The Generalized Dirichlet to Neumann Map for the KdV Equation on the Half-Line

P.A. Treharne · A.S. Fokas

Abstract For the two versions of the KdV equation on the positive half-line an initial-boundary value problem is well posed if one prescribes an initial condition plus either one boundary condition if q_t and q_{xxx} have the same sign (KdVI) or two boundary conditions if q_t and q_{xxx} have opposite sign (KdVII). Constructing the generalized Dirichlet to Neumann map for the above problems means characterizing the unknown boundary values in terms of the given initial and boundary conditions. For example, if $\{q(x,0), q(0,t)\}$ and $\{q(x,0), q(0,t), q_x(0,t)\}$ are given for the KdVI and KdVII equations, respectively, then one must construct the unknown boundary values $\{q_x(0,t), q_{xx}(0,t)\}$ and $\{q_{xx}(0,t)\}$, respectively. We show that this can be achieved without solving for $q(x,t)$ by analysing a certain "global relation" which couples the given initial and boundary conditions with the unknown boundary values, as well as with the function $\Phi^{(t)}(t,k)$, where $\Phi^{(t)}$ satisfies the t-part of the associated Lax pair evaluated at $x = 0$. The analysis of the global relation requires the construction of the so-called Gelfand–Levitan–Marchenko triangular representation for $\Phi^{(t)}$. In spite of the efforts of several investigators, this problem has remained open. In this paper, we construct the representation for $\Phi^{(t)}$ for the first time and then, by employing this representation, we solve *explicitly* the global relation for the unknown boundary values in terms of the given initial and boundary conditions and the func-

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tion $\Phi^{(t)}$. This yields the unknown boundary values in terms of a nonlinear Volterra integral equation. We also discuss the implications of this result for the analysis of the long t-asymptotics, as well as for the numerical integration of the KdV equation.

1 Introduction

The Korteweg–de Vries (KdV) equation

$$q_t + cq_x - \lambda q_{xxx} + 6qq_x = 0, \quad c = 0, 1, \ \lambda = \pm 1, \tag{1.1}$$

appears in a wide range of physical applications. For example, it models one-dimensional small amplitude surface gravity waves propagating in a shallow channel of water, where $\lambda = 1$ corresponds to the case of dominant surface tension. This equation has a celebrated history in the field of integrability; it was for this equation that a method was first developed, later called the inverse scattering transform method, for solving the Cauchy (initial value) problem for integrable nonlinear evolution equations (Gardner et al. 1967; Fokas and Zakharov 1994). In 1968 Peter Lax realised that the key to the integrability of the KdV equation was the fact that it can be expressed as the compatibility condition of two linear eigenvalue equations (Lax 1968), now called a "Lax pair". Soon thereafter it was shown that a similar Lax pair formulation exists for the nonlinear Schrödinger equation (Zakharov and Shabat 1972), as well as for the sine-Gordon and for the modified KdV equations. Actually, the existence of a Lax pair depending on a complex spectral parameter, denoted here by k, is a defining property of integrability.

The problem of solving the KdV equation on the half-line $0 < x < \infty$, $t > 0$, was considered in Fokas (2002a) using the general methodology introduced by one of the authors in Fokas (1997) and further developed in Fokas et al. (2005). The starting point of this method is the *simultaneous* spectral analysis of the two eigenvalue equations that make up the associated Lax pair. This yields the solution of the KdV equation in terms of the solution of a matrix Riemann–Hilbert (RH) problem. This RH problem is formulated in the complex plane of the spectral parameter, the complex k-plane, and its "jump" matrix is uniquely specified in terms of appropriate spectral functions, which are defined in terms of the initial condition $q(x, 0)$ and the boundary values $q(0, t), q_x(0, t), q_{xx}(0, t)$. Thus, the solution of a well posed initial-boundary value problem for the KdV equation can be expressed in terms of a matrix RH problem with *known* jumps, provided that one can construct the generalized Dirichlet to Neumann map, i.e. one can characterize the unknown boundary values in terms of the given initial and boundary conditions. It was shown in Fokas (1997) that this can be achieved, in principle, by analysing the so-called *global relation*, which is a simple algebraic relation coupling the spectral functions in the complex k-plane. Since these functions are defined in terms of the initial condition and all the boundary values, the global relation provides an implicit characterization of the Dirichlet to Neumann map. A breakthrough in the analysis of the global relation was announced in Boutet de Monvel et al. (2003), where it was shown that the global relation for the nonlinear Schrödinger equation can be solved *explicitly* in terms of the given initial and boundary conditions, as well as in terms of $\Phi^{(t)}(t, k)$, the solution of

the t-part of the associated Lax pair evaluated at $x = 0$. Following this development, the generalized Dirichlet to Neumann map for several integrable nonlinear PDEs was analysed in Fokas (2004), but the corresponding problem for (1.1) remained open. This was due to the fact that for the explicit solution of the global relation one requires an appropriate Gelfand–Levitan–Marchenko representation for $\Phi^{(t)}(t, k)$ and, in spite of the attempts of several investigators, such a representation could not be constructed for (1.1) until now. In this paper, we first present an appropriate Gelfand–Levitan–Marchenko representation for (1.1) and then use this representation to solve the global relation explicitly.

The results of this paper together with those of Fokas (2002a) complete the analysis of the initial-boundary value problem of the KdV equation on the half-line using the machinery of integrability. This has important analytical and numerical implications: (a) Although well-posedness results for the KdV equation can be obtained using PDE techniques, long time asymptotic results can be obtained so far only through the integrability formalism. (b) Following the influential work of Engquist and Majda (1977), the consideration of the Dirichlet to Neumann map provides the basis of the so-called method of reflectionless boundary conditions which, for evolution equations, is a method for the numerical computation of the initial value problem on the line. This approach has been extended in Zheng (2006) to integrable nonlinear PDEs using the Dirichlet to Neumann results of Boutet de Monvel et al. (2003) and Fokas (2004). Actually, it was stated in Zheng (2006) that this approach could *not* be applied to the KdV equation because the problem of constructing the Dirichlet to Neumann map for (1.1) was still open. Here, we present the solution of this problem and therefore the approach of Zheng (2006) can now be applied to the KdV equation.

The paper is organized as follows. In Sect. 2 we solve the global relation and present our main result. In Sect. 3 we derive a Gelfand–Levitan–Marchenko representation for $\Phi^{(t)}(t, k)$. In Sect. 4, using the techniques developed in Boutet de Monvel et al. (2003), we show that it is possible to solve the global relation for both versions of the KdV equation explicitly. Actually, following Zheng (2006), we compute certain k-integrals and hence we carry the technique of Boutet de Monvel et al. (2003) one step further by showing that the relevant expressions can be simplified substantially, see Sect. 2. These results are discussed further in Sect. 5. In Appendix 1 we review the general methodology of Fokas (2002a), namely: We first analyse the "direct" spectral problem by defining appropriate spectral functions $\{a(k), b(k)\}$ and $\{A(k), B(k)\}$, associated with the initial condition and with the boundary values, respectively. We then formulate the "inverse" spectral problem through a matrix RH problem defined in terms of the above spectral functions. The solution of this problem yields the solution of the KdV equation with prescribed initial and boundary data, provided that the spectral functions satisfy the global relation. In Appendix 2 we discuss the generalized Dirichlet to Neumann map for the linear version of (1.1) and show that in the limit of small data this map reduces to the corresponding map of the linear equation.

Remark 1.1 The usual caveat where one removes the q_x term from (1.1) by employing travelling coordinates is not available in the quarter-plane, since this transformation yields a moving boundary problem (see Bona et al. 2002). If one drops the q_x

term arbitrarily (i.e. take $c = 0$ in (1.1)), the resulting initial-boundary value problem can be treated by a considerably simplified version of the analysis that is developed here.

1.1 Notations and Assumptions

We use the following notations and assumptions:

- Subscripts with respect to x and t denote partial derivatives, for example

$$q_t = \frac{\partial q}{\partial t}, \qquad q_{xxx} = \frac{\partial^3 q}{\partial t^3}.$$

- $\overline{f(k)}$ denotes the complex conjugate of the function $f(k)$.
- $q(x, 0) = q_0(x)$ denotes the initial condition. It is assumed that the initial condition belongs to the Schwarz space on the half-line, which will be denoted by $S(\mathbb{R}^+)$.
- We denote the boundary values at $x = 0$ by

$$q(0, t) = g_0(t), \qquad q_x(0, t) = g_1(t), q \quad q_{xx}(0, t) = g_2(t). \tag{1.2}$$

  A subset of these functions is prescribed as boundary conditions. We assume that these boundary conditions are sufficiently smooth and are compatible with $q_0(x)$ at $x = t = 0$.
- $A_{11}, A_{12}, A_{21}, A_{22}$ denote the $(11), (12), (21), (22)$ entries, respectively, of the 2×2 matrix A.
- $\overline{D}$ denotes the closure of the domain D in the complex k-plane.
- The dispersion relation $\omega(k)$ is defined by

$$\omega(k) = 4\lambda k^3 + ck, \quad \lambda = \pm 1, \ c = 0, 1. \tag{1.3}$$

2 The Main Result

Given the initial condition $q_0(x)$, define the spectral functions $a(k)$ and $b(k)$ by

$$a(k) = \psi_2(0, k), \qquad b(k) = \psi_1(0, k), \quad \text{Im}\, k \le 0, \tag{2.1}$$

where the vector $(\psi_1(x, k), \psi_2(x, k))$ satisfies the following ODE, which is uniquely specified in terms of $q_0(x)$:

$$\frac{\partial \psi_1}{\partial x} - 2ik\psi_1 = \frac{\lambda}{2ik} q_0(x)(\psi_1 + \psi_2),$$

$$\frac{\partial \psi_2}{\partial x} = -\frac{\lambda}{2ik} q_0(x)(\psi_1 + \psi_2), \tag{2.2a}$$

where

$$0 < x < \infty, \qquad \text{Im}\, k \le 0,$$

and

$$\lim_{x \to \infty} \psi_1(x, k) = 0, \qquad \lim_{x \to \infty} \psi_2(x, k) = 1. \tag{2.2b}$$

