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Turing Instabilities at Hopf Bifurcation

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Abstract Turing–Hopf instabilities for reaction-diffusion systems provide spatially inhomogeneous time-periodic patterns of chemical concentrations. In this paper we suggest a way for deriving asymptotic expansions to the limit cycle solutions due to a Hopf bifurcation in two-dimensional reaction systems and we use them to build convenient normal modes for the analysis of Turing instabilities of the limit cycle. They extend the Fourier modes for the steady state in the classical Turing approach, as they include time-periodic fluctuations induced by the limit cycle. Diffusive instabilities can be properly considered because of the non-catastrophic loss of stability that the steady state shows while the limit cycle appears. Moreover, we shall see that instabilities may appear even though the diffusion coefficients are equal. The obtained normal modes suggest that there are two possible ways, one weak and the other strong, in which the limit cycle generates oscillatory Turing instabilities near a Turing–Hopf bifurcation point. In the first case slight oscillations superpose over a dominant steady inhomogeneous pattern. In the second, the unstable modes show an intermittent switching between complementary spatial patterns, producing the effect known as twinkling patterns.

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1 Introduction

After Turing's paper (Turing 1952) a great amount of work has been done looking at the conditions for diffusive instability in reaction-diffusion systems and their connections with pattern formation in a wide variety of applications. In this area, many topical features have been studied in chemical systems (Schuman and Tóth 2003; Vastano et al. 1987), but especially in the mathematical model of morphogenesis (Maini et al. 1997). The Hopf bifurcation (HB) arises in numerous contexts and is widely quoted (Marsden and McCracken 1976; Kuznetsov 1998; Edelstein-Keshet 1988). It features the sudden appearance of a small limit cycle surrounding the steady state when a parameter in the system varies slightly beyond a certain threshold, called the HB point. More recently, several studies have explored the simultaneous appearance of HB and Turing instabilities in different scenarios: chemical reactions (Yang et al. 2005), semiconductor physics (Just et al. 2001; Meixner et al. 1997), and prey–predator systems (Baurmann et al. 2007). Diffusive instabilities generated by the limit cycle are often called Turing–Hopf (TH) instabilities or bifurcations, which eventually result in oscillatory inhomogeneous patterns. For instance, in Baurmann et al. (2007) the behavior of a predator–prey system showing oscillatory patterns as a consequence of TH instabilities in the neighborhood of a TH point was studied and it was concluded that these instabilities can be considered as one important mechanism for the appearance of complex spatiotemporal dynamics in dynamic population models. Just et al. (2001) studies patterning phenomena in semiconductor heterostructures, in which voltage across the heterostructure and an internal degree of freedom play the roles of activator and inhibitor respectively. In many applied problems applications of TH instabilities have been found, but still show the lack of an appropriate normal modes theory.

Some questions arise naturally from the study of TH instabilities for a reaction-diffusion system under the additional assumption that the set of parameters remains close to a TH bifurcation point. One may consider the following: In which way do the oscillations of the stable limit cycle impact on the formation of diffusive instabilities? What are the conditions leading presumably to an intermittent behavior of the ultimate spatiotemporal pattern? Does the oscillatory behavior of the spatiotemporal instability have the same frequency as the limit cycle? To address these questions we shall introduce appropriate normal modes in the stability analysis of the limit cycle.

In this paper we shall be concerned with an analytical approach to the formation of spatiotemporal patterns at the onset of TH instabilities. To do so, we derive first an asymptotic expansion of the limit cycle solution $(\overline{u}(t), \overline{v}(t))$ to the nonlinear reaction system which appears via an HB. We study the onset of diffusive instabilities of the spatially homogeneous periodic solution $(\overline{u}(t), \overline{v}(t))$ to the following system:

$$\begin{aligned} u_t &= D_u \Delta u + f(u, v; a), \\ v_t &= D_v \Delta v + g(u, v; a) \end{aligned} \tag{1}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

This cycle solution appears while the parameter a in the reaction part of the system varies, considering now the fact that reactants can diffuse within the spatial (time-independent) bounded region Ω with regular boundary. We denote the Laplacian operator by Δ. It is well known that patterns will depend not only on the reaction dynamics, but also on geometrical features of Ω including their spatial dimension. In this paper we do not impose any restriction either to the shape or to the spatial dimension of the domain Ω, but practical limitations would arise if one looks for eigenvalues and eigenvectors to the Laplacian operator in general domains. At least, many interesting inhomogeneous patterns, consisting either of squares, hexagons, stripes or their combinations, have been studied in easy-shaped domains when $\dim \Omega = 2$. Here u, v are the profiles of component concentrations under diffusion and a is a (scalar or vector) parameter. For simplicity, we shall assume that f and g are analytical in a neighborhood of the isolated steady state. All parameters and variables in this paper are considered dimensionless. For bounded spatial domains and natural boundary conditions it is known from Henry (1981) and Leiva (1996) that the non-constant spatially homogeneous periodic solution to (1) and (2) is orbitally stable if (D_u, D_v) belongs to a certain open neighborhood of the bisectrix of the first quadrant in the Cartesian product of diffusion coefficients while the nonzero Flocquet exponent of the linearized system is negative. But the periodic solution would be unstable for any pair of diffusion coefficients if the nonzero Flocquet exponent were positive. Unfortunately the sign of this exponent using our asymptotic expansions cannot be determined. The study of TH instabilities is frequently done by determining the region of the parameter space at which bifurcations and instabilities coexist. In this paper we are considering the parameters lying on a neighborhood of a codimension-two TH point, at which the asymptotic expansion of the limit cycle and the Turing analysis of diffusive instabilities are valid. In a natural way, we describe the diffusive linear instabilities through modified Fourier normal modes, here called the extended normal modes, but which are identified in the following simply as the modes. They are valid only in the presence of the limit cycle, i.e. close to an HB point. These modes, being unstable or not, show interactions between a spatial pattern (the spatial eigenfunction) and the oscillations due to the cycle. The knowledge of such modes would serve to get a better representation of the ultimate oscillatory patterns and the way in which Turing instabilities and HB interact at the onset of instability. As we shall see, these modes suggest two possible ways in which the limit cycle might generate Turing instabilities. Naturally, it will depend on the set of parameters. In the one, the amplified function results in a superposition of slight time-periodic oscillations, with the same frequency of the cycle solution, over a prevalent Turing pattern given by the spatial eigenfunction. In the second type it is amplified as an alternated switching between "complementary" spatial patterns, with different frequency than the cycle solution. We have conceived here the modes on the basis of the asymptotic expansion, via averaging techniques, to the stable limit cycle solution via a supercritical HB in the reaction system. As in Turing's mathematical procedure, we can have only a presumption about the ultimate pattern toward which the destabilized solution converges. In fact, by the techniques developed in this paper, we cannot give any proof about whether or not the ultimate pattern is time-periodic, or about the length of time at which the fluctuating instabilities can be observed. We assume that the ultimate pattern emerges (see Murray 2001) due to the boundedness of the unstable modes by the nonlinear reaction terms in (1). And, we expect that spatiotemporal oscillations could be observed at least for a while, so that the resulting solution moves closely to some pattern, being oscillatory or not. We hypothesize that, between all destabilizing extended modes, those that grow fastest will have a prevailing influence, relaying their spatiotemporal structure on the resulting time-periodic inhomogeneous pattern. One of the most widely studied models for Turing instabilities is Schnakenberg's model, which is based on a hypothetical mechanism of autocatalytic reactions. We use this example to show how our modes work.

Currently, many applications of Turing instabilities have been intensively studied for singularly perturbed reaction-diffusion systems (see for instance Ward 2006; Ward and Wei 2003; Wei and Winter 2003) but such patterns are so far from the TH point that our modes are not valid in that context. On the other hand, the procedure developed here leading to the construction of the extended mode, can be taken when the non-flux conditions at the boundary are substituted with the Dirichlet, Robin or space-periodic conditions. In the third of the above conditions the bifurcation analysis should be done again at the spatial eigenvalue $\lambda = 0$.

The plan of the paper is as follows. In Sect. 2 we give the outline of the results. In Sect. 3 elementary results about the stability analysis of limit cycles and steady states can be found. Section 4 is devoted to the HB and, particularly, we have derived there an asymptotic expansion of the limit cycle. We shall use this expansion in the derivation of the extended modes. However, it can also be used without the ultimate interest in the analysis of diffusive instabilities. In Sect. 5 we perform an asymptotic treatment of Turing instabilities for the limit cycle. In this section the extended modes are derived and, on their basis, different types of interactions between cycles and spatial patterns near a TH point are discussed. Finally, in Sect. 6 we consider the example of TH instabilities to Schnakenberg's reaction-diffusion model. Conclusions are presented in Sect. 7.

2 Outline of Results

In this paper we first suggest a simple procedure for deriving a uniform asymptotic expansion of the limit cycle in the vicinity of the HB point for a two-dimensional reaction system. First, we present in Proposition 1 (Sect. 4) an algorithm allowing the reduction of the reaction system into a second order differential equation representing a weakly nonlinear oscillator in normal form. This transform of variables is, in general, nonlinear, but we prove that it is enough to consider the linear part of this transform to obtain the equation of the oscillator, preserving the required accuracy. The main idea of this procedure is that the transform of variables can be taken "close" to the appropriate linear transform in a neighborhood of the origin. Then, applying the Krylov–Bogoliubov–Mitropolski averaging method (Bogoliubov and Mitropolski

1961) to the oscillator, in Theorem 1 (Sect. 4) we use a discriminant function to determine the appearance of an HB, subcritical or supercritical, as in the Andronov–Hopf normal form (Kuznetsov 1998). It can be seen in Corollary 1 that the existence of at least one nonzero term with even degree in the expansion of the discriminant function is a necessary and sufficient condition to determine subcritical or supercritical bifurcations, in accordance with the sign of the coefficient in the lowest even degree nonzero term. We conclude that terms with even degree in the Taylor expansion of f and g on the right side of (1) do not have any contribution to the appearance of an HB. Only appropriate interactions within the terms having odd degree in the expansions of f and g can contribute to the appearance of an HB.

Using the asymptotic expansion to the limit cycle we shall build appropriate normal modes in Proposition 5 (Sect. 5). The extended modes, which can be unstable or not, allow the study of the central matter in this paper: the appearance of Turing instabilities of the stable limit cycle. In the classic theory, the steady patterns are associated with the region of positiveness of the spatial eigenfunction participating in the unstable Fourier mode. We use the same idea to conclude in Theorem 2 (Sect. 5) that our modes suggest two possible ways in which the limit cycle might generate Turing instabilities. In the one, the amplified function is the product of the spatial eigenfunction by a time-periodic function with average close to one, leading to a superposition of slight time-periodic oscillations with the same frequency of the cycle over a dominant Turing pattern. In the second, the amplified function is the product of the spatial eigenfunction by a time-periodic function with zero average, so the product alternates the sign in each period. Consequently, an intermittent switching between the Turing pattern and its "complementary" is expected. The frequency of these oscillations is different from the frequency of the cycle solution.

