $F_\varphi(M, m) \leq 0$  for all  $(M, m)$  with equality if and only if  $(M, m) = (M_\varphi, m_\varphi)$ ; see Fig. 1.
- If  $w : S^1 \rightarrow \mathbb{R}$  is such that  $H_i[w]$  is close to  $H_i[\varphi]$ ,  $i = 0, 1, 2$ , where  $H_0, H_1, H_2$  are the conservation laws of (1.1) given by

$$\begin{aligned} H_0[u] &= \int u \, dx, & H_1[u] &= \frac{1}{2} \int yu \, dx, \\ H_2[u] &= \int \left( \mu(u)u^2 + \frac{1}{2}uu_x^2 \right) dx, \end{aligned} \tag{2.2}$$

then the function  $F_w$  is a small perturbation of  $F_\varphi$ .

Using the correspondence  $w \mapsto F_w$ , the stability of the peakon is proved as follows. If  $u$  is a solution starting close to the peakon  $\varphi$ , the conserved quantities  $H_i[u]$  are close to  $H_i[\varphi]$ ,  $i = 0, 1, 2$ , and hence  $F_{u(t)}$  is a small perturbation of  $F_\varphi$  for any  $t \in [0, T)$ . This implies that the set where  $F_{u(t)} \geq 0$  is contained in a small neighborhood of  $(M_\varphi, m_\varphi)$  in  $\mathbb{R}^2$  for any  $t \in [0, T)$ . We conclude from (2.1) that  $(M_{u(t)}, m_{u(t)})$  stays close to  $(M_\varphi, m_\varphi)$  for all times. The proof is completed by noting that if the maximum of  $u$  stays close to the maximum of the peakon, then the shape of the whole wave remains close to that of the peakon.

Our proof is inspired by Lenells (2004a), where the stability of the periodic peakons of the Camassa–Holm equation is proved.<sup>1</sup> The approach here is similar, but there are differences. The main difference is that in Lenells (2004a) the function  $F_u$  associated with a solution  $u(x, t)$  could be chosen to be independent of time, whereas here the function  $F_{u(t)}$  depends on time. Indeed, our definition of the function  $F_{u(t)}(M, m)$  involves the  $L^2$ -norm  $\|u(t)\|_{L^2(S^1)}$ , which is not conserved in time. However, since this norm is controlled by the conservation law  $H_1$ , we can ensure that it remains bounded for all times. This turns out to be enough to ascertain that the function  $F_{u(t)}$ , despite its time dependence, remains close to  $F_\varphi$  for all  $t \in [0, T)$ .

### 3 Proof of Stability

We will identify  $S^1$  with the interval  $[0, 1)$  and view functions on  $S^1$  as periodic functions on the real line of period one. For an integer  $n \geq 1$ , we let  $H^n(S^1)$  denote

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<sup>1</sup>The proof in Lenells (2004a) is in turn inspired by the proof of stability of the Camassa–Holm peakons on the line presented in Constantin and Strauss (2000).

the Sobolev space of all square integrable functions  $f \in L^2(S^1)$  with distributional derivatives  $\partial_x^i f \in L^2(S^1)$  for  $i = 1, \dots, n$ . The norm on  $H^n(S^1)$  is given by

$$\|f\|_{H^n(S^1)}^2 = \sum_{i=0}^n \int_{S^1} (\partial_x^i f)^2(x) \, dx.$$

Equation (1.1) can be rewritten in the following form:

$$u_t + uu_x + A^{-1} \partial_x \left( 2\mu(u)u + \frac{1}{2}u_x^2 \right) = 0, \tag{3.1}$$

where  $A = \mu - \partial_x^2$  is an isomorphism between  $H^s(S^1)$  and  $H^{s-2}(S^1)$ ; see Khesin et al. (2008). By a *weak solution*  $u$  of (1.1) on  $[0, T)$  with  $T > 0$ , we mean a function  $u \in C([0, T); H^1(S^1))$  such that (3.1) holds in the distributional sense and the functionals  $H_i[u]$ ,  $i = 0, 1, 2$ , defined in (2.2) are independent of  $t \in [0, T)$ . The peakons defined in (1.5) are weak solutions in this sense (Lenells et al. 2010). Our aim is to prove the following precise reformulation of the theorem stated in the introduction.