The global relation associated with the KdV equation is given by (see Fokas 2002a)

$$a(k)B(t, k) - b(k)A(t, k) = e^{2i\omega(k)t}c(t, k), \quad \operatorname{Im} k \leq 0, \ 0 < t < \infty, \tag{2.3}$$

where $\omega(k)$ is defined by (1.3), $c(t, k)$ is an analytic function of k for $\operatorname{Im} k < 0$ which is of $O(1/k)$ as $k \to \infty$, and the functions $\{A(t, k), B(t, k)\}$ are defined by

$$A(t, k) = \overline{\Phi_2(t, \bar{k})}, \qquad B(t, k) = -e^{2i\omega(k)t}\Phi_1(t, k), \tag{2.4}$$

where the vector $(\Phi_1(t, k), \Phi_2(t, k))^{\mathrm{T}}$ satisfies the following ODE, which is uniquely specified in terms of $\{g_0(t), g_1(t), g_2(t)\}$:

$$
\begin{aligned}
\frac{\partial \Phi_1}{\partial t} + 2i\omega(k)\Phi_1 &= 2ikg_0(t)\Phi_2 + g_1(t)\Phi_2 \\
&\quad + \frac{1}{2ik}\big(g_2(t) - \lambda c g_0(t) - 2\lambda g_0(t)^2\big)(\Phi_1 + \Phi_2), \\
\frac{\partial \Phi_2}{\partial t} &= -2ikg_0(t)\Phi_1 + g_1(t)\Phi_1 \\
&\quad - \frac{1}{2ik}\big(g_2(t) - \lambda c g_0(t) - 2\lambda g_0(t)^2\big)(\Phi_1 + \Phi_2),
\end{aligned}
\tag{2.5a}
$$

where

$$0 < t < \infty, \qquad k \in \mathbb{C}, \tag{2.5b}$$

and

$$\Phi_1(0, k) = 0, \qquad \Phi_2(0, k) = 1. \tag{2.5c}$$

In the remainder of this section we give our main results and show that, for both versions of the KdV equation, the global relation (2.3) can be solved *explicitly* for the unknown boundary values, i.e. we give the generalized Dirichlet to Neumann map for (1.1). These results will be proved in Sects. 3 and 4 by using a Gelfand–Levitan–Marchenko triangular representation for the spectral functions $A(t, k)$ and $B(t, k)$.

2.1 The Gelfand–Levitan–Marchenko Representation of Φ_1 and Φ_2

Proposition 2.1 *The vector $(\Phi_1(t, k), \Phi_2(t, k))^{\mathrm{T}}$ has the following Gelfand–Levitan–Marchenko representation*

$$
\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-t}^{t} \begin{pmatrix} F_1(t, s, k) \\ F_2(t, s, k) \end{pmatrix} e^{i\omega(k)(t-s)\sigma_3} \, ds,
$$
$$
\sigma_3 = \operatorname{diag}(1, -1), \tag{2.6}
$$

where

$$F_1(t, s, k) = -kB_1 + C_1 + \frac{1}{4\lambda}g_0(t)A_1 - \frac{1}{k}\left[D_1 - \frac{1}{4\lambda}g_0(t)B_2 - \frac{1}{8i\lambda}g_1(t)A_1\right],$$

$$F_2(t, s, k) = k^2 A_1 + kB_2 + C_2 + \frac{1}{k}\left[D_1 - \frac{1}{4\lambda}g_0(t)B_2 - \frac{1}{8i\lambda}g_1(t)A_1\right],$$

and the six scalar functions $\{A_1(t, s), B_1(t, s), B_2(t, s), C_1(t, s), C_2(t, s), D_1(t, s)\}$
satisfy the following well-posed characteristic-value problem:

$$(\partial_t + \partial_s)A_1 = 2ig_0 B_1 + 8i\lambda D_1, \tag{2.7a}$$

$$(\partial_t - \partial_s)B_1 = \left(G_2 - \frac{1}{2i\lambda}g_0\right)A_1 - g_1 B_2 - 2ig_0 C_2, \tag{2.7b}$$

$$(\partial_t + \partial_s)B_2 = \left(G_2 + \frac{1}{2i\lambda}g_0^2\right)A_1 - g_1 B_1 - 2ig_0 C_1, \tag{2.7c}$$

$$(\partial_t - \partial_s)C_1 = -\frac{1}{4\lambda}(\dot{g}_0 + g_0 g_1 + g_1)A_1 + \left(G_2 + \frac{1}{2i\lambda}g_0^2\right)B_1$$
$$+ \left(\frac{1}{2i\lambda}g_0 - G_2 + \frac{1}{2i\lambda}g_0^2\right)B_2 + g_1 C_2, \tag{2.7d}$$

$$(\partial_t + \partial_s)C_2 = -G_2 B_1 + \left(G_2 + \frac{1}{2i\lambda}g_0^2\right)B_2 + g_1 C_1 + (2i + 2ig_0)D_1, \tag{2.7e}$$

$$(\partial_t - \partial_s)D_1 = \frac{1}{8i\lambda}\left(\dot{g}_1 + g_1^2 + 4ig_0 G_2 + \frac{1}{\lambda}g_0^3\right)A_1 + \frac{1}{2\lambda}g_0 \partial_s B_2$$
$$+ \frac{1}{4\lambda}(\dot{g}_0 + g_0 g_1)B_2 + \left(G_2 + \frac{1}{2i\lambda}g_0^2\right)C_1 + G_2 C_2, \tag{2.7f}$$

where $\dot{g}_0 = \frac{d}{dt}g_0$, $\dot{g}_1 = \frac{d}{dt}g_1$ *and* $G_2 = \frac{i}{2}(g_2 - \lambda c g_0 - 2\lambda g_0^2)$, *with boundary conditions*

$$B_1(t, t) = -ig_0(t), \qquad C_1(t, t) = \frac{1}{2}g_1(t),$$
$$D_1(t, t) = \frac{i}{4}\left(g_2(t) - \lambda c g_0(t) - 2\lambda g_0(t)^2\right), \tag{2.8}$$

and

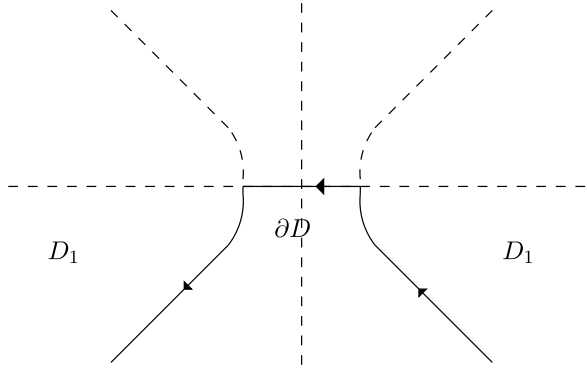$$A_1(t, -t) = B_2(t, -t) = C_2(t, -t) = 0. \tag{2.9}$$

Proof The proof of this Proposition will be given below in Sect. 3.   □

2.2 The Generalized Dirichlet to Neumann Map for KdVI

For the KdV equation (1.1) with $\lambda = -1$, an initial-boundary value problem is well posed if the initial condition and a *single* boundary condition are specified. In the

Fig. 1 The oriented boundary ∂D for the case $\lambda = -1$

case that the KdV equation models water waves the appropriate initial and boundary conditions are

$$q(x, 0) = 0, \qquad q(0, t) = g_0(t).$$

In this case the analysis of the global relation yields two algebraic relations that can be solved explicitly to give the unknown boundary values $g_1(t)$ and $g_2(t)$ in terms of the given boundary condition $g_0(t)$ and in terms of the kernel functions that appear in the Gelfand–Levitan–Marchenko representation of Φ_1 and Φ_2. We will show that

$$g_1(t) = \frac{1}{2} g_0(t) A_1(t, t)$$
$$- 3^{-1/3} \int_0^t \frac{\text{Ai}(\zeta)}{(t - \tau)^{1/3}} \left[3\mathrm{i} \frac{\partial B_1}{\partial \tau} - 4B_1 \right](t, 2\tau - t)\, \mathrm{d}\tau, \qquad (2.10a)$$

$$g_2(t) = -2g_0(t)^2 + \mathrm{i}g_0(t) B_2(t, t) - \frac{1}{2} g_1(t) A_1(t, t)$$
$$- 3^{-2/3} \int_0^t \frac{\text{Ai}'(\zeta)}{(t - \tau)^{2/3}} \left[\frac{3\mathrm{i}}{2} \frac{\partial B_1}{\partial \tau} + 2B_1 \right](t, 2\tau - t)\, \mathrm{d}\tau, \qquad (2.10b)$$

where $\text{Ai}(\zeta)$ is a solution of Airy's equation given by

$$\text{Ai}(\zeta) = \frac{1}{2\pi} \int_{\partial D} \mathrm{e}^{\mathrm{i}(\zeta k + k^3/3)}\, \mathrm{d}k, \quad \zeta = -3^{-1/3}(t - \tau)^{2/3}, \qquad (2.11)$$

the contour ∂D, shown in Fig. 1, is the oriented boundary of the domain D_1 defined by
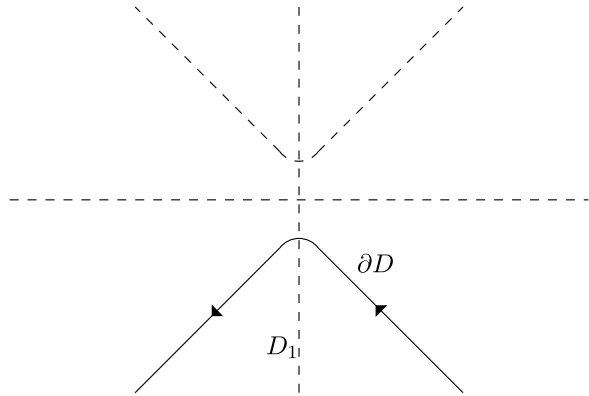
$$D_1 = \left\{ k \in \mathbb{C}, \; \text{Im}\, k < 0 \cap \text{Im}\, \omega(k) > 0 \right\},$$

and $\omega(k)$ is defined by (1.3) with $\lambda = -1$.