3 Preliminaries

Let us first briefly review some basic questions about stability of periodic solutions in the frame of ordinary and partial differential equations, and finally, about the Turing instability of the steady state.

3.1 Stability Analysis of the Limit Cycle

For a dynamical system on the plane

$$\dot{Y} = F(Y, a) \tag{3}$$

an isolated non-punctual closed orbit is called *limit cycle*. A source of limit cycle formation is HB. The appearance of such solutions to (3) depending on a real parameter a would happen when the parameter takes values slightly beyond a certain threshold value a_0, called the *HB point* (Marsden and McCracken 1976). The situation would be sketched as follows: for $a < a_0$ the vector steady state Y_0 defined by $F(Y_0, a) = 0$ is an unstable focus, but for $a > a_0$ this steady state is stable focus. So, the behavior of the solutions changes abruptly at the bifurcation value. The reader can find a remarkable study of the HB in Marsden and McCracken (1976), or the compact version in Edelstein-Keshet (1988).

The question of the stability of the cycle can be reduced by the standard procedure to the stability of the trivial solution of linear systems with periodic coefficients: more precisely, the Flocquet theory (Verhulst 1990). Let $\Theta(t)$ be a non-trivial T-periodic vector solution to (3). Then, making the substitution $Y=\Theta(t)+Z(t)$ and linearizing about $\Theta(t)$ we obtain the following linear equation with a periodic matrix for the vector perturbation:

$$\dot{Z}=J_{\Theta}(t)Z \tag{4}$$

where $J_{\Theta}(t)$ is the Jacobian matrix of the right side term F evaluated on the periodic solution $\Theta(t)$. The linear periodic system in (4) has the non-trivial periodic solution $\dot{\Theta}(t)$, so one of the Flocquet exponents of (4) is zero. It is known that the expression of the nonzero Flocquet exponent in terms of the mean value of the trace of the Jacobian matrix evaluated at the cycle solution is

$$\rho_2=\frac{1}{T}\int_0^{\mathrm{T}}\left(\frac{\partial f(\Theta(s))}{\partial y_1}+\frac{\partial g(\Theta(s))}{\partial y_2}\right)ds. \tag{5}$$

According to the linear theory, if the Flocquet exponent ρ_2 is negative ($\rho_2=-\upsilon$) and $Y(t)$ is a solution to (3) with initial condition $Y(t_0)$ close enough to $\Theta(t_1)$, then

$$\left\|Y(t)-\Theta(t+\phi_0)\right\|\le C\exp(-\upsilon t)$$

for $t>0$, being ϕ_0 certain real constant and $C>0$. Then, we say that $\Theta(t)$ is *orbitally asymptotically stable*.

3.2 Orbital Stability in Parabolic PDE

Assume that (3) has a limit cycle solution $\Theta(t)$ due to HB as the parameter τ_a varies, then $\Theta(t)$ is also a spatially homogeneous solution to (1) and (2). The concept of an orbitally stable solution to the system of (1) and (2) results in a natural extension of the precedent concept to an infinite dimensional dynamical system defined in an appropriate Hilbert space. More exactly, given a bounded domain $\Omega\subset\mathbb{R}^n$, let $\mathcal{A}$ be the non-negative self-adjoint linear operator in $L_2(\Omega)$ defined in the dense subset $\mathcal{D}(\mathcal{A})=\{\phi\in H^2(\Omega)|\frac{\partial\phi}{\partial n}=0 \text{ on } \partial\Omega\}$ by the correspondence $\mathcal{A}u=-\Delta u$. Here $H^2(\Omega)$ is the usual Sobolev space and the boundary condition is understood in the sense of traces. Let us consider in $\mathcal{D}(\mathcal{A})$ the equation

$$Y_t=-D\mathcal{A}Y+F(Y) \tag{6}$$

where D represents the diagonal matrix of diffusion coefficients, and the function F represents the reaction without diffusion. A solution to (6) is a continuous function defined in an open subset of $\mathbb{R}$ with values in $[\mathcal{D}(\mathcal{A})]^2\subset\mathbf{H}=[L_2(\Omega)]^2$, which satisfy the equation in a mild sense (Leiva 1996). In what follows, we will identify a solution to (6) as a solution to the boundary value problem of (1) and (2). Let us take an appropriate norm in $\mathcal{D}(\mathcal{A})$, say $\|\cdot\|$.

Definition 1 The spatially homogeneous periodic solution $\Theta(t)$ to (6) is *orbitally asymptotically stable* if there are positive constants ρ, δ, μ such that, for every solution $\Psi(t)$ to (6) satisfying

$$\min_t \left\| \Psi(t_0) - \Theta(t) \right\| \le \rho$$

for some t_0, then exists $h \in \mathbb{R}$ such that

$$\left\| \Psi(t+h) - \Theta(t) \right\| \le \delta \exp(-\mu t)$$

for all $t \ge t_0$. If $\mu = 0$, the solution $\Theta(t)$ is said to be *neutrally stable*. In any other case the solution is said to be *unstable*.

Let us recall about a general result quoted in Leiva (1996) summarized as follows.

Theorem (Diffusive Stability) *Let ρ_1, ρ_2 be the Floquet exponents for the periodic linear system of* (4), *with $\rho_1 = 0$ being simple. It can be concluded that*: (a) *If $\rho_2 > 0$, then $\Theta(t)$ is unstable for* (1). (b) *If $\rho_2 < 0$ and $0 \le |D_u - D_v|$ is small enough, then $\Theta(t)$ is orbitally asymptotically stable for* (1).

The solution $\Theta(t)$ in the above theorem is not associated necessarily to a limit cycle solution. We shall obtain in Sect. 4.1 an asymptotic expansion to $\Theta(t)$ at a supercritical HB, but substituting this expansion into (5) we cannot be conclusive about the sign of ρ_2. At least, if for instance it happens that $\rho_2 > 0$, this fact gives us the idea that the limit cycle would generate Turing instabilities even though the diffusion coefficients are equal.

3.3 Turing Instabilities of the Stable Steady State

In Turing's seminal paper (Turing 1952) a stable steady state to the reaction process without diffusion was assumed, allowing a separate analysis of the destabilizing influence of diffusion. Turing showed that dissimilar diffusion coefficients of the participating reactants would destabilize the steady state of the reaction kinetics. Such instabilities lead to steady spatially varying profiles in the reactant concentrations which are called *patterns*. The mathematical basis for these assertions is the stability analysis of spatially homogeneous solutions with compact orbit, so leading to spectral analysis. The standard Turing procedure for the stability analysis is done for systems with two chemical reactants considering *normal modes* of the type

$$Z(x,t) = \exp(\sigma t) U_k(x) R \tag{7}$$

as non-trivial solutions to the linearized equation

$$\frac{\partial Z}{\partial t} = D \Delta Z + J_a Z \tag{8}$$

where $U_k(x)$ are eigenfunctions associated to the eigenvalue λ_k ($k \in \mathbb{N}$) of the unbounded non-negative linear operator $(-\Delta)$ with Neumann boundary conditions at $\partial\Omega$. We shall call these λ_k the *spatial eigenvalues*, and the $U_k(x)$ are called the *spatial eigenfunctions*. Perturbations in the form of (7) allow us to study the stability on bounded domains, due to the fact that any small perturbation Z can be expanded in terms of such basic functions

$$Z(x,t)=\sum_{k=1}^{\infty}\exp(\sigma_k t)U_k(x)R_k, \tag{9}$$

that is, in Fourier series. In this series, eigenvalues are eventually repeated several times according with their (finite) multiplicity. The R_k are determined by the Fourier development of the initial condition. In (7) the temporal eigenvalues σ are determined by the second degree equation

$$\det(J_a-\lambda_k D-\sigma I)=0 \tag{10}$$

with R being a nonzero σ-eigenvector of the matrix $(J_a-\lambda_k D)$. Here J_a, D and I are 2×2 matrixes, which are the Jacobian matrix at the spatially homogeneous steady-state solution to (1), the diagonal matrix of diffusion coefficients and the identity matrix, respectively. Note that $\sigma=\sigma(\lambda_k)$. Now, let us take in the rest of this paragraph the subindex k as the wavevector, in order to simplify notations. Feasible wavevectors k in (7) are determined by the boundary conditions. For instance, if we consider the parallelepiped $\Omega=[0,L_1]\times\cdots\times[0,L_n]$, $x=(x_1,\dots,x_n)$, the eigenfunctions in (9) can be taken as: $U_k(x)=\prod_{j=1}^{n}\cos k_j x_j$ where $k=(k_1,\dots,k_n)$ and $k_j=(\frac{\pi}{L_j}p_j)$, for such $p_j\in\mathbb{N}$. The corresponding spatial eigenvalue to such an eigenfunction will be $\lambda_k=|k|^2=(k_1^2+\cdots+k_n^2)$.

As usual in this analysis, rather than to study the general shape of the neutral stability curves on the plane of diffusion parameters, it will be most convenient to identify the value of the ratio D_v/D_u for which the steady state becomes locally unstable. The evolution to a spatially patterned state, as D_v/D_u was varied, is the basic process generating spatial pattern in biology and chemistry (Maini et al. 1997). It is natural to expect that the behavior of solutions near the boundary has a profound effect on mode selection and robustness of patterning, as the spatial eigenvalues λ_k will depend on the kinds of the boundary conditions and on the shape of the domain. The Dirichlet, Robin and periodic boundary conditions also have to be considered in pattern formation (Golubitsky et al. 2000). A quite different scenario, not treated here, takes place when the reaction-diffusion system is considered in an unbounded domain (van der Ploeg and Doelman 2005; Sandstede and Scheel 2001).

