Gevrey Regularity for the Attractor of the 3D Navier–Stokes–Voight Equations

Varga K. Kalantarov · Boris Levant · Edriss S. Titi

Abstract Recently, the Navier–Stokes–Voight (NSV) model of viscoelastic incompressible fluid has been proposed as a regularization of the 3D Navier–Stokes equations for the purpose of direct numerical simulations. In this work, we prove that the global attractor of the 3D NSV equations, driven by an analytic forcing, consists of analytic functions. A consequence of this result is that the spectrum of the solutions of the 3D NSV system, lying on the global attractor, have exponentially decaying tail, despite the fact that the equations behave like a damped hyperbolic system, rather than the parabolic one. This result provides additional evidence that the 3D NSV with the small regularization parameter enjoys similar statistical properties as the 3D Navier–Stokes equations. Finally, we calculate a lower bound for

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the exponential decaying scale—the scale at which the spectrum of the solution start to decay exponentially, and establish a similar bound for the steady state solutions of the 3D NSV and 3D Navier–Stokes equations. Our estimate coincides with the known bounds for the smallest length scale of the solutions of the 3D Navier–Stokes equations, established earlier by Doering and Titi.

Keywords Navier–Stokes–Voight equations · Navier–Stokes equations · Global attractor · Regularization of the Navier–Stokes equations · Turbulence models · Viscoelastic models · Gevrey regularity

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1 Introduction

We consider the Navier–Stokes–Voight (NSV) model of viscoelastic fluid which is governed by the system of equations

$$u_t - \nu \triangle u - \alpha^2 \triangle u_t + (u \cdot \nabla)u + \nabla p = f, \tag{1a}$$

$$\nabla \cdot u = 0, \tag{1b}$$

$$u(x, 0) = u^{\text{in}}(x), \tag{1c}$$

in $\Omega = [0, L]^3 \subset \mathbb{R}^3$, equipped with the periodic boundary conditions. $u(x, t)$ represents the velocity field, p is the pressure, $\nu > 0$ stands for kinematic viscosity, f is the forcing, and finally, α is a real positive length scale parameter, for which the ratio $\frac{\alpha^2}{\nu}$ characterizes the response time that is required for the fluid to respond to the applied force. The system (1) was first studied by Oskolkov, who introduced the NSV equations (see Oskolkov 1973, 1980) as a model of motion of linear, viscoelastic fluid.

Recently, in Cao et al. (2006), the 3D Navier–Stokes–Voight equations were suggested as a regularization model for the 3D Navier–Stokes equations, where α is considered a small regularization parameter. First, it was recognized that the inviscid ($\nu = 0$) version of the NSV system (1) coincides with the inviscid simplified Bardina subgrid scale model of turbulence. The viscous simplified Bardina model was introduced and studied in Layton and Lewandowski (2006) (see also Bardina et al. 1980, and Berselli et al. 2006 for the original Bardina model). In Cao et al. (2006), the viscous and inviscid simplified Bardina model were shown to be globally well posed. Moreover, it was also shown that the viscous simplified Bardina model has a finite dimensional global attractor, and the energy spectrum was investigated in Cao et al. (2006). Viewed from the numerical analysis point of view, the authors of Cao et al. (2006) proposed the inviscid simplified Bardina model (or equivalently the inviscid NSV equations) as an inviscid regularization (because no additional viscosity or hyperviscosity is introduced) of the 3D Euler equations, subject to periodic boundary conditions. Motivated by this observation, the system (1) was also proposed in

Cao et al. (2006) as a regularization, of the 3D Navier–Stokes equations for the purpose of direct numerical simulations for both the periodic and the no-slip Dirichlet boundary conditions.

The addition of the $-\alpha^2 \triangle u_t$ term has two main effects. First, it regularizes the equation in a way that the three-dimensional system (1) becomes now globally well posed (see Cao et al. 2006; Oskolkov 1973). On the other hand, as was noted in Kalantarov and Titi (2007), it changes the parabolic character of the original Navier–Stokes equations. Therefore, one does not observe any immediate smoothing of the solutions, as expected in parabolic PDEs. We also remark that this type of inviscid regularization has been recently used for the two-dimensional surface quasi-geostropic model (Khouider and Titi 2007). In particular, necessary and sufficient conditions for the formation of singularity were presented in terms of regularizing parameter.

The long-time dynamics of the system (1) has been studied in Kalantarov (1986) and Kalantarov and Titi (2007), where the existence of the finite-dimensional global attractor of the system has been established. Moreover, upper bounds for the number of determining modes, and the fractal dimension of the global attractor of the 3D NSV model where derived in Kalantarov and Titi (2007). In particular, it was shown that the attractor lies in the bounded subset of the Sobolev space $H^1(\Omega)$, whenever the forcing term $f \in L^2(\Omega)$.

In this work, we show that the global attractor of the 3D NSV model consists of the real analytic functions, whenever the forcing term f is analytic. The idea is to construct an asymptotic approximation $v(x, t)$ to the solution $u(x, t)$ of the system (1) satisfying

$$\lim_{t \to \infty} \left\| v(\cdot, t) - u(\cdot, t) \right\|_{L^2(\Omega)} = 0,$$

and show that $v(x, t)$ lies in certain Gevrey class—a subspace of the real analytic functions. Functions belonging to Gevrey regularity class are characterized by the exponential decay of the tail of their Fourier coefficients. Our method of the proof—splitting of $v(x, t)$ into higher and lower Fourier components has been used before in the context of the weakly damped driven nonlinear Schrödinger equation in Oliver and Titi (1998) and a model of Bénard convection in a porous medium in Oliver and Titi (2000) (see also Goubet 1996). Recently, the authors of Chueshov et al. (2004) followed the same methods to prove the Gevrey regularity of the global attractor of the generalized Benjamin–Bona–Mahony equation. For different methods in the abstract settings for wide class of equations, see Hale and Raugel (2003).

An important consequence of our result is that the solutions of the 3D NSV system (1) lying on the global attractor posses a dissipation range, despite the fact that the equations behave like the damped hyperbolic system, rather than the parabolic equation. This fact provides additional evidence that (1) with the small regularization parameter α, can indeed be used as a model to study the statistical properties of turbulent solutions of the 3D Navier–Stokes equations, a subject of ongoing research.

Finally, we obtain bounds for the exponential decaying length scale of the general solutions of the NSV system lying on the attractor. The obtained estimate is similar to the bounds for the smallest length scale in the turbulent flow that was previously calculated for the solutions of the 3D Navier–Stokes equations in Doering and Titi (1995). In addition, using the techniques introduced in Oliver and Titi (2001), we

estimate the exponential decaying scale of the stationary solutions of the 3D NSV and 3D Navier–Stokes equations. Our bounds coincide with those obtained in this paper for the general solutions of the NSV system lying on the global attractor, and for those of the 3D Navier–Stokes equations reported in Doering and Titi (1995).