2.3 The Generalized Dirichlet to Neumann Map for KdVII

For the KdV equation (1.1) with $\lambda = 1$, the initial-boundary value problem is well posed if the initial condition and *two* independent boundary conditions are specified.

Fig. 2 The oriented boundary ∂D for the case $\lambda = 1$

In the case that the KdV equation models water waves with dominant surface tension the appropriate initial and boundary conditions are

$$q(x, 0) = 0, \qquad q(0, t) = g_0(t), \qquad q_x(0, t) = g_1(t).$$

In this case the global relation can be solved explicitly for the unknown boundary value $g_2(t)$ in terms of the given boundary conditions and in terms of the kernel functions that appear in the Gelfand–Levitan–Marchenko representation of Φ_1 and Φ_2. We will show that

$$
\begin{aligned}
g_2(t) = {} & c g_0(t) + 2 g_0(t)^2 - i g_0(t) B_2(t, t) - \frac{1}{2} g_1(t) A_1(t, t) \\
& + 3^{-2/3} \int_0^t \frac{\text{Ai}'(\zeta)}{(t - \tau)^{2/3}} \left[\frac{3i}{2} \frac{\partial B_1}{\partial \tau} + 2 B_1 \right] (t, 2\tau - t) \, d\tau \\
& - 3^{-1/3} \int_0^t \frac{\text{Ai}(\zeta)}{(t - \tau)^{1/3}} \left[6 \frac{\partial C_1}{\partial \tau} + 8i C_1 + g_0(t) \left(\frac{3}{2} \frac{\partial A_1}{\partial \tau} + 2i A_1 \right) \right] \\
& \times (t, 2\tau - t) \, d\tau,
\end{aligned}
\tag{2.12}
$$

where $\text{Ai}(\zeta)$ is given by (2.11) but with $\zeta = 3^{-1/3}(t - \tau)^{2/3}$, the contour ∂D, shown in Fig. 2, is the oriented boundary of the domain

$$D_1 = \left\{ k \in \mathbb{C}, \, \text{Im} \, k < 0 \cap \text{Im} \, \omega(k) > 0 \right\},$$

and $\omega(k)$ is defined by (1.3) with $\lambda = 1$.

Remark 2.2 The analogues of (2.10) and (2.12) in the case that $q(x, 0) \neq 0$ are given in (4.5) and (4.6). These formulae contain certain integrals in the complex k-plane involving the spectral functions $a(k)$, $b(k)$ (which are known) and the kernel functions that appear in the Gelfand–Levitan–Marchenko representation of $\Phi_1(t, k)$ and $\Phi_2(t, k)$.

Remark 2.3 The solution of the global relation given in (2.10) and (2.12) is expressed in terms of the given boundary conditions and in terms of the kernel functions that appear in the Gelfand–Levitan–Marchenko representation of $\Phi_1(t, k)$ and $\Phi_2(t, k)$. We note that it is also possible to rewrite (2.10) and (2.12) in terms of $\Phi_1(t, k)$ and $\Phi_2(t, k)$ only (i.e. without reference to the kernel functions), but the resulting formulas are more complicated than those given here (see Fokas 2004).

Remark 2.4 If $c = 0$ in (1.1), i.e. if the q_x term is absent, the formulae (2.10) and (2.12) simplify further and, instead of the Airy function (2.11), they now involve the gamma function. For instance, if $q(x, 0) = 0$, $c = 0$, $\lambda = -1$, the unknown boundary values $g_1(t)$ and $g_2(t)$ can be expressed in terms of $q(0, t) = g_0(t)$ by the following formulae:

$$g_1(t) = \frac{1}{2} g_0(t) A_1(t, t)$$

$$+ \frac{1}{6i\pi} \Gamma\left(\frac{1}{3}\right) \int_0^t \frac{1}{(t - \tau)^{1/3}} \left[3i \frac{\partial B_1}{\partial \tau} - 4B_1\right](t, 2\tau - t)\, d\tau, \qquad (2.13a)$$

$$g_2(t) = -2g_0(t)^2 + ig_0(t) B_2(t, t) - \frac{1}{2} g_1(t) A_1(t, t)$$

$$- \frac{1}{6i\pi} \Gamma\left(\frac{2}{3}\right) \int_0^t \frac{1}{(t - \tau)^{2/3}} \left[\frac{3i}{2} \frac{\partial B_1}{\partial \tau} + 2B_1\right](t, 2\tau - t)\, d\tau, \qquad (2.13b)$$

where $\{A_1, B_1, B_2, C_1, C_2, D_1\}$ satisfy the hyperbolic system (2.7–2.9).

Remark 2.5 It is shown in Appendix 2 that, in the linear limit of small boundary data, the expressions in (2.10) and (2.12) reduce to the *same* representations that are obtained by solving the global relation associated with the linearized KdV equation

$$q_t + cq_x - \lambda q_{xxx} = 0. \qquad (2.14)$$

3 The Gelfand–Levitan–Marchenko Representation of $\Phi^{(t)}(t, k)$

In this section we verify Proposition 2.1 directly. Let the matrix $\Phi^{(t)}(t, k)$ be defined in terms of (Φ_1, Φ_2) by

$$\Phi^{(t)}(t, k) = \begin{pmatrix} \overline{\Phi_2(t, \bar{k})} & \Phi_1(t, k) \\ \overline{\Phi_1(t, \bar{k})} & \Phi_2(t, k) \end{pmatrix}. \qquad (3.1)$$

Then (2.5) imply that this matrix satisfies the equation

$$\partial_t \Phi^{(t)}(t, k) + i\omega(k)\left[\sigma_3, \Phi^{(t)}(t, k)\right] = \left[k Q_0(t) + Q_1(t) + \frac{1}{k} Q_2(t)\right] \Phi^{(t)}(t, k), \qquad (3.2a)$$

where

$$Q_0(t) = G_0(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad Q_1(t) = g_1(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Q_2(t) = G_2(t) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

(3.2b)

and

$$G_0(t) = -2ig_0(t), \qquad G_2(t) = \frac{i}{2}\big(g_2(t) - \lambda c g_0(t) - 2\lambda g_0(t)^2\big). \qquad (3.2c)$$

Substituting the representation (cf. Boutet de Monvel et al. 2003 and Fokas 2004)

$$\Phi^{(t)}(t, k) = I + \int_{-t}^{t} \left(k^2 \tilde{A}(t, s) + k\tilde{B}(t, s) + \tilde{C}(t, s) + \frac{1}{k}\tilde{D}(t, s) \right) e^{i\omega(k)(t-s)\sigma_3} \, ds,$$

(3.3)

into equation (3.2a) we obtain an integral equation involving the matrix-valued functions $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$. By multiplying the following identity (obtained by integration by parts) by k^2

$$\int_{-t}^{t} k^3 F(s) e^{i\omega(k)(t-s)\sigma_3} \, ds$$

$$= \frac{i\lambda}{4}\big(F(t)\sigma_3 - F(-t)\sigma_3 e^{2i\omega(k)t\sigma_3}\big)$$

$$- \frac{\lambda}{4} \int_{-t}^{t} \big(kF(s) + iF_s(s)\sigma_3\big) e^{i\omega(k)(t-s)\sigma_3} \, ds, \qquad (3.4)$$

we can eliminate the integral terms involving k^5. Then, using the identity obtained from multiplying (3.4) by k, as well as (3.4) itself, we can eliminate the integral terms involving k^4 and k^3. Finally, by using the identity obtained from multiplying (3.4) by $1/k$ we can eliminate the integral terms involving k^2. The remaining terms involve k, 1, $1/k$, and $1/k^2$. The latter term involves the product $Q_2\tilde{D}$, which is equal to zero if the matrix $\tilde{D}$ has the following form

$$\tilde{D}(t, s) = \tilde{D}_1(t, s) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \qquad (3.5)$$

Equating to zero the coefficients of the terms involving k, 1, $1/k$ we obtain the following equations:

$$\tilde{B}_t + \sigma_3 \tilde{B}_s \sigma_3 = Q_0 \tilde{C} + Q_1 \tilde{B} + Q_2 \tilde{A} - \frac{1}{4\lambda} Q_0 \tilde{A}, \qquad (3.6a)$$

$$\tilde{C}_t + \sigma_3 \tilde{C}_s \sigma_3 - \frac{1}{4\lambda}\big(\tilde{A}_t + \sigma_3 \tilde{A}_s \sigma_3\big)$$

$$= Q_0 \tilde{D} + Q_1 \tilde{C} + Q_2 \tilde{B} - \frac{1}{4\lambda}\big(Q_0 \tilde{B} + Q_1 \tilde{A}\big) + \frac{1}{4i\lambda} Q_0 \tilde{A}_s \sigma_3, \qquad (3.6b)$$

$$\tilde{D}_t + \sigma_3 \tilde{D}_s \sigma_3 + \frac{1}{4i\lambda}\left(\tilde{A}_{ts}\sigma_3 + \sigma_3 \tilde{A}_{ss}\right)$$

$$= Q_1 \tilde{D} + Q_2 \tilde{C} + \frac{1}{4i\lambda}\left(Q_0 \tilde{B}_s + Q_1 \tilde{A}_s\right)\sigma_3, \tag{3.6c}$$