The appearance of Turing instabilities for a given λ_k corresponds to the existence of a positive root σ to (10). In the stable steady state case we have $\tau_a<0$, hence

$$\tau_T<0 \tag{11}$$

for any eigenvalue λ_k. Then, we require

$$\delta_T<0 \tag{12}$$

to be the appearance of TI. Here

$$\tau_T=\mathrm{trace}(J_a-\lambda_k D)=\tau_a-\lambda_k(D_u+D_v), \tag{13}$$

$$\delta_T = \det(J_a - \lambda_k D) = \delta_a - \lambda_k\big(D_u j_{22}^a + D_v j_{11}^a\big) + \lambda_k^2 D_u D_v \tag{14}$$

are functions of the eigenvalue λ_k. The positive values of λ_k for which $\delta_T < 0$ are located in the open interval $\Lambda =]\lambda_-, \lambda_+[$ where $\delta_T(\lambda_\pm) = 0$. They are

$$\lambda_\pm = \frac{(D_v j_{11}^a + D_u j_{22}^a) \pm \sqrt{(D_v j_{11}^a + D_u j_{22}^a)^2 - 4D_u D_v \delta_a}}{2D_u D_v}. \tag{15}$$

The interval Λ is a non-void subset of $\mathbb{R}_+$ only if the diffusion coefficients are different enough. The region in the parameter space at which diffusive instabilities are expected (see (12)) is called the *cone of negativeness* of δ_T. Then, it will be useful to introduce the (bifurcation) parameter

$$d = \frac{D_v}{D_u}$$

to determine *threshold diffusion ratios* for the appearance of instabilities. These ratios are given by the formula

$$d_\pm = \frac{(\delta_a - j_{12}^a j_{21}^a) \pm \sqrt{-2\delta_a j_{12}^a j_{21}^a}}{(j_{11}^a)^2}. \tag{16}$$

From (12) and (14) it can be concluded that

$$d j_{11}^a + j_{22}^a > 0 \tag{17}$$

is necessary to $\emptyset \neq \Lambda \subset \mathbb{R}_+$. In the Fourier expansion of disturbances only the terms corresponding to eigenvalues $\lambda_k \in \Lambda$ can contribute to the appearance of instabilities taking into account that $\Re e\sigma(\lambda_k) > 0$. Further details about patterning in two-species chemical interactions can be found in Murray (2001) or Edelstein-Keshet (1988, Chap. 11). The end result of Turing's method (see Murray 2003) is the assumption that the small amplitude instabilities

$$Z(x,t) \approx \sum_{\lambda_k \in \Lambda} \exp(\sigma_k t) U_k(x) R_k$$

are bounded by the nonlinear terms and evolve to a spatially inhomogeneous stationary solution to (1) and (2), which is a Turing pattern.

4 The Hopf Bifurcation Revisited

In this section we present the HB in a general reaction system with two reactants. Let us consider the system equation (3) whose reaction follows the law:

$$\begin{cases} \dot{u} = f(u, v; a), \\ \dot{v} = g(u, v; a) \end{cases} \tag{18}$$

in which f and g are assumed analytical in a vicinity of the steady state given by:

$$P_a = \big(u_0(a); v_0(a)\big) \tag{19}$$

so,

$$\begin{cases} f(u_0(a), v_0(a); a) = 0, \\ g(u_0(a), v_0(a); a) = 0. \end{cases}$$

We denote by $J_a = (j_{ij}^a)$ the Jacobian matrix of (18) at the steady-state equation (19), and

$$\delta_a = \det(J_a), \tag{20}$$

$$\tau_a = \operatorname{trace}(J_a). \tag{21}$$

We assume that the eigenvalues of J_a are complex conjugate numbers for all values of a, so

$$\tau_a^2 - 4\delta_a < 0 \tag{22}$$

and further, we assume the existence of a bifurcation value a_* in which the trace τ_a vanishes to change in sign. Then, for any value a in some open neighborhood of a_* we have

$$\delta_a > 0. \tag{23}$$

Further, if $\tau_a < 0$ (respectively $\tau_a > 0$) then our steady state is a stable (respectively unstable) focus. To be more precise, we will consider τ_a as the *intrinsic* bifurcation parameter with bifurcation value $\tau_a = 0$. We will refer to a *subcritical* bifurcation if the limit cycle appears for close to zero but with negative values of τ_a, and a *supercritical* bifurcation if it appears for positive small values of τ_a. Throughout this paper, any reference to a supercritical or to a subcritical HB will relate to an intrinsic one in the above sense. Furthermore, from (22) follows $j_{12}^a \cdot j_{21}^a < 0$ and we do not loose generality assuming $j_{12}^a > 0$.

4.1 Averaging Hopf Periodic Solutions to the Reaction System

Let us rewrite (18) near the steady-state equation (19) as

$$\dot{X} = \mathcal{F}(X) = J_a X + \Psi(X) \tag{24}$$

where $X = (U(t), V(t))^{\mathrm{T}}$ are the new variables in (18), representing the translation of the steady state to the origin, and the vector function Ψ is given by the Taylor expansion of the difference

$$\Psi\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} f(u_0(a)+U, v_0(a)+V; a) \\ g(u_0(a)+U, v_0(a)+V; a) \end{pmatrix} - J_a\begin{pmatrix} U \\ V \end{pmatrix}.$$

Hence, $\Psi(X)$ contains all nonlinearities. Let us do, in (24), an invertible analytical transform of coordinates of a neighborhood of the origin onto another

$$Y = \mathcal{H}(X) = \Gamma X + \mathcal{G}(X) \tag{25}$$

driven by a non-singular matrix Γ, and the analytic vector function:

$$\mathcal{G}(X) = \begin{pmatrix} \overline{\varphi}(U,V) \\ \overline{\psi}(U,V) \end{pmatrix} = \begin{pmatrix} \sum_{n=2}^{+\infty} \sum_{i+j=n} \overline{\varphi}_{ij} U^i V^j \\ \sum_{n=2}^{+\infty} \sum_{i+j=n} \overline{\psi}_{ij} U^i V^j \end{pmatrix} \tag{26}$$

which is assumed to have a positive radius of convergence. Note that, in the Inverse Function Theorem (Rudin 1976), the existence of the inverse is warranted only being Γ non-singular and $\mathcal{H}$ with smooth continuous derivatives. We shall denote the inverse

$$X = \mathcal{H}^{-1}(Y) = \Gamma^{-1} Y + \mathcal{K}(Y) \tag{27}$$

where

$$\mathcal{K}(Y) = \begin{pmatrix} \underline{\varphi}(z,\dot{z}) \\ \underline{\psi}(z,\dot{z}) \end{pmatrix} = \begin{pmatrix} \sum_{n=2}^{+\infty} \sum_{i+j=n} \underline{\varphi}_{ij} z^i (\dot{z})^j \\ \sum_{n=2}^{+\infty} \sum_{i+j=n} \underline{\psi}_{ij} z^i (\dot{z})^j \end{pmatrix}. \tag{28}$$

If $\mathcal{H}$ is such that

$$Y = \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \tag{29}$$

being $z(t)$ an unknown function, the integration of the system in (18) can be reduced to the integration of a second order differential equation in the variable z. As we shall see, for a given vector field, rather than the exact expressions of the functions $\overline{\varphi}(U,V)$, $\overline{\psi}(U,V)$, $\underline{\varphi}(z,\dot{z})$ and $\underline{\psi}(z,\dot{z})$ it will only be necessary to get a few appropriate coefficients in their Taylor developments to preserve the required accuracy.

Proposition 1 *Let us assume that* (22) *holds. Then, there exists an invertible transform of variables in* (25) *such that* (29) *holds. The matrix Γ is any non-trivial linear combination of*

$$\left\{ \begin{pmatrix} 1 & 0 \\ j_{11}^a & j_{12}^a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ j_{21}^a & j_{22}^a \end{pmatrix} \right\}. \tag{30}$$

The function z in (29) *satisfies the following second order equation*:

$$\ddot{z} - \tau_a \dot{z} + \delta_a z = G(z,\dot{z}) \tag{31}$$

where the right-hand side in (31) *does not involve linear terms in z, $\dot{z}$. More precisely*,

$$\begin{aligned} G(z,\dot{z}) = \Pi_2 \big\{ & \Gamma \big(J_a \mathcal{K}(Y) + \Psi\big(\mathcal{H}^{-1} Y\big)\big) \\ & + \langle \mathtt{grad}_X \mathcal{G}, \mathcal{F} \rangle \big(\mathcal{H}^{-1} Y\big) \big\}, \end{aligned} \tag{32}$$

Π_2 being the standard projector over the second component.

Proof The proof follows directly substituting (25) into (24). $\mathcal{H}$ in (25) defines an invertible transform of variables because Γ, which is the Jacobian of $\mathcal{H}$ at the origin, is assumed invertible (Rudin 1976). Further, $\mathcal{H}$ verifies (29) if and only if

$$\frac{d}{dt}(\Pi_1\mathcal{H}) = \Pi_2\mathcal{H}. \tag{33}$$

Equation (33) implies that the components γ_{ij} of Γ verify the following "concordance" condition with the Jacobian of the system at the steady state:

$$J_a^{\mathrm{T}}\begin{pmatrix}\gamma_{11}\\ \gamma_{12}\end{pmatrix} = \begin{pmatrix}\gamma_{21}\\ \gamma_{22}\end{pmatrix}. \tag{34}$$

The above condition is equivalent to the fact that the matrix Γ is a linear combination of the matrixes in (30). Furthermore, due to (22) the eigenvalues of matrix J_a^{T} are complex, then $(\gamma_{11}, \gamma_{12})^{\mathrm{T}}$ is not an eigenvector of J_a^{T} and it follows that $\det\Gamma \neq 0$. The generators of Γ are obtained solving the linear system of (34). For any matrix Γ satisfying (34), we get

$$\Gamma J_a \Gamma^{-1} = \begin{pmatrix}0 & 1\\ -\delta_a & \tau_a\end{pmatrix}.$$

For a given $\overline{\varphi}(U, V)$, the function $\overline{\psi}(U, V)$ in (26) is determined from the equality

$$\overline{\psi}(U, V) = \langle \mathtt{grad}_X\overline{\varphi}, \mathcal{F}\rangle = \frac{\partial\overline{\varphi}}{\partial U}\dot{U} + \frac{\partial\overline{\varphi}}{\partial V}\dot{V}, \tag{35}$$

and it follows that (25) represents an invertible change of coordinates satisfying (33). In this process, one can consider that $\overline{\psi}$ is *engendered* by $\overline{\varphi}$. All the coefficients of $\overline{\psi}(U, V)$ are obtained in unique way if we substitute the second component in (26) and $\overline{\varphi}(U, V)$ into (35). Note that taking $\overline{\varphi}$ as a polynomial function, it follows that $\overline{\psi}$ will also be a polynomial.

