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Discrete Nonholonomic Lagrangian Systems on Lie Groupoids

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Abstract This paper studies the construction of geometric integrators for nonholonomic systems. We develop a formalism for nonholonomic discrete Euler–Lagrange equations in a setting that permits to deduce geometric integrators for continuous nonholonomic systems (reduced or not). The formalism is given in terms of Lie groupoids, specifying a discrete Lagrangian and a constraint submanifold on it. Additionally, it is necessary to fix a vector subbundle of the Lie algebroid associated to the Lie groupoid. We also discuss the existence of nonholonomic evolution operators in terms of the discrete nonholonomic Legendre transformations and in terms

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of adequate decompositions of the prolongation of the Lie groupoid. The characterization of the reversibility of the evolution operator and the discrete nonholonomic momentum equation are also considered. Finally, we illustrate with several classical examples the wide range of application of the theory (the discrete nonholonomic constrained particle, the Suslov system, the Chaplygin sleigh, the Veselova system, the rolling ball on a rotating table and the two wheeled planar mobile robot).

Keywords Discrete Mechanics · Nonholonomic Mechanics · Lie groupoids · Lie algebroids · Reduction · Nonholonomic momentum map

Mathematics Subject Classification (2000) 17B66 · 22A22 · 37J60 · 37M15 · 70F25

1 Introduction

In the paper by Moser and Veselov (1991), dedicated to the complete integrability of certain dynamical systems, the authors proposed a discretization of the tangent bundle TQ of a configuration space Q replacing it by the product $Q \times Q$, approximating a tangent vector on Q by a pair of "close" points (q_0, q_1). In this sense, the continuous Lagrangian function $L : TQ \to \mathbb{R}$ is replaced by a discretization $L_{\mathrm{d}} : Q \times Q \to \mathbb{R}$. Then, applying a suitable variational principle, it is possible to derive the discrete equations of motion. In the regular case, one obtains an evolution operator, a map that assigns to each pair (q_{k-1}, q_k) a pair (q_k, q_{k+1}), sharing many properties with the continuous system, in particular, symplecticity, momentum conservation and a good energy behavior. We refer to Marsden and West (2001) for an excellent review in discrete Mechanics (on $Q \times Q$) and its numerical implementation.

On the other hand, in Moser and Veselov (1991), Veselov and Veselova (1989), the authors also considered discrete Lagrangians defined on a Lie group G where the evolution operator is given by a diffeomorphism of G.

All the above examples led A. Weinstein (1996) to study discrete mechanics on Lie groupoids. A Lie groupoid Γ is a geometric structure that naturally generalizes the concept of a Lie group, where now not all elements are composable. The product of a pair $g_1 g_2$ of elements is only defined on the set of composable pairs $\Gamma_2 = \{(g, h) \in \Gamma \times \Gamma \mid \beta(g) = \alpha(h)\}$ where $\alpha : \Gamma \to M$ and $\beta : \Gamma \to M$ are the source and target maps over a base manifold M. Its infinitesimal version is the Lie algebroid $E_\Gamma \to M$, which is the restriction of the vertical bundle of α to the submanifold of the identities. Lie groupoids include as particular examples the case of Cartesian products $Q \times Q$ as well as Lie groups and other examples as Atiyah or action Lie groupoids (Mackenzie 2005). Therefore, mechanics on Lie groupoids permit us to analyze simultaneously all the situations that habitually appear after reduction by symmetries, and the relation between them is a consequence of the naturalness of the formalism (reduction by groupoid morphisms).

In a recent paper (Marrero et al. 2006), we studied discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids, deriving from a variational principle the discrete Euler–Lagrange equations. We also introduced a symplectic 2-section (which is preserved by the Lagrangian evolution operator) and defined the Hamiltonian evolution operator, in terms of the discrete Legendre transformations, which is a symplectic map with respect to the canonical symplectic 2-section on the prolongation of the dual of the Lie algebroid of the given groupoid. These techniques include as particular cases the classical discrete Euler–Lagrange equations, the discrete Euler–Poincaré equations (see Bobenko and Suris 1999a, 1999b; Marsden et al. 1999a, 1999b) and the discrete Lagrange–Poincaré equations. In fact, the results in Marrero et al. (2006) may be applied in the construction of geometric integrators for continuous Lagrangian systems which are invariant under the action of a symmetry Lie group (see also Jalnapurkar et al. 2006 for the particular case when the symmetry Lie group is abelian).