Equating the coefficients of the boundary terms involving $k^2, k, 1, 1/k$ we find the following boundary conditions:

$$\tilde{A}(t,t) - \sigma_3 \tilde{A}(t,t)\sigma_3 = 0, \tag{3.7a}$$

$$\tilde{B}(t,t) - \sigma_3 \tilde{B}(t,t)\sigma_3 = Q_0, \tag{3.7b}$$

$$\tilde{C}(t,t) - \sigma_3 \tilde{C}(t,t)\sigma_3 = Q_1 - \frac{1}{4i\lambda}Q_0 \tilde{A}(t,t)\sigma_3, \tag{3.7c}$$

$$\tilde{D}(t,t) - \sigma_3 \tilde{D}(t,t)\sigma_3 - \frac{1}{4i\lambda}\left(\tilde{A}_t(t,t)\sigma_3 + \sigma_3 \tilde{A}_s(t,t)\right)$$

$$= Q_2 - \frac{1}{4i\lambda}\left(Q_0 \tilde{B}(t,t) + Q_1 \tilde{A}(t,t)\right)\sigma_3; \tag{3.7d}$$

and

$$\tilde{A}(t,-t) + \sigma_3 \tilde{A}(t,-t)\sigma_3 = 0, \tag{3.8a}$$

$$\tilde{B}(t,-t) + \sigma_3 \tilde{B}(t,-t)\sigma_3 = 0, \tag{3.8b}$$

$$\tilde{C}(t,-t) + \sigma_3 \tilde{C}(t,-t)\sigma_3 = \frac{1}{4i\lambda}Q_0 \tilde{A}(t,-t)\sigma_3, \tag{3.8c}$$

$$\tilde{D}(t,-t) + \sigma_3 \tilde{D}(t,-t)\sigma_3 + \frac{1}{4i\lambda}\left(\tilde{A}_t(t,-t)\sigma_3 + \sigma_3 \tilde{A}_s(t,-t)\right)$$

$$= \frac{1}{4i\lambda}\left(Q_0 \tilde{B}(t,-t) + Q_1 \tilde{A}(t,-t)\right)\sigma_3. \tag{3.8d}$$

We choose the matrices $\tilde{A}, \tilde{B}, \tilde{C}$ in the following form,

$$\tilde{A}(t,s) = A_1(t,s)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{B}(t,s) = \begin{pmatrix} -B_2 & -B_1 \\ B_1 & B_2 \end{pmatrix},$$

$$\tilde{C}(t,s) = \begin{pmatrix} C_2 & \tilde{C}_1 \\ \tilde{C}_1 & C_2 \end{pmatrix}. \tag{3.9}$$

Substituting (3.5) and (3.9) into equations (3.6) we find the following differential equations:

$$(\partial_t - \partial_s)B_1 = \left(G_2 - \frac{1}{4\lambda}G_0\right)A_1 - g_1 B_2 + G_0 C_2, \tag{3.10a}$$

$$(\partial_t + \partial_s)B_2 = G_2 A_1 - g_1 B_1 + G_0 \tilde{C}_1, \tag{3.10b}$$

$$(\partial_t - \partial_s)\tilde{C}_1 = \frac{1}{4i\lambda}G_0\partial_s A_1 - \frac{1}{4\lambda}g_1 A_1 + G_2 B_1 + \left(\frac{1}{4\lambda}G_0 - G_2\right)B_2$$
$$+ g_1 C_2 - G_0 \tilde{D}_1, \tag{3.10c}$$

$$(\partial_t + \partial_s)C_2 = \frac{1}{4\lambda}(\partial_t + \partial_s)A_1 + \left(\frac{1}{4\lambda}G_0 - G_2\right)B_1 + G_2 B_2$$
$$+ g_1 \tilde{C}_1 - G_0 \tilde{D}_1, \tag{3.10d}$$

$$(\partial_t - \partial_s)\tilde{D}_1 = \frac{1}{4i\lambda}g_1\partial_s A_1 - \frac{1}{4i\lambda}G_0\partial_s B_2 + G_2(\tilde{C}_1 + C_2) - g_1\tilde{D}_1, \tag{3.10e}$$

$$(\partial_t + \partial_s)\tilde{D}_1 = \frac{1}{4i\lambda}(\partial_t + \partial_s)\partial_s A_1 + \frac{1}{4i\lambda}G_0\partial_s B_1 + G_2(\tilde{C}_1 + C_2) - g_1\tilde{D}_1. \tag{3.10f}$$

Furthermore, (3.7) and (3.8) yield the following boundary conditions:

$$2B_1(t,t) = G_0(t), \tag{3.11a}$$

$$2\tilde{C}_1(t,t) = g_1(t) - \frac{1}{4i\lambda}G_0(t)A_1(t,t), \tag{3.11b}$$

$$2\tilde{D}_1(t,t) = G_2(t) - \frac{1}{4i\lambda}g_1(t)A_1(t,t) + \frac{1}{4i\lambda}G_0(t)B_2(t,t), \tag{3.11c}$$

$$-\frac{1}{4i\lambda}(\partial_t + \partial_s)A_1(t,t) = -G_2 + \frac{1}{4i\lambda}G_0(t)B_1(t,t), \tag{3.11d}$$

and

$$A_1(t,-t) = 0, \tag{3.12a}$$

$$B_2(t,-t) = 0, \tag{3.12b}$$

$$C_2(t,-t) = 0, \tag{3.12c}$$

$$2\tilde{D}_1(t,-t) - \frac{1}{4i\lambda}(\partial_t + \partial_s)A_1(t,-t) = \frac{1}{4i\lambda}G_0 B_1(t,-t), \tag{3.12d}$$

$$0 = -\frac{1}{4i\lambda}g_1 A_1(t,-t) + \frac{1}{4i\lambda}G_0 B_2(t,-t). \tag{3.12e}$$

By subtracting (3.10e) from (3.10f), we find the equation

$$\left\{(\partial_t + \partial_s)A_1 - g_1 A_1 - G_0(B_1 + B_2) - 8i\lambda\tilde{D}_1\right\}_s = 0. \tag{3.10f'}$$

Similarly, by adding (3.11c) and (3.11d), we find the boundary condition

$$(\partial_t + \partial_s)A_1(t,t) - g_1(t)A_1(t,t)$$
$$+ G_0(t)\big(B_1(t,t) + B_2(t,t)\big) - 8i\lambda\tilde{D}_1(t,t) = 0. \tag{3.11d'}$$

In summary, we have derived the following result: Let $\Phi^{(t)}(t,k)$ be expressed in terms of the matrices $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ by (3.3). Let these matrices have the form given

by (3.5) and (3.9). Then $\Phi^{(t)}(t,k)$ satisfies (3.2) provided that the scalar functions $\{A_1, B_1, B_2, \tilde{C}_1, C_2, \tilde{D}_1\}$ satisfy the differential equations (3.10a–3.10e), (3.10f′) together with the boundary conditions (3.11a–3.11c), (3.11d′) at $s = t$, and the boundary conditions (3.12) at $s = -t$.

We will now show that the above differential equations and boundary conditions are satisfied provided that the six functions $\{A_1, B_1, B_2, \tilde{C}_1, C_2, \tilde{D}_1\}$ satisfy the differential equations (3.10a–3.10e), the differential equation

$$(\partial_t + \partial_s)A_1 = g_1 A_1 - G_0(B_1 + B_2) + 8i\lambda \tilde{D}_1, \qquad (3.10f'')$$

as well as the boundary conditions (3.11a–3.11c) and (3.12a–3.12c). Indeed, if (3.10f″) is valid then (3.10f′) is also valid. Furthermore, evaluating (3.10f′) at $s = t$, it follows that (3.11d′) is also valid. In addition, evaluating (3.10f″) at $s = -t$ and dividing the resulting equation by $1/4i\lambda$ we find

$$2\tilde{D}_1(t,-t) - \frac{1}{4i\lambda}(\partial_t + \partial_s)A_1(t,-t)$$

$$= \frac{1}{4i\lambda}G_0\big(B_1(t,-t) + B_2(t,-t)\big) - \frac{1}{4i\lambda}g_1 A_1(t,-t),$$

which, taking into consideration (3.12a) and (3.12b), yields (3.12d). Finally, (3.12a) and (3.12b) imply that (3.12e) is identically satisfied.

It is convenient to replace $(\partial_t + \partial_s)A_1$ in the RHS of (3.10d) by the RHS of (3.10f″); thus (3.10d) can be replaced by the equation

$$(\partial_t + \partial_s)C_2 = \frac{1}{4\lambda}g_1 A_1 - G_2 B_1 + \left(G_2 - \frac{G_0}{4\lambda}\right)B_2 + g_1 \tilde{C}_1 + (2i - G_0)\tilde{D}_1. \qquad (3.10d')$$

Equations (3.11b) and (3.11d′) suggest the following change of variables

$$\tilde{C}_1(t,s) = C_1(t,s) - \frac{1}{8i\lambda}G_0(t)A_1(t,s),$$

$$\tilde{D}_1(t,s) = D_1(t,s) - \frac{1}{8i\lambda}g_1(t)A_1(t,s) + \frac{1}{8i\lambda}G_0(t)B_2(t,s).$$

This yields the required result.