All the coefficients of the components of $\mathcal{K}(Y)$ will be determined equating the corresponding terms in

$$\begin{aligned} Y &= \mathcal{H}\circ\mathcal{H}^{-1}(Y) = \Gamma\mathcal{H}^{-1}(Y) + \mathcal{G}\big(\mathcal{H}^{-1}(Y)\big)\\ &= Y + \Gamma\big(\mathcal{K}(Y)\big) + \mathcal{G}\big(\mathcal{H}^{-1}(Y)\big). \end{aligned} \tag{36}$$

This procedure leads to a hierarchy of equations from which all of the required coefficients in the inverse transform can be determined. More precisely, for each pair (i, j) such that $i + j \geq 2$, a pair of coefficients $\underline{\varphi}_{ij}$ and $\underline{\psi}_{ij}$ from a linear algebraic system with matrix Γ will be univocally determined. The functions $\underline{\varphi}$ and $\underline{\psi}$ are not necessarily polynomials even though $\overline{\varphi}$ and $\overline{\psi}$ are, but $\underline{\varphi}$ and $\underline{\psi}$ have positive radius of convergence if $\overline{\varphi}$ and $\overline{\psi}$ have also. Finally, computing derivatives in (25) and substituting (27) follows

$$\begin{aligned} \dot{Y} &= \Gamma\big(J_a X + \Psi(X)\big) + \langle \mathtt{grad}_X\mathcal{G}, \mathcal{F}\rangle\\ &= \big(\Gamma J_a\Gamma^{-1}\big)Y + \Gamma\big(J_a\mathcal{K}(Y) + \Psi\big(\mathcal{H}^{-1}Y\big)\big) + \langle \mathtt{grad}_X\mathcal{G}, \mathcal{F}\rangle\big(\mathcal{H}^{-1}Y\big) \end{aligned}$$

and taking second components we get (31). □

Remark 1 Let f and g in (18) be polynomial functions with degree M. The degree of $\overline{\varphi}$ larger than M can be taken and the coefficients $\overline{\varphi}_{ij} = 0$ in (26) can be selected at least while $i + j \leq M$. Then, $\overline{\psi}$ is determined by (35) and consequently $\overline{\psi}_{ij} = 0$ at least while $i + j \leq M$. Furthermore, from (36) follows $\underline{\varphi}_{ij} = 0$ and $\underline{\psi}_{ij} = 0$ while $i + j \leq M$. So, in a neighborhood of the origin the transform H is "close enough" to the matrix transform Γ and the function in (32) can be approximately represented as

$$G(z, \dot{z}) = \Pi_2\{\Gamma(\Psi(\Gamma^{-1}Y))\} + O(\|Y\|^{M+1}). \tag{37}$$

For instance, a suitable transform is that engendered by $\overline{\varphi} = U^{M+1}$.

From (31) we look for an oscillation with positive and small, but finite, amplitude ε. The small parameter ε will be "identified" later. Taking in (31) the change of variables

$$z(t) = \varepsilon\varsigma(t), \tag{38}$$

follows the equation of a weakly nonlinear oscillator in *normal form*:

$$\ddot{\varsigma} - \tau_a\dot{\varsigma} + \delta_a\varsigma = \varepsilon G(\varsigma, \dot{\varsigma}; \varepsilon). \tag{39}$$

Then, the cycle solution to (18) will correspond to the non-trivial periodic solution to (39). It is known that averaging methods provide an algorithm for preparing HB problems (Marsden and McCracken 1976, Sect. 4C), and we shall apply the Krylov–Bogoliubov–Mitropolski averaging method (Sanders and Verhulst 1985; Bogoliubov and Mitropolski 1961) in order to find an asymptotic expansion to the solution. So, let us consider the new variables $r = r(t)$ and $\theta = \theta(t)$ defined as follows:

$$\varsigma = r\cos(t + \theta), \tag{40}$$

$$\dot{\varsigma} = -r\sin(t + \theta); \tag{41}$$

then, the corresponding averaged equations are

$$\dot{r} = -\frac{1}{2\pi}\int_0^{2\pi} \sin\phi\{-\tau_a r\sin\phi + \varepsilon G(r\cos\phi, -r\sin\phi; \varepsilon)\}\, d\phi, \tag{42}$$

$$\dot{\theta} = -\frac{1}{2\pi r}\int_0^{2\pi} \cos\phi\{-\tau_a r\sin\phi + \varepsilon G(r\cos\phi, -r\sin\phi; \varepsilon)\}\, d\phi \tag{43}$$

and finally,

$$\dot{r} = \frac{r}{2}\{\tau_a - p(r; \varepsilon)\}, \tag{44}$$

$$\dot{\theta} = q(r; \varepsilon) \tag{45}$$

in which

$$p(r; \varepsilon) = \frac{\varepsilon}{\pi r}\int_0^{2\pi} \sin\phi G(r\cos\phi, -r\sin\phi; \varepsilon)\, d\phi, \tag{46}$$

$$q(r;\varepsilon)=-\frac{\varepsilon}{2\pi r}\int_0^{2\pi}\cos\phi G(r\cos\phi,-r\sin\phi;\varepsilon)\,d\phi. \tag{47}$$

Proposition 2 *Functions $p(r;\varepsilon)$ and $q(r;\varepsilon)$ have at least order $O(\varepsilon^2)$ or, equivalently, $p(r;\varepsilon)/r^2$ and $q(r;\varepsilon)/r^2$ have a finite limit as $r\to 0$. Moreover, the Taylor expansions of $p(r;\varepsilon)$ and $q(r;\varepsilon)$ must not contain odd powers of r.*

Proof Substituting into (46) and (47) the polynomial expansion of the function G given in (37) we will find, in each term, factors of the form

$$\int_0^{2\pi}\sin^m\phi\cos^n\phi\,d\phi \tag{48}$$

which are nonzero only for some pairs (m,n) such that $m+n$ is an even integer. In particular, the first assertion follows from the fact that the polynomial expansion of $G(r\cos\phi,-r\sin\phi;\varepsilon)$ does not have linear terms and quadratic terms have projection zero over the subspace of $L_2(0,\pi)$ spanned by $\{\sin\phi,\cos\phi\}$. □

Proposition 3 *If the function $p(r;\varepsilon)$ is not identically zero, there must exist a positive integer N and a positive real number r_0 such that $p(r,\varepsilon)$ has the non-trivial Taylor expansion*:

$$p(r;\varepsilon)=\omega\varepsilon^{2N}r_0^{-2N}r^{2N}+O\big(\varepsilon^{2N+2}r^{2N+2}\big) \tag{49}$$

where $\omega=+1$ or -1. Let us assume the existence of a positive root r to the equation

$$p(r;\varepsilon)-\tau_a=0 \tag{50}$$

for positive (respectively negative) values τ_a sufficiently close to zero. Then, up to the leading term, the root to (50) has the form

$$r=r_0\left(\frac{|\tau_a|}{\varepsilon^{2N}}\right)^{\frac{1}{2N}}+O\big(\varepsilon^2\big). \tag{51}$$

Proof The first assertion follows directly from (46) and the polynomial expansion of $G(z,\dot{z};\varepsilon)$. From (49), to the existence of a positive root to (50) in the form of (51), it is necessary and sufficient that $\omega=\mathtt{sign}\,\tau_a$. □

Let us now make the association between the bifurcation parameter and the small amplitude of the periodic oscillations. From here, we will take the following

Definition 2 Let us assume the existence of the positive root to (51) either for positive or for negative values of τ_a. Then, the small parameter ε, heuristically introduced in (38), will be taken as

$$\varepsilon^{2N}=|\tau_a|. \tag{52}$$

Remark 2 From (52) it follows that the root in (51) can now be written as

$$r = r_0 + O\left(|\tau_a|^{\frac{1}{N}}\right). \tag{53}$$

Due to Proposition 3 the function p in (46) will play the role of a "discriminant" with respect to the HB, as we shall see in the next theorem. Let us now consider the following approach to the well-known Hopf bifurcation theorem for the case $n = 2$. We will approximate the periodic solution to (39) via (40) and (41) with the periodic solution to the averaged equations (42) and (43), respectively. Our approach follows, from this point, in a similar way to that in the study of the so-called Andronov–Hopf bifurcations (Kuznetsov 1998, Chaps. 2 and 3), in which the stability and bifurcation analysis is done by studying the separate equation for the radius in polar coordinates.

Theorem 1 (Averaging in Hopf bifurcation) *Let us assume that* (22) *holds and that* (50) *has the root to* (51) *for positive* (*respectively*, *negative*) *but sufficiently close to zero values of the bifurcation parameter* τ_a. *Then*, *a limit cycle to the system in* (18) *appears*. *This bifurcation is supercritical if the root exists for* $0 < \tau_a \ll 1$, *and subcritical if the root exists for* $0 < -\tau_a \ll 1$. *Furthermore*, *the limit cycle is orbitally asymptotically stable* (*respectively*, *unstable*) *if and only if the bifurcation is supercritical* (*respectively*, *subcritical*).

Proof The existence of a positive root to (50) is equivalent to the existence of a circle limit cycle to the averaged system equations (44) and (45). If the steady state is stable (unstable) then the cycle must be unstable (stable). In other words, the sign of τ_a must be equal to the sign of $p(r;\varepsilon)$ being $0 < r < r_0$ and $|\tau_a|$ small enough. □

Corollary 1 *Let us assume that* (22) *holds*. *A necessary* (*but not sufficient*) *condition to the appearance of an HB is the existence of nonzero terms of odd order in the expansion of the polynomial functions* f *or* g.

Proof From (37) it follows that only the existence of odd-order terms in the polynomials f or g having nonzero coefficients would lead to the existence of (even order) terms having nonzero coefficients in the expansion of (49) and, consequently, to the possible existence of a root to (50). Moreover, the condition is obviously not sufficient, because the numbers in (48) are involved too. □

In Just et al. (2001) the authors derived amplitude equations using a weakly nonlinear analysis near a codimension-two TH point, provided periodic boundary conditions, and considered a small parameter which has, in that context, the same meaning of $(\tau_a)^{1/2}$. They observed, for the system studied in that paper, that the HB is solely determined by the third-order terms, and this fact agrees with our conclusion in Corollary 1.

Remark 3 We can also conclude that, if the function $G(z,\dot{z})$ defined in (37) has a polynomial expansion conformed exclusively by terms of even degree, then neither supercritical nor subcritical bifurcations will appear. Moreover, it follows that an

HB will not appear in dimension two if the polynomial system of (18) has at most quadratic components. Corollary 1 is in accordance with the topological normal form for the HB (Kuznetsov 1998, Theorem 3.4). We point out that Corollary 1 only concerns the HB. For instance, there are examples of quadratic dynamical systems of two components in which a limit cycle appears, but as a consequence of a homoclinic bifurcation (Kuznetsov 1998, Chap. 6). Corollary 1 also agrees with the more general question of the nonexistence of limit cycles in two-species second order kinetics (see Schuman and Tóth 2003 and references therein). Note that Wilhelm and Heinrich (1996) gives an example of a three-dimensional system of degree two which exhibits the HB.