**Theorem 3.1** *For every  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $u \in C([0, T); H^1(S^1))$  is a weak solution of (1.1) with*

$$\|u(\cdot, 0) - c\varphi\|_{H^1(S^1)} < \delta$$

then

$$\|u(\cdot, t) - c\varphi(\cdot - \xi(t) + 1/2)\|_{H^1(S^1)} < \epsilon \quad \text{for } t \in [0, T),$$

where  $\xi(t) \in \mathbb{R}$  is any point where the function  $u(\cdot, t)$  attains its maximum.

The proof of Theorem 3.1 will proceed through a series of lemmas. The first lemma summarizes the properties of the peakon. For simplicity, we henceforth take  $c = 1$ .

**Lemma 3.2** *The peakon  $\varphi(x)$  is continuous on  $S^1$  with peak at  $x = \pm 1/2$ . The extrema of  $\varphi$  are*

$$M_\varphi = \varphi(1/2) = 1, \quad m_\varphi = \varphi(0) = \frac{23}{26}.$$

Moreover,

$$\lim_{x \uparrow 1/2} \varphi_x(x) = \frac{6}{13}, \quad \lim_{x \downarrow -1/2} \varphi_x(x) = -\frac{6}{13},$$

and

$$H_0[\varphi] = \frac{12}{13}, \quad H_1[\varphi] = \max_{x \in S^1} \varphi_x = \frac{6}{13}, \quad H_2[\varphi] = \frac{9024}{10985}.$$

*Proof* These properties follow easily from the definition (1.5) of  $\varphi$  and the definition (2.2) of  $\{H_i\}_1^3$ . For example,

$$H_0[\varphi] = \int_{-1/2}^{1/2} \frac{12x^2 + 23}{26} dx = \frac{12}{13}. \quad \square$$

We define the  $\mu$ -inner product  $\langle \cdot, \cdot \rangle_\mu$  and the associated  $\mu$ -norm  $\| \cdot \|_\mu$  by

$$\begin{aligned} \langle u, v \rangle_\mu &= \mu(u)\mu(v) + \int_{S^1} u_x v_x dx, & \|u\|_\mu^2 &= \langle u, u \rangle_\mu = 2H_1[u], \\ u, v &\in H^1(S^1), \end{aligned} \tag{3.2}$$

and consider the expansion of the conservation law  $H_1$  around the peakon  $\varphi$  in the  $\mu$ -norm. The following lemma shows that the error term in this expansion is given by 12/13 times the difference between  $\varphi$  and the perturbed solution  $u$  at the point of the peak.

**Lemma 3.3** *For every  $u \in H^1(S^1)$  and  $\xi \in \mathbb{R}$ ,*

$$H_1[u] - H_1[\varphi] = \frac{1}{2} \|u - \varphi(\cdot - \xi)\|_\mu^2 + \frac{12}{13} (u(\xi + 1/2) - M_\varphi).$$

*Proof* We compute

$$\begin{aligned} \frac{1}{2} \|u - \varphi(\cdot - \xi)\|_\mu^2 &= H_1[u] + H_1[\varphi(\cdot - \xi)] - \mu(u)\mu(\varphi) - \int_{S^1} u_x(x)\varphi_x(x - \xi) dx \\ &= H_1[u] + H_1[\varphi] - \mu(u)\mu(\varphi) + \int_{S^1} u(x + \xi)\varphi_{xx}(x) dx. \end{aligned}$$

Since

$$\varphi_{xx} = \frac{12}{13} - \frac{12}{13} \delta(x - 1/2), \tag{3.3}$$

we find

$$\int_{S^1} u(x + \xi)\varphi_{xx}(x) dx = \frac{12}{13} \int_{S^1} u(x) dx - \frac{12}{13} u(\xi + 1/2).$$

Using that  $H_0[\varphi] = \mu(\varphi) = \frac{12}{13}$ , we obtain

$$\frac{1}{2} \|u - \varphi(\cdot - \xi)\|_\mu^2 = H_1[u] - H_1[\varphi] + \frac{12}{13} (1 - u(\xi + 1/2)).$$

This proves the lemma. □

*Remark 3.4* For a wave profile  $u \in H^1(S^1)$ , the functional  $H_1[u]$  represents kinetic energy. Lemma 3.3 implies that if a wave  $u \in H^1(S^1)$  has energy  $H_1[u]$  and height  $M_u$  close to the peakon’s energy and height, then the whole shape of  $u$  is close to that of the peakon. Another physically relevant consequence of Lemma 3.3

is that, among all waves of fixed energy, the peakon has maximal height. Indeed, if  $u \in H^1(S^1) \subset C(S^1)$  is such that  $H_1[u] = H_1[\varphi]$  and  $u(\xi) = \max_{x \in S^1} u(x)$ , then  $u(\xi) \leq M_\varphi$ .