2 Preliminaries

In this paper, we will use the following notations, which are standard in the mathematical theory of the Navier–Stokes equations (see, e.g., Constantin and Foias 1988; Foias et al. 2001; Temam 2001).

Let $\Omega = [0, L]^3$. We denote by $L^p(\Omega)$, for $1 \leq p \leq \infty$, and $H^m(\Omega)$—the usual Lebesgue and Sobolev spaces of the periodic functions on Ω, respectively. Let $\mathcal{F}$ be the set of all vector trigonometric polynomials on the periodic domain Ω, and denote

$$\mathcal{V} = \left\{ \varphi \in \mathcal{F} : \nabla \cdot \varphi = 0, \text{ and } \int_\Omega \varphi(x)\, dx = 0 \right\}.$$

We set H, and V to be the closures of $\mathcal{V}$ in the $L^2(\Omega)$ and $H^1(\Omega)$ topology, respectively.

We denote by $P_\sigma : L^2 \to H$—the Helmholtz–Leray orthogonal projection operator, and by $A = -P_\sigma \triangle$—the Stokes operator subject to the periodic boundary conditions with domain $D(A) = (H^2(\Omega))^3 \cap V$. Observe that in the space-periodic case

$$Au = -P_\sigma \triangle u = -\triangle u, \quad \text{for all } u \in D(A).$$

The operator A^{-1} is a positive definite, self-adjoint, compact operator from H into H. We denote by $0 < (\frac{2\pi}{L})^3 = \lambda_1 \leq \lambda_2 \leq \cdots$ the eigenvalues of A, repeated according to their multiplicities. The eigenvalues λ_j satisfy for some dimensionless constant $c_0 > 0$,

$$\frac{j^{2/3}}{c_0} \leq \frac{\lambda_j}{\lambda_1} \leq c_0 j^{2/3}, \quad \text{for } j = 1, 2, 3, \dots.$$

In the periodic case, this observation is simple (see, e.g., Constantin and Foias 1988); however, in the general case, this is a result of the famous Weyl's formula for the case of the Stokes operator due to Métivier (see, e.g., Constantin and Foias 1988; Métivier 1978; Temam 1988).

For any $s \in \mathbb{R}$, we can define the Hilbert spaces $V_s := D(A^{s/2})$ with the inner product and norm

$$(u, v)_s = \sum_{j \in \mathbb{Z}^3} \left(u_j \cdot v_j |j|^{2s} \right), \qquad |u|_s^2 = (u, u)_s,$$

for every $u, v \in V_s$, where u_j, v_j are the corresponding Fourier coefficients of u and v, respectively. Note that $V_0 = H$. Moreover, we denote $V = V_1$, and the corresponding inner product and norm will be written for $u, v \in V$

$$\big((u, v)\big) = (u, v)_1, \qquad \|u\| = |u|_1.$$

For any $w_1, w_2 \in \mathcal{V}$, we define the following bilinear form

$$B(w_1, w_2) = P_\sigma\big((w_1 \cdot \nabla)w_2\big).$$

It can be shown (see, e.g., Constantin and Foias 1988; Temam 2001) that B can be extended to a continuous map $B : V \times V \to V'$, where $V' = V_{-1}$ is a dual space of V. In particular, for $u, v, w \in V$, there exists a constant $c > 0$, depending only on Ω, such that

$$\big|\langle B(u, v), w\rangle_{V'}\big| \le c\lambda_1^{-3/4}|u|^{1/2}\|u\|^{1/2}\|v\|\,\|w\|, \tag{2}$$

where $\langle x, y\rangle_{V'}$ denotes an action of an element $x \in V$ on the element of the dual space $y \in V'$.

Finally, using the above definitions, we write the system (1) in the following equivalent functional form

$$\dot{u} + \nu Au + \alpha^2 A\dot{u} + B(u, u) = f, \tag{3a}$$

$$u(x, 0) = u^{\text{in}}(x). \tag{3b}$$

To show that the solution of the problem (3) has an analytic asymptotic (in time) approximation, we will use the concept of the Gevrey class regularity. For a given $\tau > 0$, and $r \ge 0$, we define the Gevrey space to be

$$G_\tau^r := D\big(A^{r/2}e^{\tau A^{1/2}}\big) = \left\{u \in H : \big|A^{r/2}e^{\tau A^{1/2}}u\big|^2 = \sum_{j \in \mathbb{Z}^3}|u_j|^2|j|^{2r}e^{2\tau|j|} < \infty\right\}.$$

The space is equipped with the corresponding inner product and norm

$$(u, v)_{r,\tau} = \big(A^{r/2}e^{\tau A^{1/2}}u, A^{r/2}e^{\tau A^{1/2}}v\big) = \sum_{j \in \mathbb{Z}^3}u_j \cdot v_j|j|^{2r}e^{2\tau|j|},$$

$$|u|_{r,\tau} = \big|A^{r/2}e^{\tau A^{1/2}}u\big|,$$

for $u, v \in G_\tau^r$. One can prove that the space of real analytic functions $C^\omega(\Omega)$ has the following characterization

$$C^\omega(\Omega) = \bigcup_{\tau > 0} G_\tau^r,$$

for any $r \ge 0$ (see, e.g., Levermore and Oliver 1997). The concept of the Gevrey class regularity for showing the analyticity of the solutions of the Navier–Stokes equations was first introduced in Foias and Temam (1989), simplifying earlier proofs. Later this technique was extended to the large class of analytic nonlinear parabolic equations in Ferrari and Titi (1998).

We conclude this section by a few technical propositions that will be used in the proof of our main results. First, we will need the following estimates for the nonlinear term. The proof of the proposition is obtained by the standard interpolation estimates using the Gagliardo–Nirenberg and Ladyzhenskaya inequalities (see, e.g., Constantin and Foias 1988; Temam 2001).

Proposition 1 *The bilinear form $B(u, u)$ satisfies*:

(i) *If $u \in V$, then $B(u, u) \in V_{-1/2}$, and*

$$\left| B(u, u) \right|_{-1/2} \le c_1 \lambda_1^{-3/4} \|u\|^2. \tag{4}$$

(ii) *If $u \in V_{3/2}$, then $B(u, u) \in H$, and*

$$\left| B(u, u) \right| \le c_2 \lambda_1^{-3/4} \|u\| |u|_{3/2}. \tag{5}$$

(iii) *For any integer $m \ge 1$, if $u \in V_{m+1}$, then $B(u, u) \in V_m$, and*

$$\left| B(u, u) \right|_m \le c_m \lambda_1^{-7/8} \|u\|^{1/4} |u|_2^{3/4} |u|_{m+1}, \tag{6}$$

where $c_1, c_2, c_m > 0$ are scale invariant constants, and c_m depends on m.

Let $\lambda > 0$, denote by P_λ the H-orthogonal projection onto the span of eigenfunctions of A corresponding to eigenvalues of the magnitude less then or equal to λ. Denote $Q_\lambda = I - P_\lambda$. The following Poincaré-type inequalities hold.