Remark 4 Higher order interactions in the reaction system of (18) would lead to the existence of more than one positive root of (50). In such cases, following the idea in Theorem 1 one could conclude the existence of multiple limit cycles after the HB. We point out that roots to (50) may be not properly associated with the HB, as happens at the Bautin bifurcation normal form Kuznetsov (1998). In Hofbauer and So (1994) one can find a reference to the evidence of multiple limit cycles as a consequence of an HB, albeit for a 3D competitive predator–prey system. There, it was expected that the parameter range in which several cycles coexist is rather small, so it would be hard to find them by numerical integration. We mention that, in a quite different context, multiple Hopf bifurcations have been reported to the Navier–Stokes equations in rotating cylinders (Marques et al. 2003).

From (45) we will obtain the angular speed of the oscillation. Finally, going back to the substitutions given in (38) and (25) we shall obtain the uniform asymptotic expansion of the solution to (31). We develop the periodic solution $\Theta(t)=(\overline{u}(t),\overline{v}(t))$ to (18) which generates the limit cycle, by

$$\overline{u}(t)=u_0(a)+u_1(t)\big(|\tau_a|\big)^{\frac{1}{2N}}+O\big(|\tau_a|^{\frac{1}{N}}\big), \tag{54}$$

$$\overline{v}(t)=v_0(a)+v_1(t)\big(|\tau_a|\big)^{\frac{1}{2N}}+O\big(|\tau_a|^{\frac{1}{N}}\big) \tag{55}$$

where $(u_0(a); v_0(a))$ are given in (19), and

$$\begin{pmatrix} u_1(t)\\ v_1(t)\end{pmatrix}=r_0\Gamma^{-1}\begin{pmatrix}\cos(\varpi t)\\ -\sin(\varpi t)\end{pmatrix} \tag{56}$$

with frequency given

$$\varpi=1+q\big(r_0,\big(|\tau_a|\big)^{\frac{1}{2N}}\big) \tag{57}$$

and period $T=\frac{2\pi}{\varpi}$. Note that, up to the leading terms, the expansions of the cycle solution in (54) and (55) are uniform, as the $O(|\tau_a|^{\frac{1}{2N}})$-terms are bounded functions.

5 Turing–Hopf Instabilities

By studying the appearance of TH instabilities for the stable limit cycle solution to the reaction system due to a supercritical HB, we implicitly are considering unstable the steady state. In this situation, fortunately, for any initial condition close to the steady state the corresponding outgoing solution is bounded for all t, and so the loss of stability is non-catastrophic. These instabilities have been extensively studied near codimension-two bifurcations (see for instance Just et al. 2001; Meixner et al. 1997), meaning that the analysis is done near a point at the intersection of the manifolds $\delta_T = 0$ and $\tau_a = 0$ in the parameter space. Our analysis in this paper is also restricted to this frame. In this section we shall see an asymptotic approach to the analysis of diffusive instabilities for the stable cycle by taking into account the behavior near the unstable steady state. So, it is reasonable to study diffusive instabilities for an unstable steady state in the presence of a supercritical HB, because wavenumber-zero instabilities will not be a source of unbounded solutions. We remark that the procedure in this section does not apply for subcritical HB, because the destabilizing effects of perturbations for the unstable limit cycle after HB or, for the unstable steady state before HB, could introduce instabilities which are not properly related with pattern formation.

5.1 Turing Instabilities for the Stable Limit Cycle

The analysis of Turing instabilities near the stable steady state via the normal modes in (7) is to determine whether the growth rate σ, which is a root to (10), can ever have a positive real part. As we shall see, we will introduce appropriate normal modes for representing the TH instabilities and again the values of σ will determine the stability. In the Turing analysis of the stable steady state, being $\tau_a < 0$ follows that $\tau_T < 0$; hence, the condition for the appearance of instabilities is given by $\delta_T < 0$. But, in the presence of a limit cycle we have $\tau_a > 0$, so according to the formula in (13) the sign of τ_T would vary as λ_k does. Further, if $\tau_T \leq 0$, the condition for diffusive instabilities is $\delta_T < 0$. In addition, we shall consider the remaining possibility,

$$\tau_T > 0. \tag{58}$$

Of course, if there is a spatial eigenvalue for which (58) holds, it should be one of the lower ones. If (58) holds, for any value of δ_T there is at least one root σ to (10) having positive real part. Then, the condition of (58) itself destabilizes the system, but in dependence of δ_T will do it in different ways. Now we shall study small perturbations of the spatially homogeneous periodic solution

$$\Theta(t) = \big(\overline{u}(t), \overline{v}(t)\big) \tag{59}$$

to (1) and (2). As before, we are denoting by $\Theta(t)$ the corresponding solution to (18) via a supercritical HB, the components of which are given in (54) and (55). Denoting the corresponding perturbations by capital letters, we get

$$u(t,x) = \overline{u}(t) + U(t,x),$$
$$v(t,x) = \overline{v}(t) + V(t,x),$$

substituting which into (1) and linearizing we get the following system with periodic coefficients for the perturbations:

$$\frac{\partial Z}{\partial t} = D\Delta Z + J_\Theta(t)Z \tag{60}$$

where $Z(x,t)$ is the column vector with components U and V satisfying (2). Substituting the development of Θ into (54) and (55) into the Jacobian $J_\Theta(t)$, we get

$$J_\Theta(t) = J_a + \tau_a^{\frac{1}{2N}} J_{1/2N}(t) + O\left(\tau_a^{\frac{1}{N}}\right)$$

where

$$J_{1/2N}(t) = (\kappa_{ij}) \tag{61}$$

is a time-periodic matrix where each κ_{ij} is the differential of the partial derivative of f or g with respect to u and v evaluated at $(u_0; v_0)$; $u_1(t)$, $v_1(t)$. The first subindex concerns f and g respectively, and the second corresponds to u and v respectively. For instance,

$$\kappa_{11} = \left.\frac{\partial^2 f}{\partial u^2}\right|_{(u_0;v_0)} u_1(t) + \left.\frac{\partial^2 f}{\partial u \partial v}\right|_{(u_0;v_0)} v_1(t)$$

and $u_1(t)$, $v_1(t)$ are given in (56).

Let us assume that the solutions to (60) could be asymptotically developed in the small parameter as follows:

$$Z = Z_0(t,x) + \tau_a^{\frac{1}{2N}} Z_1(t,x) + O\left(\tau_a^{\frac{1}{N}}\right). \tag{62}$$

Then, substituting (62) into the system equation (60), we get a hierarchy of equations determining Z_j:

$$\frac{\partial Z_0}{\partial t} = D\Delta Z_0 + J_a Z_0, \tag{63}$$

which correspond to the $O(1)$ terms, and

$$\frac{\partial Z_1}{\partial t} = D\Delta Z_1 + J_a Z_1 + J_{1/2N}(t) Z_0 \tag{64}$$

which correspond to the $O(\tau_a^{\frac{1}{2N}})$ terms. The functions Z_0, Z_1 are determined from (63) and (64) considering homogeneous Neumann boundary conditions in both cases. As the solution Z_0 can be expanded in a Fourier series, we would focus only on terms with the form of (7). Corresponding to such Z_0 we expect the solution to (64) to be in the form

$$Z_1(t,x) = \exp(\sigma t) U_k(x) W_k(t) R, \tag{65}$$

where the matrix-valued function $W_k(t)$ satisfies the equation

$$\dot{W}_k = (E_k - \sigma I) W_k + J_{\frac{1}{2N}}(t) \tag{66}$$

being

$$E_k = J_a - \lambda_k D,$$

and σ being an eigenvalue of E_k, i.e. a root to (10). First note that

$$J_{\frac{1}{2N}}(t) = K\cos(\varpi t) + L\sin(\varpi t)$$

where K and L are constant matrixes. Let us look for a particular solution to (66) in the form

$$\underline{W_k}(t) = A\cos(\varpi t) + B\sin(\varpi t) \tag{67}$$

where A and B are constant matrixes. Substituting both formulae into (66) and equating the corresponding terms, a linear algebraic matrix system is obtained with a solution given by

$$A = -\big(\varpi^2 I + (E_k - \sigma I)^2\big)^{-1}\big(\varpi L + (E_k - \sigma I)K\big), \tag{68}$$

$$B = \big(\varpi^2 I + (E_k - \sigma I)^2\big)^{-1}\big(\varpi K - (E_k - \sigma I)L\big), \tag{69}$$

because the numbers $(\sigma \pm i\varpi)$ are not eigenvalues of E_k. Then, the general solution to (66) has the form

$$W_k(t) = \exp\big((E_k - \sigma I)t\big)C + \underline{W_k}(t).$$

We are particularly interested in the solution to (66) verifying $W_k(0) = 0$, which is given by

$$\overline{W_k}(t) = -\exp\big((E_k - \sigma I)t\big)A + \underline{W_k}(t). \tag{70}$$

Let us call $\widehat{\sigma}$ the other eigenvalue of E_k. The important question about the boundedness of the solution of (70) as $t \to +\infty$, or equivalently, the stability of the linear equation (66), is determined by the eigenvalues of $(E_k - \sigma I)$, which can be easily computed from σ and $\widehat{\sigma}$.

Proposition 4 *Let σ be a root of* (10) *having positive real part. The system equation* (66) *is stable if one of the following conditions holds*: (i) $\tau_T \le 0$ *and* $\delta_T < 0$; (ii) $\tau_T > 0$ *and* $\delta_T \le 0$; (iii) $\tau_T > 0$ *and* $\delta_T \ge \frac{1}{4}\tau_T^2$. *Moreover, if*: (iv) $\tau_T > 0$ *and* $0 < \delta_T < \frac{1}{4}\tau_T^2$, *then* (66) *is unstable whenever σ is the lowest root of* (10), *and it is stable otherwise.*

Proof The proof follows from the eigenvalues of the matrix $(E_k - \sigma I)$, in which σ is an eigenvalue of the matrix E_k with positive real part. In the first three referenced cases the matrix $(E_k - \sigma I)$ has one eigenvalue equal to zero and the other having non-positive real part. In (iv) $(E_k - \sigma I)$ has one eigenvalue zero and the other would be either negative or positive in dependence on whether or not σ is the greatest of the two positive eigenvalues of E_k. □

The importance of the boundedness of $\overline{W_k}(t)$ is in connection with the uniformity in the expansion between brackets in the following result.