From the perspective of geometric integration, there is a great interest in introducing new geometric techniques for developing numerical integrators. This interest arises because standard methods often introduce some spurious effects, like dissipation in conservative systems (Hairer et al. 2002; Sanz-Serna and Calvo 1994). The case of dynamical systems subjected to constraints is also of considerable interest. In particular, the case of holonomic constraints is well established in the literature of geometric integration, for instance, in simulation of molecular dynamics where the constraints may be molecular bond lengths or angles and also in multibody dynamics (see Hairer et al. 2002; Leimkuhler and Reich 2004 and references therein).

By contrast, the construction of geometric integrators for the case of nonholonomic constraints is less well understood. This type of constraints appears, for instance, in mechanical models of convex rigid bodies rolling without sliding on a surface (Neimark and Fufaev 1972). The study of systems with nonholonomic constraints goes back to the nineteenth century. The equations of motion were obtained applying either D'Alembert's principle of virtual work or Gauss's principle of least constraint. Recently, many authors have shown a new interest in that theory and also in its relation to the new developments in control theory and robotics using geometric techniques (see, for instance, Bates and Śniatycki 1992; Bloch 2003; Bloch et al. 1996; Cortés 2002; Koiller 1992; de León et al. 1997; de León and Martín de Diego 1996).

Geometrically, nonholonomic constraints are globally described by a submanifold $\mathcal{M}$ of the velocity phase space TQ. If $\mathcal{M}$ is a vector subbundle of TQ, we are dealing with linear constraints and, in the case when $\mathcal{M}$ is an affine subbundle, we are in the case of affine constraints. Lagrange–D'Alembert's or Chetaev's principles allow us to determine the set of possible values of the constraint forces only from the set of admissible kinematic states, that is, from the constraint manifold $\mathcal{M}$ determined by the vanishing of the nonholonomic constraints ϕ^i. Therefore, assuming that the dynamical properties of the system are mathematically described by a Lagrangian function $L: TQ \to \mathbb{R}$ and by a constraint submanifold $\mathcal{M}$, the equations of motion, following Chetaev's principle, are

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right]\delta q^i = 0,$$

where δq^i denotes the virtual displacements satisfying $\frac{\partial \phi^a}{\partial \dot{q}^i}\delta q^i = 0$. By using the Lagrange multiplier rule, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \bar{\lambda}_a \frac{\partial \phi^a}{\partial \dot{q}^i}, \tag{1.1}$$

with the condition $\dot{q}(t) \in \mathcal{M}$, $\bar{\lambda}_a$ being the Lagrange multipliers to be determined. Recently, J. Cortés et al. (2005) (see also Cortés and Martínez 2004; Mestdag 2005; Mestdag and Langerock 2005) proposed a unified framework for nonholonomic systems in the Lie algebroid setting, that we will use in this paper, generalizing some previous work for free Lagrangian mechanics on Lie algebroids (see, for instance, de León et al. 2005; Martínez 2001a, 2001b, 2002).

The construction of geometric integrators for (1.1) is very recent. In fact, in McLachlan and Scovel (1996) there appears as an open problem:

> ...The problem for the more general class of non-holonomic constraints is still open, as is the question of the correct analogue of symplectic integration for non-holonomically constrained Lagrangian systems...

Numerical integrators derived from discrete variational principles have proved their adaptability to many situations: collisions, classical field theory, external forces etc. (Marsden 1999; Marsden and West 2001) and it also seems very adequate for nonholonomic systems, since nonholonomic equations of motion come from Hölder's variational principle which is not a standard variational principle (Arnold 1978), but admits an adequate discretization. This is the procedure introduced by J. Cortés and S. Martínez (Cortés 2002; Cortés and Martínez 2001) and followed by other authors (Fedorov 2007; Fedorov and Zenkov 2005a, 2005b; McLachlan and Perlmutter 2006) extending, moreover, the results to nonholonomic systems defined on Lie groups (see also de León et al. 2004 for a different approach using generating functions).