The peakon  $\varphi$  satisfies the differential equation

$$\varphi_x = \begin{cases} -\frac{12}{13}\sqrt{\frac{13}{6}(\varphi - m_\varphi)}, & -1/2 < x \leq 0, \\ \frac{12}{13}\sqrt{\frac{13}{6}(\varphi - m_\varphi)}, & 0 \leq x < 1/2. \end{cases} \tag{3.4}$$

Let  $u \in H^1(S^1) \subset C(S^1)$  and write  $M = M_u = \max_{x \in S^1} \{u(x)\}$ ,  $m = m_u = \min_{x \in S^1} \{u(x)\}$ . Let  $\xi$  and  $\eta$  be such that  $u(\xi) = M$  and  $u(\eta) = m$ . Inspired by (3.4), we define the real-valued function  $g(x)$  by

$$g(x) = \begin{cases} u_x + \frac{12}{13}\sqrt{\frac{13}{6}(u - m)}, & \xi < x \leq \eta, \\ u_x - \frac{12}{13}\sqrt{\frac{13}{6}(u - m)}, & \eta \leq x < \xi + 1, \end{cases}$$

and extend it periodically to the real line. We compute

$$\begin{aligned} \int_{S^1} g^2(x) dx &= \int_\xi^\eta \left(u_x + \frac{12}{13}\sqrt{\frac{13}{6}(u - m)}\right)^2 dx \\ &\quad + \int_\eta^{\xi+1} \left(u_x - \frac{12}{13}\sqrt{\frac{13}{6}(u - m)}\right)^2 dx \\ &= \int_\xi^\eta u_x^2 dx + \frac{24}{13} \int_\xi^\eta u_x \sqrt{\frac{13}{6}(u - m)} dx + \frac{144}{169} \int_\xi^\eta \frac{13}{6}(u - m) dx \\ &\quad + \int_\eta^{\xi+1} u_x^2 dx - \frac{24}{13} \int_\eta^{\xi+1} u_x \sqrt{\frac{13}{6}(u - m)} dx \\ &\quad + \frac{144}{169} \int_\eta^{\xi+1} \frac{13}{6}(u - m) dx. \end{aligned}$$

Notice that

$$\frac{d}{dx} \left[ 8\sqrt{\frac{2}{39}}(u - m)^{3/2} \right] = \frac{24}{13} u_x \sqrt{\frac{13}{6}(u - m)}.$$

Hence,

$$\int_\xi^\eta u_x \sqrt{\frac{13}{6}(u - m)} dx = - \int_\eta^{\xi+1} u_x \sqrt{\frac{13}{6}(u - m)} dx$$

and

$$\frac{24}{13} \int_\xi^\eta u_x \sqrt{\frac{13}{6}(u - m)} dx = \left[ 8\sqrt{\frac{2}{39}}(u - m)^{3/2} \right]_\xi^\eta = -8\sqrt{\frac{2}{39}}(M - m)^{3/2}.$$

We conclude that

$$\frac{1}{2} \int_{S^1} g^2(x) \, dx = H_1[u] - \frac{1}{2} \mu(u)^2 - 8\sqrt{\frac{2}{39}}(M-m)^{3/2} + \frac{12}{13}(\mu(u) - m). \quad (3.5)$$

In the same way, we compute

$$\begin{aligned} & \int_{S^1} u g^2(x) \, dx \\ &= \int_{\xi}^{\eta} u \left( u_x + \frac{12}{13} \sqrt{\frac{13}{6}(u-m)} \right)^2 dx + \int_{\eta}^{\xi+1} u \left( u_x - \frac{12}{13} \sqrt{\frac{13}{6}(u-m)} \right)^2 dx \\ &= \int_{\xi}^{\eta} u u_x^2 dx + \frac{24}{13} \int_{\xi}^{\eta} u u_x \sqrt{\frac{13}{6}(u-m)} dx + \frac{144}{169} \int_{\xi}^{\eta} u \frac{13}{6}(u-m) dx \\ & \quad + \int_{\eta}^{\xi+1} u u_x^2 dx - \frac{24}{13} \int_{\eta}^{\xi+1} u u_x \sqrt{\frac{13}{6}(u-m)} dx + \frac{144}{169} \int_{\eta}^{\xi+1} u \frac{13}{6}(u-m) dx. \end{aligned}$$