Proposition 2 *Let $\bar{v} \in P_\lambda G_\tau^{r+1}$, and $\hat{v} \in Q_\lambda G_\tau^{r+1}$. Then*

$$|\bar{v}|_{r+1,\tau} \le e^{\tau \lambda^{1/2}} |\bar{v}|_{r+1}, \quad and \quad |\hat{v}|_{r,\tau} \le \lambda^{-1/2} |\hat{v}|_{r+1,\tau}. \tag{7}$$

We will also need an estimate for the nonlinear term in the Gevrey space. Similar inequalities can be found in Foias et al. (2001) (see also Doering and Titi 1995; Foias and Temam 1989).

Proposition 3 *For any $\tau > 0$, $u, w \in G_\tau^2$, and $v \in G_\tau^1$, the following inequality holds*

$$\left| \left(B(u, v), w \right)_{1,\tau} \right| \le C_1 \lambda_1^{-3/4} |u|_{1,\tau}^{1/2} |u|_{2,\tau}^{1/2} |v|_{1,\tau} |w|_{2,\tau}, \tag{8}$$

for some scale invariant constant $C_1 > 0$.

It is not difficult to prove the following proposition using the Galerkin approximation procedure (see Kalantarov and Titi 2007).

Proposition 4 *Let $s \in \mathbb{R}$. Assume that $g(t) \in L^\infty([0, T], V_{s-2})$, for some $0 < T < \infty$. Then the linear problem*

$$z_t + \nu A z + \alpha^2 A z_t = g(t), \quad z(0) = 0,$$

has a unique solution $z(t) \in C([0, T], V_s)$. In addition, the following estimate holds,

$$\left| z(t) \right|_s \le \frac{\|g(t)\|_{L^\infty([0,T], V_{s-2})}}{\alpha \nu \sqrt{d_0}}, \tag{9}$$

for all $t \in [0, T]$, and $d_0 = (\frac{1}{\lambda_1} + \alpha^2)^{-1}$.

We will use the following small proposition that we state here without a proof.

Proposition 5 *Consider a nonnegative function $\varphi(t)$ satisfying for all $t \geq t_0$, for some t_0, the following inequality*

$$\frac{d\varphi}{dt} \leq -a\varphi + b\varphi^{3/2} + c, \quad \varphi(t_0) = 0,$$

where the positive real coefficients a, b, c obey

$$bc^{1/2} < \left(\frac{a}{2}\right)^{3/2}. \tag{10}$$

Then for all $t \geq t_0$

$$\varphi(t) \leq \frac{2C}{a}.$$

Finally, we will need the following Lemma from Jones and Titi (1992) (see also Foias et al. 2001).

Lemma 1 *Let $a(t)$ and $b(t)$ be locally integrable functions on $(0, \infty)$ which satisfy for some $T > 0$ the conditions*

$$\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} a(\tau)\,d\tau > 0,$$

$$\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} a^-(\tau)\,d\tau < \infty,$$

$$\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} b^+(\tau)\,d\tau = 0,$$

where $a^- = \max\{-a, 0\}$ and $b^+ = \max\{b, 0\}$. Suppose that $\phi(t)$ is a nonnegative, absolutely continuous function on $[0, \infty)$ that satisfies the following inequality almost everywhere on $[0, \infty)$,

$$\phi'(t) + a(t)\phi(t) \leq b(t).$$

Then $\phi(t) \to 0$, as $t \to \infty$.

3 Asymptotic Approximation in V_m

The question of global existence and uniqueness of (1) was first studied in Oskolkov (1973) (the inviscid case, $\nu = 0$, was studied in Cao et al. 2006). It was shown that for every $u^{\text{in}} \in V$, the solution of the system (1) is globally well posed, and satisfies $u(x, t) \in L^\infty([0, \infty), V)$. In this section, we construct an asymptotic approximation of the solution of (1) in the space V_m, for every $m \geq 2$. The result can be stated as follows.

Theorem 1 *Let $m \geq 2$ be an integer. Consider a solution $u(x, t)$ of the NSV system* (1), *corresponding to the initial condition $u^{\text{in}} \in V$ with the forcing $f \in V_{m-2}$. There exists a function*

$$v^{(m)}(t) \in L^\infty\big([0, \infty), V_m\big),$$

satisfying

$$\lim_{t \to \infty} \big\| u(t) - v^{(m)}(t) \big\| = 0.$$

Proof Let us fix $m \geq 2$, and let $u^{\text{in}} \in V$. First, let us write the solution $u(t) = v(t) + w(t)$, where $v(t)$ and $w(t)$ satisfy the coupled system

$$v_t + \nu A v + \alpha^2 A v_t = f - B(u, u), \quad v(0) = 0, \tag{11a}$$

$$w_t + \nu A w + \alpha^2 A w_t = 0, \quad w(0) = u^{\text{in}}. \tag{11b}$$

This decomposition has been used in Kalantarov and Titi (2007). First, by using the fact that $u(x, t) \in L^\infty([0, \infty), V)$, and applying subsequently the first part of Proposition 1 and Proposition 4 to (11a), we conclude that

$$v(t) \in L^\infty\big([0, \infty), V_{3/2}\big). \tag{12}$$

Next, from (11b), we immediately get

$$\big| w(t) \big|^2 + \alpha^2 \big\| w(t) \big\|^2 \leq e^{-\nu d_0 t} \big(\big| u^{\text{in}} \big|^2 + \alpha^2 \big\| u^{\text{in}} \big\|^2 \big), \tag{13}$$

where $d_0 = (\frac{1}{\lambda_1} + \alpha^2)^{-1}$. Therefore, $v(x, t)$ is an asymptotic (in time) approximation of $u(x, t)$, namely

$$\lim_{t \to \infty} \big\| u(t) - v(t) \big\| = \lim_{t \to \infty} \big\| w(t) \big\| = 0.$$

At the next step, let us consider $v^{(2)}(x, t)$—the solution of the following equation

$$v_t^{(2)} + \nu A v^{(2)} + \alpha^2 A v_t^{(2)} = f - B(v, v), \quad v^{(2)}(0) = 0. \tag{14}$$

According to the second part of Proposition 1, the right-hand side of (14) is in $L^\infty([0, \infty), H)$. Therefore, applying Proposition 4, we conclude that the unique solution of (14) satisfies

$$v^{(2)}(t) \in L^\infty\big([0, \infty), V_2\big). \tag{15}$$

Denote $z^{(2)} = v^{(2)} - v$, which satisfies

$$z_t^{(2)} + \nu A z^{(2)} + \alpha^2 A z_t^{(2)} = B(u, u - v) + B(u - v, v), \quad z^{(2)}(0) = 0. \tag{16}$$