3.1 The Spectral Functions $A(t,k)$, $B(t,k)$

Equation (3.1), together with (3.2) and the integral representation for $\Phi^{(t)}(t,k)$ in (3.3), imply the following expression for Φ_1:

$$\Phi_1(t,k) = e^{-i\omega(k)t}\int_{-t}^{t}\left[-kB_1(t,s) + \tilde{C}_1(t,s) - \frac{1}{k}\tilde{D}_1(t,s)\right]e^{i\omega(k)s}\,ds$$

$$= \int_0^t\left[-2kB_1(t,2\tau - t) + 2C_1(t,2\tau - t)\right.$$

$$\left. + \frac{1}{2\lambda}g_0(t)A_1(t,2\tau - t)\right]e^{2i\omega(k)(\tau - t)}\,d\tau,$$

$$-\frac{1}{k}\int_0^t\left[2D_1(t,2\tau-t)-\frac{1}{2\lambda}g_0(t)B_2(t,2\tau-t)\right.$$

$$\left.-\frac{1}{4\mathrm{i}\lambda}g_1(t)A_1(t,2\tau-t)\right]\mathrm{e}^{2\mathrm{i}\omega(k)(\tau-t)}\,\mathrm{d}\tau,$$

where we have used the change of variables $s\mapsto 2\tau-t$. The definition of $B(t,k)$ (2.4) then yields

$$B(t,k)=2k\hat{B}_1-\left(2\hat{C}_1+\frac{1}{2\lambda}g_0(t)\hat{A}_1\right)$$

$$+\frac{1}{k}\left(2\hat{D}_1-\frac{1}{2\lambda}g_0(t)\hat{B}_2-\frac{1}{4\mathrm{i}\lambda}g_1(t)\hat{A}_1\right),\qquad (3.13a)$$

where

$$\hat{N}(t,k)=\int_0^t N(t,2\tau-t)\mathrm{e}^{2\mathrm{i}\omega(k)\tau}\,\mathrm{d}\tau.\qquad (3.13b)$$

An analogous expression can also be written for $A(t,k)$:

$$A(t,k)=1+2k^2\hat{A}_1-2k\hat{B}_2+2\hat{C}_2$$

$$-\frac{1}{k}\left(2\hat{D}_1-\frac{1}{2\lambda}g_0(t)\hat{B}_2-\frac{1}{4\mathrm{i}\lambda}g_1(t)\hat{A}_1\right).\qquad (3.13c)$$

4 The Generalized Dirichlet to Neumann Map

As was noted earlier (see also Appendix 1), given $q_0(x)$ and a "proper"[1] subset of the boundary values $\{g_0(t),g_1(t),g_2(t)\}$ we can characterize the unknown boundary values by the requirement that the spectral functions $\{A(t,k),B(t,k)\}$ satisfy the global relation (2.3). For the Riemann–Hilbert problem formulated in Theorem 6.3 (see Appendix 1), the spectral functions $\{A(t,k),B(t,k)\}$ are needed in the domain

$$D_3=\left\{k\in\mathbb{C},\ \mathrm{Im}\,k>0\cap\mathrm{Im}\,\omega(k)>0\right\}.$$

However, since (2.3) is valid for $\mathrm{Im}\,k\leq 0$ we must transform the global relation from the lower half k-plane into D_3. We do this by using the invariance properties of the exponential $\mathrm{e}^{2\mathrm{i}\omega(k)t}$.

For the case $\lambda=-1$, the two non-trivial roots of the dispersion relation $\omega(p)=\omega(k)$ are such that $k\in D_3\Rightarrow p_+(k),p_-(k)\in D_1$ where D_1 is defined as

$$D_1=\left\{k\in\mathbb{C},\ \mathrm{Im}\,k<0\cap\mathrm{Im}\,\omega(k)>0\right\}.$$

---

[1]"Proper" here means those boundary conditions for which the initial-boundary value problem is at least linearly well posed. For example, if $\lambda=-1$ the boundary value problem with $q(0,t)$ given is well posed. Similarly, if $\lambda=1$ the boundary value problem with $\{q(0,t),q_x(0,t)\}$ given is well posed.
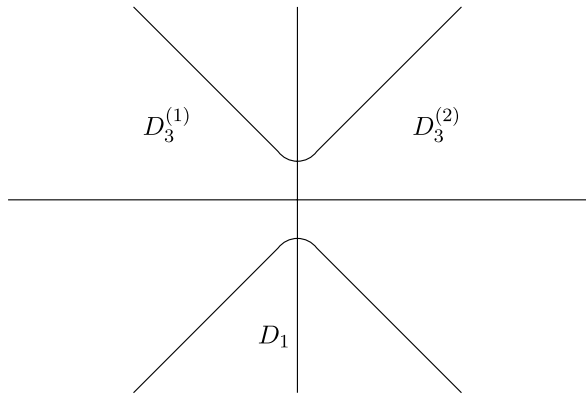
Evaluating the global relation (2.3) at p_+ and p_- we thus find two equations for the spectral functions $A(t, k)$ and $B(t, k)$, which are valid for $k \in D_3$

$$a(p_+)B(t, p_+) - b(p_+)A(t, p_+) = e^{2i\omega(k)t} c(t, p_+), \tag{4.1a}$$

$$a(p_-)B(t, p_-) - b(p_-)A(t, p_-) = e^{2i\omega(k)t} c(t, p_-), \qquad k \in D_3. \tag{4.1b}$$

These two equations give two independent relations coupling the known spectral functions $\{a(k), b(k)\}$ with the unknown spectral functions $\{A(t, k), B(t, k)\}$. If we specify appropriate boundary conditions, for instance $q(0, t) = g_0(t)$, then we can use (4.1) to determine the unknown boundary values $g_1(t)$ and $g_2(t)$ in terms of the given initial and boundary conditions.

An analogous situation also exists for the case $\lambda = 1$. In this case the two non-trivial roots of the dispersion relation $\omega(\nu) = \omega(k)$ have the following properties:

$$k \in D_3^{(1)} \quad \Rightarrow \quad \nu_+(k) \in D_3^{(2)}, \quad \nu_-(k) \in D_1,$$
$$k \in D_3^{(2)} \quad \Rightarrow \quad \nu_+(k) \in D_1, \quad \nu_-(k) \in D_3^{(1)},$$

where $D_3^{(1)}$, $D_3^{(2)}$ are shown in Fig. 3.

If we adopt the notation whereby $\nu(k) = \{\nu_+(k), \nu_-(k)\}$ is defined such that $k \in D_3 \Rightarrow \nu(k) \in D_1$, then from (2.3) we have

$$a(\nu)B(t, \nu) - b(\nu)A(t, \nu) = e^{2i\omega(k)t} c(t, \nu), \quad k \in D_3. \tag{4.2}$$

This equation is a single relation coupling the known spectral functions $\{a(\nu), b(\nu)\}$ with the unknown spectral functions $\{A(t, \nu), B(t, \nu)\}$. If we specify appropriate boundary conditions, for instance $q(0, t) = g_0(t)$ and $q_x(0, t) = g_1(t)$, then we can use (4.2) to determine the unknown boundary value $g_2(t)$ in terms of the given initial and boundary conditions.

Before proceeding to solve (4.1) and (4.2) for the unknown boundary values, we first introduce certain integral identities that will be required for our analysis:

$$\int_{\partial D} \left[\int_0^{t'} e^{2i\omega(k)(\tau - t')} K(t, \tau)\, d\tau \right] dk = -\frac{\pi}{(3\lambda)^{1/3}} \int_0^{t'} \frac{\mathrm{Ai}(\zeta)}{(t' - \tau)^{1/3}} K(t, \tau)\, d\tau,$$

(4.3a)

$$\int_{\partial D} k \left[\int_0^{t'} e^{2i\omega(k)(\tau - t')} K(t, \tau)\, d\tau \right] dk = \frac{i\pi}{2(3\lambda)^{2/3}} \int_0^{t'} \frac{\mathrm{Ai}'(\zeta)}{(t' - \tau)^{2/3}} K(t, \tau)\, d\tau,$$

(4.3b)

where $\mathrm{Ai}(\zeta)$ is the solution of Airy's equation given in (2.11), $K(t, \tau)$ is a smooth function of the arguments indicated,

$$0 < t' < t,$$

and the contour ∂D is the oriented contour shown in Fig. 1. In order to prove the first identity we first interchange the order of integration and then introduce new variables

$$s = -2(3\lambda)^{1/3} (t' - \tau)^{1/3} k, \qquad \zeta = (3\lambda)^{-1/3} (t' - \tau)^{2/3}$$

to evaluate the k-integral. The result then follows from (2.11). Similar considerations apply to prove the second identity above.

We will also make use of the following integral identities:

$$\int_{\partial D} \omega'(k) \left[\int_0^t e^{2i\omega(k)(\tau - t')} K(t, \tau)\, d\tau \right] dk = \pi K(t, t'),$$

(4.4a)

$$\int_{\partial D} \omega(k) \left[\int_0^t e^{2i\omega(k)(\tau - t')} K(t, \tau)\, d\tau \right] dk$$

$$= -\frac{i\pi}{2(3\lambda)^{1/3}} \int_0^{t'} \frac{\mathrm{Ai}(\zeta)}{(t' - \tau)^{1/3}} \frac{\partial K(t, \tau)}{\partial \tau}\, d\tau,$$

(4.4b)

$$\int_{\partial D} k\omega(k) \left[\int_0^t e^{2i\omega(k)(\tau - t')} K(t, \tau)\, d\tau \right] dk$$

$$= \frac{\pi}{4(3\lambda)^{2/3}} \int_0^{t'} \frac{\mathrm{Ai}'(\zeta)}{(t' - \tau)^{2/3}} \frac{\partial K(t, \tau)}{\partial \tau}\, d\tau.$$

(4.4c)

Identity (4.4a) follows from the usual Fourier transform identity by mapping ∂D to the real axis. In order to derive expression (4.4b) we first write the LHS as follows

$$\int_{\partial D} \left[\omega(k) \int_0^{t'} e^{2i\omega(k)(\tau - t')} K(t, \tau)\, d\tau - \frac{K(t, t')}{2i} \right] dk.$$

This ensures that the integral is well defined. We next perform integration-by-parts on the t-integral to obtain

$$\int_{\partial D} \frac{1}{2i} \left[K(t, t') - \int_0^{t'} e^{2i\omega(k)(\tau - t')} \partial K(t, \tau) \partial \tau \, d\tau - K(t, t') \right] dk.$$

The two terms involving $K(t, t')$ cancel and, by interchanging the order of integration and using the identities in (4.3), we can evaluate the k-integral to obtain the expression (4.4b). Similar considerations apply to prove the third identity (4.4c).