Since

$$\frac{d}{dx} \left[ \frac{8}{5} \sqrt{\frac{2}{39}}(u-m)^{3/2}(2m+3u) \right] = \frac{24}{13} u u_x \sqrt{\frac{13}{6}(u-m)},$$

we find that

$$\int_{\xi}^{\eta} u u_x \sqrt{\frac{13}{6}(u-m)} dx = - \int_{\eta}^{\xi+1} u u_x \sqrt{\frac{13}{6}(u-m)} dx$$

and

$$\frac{24}{13} \int_{\xi}^{\eta} u u_x \sqrt{\frac{13}{6}(u-m)} dx = -\frac{8}{5} \sqrt{\frac{2}{39}}(M-m)^{3/2}(2m+3M).$$

Therefore,

$$\begin{aligned} \frac{1}{2} \int_{S^1} u g^2(x) \, dx &= H_2[u] - \left( H_0[u] - \frac{12}{13} \right) \int_{S^1} u^2 dx - \frac{12}{13} m H_0[u] \\ & \quad - \frac{8}{5} \sqrt{\frac{2}{39}}(M-m)^{3/2}(2m+3M). \end{aligned} \quad (3.6)$$

Combining (3.6) with (3.5), we obtain

$$\begin{aligned} H_2[u] &= \frac{1}{2} \int_{S^1} u g^2(x) \, dx + \left( H_0[u] - \frac{12}{13} \right) \int_{S^1} u^2 dx + \frac{12}{13} m H_0[u] \\ & \quad + \frac{8}{5} \sqrt{\frac{2}{39}}(M-m)^{3/2}(2m+3M) \\ &\leq \frac{M}{2} \int_{S^1} g^2(x) \, dx + \left( H_0[u] - \frac{12}{13} \right) \int_{S^1} u^2 dx + \frac{12}{13} m H_0[u] \end{aligned}$$



$$\begin{aligned}
 & + \frac{8}{5} \sqrt{\frac{2}{39}} (M - m)^{3/2} (2m + 3M) \\
 = & M \left[ H_1[u] - \frac{1}{2} \mu(u)^2 - 8 \sqrt{\frac{2}{39}} (M - m)^{3/2} + \frac{12}{13} (\mu(u) - m) \right] \\
 & + \left( H_0[u] - \frac{12}{13} \right) \int_{S^1} u^2 \, dx + \frac{12}{13} m H_0[u] \\
 & + \frac{8}{5} \sqrt{\frac{2}{39}} (M - m)^{3/2} (2m + 3M). \tag{3.7}
 \end{aligned}$$

We have actually proved the following lemma.

**Lemma 3.5** *For any positive  $u \in H^1(S^1)$ , define a function*

$$F_u : \{(M, m) \in \mathbb{R}^2 : M \geq m > 0\} \rightarrow \mathbb{R}$$

by

$$\begin{aligned}
 F_u(M, m) = & M \left[ H_1[u] - \frac{1}{2} H_0[u]^2 - 8 \sqrt{\frac{2}{39}} (M - m)^{3/2} + \frac{12}{13} (H_0[u] - m) \right] \\
 & + \left( H_0[u] - \frac{12}{13} \right) \int_{S^1} u^2 \, dx + \frac{12}{13} m H_0[u] \\
 & + \frac{8}{5} \sqrt{\frac{2}{39}} (M - m)^{3/2} (2m + 3M) - H_2[u].
 \end{aligned}$$

Then

$$F_u(M_u, m_u) \geq 0,$$

where  $M_u = \max_{x \in S^1} \{u(x)\}$  and  $m_u = \min_{x \in S^1} \{u(x)\}$ .

Note that the function  $F_u$  depends on  $u$  only through the three conservation laws  $H_0[u]$ ,  $H_1[u]$ , and  $H_2[u]$ , and the  $L^2$ -norm of  $u$ .

The next lemma highlights some properties of the function  $F_\varphi(M, m)$  associated to the peakon. The graph of  $F_\varphi(M, m)$  is shown in Fig. 2.