According to Proposition 4, (16) has a unique solution $z^{(2)}(t) \in L^\infty([0, \infty), V_{3/2})$. This is because $u \in L^\infty([0, \infty), V)$, and v satisfies (12). Therefore, we can take an inner product of (16) with z. Using inequality (2), we get

$$\frac{1}{2}\frac{d}{dt}\left(\left|z^{(2)}(t)\right|^2+\alpha^2\left\|z^{(2)}(t)\right\|^2\right)+\nu\left\|z^{(2)}(t)\right\|^2$$

$$=\left(B\big(u(t),u(t)-v(t)\big),z^{(2)}(t)\right)+\left(B\big(u(t)-v(t),v(t)\big),z^{(2)}(t)\right)$$

$$\leq c\left\|u(t)-v(t)\right\|\left(\left\|u(t)\right\|+\left\|v(t)\right\|\right)\left\|z^{(2)}(t)\right\|$$

$$\leq\frac{c^2}{2\nu}\left\|u(t)-v(t)\right\|^2\left(\left\|u(t)\right\|+\left\|v(t)\right\|\right)^2+\frac{\nu}{2}\left\|z^{(2)}(t)\right\|^2,\tag{17}$$

where the last relation follows from Young's inequality. Finally, we get

$$\frac{d}{dt}\left(\left|z^{(2)}(t)\right|^2+\alpha^2\left\|z^{(2)}(t)\right\|^2\right)+\frac{\nu d_0}{2}\left(\left|z^{(2)}(t)\right|^2+\alpha^2\left\|z^{(2)}(t)\right\|^2\right)$$

$$\leq\frac{c^2}{2\nu}\left\|u(t)-v(t)\right\|^2\left(\left\|u(t)\right\|+\left\|v(t)\right\|\right)^2.$$

Using the fact that $u(t),v(t)$ are bounded uniformly in time in the V norm, and that

$$\left\|u(t)-v(t)\right\|=\left\|w(t)\right\|\to 0,\quad\text{as }t\to\infty,$$

we conclude, after applying Lemma 1 that

$$\lim_{t\to\infty}\left\|z^{(2)}(t)\right\|=\lim_{t\to 0}\left\|v(t)-v^{(2)}(t)\right\|=\lim_{t\to 0}\left\|u(t)-v^{(2)}(t)\right\|=0.$$

We can continue by induction. Fix $2\leq n\leq m$, and assume that we have constructed $v^{(j)}(t)\in L^\infty([0,\infty),V_j)$, for $j=2,3,\dots,n-1$, such that for any j

$$\lim_{t\to\infty}\left\|v^{(j-1)}(t)-v^{(j)}(t)\right\|=\lim_{t\to 0}\left\|u(t)-v^{(j)}(t)\right\|=0.\tag{18}$$

Let consider the following equation

$$v_t^{(n)}+\nu Av^{(n)}+\alpha^2 Av_t^{(n)}=f-B\big(v^{(n-1)},v^{(n-1)}\big),\quad v^{(n)}(0)=0.\tag{19}$$

Then according to Proposition 4, and due to the estimates on the nonlinear term of Proposition 1, the unique solution $v^{(n)}(t)$ of (19) satisfies $v^{(n)}(t)\in L^\infty([0,\infty),V_n)$. Moreover, denote $z^{(n)}=v^{(n)}-v^{(n-1)}$, satisfying

$$z_t^{(n)}+\nu Az^{(n)}+\alpha^2 Az_t^{(n)}=B\big(v^{(n-2)},v^{(n-2)}-v^{(n-1)}\big)$$

$$+B\big(v^{(n-2)}-v^{(n-1)},v^{(n-1)}\big),\quad z^{(n)}(0)=0.$$

Taking the inner product of the last equation with $z^{(n)}(t)$ and using Proposition 1 and relation (18), we can show by Lemma 1 that

$$\lim_{t\to\infty}\left\|z^{(n)}(t)\right\|=\lim_{t\to 0}\left\|v^{(n-2)}(t)-v^{(n-1)}(t)\right\|=\lim_{t\to 0}\left\|u(t)-v^{(n)}(t)\right\|=0,$$

finishing the proof of the theorem. $\square$

It can be proved (see Kalantarov and Titi 2007) that the solution of the NSV equations (1) satisfies for all $t \geq 0$

$$\|u(t)\|^2 \leq \frac{e^{-\nu d_0 t}}{\alpha^2}\left(|u^{\text{in}}|^2 + \alpha^2\|u^{\text{in}}\| - \frac{|f|_{-1}^2}{\nu^2 d_0}\right) + \frac{|f|_{-1}^2}{\alpha^2 \nu^2 d_0}. \tag{20}$$

Therefore, there exists t_0, depending on $|u^{\text{in}}|$, $\|u^{\text{in}}\|$, $|f|_{-1}$, ν, α, and λ_1, such that for all $t \geq t_0$

$$\|u(t)\| \leq M_1 := \frac{2|f|_{-1}}{\alpha \nu \sqrt{d_0}}. \tag{21}$$

The following lemma gives similar bounds for the asymptotic (in time) approximations $v^{(m)}(x, t)$ in the corresponding norms.

Lemma 2 *Let $f \in V_{m-2}$. Consider $t_0 \geq 0$, such that the solution of the NSV equations (1) satisfies the inequality (21) for all $t \geq t_0$. Then the following statements are true*:

(i) *The function $v(x, t) \in L^\infty([0, \infty), V_{3/2})$, constructed in Theorem 1, satisfies for all $t \geq t_0$*

$$\big|v(t)\big|_{3/2} \leq M_{3/2} := \frac{1}{\alpha \nu \sqrt{d_0}}\big(|f|_{-1/2} + c_1 \lambda_1^{-3/4} M_1^2\big). \tag{22}$$

(ii) *The function $v^{(2)}(x, t) \in L^\infty([0, \infty), V_2)$, constructed in Theorem 1, satisfies for all $t \geq t_0$*

$$\big|v^{(2)}(t)\big|_2 \leq M_2 := \frac{1}{\alpha \nu \sqrt{d_0}}\big(|f| + c_2 \lambda_1^{-3/4} M_1 M_{3/2}\big). \tag{23}$$

(iii) *Let $m > 2$ be an integer. The function $v^{(m)}(x, t) \in L^\infty([0, \infty), V_m)$, constructed in Theorem 1, satisfies for all $t \geq t_0$*

$$\big|v^{(m)}(t)\big|_m \leq M_m := \frac{1}{\alpha \nu \sqrt{d_0}}\big(|f|_{m-2} + c_m \lambda_1^{-7/8} M_1^{1/4} M_2^{3/4} M_{m-1}\big). \tag{24}$$

Proof Recall that $v(t)$ satisfies (11), $v^{(2)}(t)$ satisfies (14). In general, $v^{(m)}(t)$, for $m > 2$, satisfies (19). Therefore, the proof of the lemma is an immediate application of Proposition 4, in particular relation (9), and the inequalities of Proposition 1. □

4 Asymptotic Approximation in the Gevrey Space G_τ^1

The results of the previous section show that with a smooth enough forcing the global attractor of the system (1) lies in $C^\infty(\Omega)$, whenever f is $C^\infty(\Omega)$. However, our goal is to show that the global attractor is real analytic, whenever f is real analytic. For this purpose, we use the idea of Oliver and Titi (1998, 2000), to construct the asymptotic approximation of the solution of (1) in the Gevrey class G_τ^2, for some $\tau > 0$.