Proposition 5 *Let us assume the existence of a supercritical HB as in Theorem 1, and let $0<\tau_a\ll 1$. Then, the extended normal modes*

$$Z(x,t)=\exp(\sigma t)U_k(x)\big\{I+\tau_a^{\frac{1}{2N}}\overline{W_k}(t)+O\big(\tau_a^{\frac{1}{N}}\big)\big\}R \tag{71}$$

are asymptotic expansions of solutions to (60) or, more exactly, they are normal mode disturbances corresponding to the spatial eigenvalue λ_k in the stability analysis of $\Theta(t)$ (59) as a spatially homogeneous solution to (1) and (2).

Proof From (22) follows (23), and (39) corresponds to a weak oscillator so the procedure in Sect. 4.1 can be followed step by step. Let the solution Z_0 to (63) have the form of (7), in which the exponent σ is an eigenvalue of the matrix E_k. As the initial datum does not depend on τ_a, the initial condition for Z_1 must be taken equal to zero, and it follows that the initial condition to (66) is $W_k(0)=0$. Hence, the appropriate function $W_k(t)$ in (65) is the $\overline{W_k}(t)$ given in (70). □

Now,

Proposition 6 *The expansion between brackets in (71) is uniform up to the leading term or can be easily transformed into a uniform one.*

Proof This expansion is uniform whenever $\overline{W_k}(t)$ be bounded for $t>0$. On the contrary, the time-dependent $O(\tau_a^{\frac{1}{2N}})$-term in (71) shows an exponential growth that could cause a loss of uniformity in the expansion inside the brackets. In accordance to Proposition 4 there is only one situation at which $\overline{W_k}(t)$ is unbounded, which occurs when the real roots to (10) are $0<\sigma<\widehat{\sigma}$. However, in this last situation we could transform (71) in order to return to the uniformity with the extraction of the exponential as

$$\begin{aligned} Z(x,t) &= \exp(\sigma t)U_k(x)\exp\big((\widehat{\sigma}-\sigma)t\big) \\ &\quad\cdot\big\{e(t)+\tau_a^{\frac{1}{2N}}b(t)+O\big(\tau_a^{\frac{1}{N}}\big)\big\}R \\ &= \exp(\widehat{\sigma}t)U_k(x)\big\{\tau_a^{\frac{1}{2N}}b(t)+e(t)+O\big(\tau_a^{\frac{1}{N}}\big)\big\}R \end{aligned} \tag{72}$$

where $e(t)$ and $b(t)$ are matrix functions, the first of which has a norm tending exponentially to zero and the second is convergent as $t\to+\infty$. □

As in the former Turing analysis, the appearance of instabilities depends on the eigenvalue σ which is connected with the remaining parameters and the spatial eigenvalues through (10). The extended modes, being unstable or not, represent small disturbances only for $0<\tau_a\ll 1$, near an HB point. Far from the codimension-two TH point in the parameter space different kinds of instabilities would appear (see Yang et al. 2005; Ward 2006). We implicitly assumed the existence of a bounding domain (see Murray 2001) in the phase space of the reaction system which simultaneously contains the steady state, the limit cycle and the solutions when diffusion is included.

Furthermore, if TH instabilities associated with the spatial eigenvalue λ_k appear, we shall consider that the linearly unstable extended modes are bounded by the nonlinear terms and from these interactions emerges the ultimate steady or spatiotemporal pattern.

In the classical Turing analysis an essential assumption is that $\tau_a < 0$, which ensures the stability of the steady state in the absence of diffusion. This assumption and (17) imply together that Turing instabilities would appear only if $d \neq 1$. But, we should consider $0 < \tau_a \ll 1$ while the limit cycle exists, so we would expect the appearance of TH instabilities even if $d = 1$, i.e., even if $D_u = D_v$. This phenomenon was observed in Vastano et al. (1987) in connection with the Gray–Scott model. The sign of τ_T (see (13)) becomes relevant in the study of these instabilities, because $\tau_a > 0$ holds. For instance, real or even complex roots with positive real part to (10) always appear if $\tau_T > 0$, and it would be interesting to study the way in which the oscillations due to the limit cycle are transferred to the resulting diffusive instabilities. We shall address this question by an analysis using the extended modes.

Definition 3 Let us assume that the reaction part in (1) admits a supercritical HB and let $0 < \tau_a \ll 1$. Then, TH instabilities generated by the limit cycle arise if $\Re e(\sigma) > 0$. We shall call this *a weak TH instability* if there is at least one real root $\sigma > 0$ to (10). If the roots to (10) are complex conjugated $\sigma = \sigma_r \pm i\sigma_i$ with $\sigma_r > 0$, then we shall call it *a strong TH instability*.

The following theorem shows the reason for the above definition.

Theorem 2 *Let λ_k be a given positive spatial eigenvalue*; τ_T *and* δ_T *are given in* (13) *and* (14) *respectively. Assume further that the reaction system has a limit cycle via an HB. If* $\tau_T \leq 0$, $\delta_T < 0$, *then TH instabilities appear and they are weak. If* $\tau_T > 0$, *instabilities appear and they are weak provided* $\tau_T^2 - 4\delta_T \geq 0$, *while they are strong if* $\tau_T^2 - 4\delta_T < 0$. *If the diffusion coefficients are equal* ($d = 1$) *or close enough to each other, only strong TH instabilities would appear.*

Proof Let us recall that the appearance of a limit cycle due to a supercritical HB means that $0 < \tau_a \ll 1$, so this inequality does not necessarily contradict (17) and we would expect instabilities even if $d = 1$. Furthermore, such instabilities are weak or strong depending on whether the destabilizing σ is real and positive or complex with positive real part. If $\tau_T \leq 0$ and $\delta_T < 0$, the destabilizing values of σ are real. If $\tau_T > 0$, the values of σ would be real or complex depending on whether $\tau_T^2 - 4\delta_T \geq 0$ or not. The last assertion is a consequence of (22) and the equation

$$\tau_T^2 - 4\delta_T = \left(\tau_a^2 - 4\delta_a\right) + 2\lambda_k\left(j_{11}^a - j_{22}^a\right)(D_v - D_u) + \lambda_k^2(D_v - D_u)^2, \tag{73}$$

the right side of which is negative if $\lambda_k(D_v - D_u) = 0$, followed with the continuity argument. □

Then, we shall see two possible ways in which the limit cycle generates Turing instabilities near a TH point. Weak instabilities, appearing while $\overline{W_k}(t)$ is stable, are depicted as slight oscillations with the frequency of the cycle solution over a prevalent inhomogeneous pattern associated with the set of positiveness of the spatial eigenfunction, and these oscillations become greater with the increment of the amplitude of the limit cycle, i.e. with the increment of τ_a. From (72) it can be concluded that, with $\overline{W_k}(t)$ being unstable, the relative influence of the oscillatory behavior in the resultant matrix between brackets dwindles with time. Further, as time passes the reinforced function moves forward with a prevalent (but slight) steady inhomogeneous pattern which is not exclusively associated with the spatial eigenfunction. On the other hand, under strong instabilities the extended modes show reinforced time-periodic oscillations with frequency σ_i, like an alternate switching between the steady pattern, associated with the set of positiveness of the eigenfunction $U_k(x)$, and its "complementary" pattern, associated with the set of negativeness of the same eigenfunction. In this situation, the real part of (71) can be represented asymptotically as

$$Z(x,t) = \exp(\sigma_r t)\left\{\cos(\sigma_i t)U_k(x) + O\left(\tau_a^{\frac{1}{2N}}\right)\right\}R. \tag{74}$$

Recall that the steady Turing pattern associated with an unstable normal mode in (7) is conveniently depicted (see Murray 2003, §2.4) by the spatial region at which $U_k(x) > 0$. Let us do the same for TH patterns on the presumption that a dominant mode drives the behavior of the ultimate periodic inhomogeneous pattern. We suggest to sketch in TH patterns if the instabilities are weak and $\overline{W_k}(t)$ stable, by considering the oscillations of the function

$$U_k(x)\left\| I + \tau_a^{\frac{1}{2N}}\overline{W_k}(t)\right\| \tag{75}$$

having the frequency of the cycle. If TH instabilities are weak and $\overline{W_k}(t)$ is unstable, it will be more successful to take

$$U_k(x)\left\langle\left(\tau_a^{\frac{1}{2N}} b(t) + e(t)\right)v, v\right\rangle \tag{76}$$

where v is an appropriate nonzero vector (for instance, say that $\lim_{t\to+\infty} b(t)v \neq 0$) and the brackets represents the scalar product in $\mathbb{R}^2$. From (72) it can be noted that the oscillatory behavior in this situation dwindles with time. For strong TH instabilities we would depict the oscillatory pattern by the set of positiveness of

$$\cos(\sigma_i t)U_k(x) \tag{77}$$

oscillating with the frequency $\sigma_i = \sqrt{\delta_T - \tau_T^2/4}$ which is different from the frequency of the limit cycle.

We conclude that weak TH instabilities, being $\overline{W_k}(t)$ stable, provide small periodic changes in the intensity of a dominant steady Turing pattern; if $\overline{W_k}(t)$ be unstable, from (76) one would expect that a steady inhomogeneous pattern dominated initially ($\|e(0)\| = 1 \gg \tau_a^{\frac{1}{2N}}$) but it were followed, after a while, with another slighter and presumably different steady pattern. If the instabilities were strong, the set of positiveness of the resulting modes alternated in each period with its complementary set. It can be also concluded from (58) that while $\tau_T > 0$, within which the instabilities are strong, it could be associated only with the simpler pattern structures, that is, for those eigenfunctions $U_k(x)$ corresponding to lower spatial eigenvalues. For regular tessellation patterns the analysis above can be done in each cell or in appropriate cell structured units, under dependence of the model. For instance, in a cell of a hexagonal lattice it is known that the lowest spatial eigenvalue is associated with a spatial pattern which resembles an eye, and the eigenfunctions U_k for a hexagonal cell inscribed within a circle of non-dimensional radius $R \approx 2$ are given in Murray (2003, p. 98) provided the spatial eigenvalues $\rho_k = (k\pi)^2$ for $k \in \mathbb{Z}$. We recall that in a practical problem a length scale is selected (for instance, $S = (\text{diffusivity} \times \text{time scale})^{1/2}$) and the spatial eigenvalues depend on the non-dimensional size of it. It can be noted that the relation $\lambda_k L^2 = \widehat{\lambda}_k \widehat{L}^2$ holds if λ_k and $\widehat{\lambda}_k$ are the spatial eigenvalues for two similar domains with non-dimensional characteristic lengths L and $\widehat{L}$ respectively. If we have a large enough non-dimensional characteristic length in Ω, the lowest positive spatial eigenvalue λ_1 would be so small that $\tau_T > 0$. If in addition $\tau_T^2 - 4\delta_T < 0$ holds, then TH instabilities associated with this eigenvalue must appear inducing a "twinkling eye" pattern like the one referenced in Yang and Epstein (2003).