In this paper, we tackle the problem from the unifying point of view of Lie groupoids (see Cortés et al. 2005 for the continuous case). This technique permits us to recover all the previous methods in the literature (Cortés and Martínez 2001; Fedorov and Zenkov 2005a; McLachlan and Perlmutter 2006) and consider new cases of great importance in nonholonomic dynamics. For instance, using action Lie groupoids, we may discretize LR-nonholonomic systems such as the Veselova system or using Atiyah Lie groupoids we find discrete versions for the reduced equations of nonholonomic systems with symmetry. Neither case had been previously considered in the literature. Moreover, our procedure results in a set of reduced discrete nonholonomic equations which define an algorithm on the reduced space that is shown to be equivalent to the discrete nonholonomic equations defined in Cortés and Martínez (2001) in the sense of reconstruction. These equations are defined in spaces with less degrees of freedom than the traditional banal groupoid (where the constructions in Cortés and Martínez 2001 were developed) and we expect that the proposed methods will be more competitive in numerical experiments.

The paper is structured as follows. In Sect. 2 we review some basic results on Lie algebroids and Lie groupoids. In particular, we describe the prolongation of a Lie groupoid (Saunders 2004), which has a double structure of Lie groupoid and Lie algebroid. Then, we briefly expose the geometric structure of discrete unconstrained mechanics on Lie groupoids: Poincaré–Cartan sections, Legendre transformations and so on. The main results of the paper appear in Sect. 3, where the geometric structure of discrete nonholonomic systems on Lie groupoids is considered. In particular, given a discrete Lagrangian $L_{\mathrm{d}} : \Gamma \to \mathbb{R}$ on a Lie groupoid Γ, a constraint distribution $\mathcal{D}_{\mathrm{c}}$ in the Lie algebroid E_Γ of Γ and a discrete constraint submanifold $\mathcal{M}_{\mathrm{c}}$ in Γ, we obtain the nonholonomic discrete Euler–Lagrange equations from a discrete generalized Hölder's principle (see Sect. 3.1). In addition, we characterize the regularity of the nonholonomic system in terms of the nonholonomic Legendre transformations and decompositions of the prolongation of the Lie groupoid. In the case when the system is regular, we can define the nonholonomic evolution operator. An interesting situation, studied in Sect. 3.4, is that of reversible discrete nonholonomic Lagrangian systems, where the Lagrangian and the discrete constraint submanifold are invariants with respect to the inversion of the Lie groupoid. The particular example of reversible systems in the pair groupoid $Q \times Q$ was first studied in McLachlan and Perlmutter (2006). We also define the discrete nonholonomic momentum map. We discuss several examples, including their regularity and their reversibility, to give an idea of the breadth and flexibility of the proposed formalism. The examples include:

- Discrete holonomic Lagrangian systems on a Lie groupoid, which are a generalization of the Shake algorithm for holonomic systems (Hairer et al. 2002; Leimkuhler and Reich 2004; Marsden and West 2001).
- Discrete nonholonomic systems on the pair groupoid, recovering the equations first considered in Cortés and Martínez (2001). An explicit example of this situation is the discrete nonholonomic constrained particle.
- Discrete nonholonomic systems on Lie groups, where the equations that are obtained are the so-called discrete Euler–Poincaré–Suslov equations (see Fedorov and Zenkov 2005a). We remark that, although our equations coincide with those in Fedorov and Zenkov (2005a), the technique developed in this paper is different to the one in that paper. Two explicit examples which we describe here are the Suslov system and the Chaplygin sleigh.
- Discrete nonholonomic Lagrangian systems on an action Lie groupoid. This example is quite interesting since it allows us to discretize a well-known nonholonomic LR-system: the Veselova system (see Veselov and Veselova 1989; see also Fedorov and Jovanovic 2004). For this example, we obtain a discrete system that is not reversible and we show that the system is regular in a neighborhood around the manifold of units.
- Discrete nonholonomic Lagrangian systems on an Atiyah Lie groupoid. With this example, we are able to discretize reduced systems, in particular, we concentrate on the example of the discretization of the equations of motion of a rolling ball without sliding on a rotating table with constant angular velocity.
- Discrete Chaplygin systems, which are regular systems $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ on the Lie groupoid $\Gamma \rightrightarrows M$, for which $(\alpha, \beta) \circ i_{\mathcal{M}_{\mathrm{c}}} : \mathcal{M}_{\mathrm{c}} \to M \times M$ is a diffeomorphism and $\rho \circ i_{\mathcal{D}_{\mathrm{c}}} : \mathcal{D}_{\mathrm{c}} \to TM$ is an isomorphism of vector bundles, (α, β) being the source and target of the Lie groupoid Γ and ρ being the anchor map of the Lie algebroid E_Γ. This example includes a discretization of the two-wheeled planar mobile robot.

We conclude our paper with future lines of work.

2 Discrete Unconstrained Lagrangian Systems on Lie Groupoids

2.1 Lie Algebroids

A *Lie algebroid* E over a manifold M is a real vector bundle $\tau : E \to M$ together with a Lie bracket $[\![\cdot,\cdot]\!]$ on the space $\mathrm{Sec}(\tau)$ of the global cross-sections of $\tau : E \to M$ and a bundle map $\rho : E \to TM$, called *the anchor map*, such that if we also denote by $\rho : \mathrm{Sec}(\tau) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map, then

$$[\![X, fY]\!] = f[\![X, Y]\!] + \rho(X)(f)Y, \tag{2.1}$$

for $X, Y \in \mathrm{Sec}(\tau)$ and $f \in C^\infty(M)$ (see Mackenzie 2005).

If $(E, [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid over M, then the anchor map $\rho : \mathrm{Sec}(\tau) \to \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\mathrm{Sec}(\tau), [\![\cdot,\cdot]\!])$ and $(\mathfrak{X}(M), [\cdot,\cdot])$. Moreover, one may define the differential d of E as follows:

$$\begin{aligned} \mathrm{d}\mu(X_0, \dots, X_k) = {} & \sum_{i=0}^{k}(-1)^i \rho(X_i)\big(\mu\big(X_0, \dots, \widehat{X_i}, \dots, X_k\big)\big) \\ & + \sum_{i<j}(-1)^{i+j}\mu\big([\![X_i, X_j]\!], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k\big), \end{aligned} \tag{2.2}$$

for $\mu \in \mathrm{Sec}(\wedge^k \tau^*)$ and $X_0, \dots, X_k \in \mathrm{Sec}(\tau)$. d is a cohomology operator, that is, $d^2 = 0$. In particular, if $f : M \to \mathbb{R}$ is a real smooth function, then $\mathrm{d}f(X) = \rho(X)f$, for $X \in \mathrm{Sec}(\tau)$.

Trivial examples of Lie algebroids are a real Lie algebra of finite dimension (in this case, the base space is a single point) and the tangent bundle of a manifold M.

On the other hand, let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n (that is, $\dim E_x = \dim \tau^{-1}(x) = n$, for all $x \in M$) over a manifold M of dimension m and $\pi : P \to M$ be a fibration. We consider the subset of $E \times TP$

$$\mathcal{T}^E P = \big\{(a, v) \in E \times TP \,\big|\, (T\pi)(v) = \rho(a)\big\},$$

where $T\pi : TP \to TM$ is the tangent map to π. Denote by $\tau^\pi : \mathcal{T}^E P \to P$ the map given by $\tau^\pi(a, v) = \tau_P(v)$, $\tau_P : TP \to P$ being the canonical projection. If $\dim P = p$, one may prove that $\mathcal{T}^E P$ is a vector bundle over P of rank $n + p - m$ with vector bundle projection $\tau^\pi : \mathcal{T}^E P \to P$.