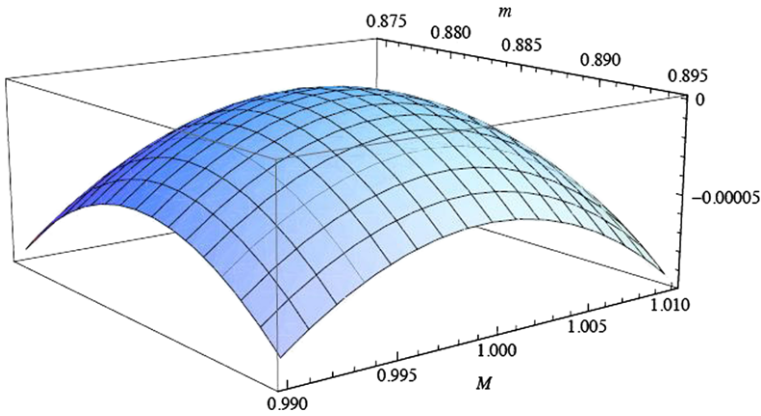
**Lemma 3.6** *For the peakon  $\varphi$ , we have*

$$F_\varphi(M_\varphi, m_\varphi) = 0,$$

$$\frac{\partial F_\varphi}{\partial M}(M_\varphi, m_\varphi) = 0, \quad \frac{\partial F_\varphi}{\partial m}(M_\varphi, m_\varphi) = 0,$$

$$\frac{\partial^2 F_\varphi}{\partial M^2}(M_\varphi, m_\varphi) = -\frac{12}{13}, \quad \frac{\partial^2 F_\varphi}{\partial M \partial m}(M_\varphi, m_\varphi) = 0, \quad \frac{\partial^2 F_\varphi}{\partial m^2}(M_\varphi, m_\varphi) = -\frac{12}{13}.$$

Moreover,  $(M_\varphi, m_\varphi)$  is the unique maximum of  $F_\varphi$ .



**Fig. 2** The graph of the function  $F_\varphi(M, m)$  near the point  $(M_\varphi, m_\varphi)$

*Proof* It follows from (3.4) that the function  $g(x)$  corresponding to the peakon is identically zero. Thus the inequality (3.7) is an equality in the case of the peakon. This means that  $F_\varphi(M_\varphi, m_\varphi) = 0$ .

On the other hand, differentiation gives

$$\begin{aligned} \frac{\partial F_u}{\partial M} &= \left[ H_1[u] - \frac{1}{2}H_0[u]^2 - 8\sqrt{\frac{2}{39}}(M - m)^{3/2} + \frac{12}{13}(H_0[u] - m) \right] \\ &\quad - 12\sqrt{\frac{2}{39}}M(M - m)^{1/2} + \frac{12}{5}\sqrt{\frac{2}{39}}(M - m)^{1/2}(2m + 3M) \\ &\quad + \frac{24}{5}\sqrt{\frac{2}{39}}(M - m)^{3/2} \\ &= \left[ H_1[u] - \frac{1}{2}H_0[u]^2 - 8\sqrt{\frac{2}{39}}(M - m)^{3/2} + \frac{12}{13}(H_0[u] - m) \right], \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F_u}{\partial m} &= 12\sqrt{\frac{2}{39}}M(M - m)^{1/2} - \frac{12}{13}M + \frac{12}{13}H_0[u] \\ &\quad + \frac{8}{5}\sqrt{\frac{2}{39}}\left[-\frac{3}{2}(M - m)^{1/2}(2m + 3M) + 2(M - m)^{3/2}\right] \\ &= \frac{12}{13}(H_0[u] - M) + 8\sqrt{\frac{2}{39}}(M - m)^{3/2}. \end{aligned}$$

Further differentiation yields

$$\frac{\partial^2 F_u}{\partial M \partial m} = -\frac{12}{13} + 12\sqrt{\frac{2}{39}}(M - m)^{1/2},$$

$$\frac{\partial^2 F_u}{\partial M^2} = \frac{\partial^2 F_u}{\partial m^2} = -12\sqrt{\frac{2}{39}}(M - m)^{1/2}.$$

To compute the derivatives of  $F_\varphi$  at  $(M_\varphi, m_\varphi)$ , take  $F_u = F_\varphi$ ,  $M = M_\varphi$ , and  $m = m_\varphi$  in the above expressions for the partial derivatives of  $F$  and use Lemma 3.2.

Next we show that  $(M_\varphi, m_\varphi)$  is the unique maximum of  $F_\varphi$ . First we have the following expression for  $F_\varphi(M, m)$ :

$$F_\varphi(M, m) = \left(\frac{150}{169} - \frac{12}{13}m\right)M + \frac{144}{169}m - \frac{16}{5}\sqrt{\frac{2}{39}}(M - m)^{5/2} - \frac{9024}{10985},$$

which has a unique critical point  $(M, m) = (M_\varphi, m_\varphi)$ . Hence it suffices to show that  $F_\varphi < 0$  on the boundary of its domain.