Remark 5 Steady-state solutions to the reaction-diffusion problems in the limit of one small diffusion coefficient have been thoroughly investigated (see for instance Ward and Wei 2003; Wei and Winter 2003; Ward 2006). In this limit situation the appearance of inner boundary layers determine finite-amplitude steady solutions featured by locally large spatial gradients which exhibit localized spatial patterns in the form of spikes, stripes and their combinations. For instance, the spike-layer solution is exponentially small away from its peak, so the classical method of matching expansions does not apply in this case. Further, the stability analysis for the localized spatial patterns is usually done using a combination of asymptotic procedures and the spectral analysis of non-local eigenvalue problems. For instance, in Yang et al. (2005) the authors studied TH bifurcations in a reaction-diffusion system for a chemical reaction, showing the coexistence of Turing patterns and segmented waves far from the codimension-two points. Such TH instabilities cannot be derived from the extended modes.

6 Turing–Hopf Instabilities to the Schnakenberg System

In this section we study Turing instabilities of the limit cycle in Schnakenberg's reaction kinetics (Schnakenberg 1979) following the procedure in the preceding section. A great deal of attention has been given in the literature to patterning for the Schnakenberg system because it has a simple structure, but it is one of the few reaction-diffusion models in morphogenesis that exhibit patterns consistent with those in experiments, and it has had strong influence on experimental design (Edelstein-Keshet 1988).

6.1 The Schnakenberg Kinetics

Let us consider the Schnakenberg system

$$\begin{cases} \dot{u} = u^2 v - u + b, \\ \dot{v} = a - u^2 v \end{cases} \tag{78}$$

in which is included a reversible reaction. This system has a single stationary point

$$(u_0, v_0) = \left(a + b, \frac{a}{(a+b)^2}\right). \tag{79}$$

Here the parameters a and b are both positive, and $b \ll a$ (Edelstein-Keshet 1988). The Jacobian matrix of the right-hand side of (78) at the steady state (u_0, v_0) is

$$J_0 = \begin{pmatrix} \frac{a-b}{a+b} & (a+b)^2 \\ -\frac{2a}{a+b} & -(a+b)^2 \end{pmatrix} = \begin{pmatrix} \xi^{-1}(2a - \xi) & \xi^2 \\ -2a\xi^{-1} & -\xi^2 \end{pmatrix} \tag{80}$$

in which we put

$$\xi = a + b. \tag{81}$$

Hence, the Jacobian determinant will be

$$\delta = \xi^2 \tag{82}$$

and, we shall consider the trace

$$\tau = \xi^{-1}(2a - \xi) - \xi^2 \tag{83}$$

as the intrinsic bifurcation parameter. Fixing $a = 0.95$ the bifurcation value $\tau = 0$ is obtained when $\xi = 0.97452$. Note that $\tau = \tau(\xi)$ is a decreasing function for $\xi > 0$.

Considering the new variables (U, V) defined by the relations

$$U = u - u_0,$$
$$V = v - v_0,$$

we reflect perturbations near the stationary point, and get the following system:

$$\begin{pmatrix} \dot{U} \\ \dot{V} \end{pmatrix} = J_0 \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} v_0 U^2 + 2u_0 UV + U^2 V \\ -v_0 U^2 - 2u_0 UV - U^2 V \end{pmatrix}, \tag{84}$$

having the steady state at $(0, 0)$.

6.2 The Limit Cycle to the Schnakenberg System

Let us now derive the asymptotic expansion of the limit cycle to the Schnakenberg system. From the nonlinear part in (84) it follows very naturally to consider the new variable

$$z = U + V \quad \text{and} \quad \dot{z} = -U. \tag{85}$$

The matrix

$$\Gamma = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

satisfies (30). So, from the pair of functions (U, V) can be determined $(z, \dot{z})$, and reciprocally. Let us now consider the second-order equation for the unknown z being equivalent to the system in (84):

$$\ddot{z} - \tau\dot{z} + \delta z = \gamma(\dot{z})^2 + 2\xi z\dot{z} - z(\dot{z})^2 - (\dot{z})^3 \tag{86}$$

where δ and τ are given in (82) and (83) respectively, and γ depends on a and b. Now, transforming (38) into (86), we obtain (39) in normal for, in which

$$G(\varsigma, \dot{\varsigma}; \varepsilon) = \left(\gamma(\dot{\varsigma})^2 + 2\xi\varsigma\dot{\varsigma} - \varepsilon\varsigma(\dot{\varsigma})^2 - \varepsilon(\dot{\varsigma})^3\right).$$

Let us consider the new variables $r = r(t)$ and $\theta = \theta(t)$ defined as in (40) and (41). Then, from (44) and (45) we get:

$$\begin{cases} \dot{r} = \frac{r}{2}(1 - \frac{3}{4}r^2)\tau, \\ \dot{\theta} = \frac{1}{8}r^2\tau. \end{cases} \tag{87}$$

From the first equation in (87) and considering (52) follows the existence of an orbitally asymptotically stable limit cycle if $\tau > 0$, corresponding to

$$r^2 = \frac{4}{3},$$

hence

$$\dot{\theta} = \frac{\tau}{6}.$$

We finally obtain the following uniform asymptotic expansion in terms of the small parameter $\tau^{\frac{1}{2}}$ of the solution to (86):

$$z(t) = 2\sqrt{\frac{\tau}{3}}\cos\left(1 + \frac{\tau}{6}\right)t + O(\tau)$$

and, from (85) follows

$$\overline{u}(t) = u_0 + 2\sqrt{\frac{\tau}{3}}\sin\left(1 + \frac{\tau}{6}\right)t + O(\tau), \tag{88}$$

$$\overline{v}(t) = v_0 + 2\sqrt{\frac{\tau}{3}}\left(\cos\left(1 + \frac{\tau}{6}\right)t - \sin\left(1 + \frac{\tau}{6}\right)t\right) + O(\tau), \tag{89}$$

as the components of the cycle $\Theta(t) = (\overline{u}(t), \overline{v}(t))$ to (78). Here (u_0, v_0) is given in (79).

6.3 Turing Instabilities for the Periodic Solution to the Schnakenberg System

Let us now consider the boundary value problem of (1) with boundary conditions given in (2). We shall denote here by $u = u(t, x)$ and $v = v(t, x)$, $x \in \Omega$, the unknowns in (1) in which the reaction part is given in (78). Suppose we are considering small perturbations near the periodic solution $\Theta(t)$ to (1) due to HB, which are assumed to satisfy (2). Let us denote the corresponding perturbation by capital letters $Z = (U, V)^{\mathrm{T}}$; we get

$$u(t,x) = \overline{u}(t) + U(t,x),$$

$$v(t,x) = \overline{v}(t) + V(t,x),$$

where $\overline{u}(t)$ and $\overline{v}(t)$ denote the components of $\Theta(t)$ given in (88) and (89) respectively. From (87) it follows that $\Theta(t)$ is an orbitally asymptotically stable solution to (78) which appears via a supercritical HB. Then, substituting (88) and (89) into the system equation (60), we get the hierarchy of equations: (63), (64), and so on. In (63) and (64) should be considered, instead of J_a, the Jacobian J_0 given in (80). Here we are taking the period T of the cycle as the time scale, and the length scale as $L = \sqrt{D_u T}$, which usually is a small number. For instance, let us consider the rectangle $\Omega = [0, l] \times [0, m]$ in which the non-dimensional lengths are taken to satisfy $l > m$ in order that the smallest positive spatial eigenvalue be $\lambda_{(1,0)} = \pi^2/l^2$.

Let us put some independent non-dimensional parameters in the first table, and the set of dependent parameters gather in the second table.

a	ξ	d	l
0.95	0.97199	1.5	50

τ	δ	$\tau^2 - 4\delta$	τ_T	δ_T	$\delta_T - \tau_T^2/4$
0.01	0.94476	−3.8879	1.3040×10^{-4}	0.94286	0.94286

If $l < \pi(\frac{1+d}{\tau})^{1/2} = 49.673$, only weak instabilities would appear, and for this reason let us first take $l = 50$. Note that

$$\tau - \lambda_{(1,0)}(1+d) > 0$$

for the lowest spatial eigenvalue, and further, $\delta_T - \tau_T^2/4 > 0$, so strong instabilities could appear associated with $\lambda_{(1,0)}$. This means that an intermittent pattern featured by a switching between the two half parts of the rectangle could emerge ($U_1(x_1, x_2) = \cos(\pi x_1/l)$). The frequency of this intermittence is $\sigma_i = \sqrt{0.94286} = 0.97101$, which is lower than the frequency $\varpi = 1 + 1.6667 \times 10^{-3}$ of the cycle.