A section $\tilde{X}$ of $\tau^\pi : \mathcal{T}^E P \to P$ is said to be *projectable* if there exists a section X of $\tau : E \to M$ and a vector field $U \in \mathfrak{X}(P)$ which is π-projectable to the vector field $\rho(X)$ and such that $\tilde{X}(p) = (X(\pi(p)), U(p))$, for all $p \in P$. For such a projectable section $\tilde{X}$, we will use the following notation $\tilde{X} \equiv (X, U)$. It is easy to prove that one may choose a local basis of projectable sections of the space $\mathrm{Sec}(\tau^\pi)$.

The vector bundle $\tau^\pi : \mathcal{T}^E P \to P$ admits a Lie algebroid structure $([\![\cdot,\cdot]\!]^\pi, \rho^\pi)$. Indeed, if (X, U) and (Y, V) are projectable sections, then

$$\big[\!\!\big[(X, U), (Y, V)\big]\!\!\big]^\pi = \big([\![X, Y]\!], [U, V]\big), \qquad \rho^\pi(X, U) = U.$$

$(\mathcal{T}^E P, [\![\cdot,\cdot]\!]^\pi, \rho^\pi)$ is the *E-tangent bundle to P or the prolongation of E over the fibration $\pi : P \to M$* (for more details, see de León et al. 2005).

Now, let $(E, [\![\cdot,\cdot]\!], \rho)$ (resp., $(E', [\![\cdot,\cdot]\!]', \rho')$) be a Lie algebroid over a manifold M (resp., M') and suppose that $\Psi : E \to E'$ is a vector bundle morphism over the map $\Psi_0 : M \to M'$. Then, the pair (Ψ, Ψ_0) is said to be a *Lie algebroid morphism* if

$$d\big((\Psi,\Psi_0)^*\phi'\big) = (\Psi,\Psi_0)^*\big(d'\phi'\big), \quad \text{for all } \phi' \in \mathrm{Sec}(\wedge^k(\tau')^*) \text{ and for all } k, \tag{2.3}$$

where d (resp., d') is the differential of the Lie algebroid E (resp., E'). In the particular case when $M = M'$ and $\Psi_0 = Id$ then (2.3) holds if and only if

$$[\![\Psi \circ X, \Psi \circ Y]\!]' = \Psi[\![X, Y]\!], \qquad \rho'(\Psi X) = \rho(X), \quad \text{for } X, Y \in \mathrm{Sec}(\tau)$$

(see de León et al. 2005).

2.2 Lie Groupoids

A *Lie groupoid* over a differentiable manifold M is a differentiable manifold Γ together with the following structural maps:

- A pair of submersions $\alpha : \Gamma \to M$, *the source*, and $\beta : \Gamma \to M$, *the target*. The maps α and β define the set of *composable pairs*

$$\Gamma_2 = \big\{(g,h) \in \Gamma \times \Gamma \mid \beta(g) = \alpha(h)\big\}.$$

- A *multiplication* $m : \Gamma_2 \to \Gamma$, to be denoted simply by $m(g,h) = gh$, such that
  - $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$.
  - $g(hk) = (gh)k$.
- An *identity section* $\epsilon : M \to \Gamma$ such that
  - $\epsilon(\alpha(g))g = g$ and $g\epsilon(\beta(g)) = g$.
- An *inversion map* $i : \Gamma \to \Gamma$, to be simply denoted by $i(g) = g^{-1}$, such that
  - $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$.

A Lie groupoid Γ over a set M will be simply denoted by the symbol $\Gamma \rightrightarrows M$.

On the other hand, if $g \in \Gamma$, then the *left-translation* by g and the *right-translation* by g are the diffeomorphisms

$$l_g : \alpha^{-1}\big(\beta(g)\big) \longrightarrow \alpha^{-1}\big(\alpha(g)\big), \quad h \longrightarrow l_g(h) = gh,$$
$$r_g : \beta^{-1}\big(\alpha(g)\big) \longrightarrow \beta^{-1}\big(\beta(g)\big), \quad h \longrightarrow r_g(h) = hg.$$

Note that $l_g^{-1} = l_{g^{-1}}$ and $r_g^{-1} = r_{g^{-1}}$.