On  $\{M = m > 0\}$ ,

$$F_\varphi(M, M) = -\frac{12}{13}\left(M - \frac{49}{52}\right)^2 - \frac{81}{43940} < 0.$$

On  $\{m = 0\}$ ,

$$F_\varphi(M, 0) = -\frac{16}{5}\sqrt{\frac{2}{39}}M^{5/2} + \frac{150}{169}M - \frac{9024}{10985},$$

which has a maximum at  $M = \frac{15 \cdot 5^{1/3}}{26 \cdot 22^{2/3}}$  with the value  $< -0.49$ .

When  $M \rightarrow \infty$  it is obvious that  $F_\varphi(M, m) \rightarrow -\infty$ . Therefore the lemma is proved. □

**Lemma 3.7** *We have*

$$\max_{x \in S^1} |f(x)| \leq \sqrt{\frac{13}{12}} \|f\|_\mu, \quad f \in H^1(S^1), \tag{3.8}$$

where the  $\mu$ -norm is defined in (3.2). Moreover,  $\sqrt{\frac{13}{12}}$  is the best constant, and equality holds in (3.8) if and only if  $f = c\varphi(\cdot - \xi + 1/2)$  for some  $c, \xi \in \mathbb{R}$ , i.e., if and only if  $f$  has the shape of a peakon.

*Proof* For  $x \in S^1$ , by (3.2) and (3.3), we have

$$\begin{aligned} \frac{13}{12} \langle \varphi(\cdot - x + 1/2), f \rangle_\mu &= \frac{13}{12} \mu(\varphi(\cdot - x + 1/2)) \mu(f) \\ &\quad + \frac{13}{12} \int_{S^1} \varphi'(y - x + 1/2) f'(y) \, dy \\ &= \frac{13}{12} \int_{S^1} (\mu - \partial_y^2) \varphi(y - x + 1/2) f(y) \, dy \\ &= \int_{S^1} \delta(y - x) f(y) \, dy = f(x). \end{aligned}$$

Thus, since

$$H_1[\varphi] = \frac{1}{2} \|\varphi\|_\mu^2 = \frac{6}{13},$$

we get

$$f(x) = \frac{13}{12} \langle \varphi(\cdot - x + 1/2), f \rangle_\mu \leq \frac{13}{12} \|\varphi\|_\mu \|f\|_\mu = \sqrt{\frac{13}{12}} \|f\|_\mu, \quad (3.9)$$

with equality if and only if  $f$  and  $\varphi(\cdot - x + 1/2)$  are proportional. Taking the maximum of (3.9) over  $S^1$  proves the lemma.  $\square$

*Remark 3.8* Lemma 3.7 again indicates that, among all traveling waves of fixed energy, the peakon has maximal height (see also Constantin and Strauss 2000; Lenells 2004a).

The next lemma shows that the  $\mu$ -norm is equivalent to the  $H^1(S^1)$ -norm.

**Lemma 3.9** Every  $u \in H^1(S^1)$  satisfies

$$\|u\|_\mu^2 \leq \|u\|_{H^1(S^1)}^2 \leq 3\|u\|_\mu^2. \quad (3.10)$$

*Proof* The first inequality holds because (by Jensen's inequality)

$$\mu(u)^2 \leq \int_{S^1} u^2 dx, \quad u \in H^1(S^1).$$

The second inequality holds because, by Lemma 3.7,

$$\|u\|_{H^1(S^1)}^2 \leq \max_{x \in S^1} |u(x)|^2 + \int_{S^1} u_x^2 dx \leq \left(\frac{13}{12} + 1\right) \|u\|_\mu^2. \quad \square$$

*Remark 3.10* The previous two lemmas can also be proved directly using a Fourier series argument. Indeed, for every  $f \in H^3(S^1)$  and  $\epsilon > 0$ , we have (cf. the proof of Lemma 2 in Constantin 2000)

$$\max_{x \in S^1} f^2(x) \leq \frac{\epsilon + 2}{24} \int_{S^1} f_x^2 dx + \frac{\epsilon + 2}{\epsilon} \mu(f)^2. \quad (3.11)$$

The inequality (see Lemma 2.6 in Lenells 2004a)

$$\max_{x \in S^1} |f(x)|^2 \leq \frac{\cosh(1/2)}{2 \sinh(1/2)} \|f\|_{H^1(S^1)}^2, \quad f \in H^1(S^1), \quad (3.12)$$

implies that the map  $f \mapsto \max_{x \in S^1} f(x)$  is continuous from  $H^1(S^1)$  to  $\mathbb{R}$ . Thus, since  $H^3$  is dense in  $H^1$ , Eq. (3.11) also holds for  $f \in H^1(S^1)$ . It follows that, for every  $u \in H^1(S^1)$  and every  $\epsilon > 0$ ,

$$\|u\|_\mu^2 \leq \|u\|_{H^1(S^1)}^2 \leq \frac{\epsilon + 2}{\epsilon} \mu^2(u) + \frac{\epsilon + 26}{24} \int_{S^1} u_x^2 dx. \quad (3.13)$$