4.1 The Solution of the Global Relation for $\lambda = -1$

We will now use the two global relations (4.1) to derive (2.10), i.e. an explicit expression for the unknown boundary values $g_1(t)$ and $g_2(t)$. Substituting into (4.1) the expression for $B(t, k)$ given by (3.13) and rearranging we find

$$2\hat{C}_1 - \frac{1}{2}g_0\hat{A}_1$$
$$= 2(p_+ + p_-)\hat{B}_1 - \frac{1}{p_+ - p_-}\left(p_+\frac{b(p_+)}{a(p_+)}A(t, p_+) - p_-\frac{b(p_-)}{a(p_-)}A(t, p_-) \right)$$
$$- \frac{1}{p_+ - p_-}e^{2i\omega(k)t}\left(p_+\frac{c(t, p_+)}{a(p_+)} - p_-\frac{c(t, p_-)}{a(p_-)} \right)$$

and

$$2\hat{D}_1 + \frac{1}{2}g_0\hat{B}_2 + \frac{1}{4i}g_1\hat{A}_1$$
$$= 2p_+p_-\hat{B}_1 - \frac{p_+p_-}{p_+ - p_-}\left(\frac{b(p_+)}{a(p_+)}A(t, p_+) - \frac{b(p_-)}{a(p_-)}A(t, p_-) \right)$$
$$- \frac{p_+p_-}{p_+ - p_-}e^{2i\omega(k)t}\left(\frac{c(t, p_+)}{a(p_+)} - \frac{c(t, p_-)}{a(p_-)} \right),$$

where $\{\hat{A}_1, \hat{B}_1, \hat{B}_2, \hat{C}_1, \hat{D}_1\}$ are defined by (3.13b). We first assume that $a(k) \neq 0$ for $k \in \overline{D_1}$. If we multiply these two equations by $\omega'(k)\exp[-2i\omega(k)t']$ and integrate along ∂D, the contour shown in Fig. 1, then terms involving $c(t, p_\pm)$ will vanish since $\omega'(k)c(t, p_\pm)/a(p_\pm)$ is analytic and of $O(1)$ for $\operatorname{Im} k \leq 0$, and the oscillatory term $\exp[2i\omega(k)(t - t')]$ is bounded in D_1. We use the fact that $k\omega'(k) = (3\omega(k) - 2)$ together with the integral identities (4.3) and (4.4) to obtain

$$2\pi C_1(t, 2t' - t)$$
$$= \pi g_0(t)A_1(t, 2t' - t) - \pi 3^{-1/3}\int_0^{t'} \frac{\operatorname{Ai}(\zeta)}{(t' - \tau)^{1/3}}\left[\frac{3i}{2}\frac{\partial B_1}{\partial \tau} - 2B_1 \right](t, 2\tau - t)\,d\tau$$
$$- \int_{\partial D} \omega'(k)e^{-2i\omega(k)t'}\frac{1}{p_+ - p_-}\left[p_+\frac{b(p_+)}{a(p_+)}A(t, p_+) - p_-\frac{b(p_-)}{a(p_-)}A(t, p_-) \right]dk,$$

and

$$
\begin{aligned}
2\pi D_1&\bigl(t, 2t' - t\bigr)\\
&= -\frac{\pi}{2} g_0(t) B_2\bigl(t, 2t' - t\bigr) + \frac{\pi}{4\mathrm{i}} g_1(t) A_1\bigl(t, 2t' - t\bigr) - \frac{c\pi}{2} B_1\bigl(t, 2t' - t\bigr)\\
&\quad + \frac{\pi}{2} 3^{-2/3} \int_0^{t'} \frac{\mathrm{Ai}'(\zeta)}{(t' - \tau)^{2/3}} \left[\frac{3}{2} \frac{\partial B_1}{\partial \tau} - 2\mathrm{i} B_1 \right](t, 2\tau - t)\,\mathrm{d}\tau\\
&\quad - \int_{\partial D} \omega'(k)\mathrm{e}^{-2\mathrm{i}\omega(k)t'} \frac{p_+ p_-}{p_+ - p_-} \left[\frac{b(p_+)}{a(p_+)} A(t, p_+) - \frac{b(p_-)}{a(p_-)} A(t, p_-) \right] \mathrm{d}k.
\end{aligned}
$$

Letting $t' \to t$ in these expressions and using the identities for $B_1(t, t)$, $C_1(t, t)$, and $D_1(t, t)$ in (2.8) we find

$$
\begin{aligned}
g_1(t) &= \frac{1}{2} g_0(t) A_1(t, t) - 3^{-1/3} \int_0^t \frac{\mathrm{Ai}(\zeta)}{(t - \tau)^{1/3}} \left[3\mathrm{i} \frac{\partial B_1}{\partial \tau} - 4B_1 \right](t, 2\tau - t)\,\mathrm{d}\tau\\
&\quad - \frac{1}{\pi} \int_{\partial D} \frac{\omega'(k)\mathrm{e}^{-2\mathrm{i}\omega(k)t}}{p_+ - p_-} \left[p_+ \frac{b(p_+)}{a(p_+)} A(t, p_+) - p_- \frac{b(p_-)}{a(p_-)} A(t, p_-) \right] \mathrm{d}k,
\end{aligned}
\tag{4.5a}
$$

$$
\begin{aligned}
g_2(t) &= -2g_0(t)^2 + \mathrm{i} g_0(t) B_2(t, t) - \frac{1}{2} g_1(t) A_1(t, t)\\
&\quad - 3^{-2/3} \int_0^t \frac{\mathrm{Ai}'(\zeta)}{(t - \tau)^{2/3}} \left[\frac{3\mathrm{i}}{2} \frac{\partial B_1}{\partial \tau} + 2B_1 \right](t, 2\tau - t)\,\mathrm{d}\tau\\
&\quad + \frac{2\mathrm{i}}{\pi} \int_{\partial D} \omega'(k)\mathrm{e}^{-2\mathrm{i}\omega(k)t} \frac{p_+ p_-}{p_+ - p_-} \left[\frac{b(p_+)}{a(p_+)} A(t, p_+) - \frac{b(p_-)}{a(p_-)} A(t, p_-) \right] \mathrm{d}k.
\end{aligned}
\tag{4.5b}
$$

We note that the last line in each of these expressions still has k-integrals involving the spectral functions $a(k), b(k)$, and $A(t, k)$. Depending upon the specific choice for the initial condition $q_0(x)$ it is possible to simplify these expressions further by evaluating $a(k)$ and $b(k)$ from (2.1) and then using the expression for $A(t, k)$ given in (3.13) to evaluate the k-integrals. For instance, if $q(x, 0) = 0$ then the spectral function $b(k)$ defined in (2.1) is zero. This implies that the k-integrals in the above expressions vanish and these expressions simplify to give (2.10).

Finally, we note that if $a(k)$ does have zeros for $k \in \overline{D_1}$, then the integral involving $\omega'(k)c(t, k)\exp[2\mathrm{i}\omega(k)(t - t')]/a(k)$ does not vanish, but rather it has a contribution due to the relevant residues. The simplest way to compute this contribution is to deform the contour ∂D to include the relevant zeros.

4.2 The Solution of the Global Relation for $\lambda = 1$

For the case of $\lambda = 1$, we use the single global relation (4.2) to determine the unknown boundary value $g_2(t)$. Substituting into (4.2) the expression for $B(t, k)$ given

by (3.13) and rearranging we find

$$
2\hat{D}_1 = \frac{1}{2}g_0\hat{B}_2 + \frac{1}{4i}g_1\hat{A}_1 - 2\nu^2\hat{B}_1 + \nu\left[2\hat{C}_1 + \frac{1}{2}g_0\hat{A}_1\right]
$$
$$
+ \frac{b(\nu)}{a(\nu)}A(t,\nu) + e^{2i\omega(k)t}\frac{c(t,\nu)}{a(\nu)}.
$$

We first assume that $a(\nu) \neq 0$ for $\nu \in \overline{D_1}$. If we multiply this equation by $\omega'(\nu)\exp[-2i\omega(k)t']$ and integrate along ∂D_1, then the final term on the RHS vanishes because $\omega'(\nu)c(t,\nu)/a(\nu)$ is analytic and of $O(1)$ for $\mathrm{Im}\,k \leq 0$, and the oscillatory term $\exp[2i\omega(k)(t - t')]$ is bounded in D_1. Using the fact that $k\omega'(k) = (3\omega(k) - 2)$ together with the integral identities (4.3) and (4.4) we find

$$
2\pi D_1\big(t, 2t' - t\big) = \frac{\pi}{2}g_0(t)B_2\big(t, 2t' - t\big) + \frac{\pi}{4i}g_1(t)A_1\big(t, 2t' - t\big)
$$
$$
- \frac{\pi}{2}3^{-2/3}\int_0^{t'}\frac{\mathrm{Ai}'(\zeta)}{(t'-\tau)^{2/3}}\left[\frac{3}{2}\frac{\partial B_1}{\partial\tau} - 2iB_1\right](t, 2\tau - t)\,d\tau
$$
$$
- \pi 3^{-1/3}\int_0^{t'}\frac{\mathrm{Ai}(\zeta)}{(t'-\tau)^{1/3}}\left[3i\frac{\partial C_1}{\partial\tau} - 4C_1\right.
$$
$$
+ g_0(t)\left(\frac{3i}{4}\frac{\partial A_1}{\partial\tau} - A_1\right)\bigg](t, 2\tau - t)\,d\tau
$$
$$
+ \int_{\partial D}k\omega'(k)e^{-2i\omega(k)t'}\frac{b(k)}{a(k)}A(t, k)\,dk.
$$

Letting $t' \to t$ in this equation and using the identity for $D_1(t, t)$ in (2.8) we find

$$
g_2(t) = cg_0(t) + 2g_0(t)^2 - ig_0(t)B_2(t, t) - \frac{1}{2}g_1(t)A_1(t, t)
$$
$$
+ 3^{-2/3}\int_0^t\frac{\mathrm{Ai}'(\zeta)}{(t-\tau)^{2/3}}\left[\frac{3i}{2}\frac{\partial B_1}{\partial\tau} + 2B_1\right](t, 2\tau - t)\,d\tau
$$
$$
- 3^{-1/3}\int_0^t\frac{\mathrm{Ai}(\zeta)}{(t-\tau)^{1/3}}\left[6\frac{\partial C_1}{\partial\tau} + 8iC_1\right.
$$
$$
+ g_0(t)\left(\frac{3}{2}\frac{\partial A_1}{\partial\tau} + 2iA_1\right)\bigg](t, 2\tau - t)\,d\tau
$$
$$
- \frac{2i}{\pi}\int_{\partial D}k\omega'(k)e^{-2i\omega(k)t}\frac{b(k)}{a(k)}A(t, k)\,dk. \tag{4.6}
$$

As in the previous case, this expression contains k-integrals involving the spectral functions $a(k)$, $b(k)$, and $A(t, k)$. Given a specific value for the initial condition $q_0(x)$ it is possible to simplify this formula by evaluating the k-integrals. In the particular case of $q(x, 0) = 0$ the spectral function $b(k)$ defined in (2.1) is zero and so the k-integrals in (4.6) vanish and we find (2.12).