Let us take $d = 6$, and $l = 5$, keeping the parameter values a and ξ as before. Then, $\tau_T < 0$ for every spatial eigenvalue, so only weak instabilities would appear. Further, $\delta_T = -8.6813 \times 10^{-3}$, and follows a similar procedure as the one for Turing instabilities for the steady state. Equation (15) is rewritten here in terms of ξ and d in the form

$$\lambda_{\pm} = \frac{(d\xi^{-1}(2a - \xi) - \xi^2) \pm \sqrt{(d\xi^{-1}(2a - \xi) - \xi^2)^2 - 4d\xi^2}}{2d}.$$

Now we have $\lambda_- = 0.36042$, $\lambda_+ = 0.43688$, and $\lambda_{(1,0)} = \pi^2/25 \in \Lambda$, so a dominant steady pattern associated with the left half part of the rectangle could appear, with

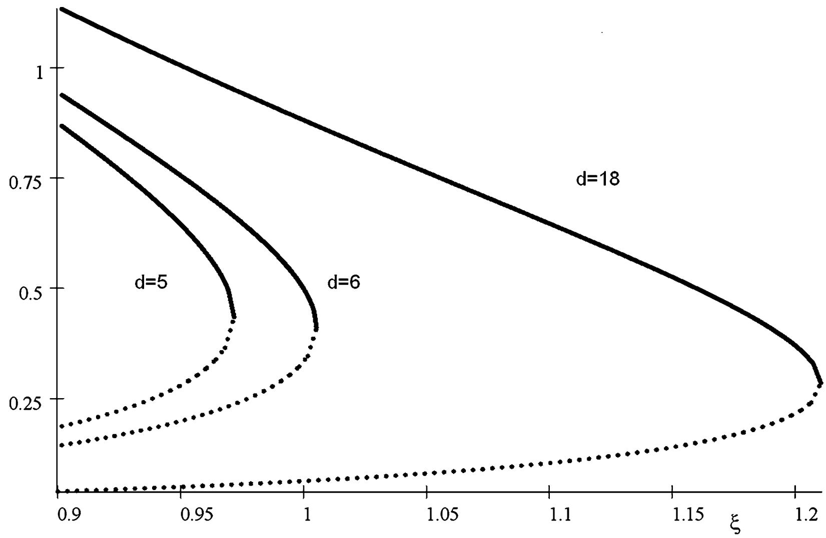

Fig. 1 λ_+ (*continuous*), λ_- (*dots*) as functions of ξ near the Hopf bifurcation

slight oscillations in its intensity. In Fig. 1 we represent λ_+ (continuous line) and λ_- (dotted line) as functions of the parameter ξ (abscissa) introduced in (81), for values $d = 5; 6; 18$. It is shown that the diameter of Λ increases as ξ decreases. So, as τ increases, one would expect that more unstable modes arise. In other words, we would expect more complex spatiotemporal patterns as the bifurcation parameter τ increases. Additionally, it is shown that Turing instabilities could arise before or even after the HB. For instance, in Fig. 1 it is shown that, being $a = 1$ and $d = 5$, we have $\Lambda \neq \emptyset$ only for $\xi < 0.96$, which means that TH instabilities would appear after HB as τ increases starting from $\tau > 0.16$.

7 Conclusions

We suggest a method for deriving an asymptotic expansion to the limit cycle for a reaction system exhibiting an HB. This procedure is simple and it is only necessary to make a transform of coordinates "close enough" to an appropriate linear one to do it. Using perturbation expansions we find appropriate normal modes which are useful in the study of diffusive instabilities generated by the limit cycle at supercritical HB. The extended modes suggest two possible ways of generation of instabilities by the limit cycle, which have been called weak and strong TH instabilities respectively. Weak ones are featured by dominant inhomogeneous steady patterns over which are superposed slight time-periodic oscillations with the same frequency of the limit cycle. While strong instabilities are featured by an intermittent switching between the inhomogeneous pattern, represented by the set of positiveness of the spatial eigenfunction, with its "complementary pattern," represented by the set of negativeness of the eigenfunction. The frequency of these oscillations is different from the frequency of the cycle. Moreover, the limit cycle would develop strong instabilities provided there are equal diffusion coefficients, if the lowest positive spatial eigenvalue is small enough. This fact makes an important difference between the required conditions for generation of diffusive instabilities about the stable limit cycle and those required about the stable steady state.

In the following table we summarize the conditions for the appearance of TH spatially inhomogeneous patterns based on the extended modes near a codimension-two TH point (i.e., laying at the intersection of the manifolds $\tau_a = 0$, and $\delta_T = 0$), and recalling that $\tau_a > 0$.

Case			THI type	Expected patterns
1	$\tau_T \le 0$,	$\delta_T < 0$,	weak	SO
2	$\tau_T > 0$,	$\delta_T \le 0$,	weak	SO
3	$\tau_T > 0$,	$0 < \delta_T < \tau_T^2/4$,	weak	steady or SO
4	$\tau_T > 0$,	$\delta_T = \tau_T^2/4$,	weak	SO
5	$\tau_T > 0$,	$\delta_T > \tau_T^2/4$,	strong	twinkling

(SO – slightly oscillatory)

We recall that in the former Turing analysis about the steady state it is not necessary to consider variations on the sign of τ_T, because from $\tau_a < 0$ follows $\tau_T < 0$. In Case 1 the conditions leading to TH instabilities are quite similar to the ones leading to Turing instabilities about the stable steady state, although slight oscillatory patterns are now expected. The other Cases only appear in association with simpler spatial structures, i.e. they corresponds to the lower eigenvalues. In Case 3, instabilities are again weak, but the patterns can be identified by slight oscillations or not, depending on whether the mode corresponds to the lowest positive root σ or not. Recall that in the latter subcase the ultimate pattern will presumably be different from that determined by the spatial eigenfunction. Case 4 assembles the points in the parameter space located at a bifurcation manifold. In the Case 5, the instabilities show intermittent spatial oscillations with frequency σ_i. We further recall the fact that τ_T is a strictly decreasing function of λ_k, so different types of TH instabilities corresponding to different eigenvalues would appear simultaneously. For regular tessellation patterns the above analysis can be done in each cell or in convenient cell structured units, depending on the model. For instance, in a cell of a hexagonal lattice, strong instabilities associated with the lowest positive eigenvalue will induce "twinkling eye" patterns, as those observed in Yang and Epstein (2003).

We cannot give a conclusive criterion to discriminate whether or not the ultimate pattern is time-periodic. It is natural to expect that in both weak and strong instabilities, oscillatory phenomena would be detected at least for a while. We assume, as usual, that the ultimate pattern emerges due to the boundedness of the unstable mode by the nonlinear reaction terms in (1). In weak instabilities the spatial structure prevails over the oscillatory and, presumably, such instabilities would lead to inhomogeneous steady patterns by the interaction with the nonlinear terms. If the parameters put the system equation (1) in the Case 3, there are two possible behaviors according to (71) or (72), and the initial conditions determine which one appears. On the other hand, strong TH instabilities presumably lead to well-defined intermittent spatiotemporal patterns with a frequency different from the one of the limit cycle.

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References

Baurmann, M., Gross, T., Feudel, U.: Instabilities in spatially extended predator–prey systems: Spatiotemporal patterns in the neighborhood of Turing–Hopf bifurcations. J. Theor. Biol. **245**(2), 220–229 (2007)

Bogoliubov, N.N., Mitropolski, Y.A.: Asymptotic Methods in the Theory of Nonlinear Oscillations. Gordon and Breach, New York (1961)

Edelstein-Keshet, L.: Mathematical Models in Biology. Birkhauser, New York (1988)

Golubitsky, M., Knobloch, E., Stewart, I.: Target patterns and spirals in planar reaction-diffusion systems. J. Nonlinear Sci. **10**, 333–354 (2000)

Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Springer, New York (1981)

Hofbauer, J., So, J.W.-H.: Multiple limit cycles for three dimensional Lotka–Volterra equations. Appl. Math. Lett. **7**(6), 65–70 (1994)

Just, W., Bose, M., Bose, S., Engel, H., Schöll, E.: Spatiotemporal dynamics near a supercritical Turing–Hopf bifurcation in a two-dimensional reaction-diffusion system. Phys. Rev. E **64**(026219), 1–12 (2001)

Kuznetsov, Yu.A.: Elements of Applied Bifurcation Theory, 2nd edn. Applied Mathematical Sciences, vol. 112. Springer, New York (1998)

Leiva, H.: Stability of a periodic solution for a system of parabolic equations. Appl. Anal. **60**, 277–300 (1996)

Maini, P.K., Painter, K.J., Chau, H.N.P.: Spatial pattern formation in chemical and biological systems. J. Chem. Soc. Faraday Trans. **93**(20), 3601–3610 (1997)

Marques, F., Gelfgat, A.Yu., Lopez, J.M.: Tangent double Hopf bifurcation in a differentially rotating cylinder FBW. Phys. Rev. Lett. E **68**(016310), 1–13 (2003)

Marsden, J.E., McCracken, M.: The Hopf Bifurcation and Its Applications. Springer, New York (1976)

Meixner, M., De Wit, A., Bose, S., Schöll, E.: Generic spatiotemporal dynamics near codimension-two Turing–Hopf bifurcations. Phys. Rev. E **55**(6), 6690–6697 (1997)

Murray, J.D.: Mathematical Biology, 3rd edn. Interdisciplinary Applied Mathematics, vol. 17. Springer, New York (2001)

Murray, J.D.: Mathematical Biology II: Spatial Models and Biomedical applications, 3rd edn. Interdisciplinary Applied Mathematics, vol. 18. Springer, New York (2003)

Rudin, W.: Principles of Mathematical Analysis, 3rd edn. McGraw-Hill, New York (1976)

Sanders, J.A., Verhulst, F.: Averaging Methods in Nonlinear Dynamical Systems. Applied Mathematical Sciences, vol. 59. Springer, New York (1985)

Sandstede, B., Scheel, A.: Essential instabilities of fronts: bifurcation and bifurcation failure. Dyn. Syst. **16**(1), 1–28 (2001)

Schnakenberg, J.: Simple chemical reactions with limit cycle behaviour. J. Theor. Biol. **81**, 389–400 (1979)

Schuman, B., Tóth, J.: No limit cycle in two species second order kinetics. Bull. Sci. Math. **127**(3), 222–230 (2003)

Turing, A.M.: The chemical basis for morphogenesis. Philos. Trans. R. Soc. Lond. B **237**, 37–72 (1952)

van der Ploeg, H., Doelman, A.: Stability of spatially periodic pulse patterns in a class of singularly perturbed reaction-diffusion equations. Indiana Univ. Math. J. **54**(5), 1219–1301 (2005)

Vastano, J.A., Pearson, J.E., Horsthemke, W., Swinney, H.L.: Chemical pattern formation with equal diffusion coefficients. Phys. Lett. A **124**(6–7), 320–324 (1987)

Verhulst, F.: Nonlinear Differential Equations and Dynamical Systems. Springer, Berlin (1990)

Ward, M.J.: Asymptotic methods for reaction-diffusion systems: past and present. Bull. Math. Biol. **68**(5), 1151–1167 (2006)

Ward, M.J., Wei, J.: Hopf bifurcations and oscillatory instabilities of spike solutions for the one-dimensional Gierer–Meinhardt model. J. Nonlinear Sci. **13**(2), 209–264 (2003)

Wei, J., Winter, M.: Asymmetric spotty patterns for the Gray–Scott model in $\mathbb{R}^2$. Stud. Appl. Math. **110**(1), 63–102 (2003)

Wilhelm, T., Heinrich, R.: Mathematical analysis of the smallest chemical reaction system with Hopf bifurcation. J. Math. Chem. **19**(2), 1–14 (1996)

Yang, L., Epstein, I.R.: Oscillatory Turing patterns in reaction-diffusion systems with two coupled layers. Phys. Rev. Lett. **90**(17), 1–4 (2003) (178303)

Yang, L., Berenstein, I., Epstein, I.R.: Segmented waves from a spatiotemporal transverse wave instability. Phys. Rev. Lett. **95**(3), 1–4 (2005) (038303)