In particular, we have (taking  $\epsilon = 1$ )

$$\|u\|_{\mu}^2 \leq \|u\|_{H^1(S^1)}^2 \leq 3\mu^2(u) + \frac{27}{24} \int_{S^1} u_x^2 \, dx \leq 3\|u\|_{\mu}^2, \tag{3.14}$$

again showing the equivalence of the two norms.

On the other hand, letting  $\epsilon = 24$  in (3.11), we recover (3.8). However, the proof we give in Lemma 3.7 provides a better idea concerning the best constant.

**Lemma 3.11** (Lenells 2004a, Lemma 2.8) *If  $u \in C([0, T]; H^1(S^1))$ , then*

$$M_{u(t)} = \max_{x \in S^1} u(x, t) \quad \text{and} \quad m_{u(t)} = \min_{x \in S^1} u(x, t)$$

are continuous functions of  $t \in [0, T)$ .

**Lemma 3.12** *Let  $u \in C([0, T]; H^1(S^1))$  be a solution of (1.1). Given a small neighborhood  $\mathcal{U}$  of  $(M_{\varphi}, m_{\varphi})$  in  $\mathbb{R}^2$ , there is a  $\delta > 0$  such that*

$$(M_{u(t)}, m_{u(t)}) \in \mathcal{U} \quad \text{for } t \in [0, T) \quad \text{if } \|u(\cdot, 0) - \varphi\|_{H^1(S^1)} < \delta. \tag{3.15}$$

*Proof* Lemma 3.6 says that  $F_{\varphi}(M_{\varphi}, m_{\varphi}) = 0$  and that  $F_{\varphi}$  has a critical point with negative definite Hessian at  $(M_{\varphi}, m_{\varphi})$ . By continuity of  $F_{\varphi}$  along with its Hessian, there is a neighborhood  $\mathcal{V} \subset D \subset \mathbb{R}^2$  around  $(M_{\varphi}, m_{\varphi})$ , where  $D$  is the domain of  $F_u$  given in Lemma 3.5, such that  $F_{\varphi}$  is concave downward with curvature bounded away from zero. This together with the boundary values of  $F_{\varphi}$  indicates that  $F_{\varphi}(M, m) \leq -\alpha < 0$  on  $D \setminus \mathcal{V}$  for some positive constant  $\alpha$  only depending on  $\mathcal{V}$ .

Suppose  $w \in H^1(S^1)$  is a small perturbation of  $\varphi$  such that  $H_i[w] = H_i[\varphi] + \epsilon_i$ ,  $i = 0, 1, 2$ . Then

$$\begin{aligned} F_w(M, m) &= F_{\varphi}(M, m) + M \left[ \epsilon_1 - H_0[\varphi]\epsilon_0 - \frac{1}{2}\epsilon_0^2 + \frac{12}{13}\epsilon_0 \right] \\ &\quad + \epsilon_0 \int_{S^1} w^2 \, dx + \frac{12}{13}m\epsilon_0 - \epsilon_2. \end{aligned}$$

Suppose  $\epsilon_1 < 6/13$  so that  $H_1[w] \leq 2H_1[\varphi]$ . Then, by Lemma 3.9,

$$\int_{S^1} w^2 \, dx \leq \|w\|_{H^1}^2 \leq 3\|w\|_{\mu}^2 = 6H_1[w] \leq 12H_1[\varphi]. \tag{3.16}$$

The point is that  $\int_{S^1} w^2 \, dx$  is bounded. Thus,  $F_w$  is a small perturbation of  $F_{\varphi}$ . The effect of the perturbation near the point  $(M_{\varphi}, m_{\varphi})$  can be made arbitrarily small by choosing the  $\epsilon_i$ 's small, that is,

$$F_w(M, m) = F_{\varphi}(M, m) + O(\epsilon_i).$$

Therefore, choosing  $\epsilon_i$  so small that  $|O(\epsilon_i)| < \alpha$ , we can conclude that the set where  $F_w \geq 0$  will be contained in  $\mathcal{V}$ .