As in the case for $\lambda = -1$, we note that if $a(k)$ does have zeros for $k \in \overline{D_1}$, then the integral involving $\omega'(k)c(t, k)\exp[2i\omega(k)(t - t')]/a(k)$ does not vanish. The simplest way to compute the contribution from this term is to deform the contour ∂D_1 to include the relevant zeros.

5 Conclusion

The generalized Dirichlet to Neumann map for integrable nonlinear evolution PDEs can be characterized by the requirement that the spectral functions $\{a(k), b(k), A(t, k), B(t, k)\}$ satisfy the so-called global relation. The Gelfand–Levitan–Marchenko representation of $\Phi^{(t)}(t, k)$ was used in Boutet de Monvel et al. (2003) and Fokas (2004) to solve the global relation *explicitly* for the nonlinear Schrödinger, the sine-Gordon, and the modified KdV equations (see also Boutet de Monvel et al. 2004). We have shown in this paper that the global relation can also be solved explicitly for both cases $\lambda = \pm 1$ of the KdV equation.

The analysis of the KdV equation in the case $\lambda = -1$ is more complicated than the analysis of the case $\lambda = 1$ since in the former case the *single* global relation (2.3) must be solved for *two* unknown boundary values. For economy of presentation we have concentrated on the case where $g_0(t)$ is prescribed as a boundary condition and, by using certain invariant transformations of the dispersion relation $\omega(k) = -4k^3 + ck$, we have constructed explicit expressions for $g_1(t)$ and $g_2(t)$. The analysis is similar for the case that, say $g_1(t)$, is prescribed.

The explicit solution of the global relation yields the unknown boundary values in terms of the given initial and boundary conditions and in terms of the kernel functions that appear in the Gelfand–Levitan–Marchenko representation of $\Phi^{(t)}(t, k)$. These kernel functions in turn satisfy a system of linear ODEs which, after inserting the expressions for the unknown boundary values, give nonlinear Volterra integral equations in which the nonlinearity now appears in a simple explicit form. The rigorous analysis of this system has not yet been carried out, however because the nonlinear integral equations are of Volterra type it should be straightforward to establish existence of the solution at least for small t or for "small" boundary conditions (Fokas 2004).

We remark that the explicit characterization of the Dirichlet to Neumann map that we have presented here appears to be particularly useful in the context of numerical simulations of soliton equations. Indeed, the result of Fokas (2004) has been used in Zheng (2006) for the implementation of the so-called method of nonreflecting boundary conditions for the numerical integration of the modified KdV equation. Actually, it was in Zheng (2006) that the formulae of Fokas (2004) were simplified by computing the k-integrals. In this paper we have extended this analysis and applied it to the KdV equation.

Finally, we note that initial-boundary value problems for integrable nonlinear PDEs can also be analysed using PDE techniques (Bona et al. 2002, 2003; Colliander and Kenig 2002). In particular we note the remarkable result of Colliander and Kenig (2002) where global well posedness is established. We consider our approach as complementary to the above approach: Having established *a priori* global existence for $q(x, t)$ and therefore for $\{q(0, t), q_x(0, t), q_{xx}(0, t)\}$, our results can be

extended globally. Therefore, our results imply an explicit characterization of the unknown boundary values which appears important both for analytical as well as computational considerations. Also, we note the Riemann–Hilbert formalism reviewed in Appendix 1 together with the powerful Deift–Zhou method (Deift and Zhou 1993) yield explicit results for the large t-asymptotics of the solution.

Appendix 1: Lax Pair, Spectral Functions and the "Inverse" Problem

In this appendix we summarize the main results for the KdV equation presented in Fokas (2002a). Details and proofs can be found in Fokas (2002a) and Fokas et al. (2005).

Equation (1.1) admits the Lax pair formulation

$$\mu_x - ik[\sigma_3, \mu] = Q(x, t, k)\mu, \tag{6.1a}$$

$$\mu_t + i\omega(k)[\sigma_3, \mu] = P(x, t, k)\mu, \quad k \in \mathbb{C}, \tag{6.1b}$$

where $\omega(k)$ is defined by (1.3), the eigenfunction $\mu(x, t, k)$ is a 2×2 matrix-valued function, and the functions $Q(x, t, k)$, $P(x, t, k)$ are defined by

$$Q(x, t, k) = \frac{\lambda}{2k}(\sigma_2 - i\sigma_3)q(x, t),$$

$$P(x, t, k) = -2k\sigma_2 q(x, t)$$

$$+ \sigma_1 q_x(x, t) + \frac{1}{2k}(\sigma_2 - i\sigma_3)\big(q_{xx}(x, t) - \lambda cq(x, t) - 2\lambda q^2(x, t)\big),$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $[\cdot, \cdot]$ denotes the usual matrix commutator

$$[\sigma_3, A] = \sigma_3 A - A\sigma_3.$$

6.1 The "Direct" Spectral Problem

The spectral functions denoted by $\{a(k), b(k), A(t, k), B(t, k)\}$, are defined in terms of the initial condition and the boundary values by the *direct map*

$$\{q_0(x), g_0(t), g_1(t), g_2(t)\} \mapsto \{a(k), b(k), A(t, k), B(t, k)\}.$$

In order to set up the map $q_0(x) \mapsto \{a(k), b(k)\}$, we first introduce the matrix-valued function $\Psi^{(x)}(x, k)$. This function is defined in terms of $q_0(x)$ as the following matrix solution of the x-part of the associated Lax pair (6.1) evaluated at $t = 0$:

$$\partial_x \Psi^{(x)}(x, k) - ik\big[\sigma_3, \Psi^{(x)}(x, k)\big] = Q_0(x, k)\Psi^{(x)}(x, k), \tag{6.2a}$$

where

$$Q_0(x, k) = \frac{\lambda}{2k}(\sigma_2 - i\sigma_3)q_0(x), \tag{6.2b}$$

$$\lim_{x \to \infty} \Psi^{(x)}(x, k) = I, \quad 0 < x < \infty, \tag{6.2c}$$

and for the first column of $\Psi^{(x)}$ we take $\operatorname{Im} k \geq 0$, whereas for the second column we take $\operatorname{Im} k \leq 0$.

Definition 6.1 The spectral functions $a(k)$ and $b(k)$ associated with the initial condition $q_0(x)$ are defined by

$$a(k) = \psi_2(0, k), \qquad b(k) = \psi_1(0, k), \quad \operatorname{Im} k \leq 0, \tag{6.3}$$

where $\psi_1(x, k)$ and $\psi_2(x, k)$ are the following components of the matrix-valued function $\Psi^{(x)}(x, k)$ defined by (6.2),

$$\Psi^{(x)}(x, k) = \begin{pmatrix} \overline{\psi_2(x, \bar{k})} & \psi_1(x, k) \\ \overline{\psi_1(x, \bar{k})} & \psi_2(x, k) \end{pmatrix}. \tag{6.4}$$

In order to set up the map $\{g_0(t), g_1(t), g_2(t)\} \mapsto \{A(t, k), B(t, k)\}$, we introduce the matrix-valued function $\Phi^{(t)}(t, k)$. This function is defined in terms of $\{g_0(t), g_1(t), g_2(t)\}$ as the following solution of the t-part of the associated Lax pair (A.1) evaluated at $x = 0$:

$$\partial_t \Phi^{(t)}(t, k) + i\omega(k)\big[\sigma_3, \Phi^{(t)}(t, k)\big] = P_0(t, k)\Phi^{(t)}(t, k), \tag{6.5a}$$

where

$$P_0(t, k) = -2\sigma_2 g_0 k + \sigma_1 g_1 + \frac{1}{2}\big(\sigma_2 - i\sigma_3\big)\big(g_2(t) - \lambda c g_0(t) - 2\lambda g_0(t)^2\big)\frac{1}{k}, \tag{6.5b}$$

and

$$\Phi^{(t)}(0, k) = I, \tag{6.5c}$$

and $k \in \mathbb{C}$.[2]

Definition 6.2 The spectral functions $A(t, k)$ and $B(t, k)$ associated with the boundary values $\{g_0(t), g_1(t), g_2(t)\}$ are defined by

$$\begin{pmatrix} -e^{-2i\omega(k)t} B(t, k) \\ \overline{A(t, \bar{k})} \end{pmatrix} = \begin{pmatrix} \Phi_1(t, k) \\ \Phi_2(t, k) \end{pmatrix}, \tag{6.6}$$

where $\Phi_1(t, k)$ and $\Phi_2(t, k)$ are the (12) and (22) components, respectively, of the matrix-valued function $\Phi^{(t)}(t, k)$ defined by (6.3).

---

[2] If $t \to \infty$, we take k such that $\operatorname{Im}\omega(k) \geq 0$ for the first column of $\Phi^{(t)}$, and k such that $\operatorname{Im}\omega(k) \leq 0$ for the second column.

6.2 The Global Relation

Although the relationship between the initial condition and the boundary values of the solution of the KdV equation is complicated, this relationship takes a surprisingly simple form when expressed in terms of the corresponding spectral functions, see (2.3). In the case $0 < t \leq \infty$, (2.3) becomes

$$a(k)B(\infty, k) - b(k)A(\infty, k) = 0, \quad \operatorname{Im} k \leq 0 \cap \operatorname{Im} \omega(k) \geq 0. \tag{6.7}$$

The derivation of the global relation can be found in Fokas (2002a).