A vector field $\tilde{X}$ on Γ is said to be *left-invariant* (resp., *right-invariant*) if it is tangent to the fibers of α (resp., β) and $\tilde{X}(gh) = (T_h l_g)(\tilde{X}_h)$ (resp., $\tilde{X}(gh) = (T_g r_h)(\tilde{X}(g))$), for $(g,h) \in \Gamma_2$.

Now, we will recall the definition of the *Lie algebroid associated with* Γ.

We consider the vector bundle $\tau : E_\Gamma \to M$, whose fiber at a point $x \in M$ is $(E_\Gamma)_x = V_{\epsilon(x)}\alpha = \mathrm{Ker}(T_{\epsilon(x)}\alpha)$. It is easy to prove that there exists a bijection between the space $\mathrm{Sec}(\tau)$ and the set of left-invariant (resp., right-invariant) vector fields on Γ. If X is a section of $\tau : E_\Gamma \to M$, the corresponding left-invariant (resp., right-invariant) vector field on Γ will be denoted $\overleftarrow{X}$ (resp., $\overrightarrow{X}$), where

$$\overleftarrow{X}(g) = (T_{\epsilon(\beta(g))} l_g)\big(X\big(\beta(g)\big)\big), \tag{2.4}$$

$$\overrightarrow{X}(g) = -(T_{\epsilon(\alpha(g))} r_g)\big((T_{\epsilon(\alpha(g))} i)\big(X\big(\alpha(g)\big)\big)\big), \tag{2.5}$$

for $g \in \Gamma$. Using the above facts, we may introduce a Lie algebroid structure $([\![\cdot,\cdot]\!], \rho)$ on E_Γ, which is defined by

$$\overleftarrow{[\![X,Y]\!]} = \big[\overleftarrow{X}, \overleftarrow{Y}\big], \qquad \rho(X)(x) = (T_{\epsilon(x)}\beta)\big(X(x)\big), \tag{2.6}$$

for $X, Y \in \mathrm{Sec}(\tau)$ and $x \in M$. Note that

$$\overrightarrow{[\![X,Y]\!]} = -\big[\overrightarrow{X}, \overrightarrow{Y}\big], \qquad \big[\overrightarrow{X}, \overleftarrow{Y}\big] = 0, \tag{2.7}$$

(for more details, see Coste et al. 1987; Mackenzie 2005).

Given two Lie groupoids $\Gamma \rightrightarrows M$ and $\Gamma' \rightrightarrows M'$, a *morphism of Lie groupoids* is a smooth map $\Phi : \Gamma \to \Gamma'$ such that

$$(g,h) \in \Gamma_2 \quad \Longrightarrow \quad \big(\Phi(g), \Phi(h)\big) \in (\Gamma')_2,$$

and

$$\Phi(gh) = \Phi(g)\Phi(h).$$

A morphism of Lie groupoids $\Phi : \Gamma \to \Gamma'$ induces a smooth map $\Phi_0 : M \to M'$ in such a way that

$$\alpha' \circ \Phi = \Phi_0 \circ \alpha, \qquad \beta' \circ \Phi = \Phi_0 \circ \beta, \qquad \Phi \circ \epsilon = \epsilon' \circ \Phi_0,$$

α, β and ϵ (resp., α', β' and ϵ') being the source, the target and the identity section of Γ (resp., Γ').