Now let  $\mathcal{U}$  be given as in the statement of the lemma. Shrinking  $\mathcal{U}$  if necessary, we infer the existence of a  $\delta' > 0$  (depending on  $\mathcal{U}$ ) such that for  $u \in C([0, T]; H^1(S^1))$  with

$$|H_i[u] - H_i[\varphi]| < \delta', \quad i = 0, 1, 2, \tag{3.17}$$

it holds that the set where  $F_{u(t)} \geq 0$  is contained in  $\mathcal{U}$  for each  $t \in [0, T)$ . By Lemma 3.5 and Lemma 3.11,  $M_{u(t)}$  and  $m_{u(t)}$  are continuous functions of  $t \in [0, T)$  and  $F_{u(t)}(M_{u(t)}, m_{u(t)}) \geq 0$  for  $t \in [0, T)$ . We conclude that for  $u$  satisfying (3.17), we have

$$(M_{u(t)}, m_{u(t)}) \in \mathcal{U} \quad \text{for } t \in [0, T) \quad \text{if } (M_{u(0)}, m_{u(0)}) \in \mathcal{U}.$$

However, the continuity of the conserved functionals  $H_i : H^1(S^1) \rightarrow \mathbb{R}$ ,  $i = 0, 1, 2$ , shows that there is a  $\delta > 0$  such that (3.17) holds for all  $u$  with

$$\|u(\cdot, 0) - \varphi\|_{H^1(S^1)} < \delta.$$

Moreover, in view of the inequality (3.12), taking a smaller  $\delta$  if necessary, we may also assume that  $(M_{u(0)}, m_{u(0)}) \in \mathcal{U}$  if  $\|u(\cdot, 0) - \varphi\|_{H^1(S^1)} < \delta$ . This proves the lemma. □

*Proof of Theorem 3.1* Let  $u \in C([0, T]; H^1(S^1))$  be a solution of (1.1) and suppose we are given an  $\epsilon > 0$ . Pick a small neighborhood  $\mathcal{U}$  of  $(M_\varphi, m_\varphi)$  such that  $|M - M_\varphi| < \frac{13\epsilon^2}{144}$  if  $(M, m) \in \mathcal{U}$ . Choose a  $\delta > 0$  as in Lemma 3.12 so that (3.15) holds. Taking a smaller  $\delta$  if necessary we may also assume that

$$|H_1[u] - H_1[\varphi]| < \frac{\epsilon^2}{12} \quad \text{if } \|u(\cdot, 0) - \varphi\|_{H^1(S^1)} < \delta.$$

Applying Lemma 3.9 and Lemma 3.3, we conclude that

$$\begin{aligned} & \|u(\cdot, t) - \varphi(\cdot - \xi(t))\|_{H^1(S^1)}^2 \\ & \leq 3 \|u(\cdot, t) - \varphi(\cdot - \xi(t))\|_\mu^2 \\ & = 6(H_1[u] - H_1[\varphi]) + \frac{72}{13}(M_\varphi - M_{u(t)}) < \epsilon^2, \quad t \in [0, T), \end{aligned}$$

where  $\xi(t) \in \mathbb{R}$  is any point where  $u(\xi(t) + 1/2, t) = M_{u(t)}$ . This completes the proof of the theorem. □

*Remark 3.13* Note that our proof of stability applies to any  $u \in C([0, T]; H^1(S^1))$  such that  $H_i[u]$ ,  $i = 0, 1, 2$ , are independent of time. The fact that  $u$  satisfies (3.1) in the distributional sense was actually never used.

### 4 Comments

Some classical solutions of (1.1) exist for all time, while others develop into breaking waves (Fu et al. 2010; Khesin et al. 2008; Lenells et al. 2010). If  $u_0 \in H^3(S^1)$ ,

then there exists a maximal time  $T = T(u_0) > 0$  such that (1.1) has a unique solution  $u \in C([0, T]; H^3(S^1)) \cap C^1([0, T]; H^2(S^1))$  with  $H_0, H_1, H_2$  conserved. For  $u_0 \in H^r(S^1)$  with  $r > 3/2$ , it is known (Lenells et al. 2010) that (1.1) has a unique strong solution  $u \in C([0, T]; H^r(S^1))$  for some  $T > 0$ , with  $H_0, H_1, H_2$  conserved. However, the peakons do not belong to the space  $H^r(S^1)$  for  $r > 3/2$ . Thus, to describe the peakons one has to study weak solutions of (1.1). The question of existence and uniqueness of weak solutions to (1.1) is still open at this point. Therefore, close to a peakon, there may exist profiles that develop into breaking waves and profiles that lead to globally existing waves. Our stability theorem is applicable in both cases up to breaking time.

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