6.3 The "Inverse" Spectral Problem

The *inverse map*

$$\left\{ a(k), \mathrm{e}^{-2\mathrm{i}\omega(k)t} b(k), A(T, k), \mathrm{e}^{-2\mathrm{i}\omega(k)t} B(T, k) \right\} \mapsto q(x, t),$$

defines the function $q(x, t)$ by

$$q(x, t) = \lim_{k \to \infty} 4k^2 \big(M(x, t, k) \big)_{12}, \quad 0 < x < \infty,\ 0 < t < T, \tag{6.8}$$

where $M(x, t, k)$ satisfies the 2×2 matrix RH problem formulated in Theorem 6.3, which we state below without proof. For conciseness we give only the solitonless case. The inclusion of solitons is a simple procedure, the details of which can be found in Fokas (2002a). An improved procedure for including solitons is presented in Boutet de Monvel and Kotlyarov (2000).

Theorem 6.3 (Fokas 2002a) *Let $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$. Define $\{a(k), b(k)\}$ by (6.3). Suppose that there exist smooth functions $\{g_0(t), g_1(t), g_2(t)\}$ satisfying $\{g_j(0) = \partial_x^j q_0(0)\}_0^2$, such that the functions $\{A(t, k), B(t, k)\}$, defined by (6.6), satisfy the global relation (2.3), where $c(t, k)$ is analytic in $\{k \in \mathbb{C} \setminus 0 | \operatorname{Im} k < 0\}$ and is of $O(1/k)$ as $k \to \infty$.*[3]

Let $M(x, t, k)$ be a solution of the following 2×2 matrix RH problem:

- *M is sectionally meromorphic in $k \in \mathbb{C} \setminus \{k | \operatorname{Im} \omega(k) = 0\}$.*
- *Along the contours $\{k | \operatorname{Im} \omega(k) = 0\}$, M satisfies the following jump conditions*

$$M_-(x, t, k) = M_+(x, t, k)J(x, t, k), \quad \operatorname{Im} \omega(k) = 0, \tag{6.9}$$

where the matrices $M_\pm(x, t, k)$ are the limit values of M as k approaches $\{\operatorname{Im} \omega(k) = 0\}$, and the jump matrix $J(x, t, k)$ is defined in terms of the spectral

---

[3]For the case $0 < t \leq \infty$, (2.3) is replaced by (6.7) and we further assume that $g_0(t), g_1(t), g_2(t)$ belong to $\mathcal{S}(\mathbb{R}^+)$.

functions $a(k)$, $b(k)$, $A(T, k)$, and $B(T, k)$ as follows:

$$J(x,t,k) = \begin{cases} \begin{pmatrix} \frac{1}{\Gamma(\bar{k})\mathrm{e}^{2i\theta}} & 0 \\ 1 \end{pmatrix}, & k \in \overline{D_1} \cap \overline{D_2}, \\ \begin{pmatrix} 1 & -\Gamma(k)\mathrm{e}^{-2i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - |\gamma(k)|^2 & \gamma(k)\mathrm{e}^{-2i\theta} \\ -\overline{\gamma(\bar{k})}\mathrm{e}^{2i\theta} & 1 \end{pmatrix} \\ \quad \times \begin{pmatrix} \frac{1}{\Gamma(\bar{k})\mathrm{e}^{2i\theta}} & 0 \\ 1 \end{pmatrix}, & k \in \overline{D_2} \cap \overline{D_3}, \\ \begin{pmatrix} 1 & -\Gamma(k)\mathrm{e}^{-2i\theta} \\ 0 & 1 \end{pmatrix}, & k \in \overline{D_3} \cap \overline{D_4}, \end{cases} \qquad (6.10)$$

where

$$\theta(x,t,k) = -kx + \omega(k)t; \qquad (6.11)$$

the functions $\gamma(k)$, $\Gamma(k)$ are given by

$$\gamma(k) = \frac{b(k)}{\overline{a(\bar{k})}}, \quad k \in \overline{D_2} \cap \overline{D_3}, \qquad (6.12a)$$

$$\Gamma(k) = \frac{B(T,k)}{\overline{a(\bar{k})d(\bar{k})}}, \quad k \in \overline{D_2} \cap \overline{D_3}, \ \overline{D_3} \cap \overline{D_4}, \qquad (6.12b)$$

with

$$d(k) = a(k)\overline{A(T,\bar{k})} - b(k)\overline{B(T,\bar{k})}, \quad k \in D_2; \qquad (6.12c)$$

and the domains D_1–D_4 are defined as:

$$\begin{aligned} D_1 &= \{k \in \mathbb{C}, \operatorname{Im} k < 0 \cap \operatorname{Im} \omega(k) > 0\}, \\ D_2 &= \{k \in \mathbb{C}, \operatorname{Im} k < 0 \cap \operatorname{Im} \omega(k) < 0\}, \\ D_3 &= \{k \in \mathbb{C}, \operatorname{Im} k > 0 \cap \operatorname{Im} \omega(k) > 0\}, \\ D_4 &= \{k \in \mathbb{C}, \operatorname{Im} k > 0 \cap \operatorname{Im} \omega(k) < 0\}. \end{aligned} \qquad (6.13)$$

•
$$M(x,t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty. \qquad (6.14)$$

• $M(x,t,k)$ *has a pole at $k = 0$ satisfying*

$$M(x,t,k) \sim \frac{if(x,t)}{k} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad k \to 0.$$

Then we have the following:

(1) $M(x,t,k)$ *exists and is unique.*
(2) $q(x,t)$ *defined in terms of $M(x,t,k)$ by* (6.8) *satisfies the KdV equation* (1.1).
(3) $q(x,t)$ *satisfies the initial condition $q(x,0) = q_0(x)$, and furthermore, it has the following boundary values*:

$$q(0,t) = g_0(t), \qquad q_x(0,t) = g_1(t), \qquad q_{xx}(0,t) = g_2(t).$$

Proof Details of the proof of this theorem can be found in Fokas (2002a).   □

Appendix 2: The Linear Limit

In this appendix we give the generalized Dirichlet to Neumann map for the linear version of (1.1). We also prove that, in the linear limit, our construction of the Dirichlet to Neumann map for the KdV equation in Sect. 2 reduces to the corresponding map in the linear case.

In the approximation of small q_0, g_0, g_1, g_2 (or small q_0 and small t) then, from Proposition 2.1, we find $B_1(t, s) = B_1(t + s)$ and thus $B_1(t, t) = B_1(2t) = -ig_0(t)$. Hence

$$B_1(t, 2\tau - t) = B_1(2\tau) = -ig_0(\tau).$$

Similar considerations apply for the other kernel functions and so we have

$$B_1(t, 2\tau - t) \approx -ig_0(\tau), \qquad A_1(t, 2\tau - t) \approx 0,$$

$$C_1(t, 2\tau - t) \approx \frac{1}{2}g_1(\tau), \qquad B_2(t, 2\tau - t) \approx 0,$$

$$D_1(t, 2\tau - t) \approx \frac{i}{4}\big[g_2(\tau) - \lambda c g_0(\tau)\big], \qquad C_2(t, 2\tau - t) \approx 0.$$

and, from (2.2),

$$a(k) \sim 1, \qquad b(k) \sim -\frac{1}{2ik}\int_0^\infty e^{-2ikx} q_0(x)\,dx.$$

7.1 The Linear Limit for KdVI ($\lambda = -1$)

Taking the linear limit of (4.5) and substituting the above expressions, we find

$$g_1(t) \approx \frac{2}{\pi}\int_D ik\omega'(k)\left[\int_0^t e^{2i\omega(k)(\tau-t)} g_0(\tau)\,d\tau\right] dk$$
$$- \frac{1}{\pi}\int_{\partial D} \omega'(k)e^{-2i\omega(k)t}\frac{1}{p_+ - p_-}\big[p_+\hat{q}_0(p_+) - p_-\hat{q}_0(p_-)\big]\,dk,$$

and

$$g_2(t) \approx \frac{4}{\pi}\int_D k^2\omega'(k)\left[\int_0^t e^{2i\omega(k)(\tau-t)} g_0(\tau)\,d\tau\right] dk$$
$$+ \frac{2i}{\pi}\int_{\partial D} \omega'(k)e^{-2i\omega(k)t}\frac{p_+ p_-}{p_+ - p_-}\big[\hat{q}_0(p_+) - \hat{q}_0(p_-)\big]\,dk,$$

where

$$\hat{q}_0(k) = \int_0^\infty e^{-2ikx} q_0(x)\,dx.$$

By using the identities (4.3) and (4.4) we see that these are the same formulae we get by solving the global relation associated with (2.14) with $\lambda = -1$, see Fokas (2002b).

7.2 The Linear Limit for KdVII ($\lambda = 1$)

Taking the linear limit of (4.6) and substituting the above expressions, we find

$$
\begin{aligned}
g_2(t) \approx c g_0(t) &+ \frac{8}{\pi} \int_D k^2 \omega'(k) \left[\int_0^t e^{2i\omega(k)(\tau - t)} g_0(\tau) \, d\tau \right] dk \\
&- \frac{4i}{\pi} \int_D k \omega'(k) \left[\int_0^t e^{2i\omega(k)(\tau - t)} g_1(\tau) \, d\tau \right] dk \\
&- \frac{4i}{\pi} \int_{\partial D} \omega'(k) e^{-2i\omega(k)t} \hat{q}_0(k) \, dk,
\end{aligned}
$$

where

$$
\hat{q}_0(k) = \int_0^\infty e^{-2ikx} q_0(x) \, dx.
$$

By using the identities (4.3) and (4.4) we see that these are the same formulae we get by solving the global relation associated with equation (2.14) with $\lambda = 1$, see Fokas (2002b).

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