Theorem 2 *Let $u(x, t)$ be a solution of the NSV system* (1), *corresponding to the initial condition $u^{\text{in}} \in V$ with the forcing $f \in G^1_{\tau_0}$, for some $\tau_0 > 0$. Let $t_0 \geq 0$ be as in Lemma* 2, *then there exists a function*

$$v^\omega(t) \in L^\infty\big([t_0, \infty), G^2_\tau\big), \tag{25}$$

for some $\tau > 0$, depending only on $|f|_{1,\tau_0}$, ν, λ_1 and α, satisfying

$$\lim_{t \to \infty} \big\| u(t) - v^\omega(t) \big\| = 0. \tag{26}$$

Proof Let $\lambda > 0$ to be chosen later. First, consider $v^{(2)}(x, t)$—an asymptotic approximation of $u(x, t)$, which is constructed in Theorem 1. Moreover, according to Lemma 1, there exists a constant $M_2 > 0$ (see relation (23)), such that

$$\big| v^{(2)}(t) \big|_2 \leq M_2, \quad \forall t \geq t_0. \tag{27}$$

Denote $\bar{v}(t) = P_\lambda v^{(2)}(t)$, and consider $\hat{v}(t)$—a solution of the following equation

$$\hat{v}_t + \nu A \hat{v} + \alpha^2 A \hat{v}_t + Q_\lambda B(\bar{v} + \hat{v}, \bar{v} + \hat{v}) = \hat{f}, \quad \hat{v}(t_0) = 0, \tag{28}$$

for $t \geq t_0$, where for notation simplicity, we denoted $\hat{f} = Q_\lambda f$. Equation (28) formally looks like a projection of the system (1) onto the higher wavenumber components, however, the low wavenumber modes $\bar{v}$ of the advection term satisfy a slightly different equation (see also Olson and Titi 2003 for such a construction for studying data assimilation). Let us denote by

$$v^\omega(t) = \bar{v}(t) + \hat{v}(t), \tag{29}$$

for $t \geq t_0$. Our goal is to show first that there exists $\tau > 0$ such that $v^\omega \in G^2_\tau$. Observe that $\bar{v}$ is just a trigonometric polynomial, and in particular, is analytic. Therefore, we need to show that we can choose λ large enough, such that $\hat{v} \in G^2_\tau$, for some $\tau > 0$. Finally, we will show that $v^\omega(x, t)$ is indeed an asymptotic approximation of $u(x, t)$.

Note, that in order to prove that the solution of (28) lies in a Gevrey class of real analytic functions, we consider the Galerkin procedure to (28). However, we omit this standard procedure, and obtain formal a priori estimates on the solutions in the relevant Gevrey space norm. Taking formally the inner product of (28) in G^1_τ with $\hat{v}$, we obtain the following inequality:

$$\frac{1}{2} \frac{d}{dt} \big(|\hat{v}|^2_{1,\tau} + \alpha^2 |\hat{v}|^2_{2,\tau} \big) + \nu |\hat{v}|^2_{2,\tau} \leq \big| (\hat{f}, \hat{v})_{1,\tau} \big| + \big| (B(\bar{v}, \bar{v}), \hat{v})_{1,\tau} \big|$$
$$+ \big| (B(\bar{v}, \hat{v}), \hat{v})_{1,\tau} \big| + \big| (B(\hat{v}, \bar{v}), \hat{v})_{1,\tau} \big|$$
$$+ \big| (B(\hat{v}, \hat{v}), \hat{v})_{1,\tau} \big|. \tag{30}$$

Next, we estimate the terms on the right-hand side of (30). First, using subsequently the Cauchy–Schwartz and Young inequalities, as well as Proposition 2, we get assuming $\tau \leq \tau_0$,

$$\big| (\hat{f}, \hat{v})_{1,\tau} \big| \leq |\hat{f}|_{1,\tau} \cdot |\hat{v}|_{1,\tau} \leq \frac{5}{4\nu\lambda} |\hat{f}|^2_{1,\tau} + \frac{\nu}{5} |\hat{v}|^2_{2,\tau}. \tag{31}$$

Next, using Proposition 3, Young inequality, and the Poincaré-type inequalities of
Proposition 2, we get the following series of estimates for all $t \geq t_0$:

$$\left|\left(B(\bar{v},\bar{v}),\hat{v}\right)_{1,\tau}\right| \leq C_1 \lambda_1^{-3/4} |\bar{v}|_{1,\tau}^{3/2} |\bar{v}|_{2,\tau}^{1/2} |\hat{v}|_{2,\tau} \leq \frac{5C_1^2 |\bar{v}|_{1,\tau}^3 |\bar{v}|_{2,\tau}}{4\nu\lambda_1^{3/2}} + \frac{\nu}{5} |\hat{v}|_{2,\tau}^2$$

$$\leq \frac{5C_1^2 e^{4\tau\lambda^{1/2}} M_1^3 M_2}{4\nu\lambda_1^{3/2}} + \frac{\nu}{5} |\hat{v}|_{2,\tau}^2, \tag{32}$$

$$\left|\left(B(\bar{v},\hat{v}),\hat{v}\right)_{1,\tau}\right| \leq C_1 \lambda_1^{-3/4} |\bar{v}|_{1,\tau}^{1/2} |\bar{v}|_{2,\tau}^{1/2} |\hat{v}|_{1,\tau} |\hat{v}|_{2,\tau}$$

$$\leq \frac{C_1 e^{\tau\lambda^{1/2}} M_1^{1/2} M_2^{1/2}}{\lambda^{1/2}\lambda_1^{3/4}} |\hat{v}|_{2,\tau}^2, \tag{33}$$

$$\left|\left(B(\hat{v},\bar{v}),\hat{v}\right)_{1,\tau}\right| \leq C_1 \lambda_1^{-3/4} |\hat{v}|_{1,\tau}^{1/2} |\hat{v}|_{2,\tau}^{3/2} |\bar{v}|_{1,\tau} \leq \frac{C_1 e^{\tau\lambda^{1/2}} M_1}{\lambda^{1/4}\lambda_1^{3/4}} |\hat{v}|_{2,\tau}^2, \tag{34}$$

$$\left|\left(B(\hat{v},\hat{v}),\hat{v}\right)_{1,\tau}\right| \leq \frac{C_1}{\lambda_1^{3/4}} |\hat{v}|_{1,\tau}^{3/2} |\hat{v}|_{2,\tau}^{3/2} \leq \frac{C_1}{\lambda^{3/4}\lambda_1^{3/4}} |\hat{v}|_{2,\tau}^3$$

$$\leq \frac{C_1}{\lambda^{3/4}\lambda_1^{3/4}\alpha^3} \left(|\hat{v}|_{1,\tau}^2 + \alpha^2 |\hat{v}|_{2,\tau}^2\right)^{3/2}. \tag{35}$$

Let us set $\tau = \min\{\lambda^{-1/2}, \tau_0\}$. Then, we will choose λ large enough satisfying

$$\max\left\{\frac{C_1 e M_1^{1/2} M_2^{1/2}}{\lambda^{1/2}\lambda_1^{3/4}}, \frac{C_1 e M_1}{\lambda^{1/4}\lambda_1^{3/4}}\right\} \leq \frac{\nu}{5}. \tag{36}$$

Using the last bounds, we are ready to substitute relations (31), (32), (33), (34),
and (35) into (30). After rearranging the terms, we get

$$\frac{1}{2}\frac{d}{dt}\left(|\hat{v}|_{1,\tau}^2 + \alpha^2 |\hat{v}|_{2,\tau}^2\right) + \frac{\nu}{5} |\hat{v}|_{2,\tau}^2$$

$$\leq \frac{C_1}{\lambda^{3/4}\lambda_1^{3/4}\alpha^3} \left(|\hat{v}|_{1,\tau}^2 + \alpha^2 |\hat{v}|_{2,\tau}^2\right)^{3/2} + \frac{5|\hat{f}|_{1,\tau}^2}{4\nu\lambda} + \frac{5C_1^2 e^4 M_1^3 M_2}{4\nu\lambda_1^{3/2}}. \tag{37}$$

Next, using Poincaré-type inequality, Proposition 2, and setting $d_2 = (\frac{1}{\lambda} + \alpha^2)^{-1}$, we
can write

$$\frac{\nu}{5} |\hat{v}|_{2,\tau}^2 \geq \frac{\nu d_2}{5} \left(\lambda^{-1} |\hat{v}|_{2,\tau}^2 + \alpha^2 |\hat{v}|_{2,\tau}^2\right) \geq \frac{\nu d_2}{5} \left(|\hat{v}|_{1,\tau}^2 + \alpha^2 |\hat{v}|_{2,\tau}^2\right). \tag{38}$$

Substituting into (37) gives us the inequality

$$\frac{1}{2}\frac{d}{dt}\big(|\hat{v}|^2_{1,\tau} + \alpha^2|\hat{v}|^2_{2,\tau}\big) \le -\frac{\nu d_2}{5}\big(|\hat{v}|^2_{1,\tau} + \alpha^2|\hat{v}|^2_{2,\tau}\big)$$

$$+ \frac{C_1}{\lambda^{3/4}\lambda_1^{3/4}\alpha^3}\big(|\hat{v}|^2_{1,\tau} + \alpha^2|\hat{v}|^2_{2,\tau}\big)^{3/2} + \frac{5|\hat{f}|^2_{1,\tau}}{4\nu\lambda}$$

$$+ \frac{5C_1^2 e^4 M_1^3 M_2}{4\nu\lambda_1^{3/2}}. \tag{39}$$

Now we can apply Proposition 5 to the function $(|\hat{v}(t)|^2_{1,\tau} + \alpha^2|\hat{v}(t)|^2_{2,\tau})$ satisfying inequality (39). Using relation (10), we conclude that $(|\hat{v}(t)|^2_{1,\tau} + \alpha^2|\hat{v}(t)|^2_{2,\tau})$ is bounded for all $t \ge t_0$, and in particular $v(t) \in L^\infty([t_0, \infty), G_\tau^2)$, whenever the following holds

$$\frac{C_1}{\lambda^{3/4}\lambda_1^{3/4}\alpha^3}\left(\frac{5|\hat{f}|^2_{1,\tau}}{4\nu\lambda} + \frac{5C_1^2 e^4 M_1^3 M_2}{4\nu\lambda_1^{3/2}}\right)^{1/2} < \left(\frac{\nu d_2}{10}\right)^{3/2}.$$

In order to satisfy the last inequality, we have to choose λ large enough, such that

$$\alpha^2\nu\lambda^{1/2}\lambda_1^{1/2}d_2 > \left(\frac{C_4|\hat{f}|^2_{1,\tau}}{\nu\lambda} + \frac{C_5 M_1^3 M_2}{\nu\lambda_1^{3/2}}\right)^{1/3}, \tag{40}$$

for some absolute constants $C_4, C_5 > 0$. For such choice of λ, we have $v(t) \in L^\infty([t_0, \infty), G_\tau^2)$, and this proves the first part of the theorem.

We are left to show that $v^\omega(x, t)$ is an asymptotic approximation of the solution $u(x, t)$ of the NSV equations (1). Let $z = u - v^\omega$, and denote $\bar{z} = P_\lambda(u - v^{(2)})$, $\hat{z} = Q_\lambda u - \hat{v}$. Clearly, by the construction and Theorem 1 that

$$\lim_{t\to\infty}\big\|P_\lambda u(t) - \bar{v}(t)\big\| = \lim_{t\to\infty}\big\|\bar{z}(t)\big\| = 0. \tag{41}$$

Therefore, to prove (26), we need to show that

$$\lim_{t\to\infty}\big\|Q_\lambda u(t) - \hat{v}(t)\big\| = \lim_{t\to\infty}\big\|\hat{z}(t)\big\| = 0.$$

Observe that $\hat{z}$ satisfies the equation

$$\hat{z}_t + \nu A\hat{z} + \alpha^2 A\hat{z}_t + Q_\lambda\big(B(u, z) + B(z, u) - B(z, z)\big) = 0, \quad \hat{z}(t_0) = Q_\lambda u(t_0).$$

Taking an inner product of the last equation with $\hat{z}$, we get

$$\frac{1}{2}\frac{d}{dt}\big(|\hat{z}|^2 + \alpha^2\|\hat{z}\|^2\big) + \nu\|\hat{z}\|^2$$

$$\le \big|(B(\hat{z}, u), \hat{z})\big| + \big|Q_\lambda\big(B(u, \bar{z}) + B(\bar{z}, u) - B(z, \bar{z}), \hat{z}\big)\big|. \tag{42}$$

The first summand on the right-hand side of (42) can be estimated as follows. Using (2), Proposition 2, and relation (21)

$$\big|\big(B(\hat{z}(t), u(t)), \hat{z}(t)\big)\big| \le c\|u(t)\|\big|\hat{z}(t)\big|^{1/2}\|\hat{z}(t)\|^{3/2} \le \frac{cM_1}{\lambda^{1/4}\lambda_1^{3/4}}\|\hat{z}(t)\|^2,$$

for $t \geq t_0$. Plugging this inequality into (42), and using relation (38), we get

$$\frac{1}{2}\frac{d}{dt}\big(|\hat{z}|^2 + \alpha^2\|\hat{z}\|^2\big) + d_2\bigg(\nu - \frac{cM_1}{\lambda^{1/4}\lambda_1^{3/4}}\bigg)\big(|\hat{z}|^2 + \alpha^2\|\hat{z}\|^2\big) \leq b(t), \qquad (43)$$

where

$$b(t) = \big|\big(B\big(u(t), \bar{z}(t)\big), \hat{z}(t)\big)\big| + \big|\big(B\big(\bar{z}(t), u(t)\big), \hat{z}(t)\big)\big| + \big|\big(B\big(z(t), \bar{z}(t)\big), \hat{z}(t)\big)\big|.$$

Applying relation (41) and using the fact that u is bounded in the V norm, we conclude that $b(t) \to 0$, as $t \to \infty$. Therefore, applying Gronwall's Lemma 1 to (43) yields

$$\lim_{t \to \infty}\big\|\hat{z}(t)\big\| = 0,$$

for λ large enough, satisfying

$$\lambda > \lambda_1^3\bigg(\frac{cM_1}{\nu}\bigg)^4. \qquad (44)$$

Summarizing, the statement of the theorem holds for λ large enough satisfying relations (36), (40), and (44).

$\square$

5 Estimating the Exponential Decaying Small Scale

As we have mentioned in the Introduction, an additional goal of this research is to provide further support for the proposal made in Cao et al. (2006) that the NSV system (1), with the small regularization parameter α, can be used as a numerical model for studying the original Navier–Stokes equations, and in particular their statistical properties. Theorem 2 actually states that the global attractor of the NSV system consists of real analytic functions $u(x, t)$, whose Fourier spectrum $\hat{u}(k, t)$ satisfies the decay estimate

$$\big|\hat{u}(k, t)\big| \leq c|k|^{-2}e^{-|k|/\lambda^{1/2}}.$$

Therefore, following the ideas of Doering and Titi (1995) (see also Henshaw et al. 1989, 1990 for a different approach), the quantity $1/\lambda^{1/2}$, can be naturally identified as the *exponential decaying length scale*, since the exponential decay of the spectrum of u is effective only at high wavenumbers satisfying $|k| > \lambda^{1/2}$.

In the case of the Navier–Stokes equations, the exponential decaying length scale, and similarly the radius of analyticity of solutions, can be identified with the *smallest effective length scale* in the turbulent flow (see, e.g., Foias et al. 2001; Doering and Titi 1995; Henshaw et al. 1989, 1990). Classical Kolmogorov theory of turbulence states that the smallest effective length scale in the flow is proportional to

$$\ell_K = \bigg(\frac{\nu^3}{\varepsilon}\bigg)^{1/4},$$

where

$$\varepsilon = \nu\langle \|u\|^2\rangle,$$

is the mean energy dissipation rate, and $\langle\cdot\rangle$ denotes either the long time average, or the ensemble average with respect to the proper invariant probability measure. In Doering and Titi (1995), it was shown that for the solution $u(t)$ of the 3D Navier–Stokes equations, as long $\|u(t)\|$ remains bounded uniformly on some interval of time $[0, T]$, the smallest length scale of the turbulent flow satisfies

$$\ell \sim L\left(\frac{\ell_K}{L}\right)^4, \tag{45}$$

where in the definition of ℓ_K, instead of the usual definition of the energy dissipation rate ε, the authors considered the largest instantaneous energy dissipation rate on the time interval $[t_1, T]$, on which the solution of the equations remains regular

$$\varepsilon_{\sup} = \sup_{t_1 \le t \le T} \nu\|v(t)\|^2.$$

In the case of the NSV system, similarly to the Navier–Stokes equations, we can define ℓ_{NSV}—the exponential decaying length scale. In other words, ℓ_{NSV} is the largest length scale below which an exponential decay of the spectrum of the solutions of the NSV system lying on the global attractor becomes effective. In this section, we would like to derive a lower bound for ℓ_{NSV}, similar to relation (45) for the 3D Navier–Stokes equations. For other estimates on a related smallest length scale (via computation of the radius of analyticity of the solutions) of the Navier–Stokes equations in 2 and 3 dimensions, see Henshaw et al. (1989, 1990), and Kukavica (1999) (see also Ilyin and Titi 2007). See also Constantin et al. (1985) and Foias et al. (2001) for other approach to this subject.

The energy of the NSV system is defined as

$$E(t) = |u(t)|^2 + \alpha^2\|u(t)\|^2,$$

which satisfies the balance

$$\frac{1}{2}\frac{d}{dt}E(t) = -\nu\|u\|^2 + (f, u).$$

We denote the mean rate of dissipation of energy for the NSV system as

$$\varepsilon = \nu\langle\|u\|^2\rangle,$$

where $\langle\cdot\rangle$ stands for the long-time average. Moreover, we have the bound

$$\varepsilon \le \varepsilon_{\sup} := \nu M_1^2.$$

In order to find a lower bound for the exponentially decaying length scale of the NSV flow, we need to estimate the value of λ, from the inequalities (36), (40), and (44) in

the proof of Theorem 2, since $\lambda^{-1/2}$ is a lower bound for the radius of analyticity of the solutions of the NSV system lying on the attractor and, therefore,

$$\ell_{\text{NSV}} \geq \lambda^{-1/2}.$$

First, note that the condition (44) is satisfied for

$$\lambda^{-1/2} \sim \frac{\nu^3}{L^3 \varepsilon_{\text{sup}}} = L\left(\frac{\ell_K}{L}\right)^4. \tag{46}$$

Moreover, for a small viscosity ν and α, we can estimate M_2, using the expressions of Lemma 2 in the following way

$$M_2 \sim C_6 \frac{M_1^3}{\alpha^2 \nu^2 \lambda_1^{5/2}},$$

where $C_6 > 0$ is an absolute constant. Therefore, the condition (36) holds if

$$\lambda^{-1/2} \sim \frac{\nu \lambda_1^{3/4}}{M_1^{1/2} M_2^{1/2}} \sim \frac{\alpha \nu^2 \lambda_1^2}{M_1^2} \sim \frac{\alpha \nu^3}{L^4 \varepsilon_{\text{sup}}} = \alpha\left(\frac{\ell_K}{L}\right)^4, \tag{47}$$

or, on the other hand if

$$\lambda^{-1/2} \sim \frac{\nu^2 \lambda_1^{3/2}}{M_1^2} \sim \frac{\nu^3}{L^3 \varepsilon_{\text{sup}}} = L\left(\frac{\ell_K}{L}\right)^4. \tag{48}$$

Finally, we are left to check when the condition (40) is satisfied. In order to do this, let us assume, as it is conventionally done, that $\hat{f} = 0$, namely, λ is chosen large enough such that the forcing f is supported on the modes less than λ^{-1}. In addition, we assume that $\lambda > \alpha^{-2}$, so that $d_2 \geq \frac{1}{2}\alpha^{-2}$. In that case, the condition (40) becomes

$$\lambda^{1/2} \sim \frac{M_1 M_2^{1/3}}{\nu^{4/3} \lambda_1} \sim \frac{M_1^2}{\nu^2 \alpha^{2/3} \lambda_1^{11/6}} \sim L^{11/3} \alpha^{-2/3} \frac{M_1^2}{\nu^2},$$

and we obtain the estimate

$$\lambda^{-1/2} \sim L^{1/3} \alpha^{2/3} \left(\frac{\ell_K}{L}\right)^4. \tag{49}$$

Combining relations (46), (47), (48), and (49), we conclude that the exponential decaying length scale of the NSV equations satisfies

$$\ell_{\text{NSV}} \geq \min\{L, \alpha, L^{1/3} \alpha^{2/3}\} \cdot \left(\frac{\ell_K}{L}\right)^4. \tag{50}$$

Note that this estimate has the same asymptotic behavior as the estimate of the characteristic length scale of the 3D Navier–Stokes equations obtained in Doering and Titi (1995), without requiring any additional assumptions on the regularity of the flow of the system (1).

6 Radius of Analyticity of Stationary Solutions

At the end of the previous section, we computed the exponential decaying length scale of the NSV model by estimating the radius of analyticity of the functions lying in the global attractor of the system. A particular example of the functions lying on the attractor are the stationary solutions of the system. The goal of this section is to show that lower bounds for the exponential decaying length scale of the stationary solutions of the NSV system are the same as those obtained in the last section for the general element of the global attractor. Observe that the NSV equations has the same stationary solutions as the 3D Navier–Stokes equations. All calculation in this section are formal and can be rigorously justified using the Galerkin approximation procedure. We are following the ideas introduced in Oliver and Titi (2001).

The steady state equation of (1) has the form

$$\nu Au + B(u, u) = f. \tag{51}$$

Note the identity, where τ is now a dummy variable in the interval $[0, \sigma]$

$$\frac{d}{d\tau}\|w\|_{1,\tau}^2 \equiv 2\|w\|_{3/2,\tau}^2, \tag{52}$$

for every time independent function $w \in G_\sigma^{3/2}$. Taking an inner product of (51) with $A^{1/2}e^{2\tau A^{1/2}}u$, we obtain

$$\nu\|u\|_{3/2,\tau}^2 \le \left|\left(B(u, u), A^{1/2}u\right)_{0,\tau}\right| + \left|\left(f, A^{1/2}u\right)_{0,\tau}\right|. \tag{53}$$

Let us assume that the forcing f is supported on the first N_f modes. Therefore, we can write

$$\left|\left(f, A^{1/2}u\right)_{0,\tau}\right| \le e^{2\tau N_f^{1/2}}N_f^{1/2}\left|(f, u)\right| \le e^{2\sigma N_f^{1/2}}N_f^{1/2}\nu\|u\|^2,$$

where the last inequality is the result of the energy conservation, and we chose a large σ to be determined later. Moreover, we can estimate

$$\left|\left(B(u, u), A^{1/2}u\right)\right|_{0,\tau} \le c\lambda_1^{3/4}\|u\|_{1,\tau}^2\|u\|_{3/2,\tau} \le \frac{c^2}{2\nu}\|u\|_{1,\tau}^4 + \frac{\nu}{2}\|u\|_{3/2,\tau}^2.$$

Let us substitute the last two inequalities into (53) to get

$$\frac{1}{2}\nu\|u\|_{3/2,\tau}^2 \le \frac{c^2\lambda_1^{3/2}}{2\nu}\|u\|_{1,\tau}^4 + e^{2\sigma N_f^{1/2}}N_f^{1/2}\nu\|u\|^2, \tag{54}$$

which we can rewrite

$$\|u\|_{3/2,\tau}^2 \le \frac{c^2\lambda_1^{3/2}}{\nu^2}\|u\|_{1,\tau}^4 + 2e^{2\sigma N_f^{1/2}}N_f^{1/2}\|u\|^2. \tag{55}$$

Applying to the last inequality, the identity (52), we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|u\|_{1,\tau}^2 \le \frac{c^2\lambda_1^{3/2}}{\nu^2}\|u\|_{1,\tau}^4 + 2e^{2\sigma N_f^{1/2}}N_f^{1/2}\|u\|^2.$$

Denote

$$y(\tau) = \frac{2c^2\lambda_1^{3/2}}{\nu^2}\|u\|_{1,\tau}^2, \qquad F := \frac{2\sqrt{2}c\lambda_1^{3/4}}{\nu}e^{\sigma N_f^{1/4}}N_f^{1/4}\|u\|.$$

Therefore, $y(\tau)$ satisfies

$$\dot{y} \leq y^2 + F^2 \leq (y+F)^2.$$

Once again, denote, $z = y + F$, which satisfies

$$\dot{z} \leq z^2,$$

therefore,

$$z(\tau) = y(\tau) + F \leq \left(z^{-1}(0) - \tau\right)^{-1}.$$

"Blow-up time" is

$$\tau_{\mathrm{B}} > z^{-1}(0) = \frac{1}{\frac{2c^2\lambda_1^{3/2}}{\nu^2}\|u\|_{1,\tau}^2 + \frac{2\sqrt{2}c\lambda_1^{3/4}}{\nu}e^{\sigma N_f^{1/4}}N_f^{1/4}\|u\|} \geq C\frac{\nu^2}{\lambda_1^{3/2}\|u\|^2} \geq L\left(\frac{\ell}{L}\right)^4,$$

where we used the fact that u is a steady state of the system (1), and hence lies in the global attractor. As a result, we get $u \in G_\tau^2$ for all $\tau < \tau_{\mathrm{B}}$. Therefore, we showed that the exponential decaying length scale of the stationary solutions of the NSV system, satisfies the same bound (50) as the general solution of the NSV equations lying on the global attractor. Moreover, this bound also holds for the smallest length scales of the stationary solutions of the Navier–Stokes equations—similar to the general bound obtained in Doering and Titi (1995).

7 Conclusions

We prove that the elements of the global attractor of the 3D NSV equations (1) with periodic boundary conditions, driven by an analytic forcing, are analytic. A consequence of this result is that the solutions of the 3D NSV system lying on the global attractor have exponentially decaying spectrum, despite the fact that the addition of the $-\alpha^2\triangle u_t$ term changes the parabolic character of the original Navier–Stokes equation, which now starts to behave similar to a damped hyperbolic system.

An important consequence of our result is that the solutions of the 3D NSV system (1) lying on the global attractor posses a dissipation range—an exponentially decaying spectrum. This fact provides additional evidence that (1) with the small regularization parameter α enjoys similar statistical properties of the 3D Navier–Stokes equations, a fact that was first suggested in Cao et al. (2006).

Finally, following the ideas of Oliver and Titi (2001), we have computed a lower bound of the radius of analyticity of the steady state solution of the NSV and Navier–Stokes equations. The bound coincides with the one obtained for the general solutions of the system (1) lying on the global attractor.

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