Suppose that (Φ, Φ_0) is a morphism between the Lie groupoids $\Gamma \rightrightarrows M$ and $\Gamma' \rightrightarrows M'$ and that $\tau : E_\Gamma \to M$ (resp., $\tau' : E_{\Gamma'} \to M'$) is the Lie algebroid of Γ (resp., Γ'). Then, if $x \in M$ we may consider the linear map $E_x(\Phi) : (E_\Gamma)_x \to (E_{\Gamma'})_{\Phi_0(x)}$ defined by

$$E_x(\Phi)(v_{\epsilon(x)}) = (T_{\epsilon(x)}\Phi)(v_{\epsilon(x)}), \quad \text{for } v_{\epsilon(x)} \in (E_\Gamma)_x. \tag{2.8}$$

In fact, we have that the pair $(E(\Phi), \Phi_0)$ is a morphism between the Lie algebroids $\tau : E_\Gamma \to M$ and $\tau' : E_{\Gamma'} \to M'$ (see Mackenzie 2005).

Trivial examples of Lie groupoids are Lie groups and the pair or banal groupoid $M \times M$, M being an arbitrary smooth manifold. The Lie algebroid of a Lie group Γ is just the Lie algebra $\mathfrak{g}$ of Γ. On the other hand, the Lie algebroid of the pair (or banal) groupoid $M \times M$ is the tangent bundle TM to M.

Apart from the Lie algebroid E_Γ associated with a Lie groupoid $\Gamma \rightrightarrows M$, other interesting Lie algebroids associated with Γ are the following ones:

The E_Γ-Tangent Bundle to E^*_Γ Let $\mathcal{T}^{E_\Gamma}E^*_\Gamma$ be the E_Γ-tangent bundle to E^*_Γ, that is,

$$\mathcal{T}^{E_\Gamma}_{\Upsilon_x}E^*_\Gamma=\{(v_x,X_{\Upsilon_x})\in(E_\Gamma)_x\times T_{\Upsilon_x}E^*_\Gamma \mid (T_{\Upsilon_x}\tau^*)(X_{\Upsilon_x})=(T_{\epsilon(x)}\beta)(v_x)\}$$

for $\Upsilon_x\in(E^*_\Gamma)_x$, with $x\in M$. As we know, $\mathcal{T}^{E_\Gamma}E^*_\Gamma$ is a Lie algebroid over E^*_Γ.

We may introduce the canonical section Θ of the vector bundle $(\mathcal{T}^{E_\Gamma}E^*_\Gamma)^*\to E^*_\Gamma$ as follows:

$$\Theta(\Upsilon_x)(a_x,X_{\Upsilon_x})=\Upsilon_x(a_x),$$

for $\Upsilon_x\in(E^*_\Gamma)_x$ and $(a_x,X_{\Upsilon_x})\in\mathcal{T}^{E_\Gamma}_{\Upsilon_x}E^*_\Gamma$. Θ is called the *Liouville section associated with* E_Γ. Moreover, we define *the canonical symplectic section* Ω associated with E_Γ by $\Omega=-d\Theta$, where d is the differential on the Lie algebroid $\mathcal{T}^{E_\Gamma}E^*_\Gamma\to E^*_\Gamma$. It is easy to prove that Ω is nondegenerate and closed, that is, it is a symplectic section of $\mathcal{T}^{E_\Gamma}E^*_\Gamma$ (see de León et al. 2005).

Now, if Z is a section of $\tau:E_\Gamma\to M$, then there is a unique vector field Z^{*c} on E^*_Γ, *the complete lift of* X *to* E^*_Γ, satisfying the two following conditions:

(i) Z^{*c} is τ^*-projectable on $\rho(Z)$
(ii) $Z^{*c}(\widehat{X})=\widehat{[\![Z,X]\!]}$

for $X\in\mathrm{Sec}(\tau)$ (see de León et al. 2005). Here, if X is a section of $\tau:E_\Gamma\to M$, then $\widehat{X}$ is the linear function $\widehat{X}\in C^\infty(E^*)$ defined by

$$\widehat{X}(a^*)=a^*\big(X\big(\tau^*(a^*)\big)\big),\quad \text{for all } a^*\in E^*.$$

Using the vector field Z^{*c}, one may introduce *the complete lift* $Z^{*\mathbf{c}}$ of Z as the section of $\tau^{\tau^*}:\mathcal{T}^{E_\Gamma}E^*_\Gamma\to E^*_\Gamma$ defined by

$$Z^{*\mathbf{c}}(a^*)=\big(Z\big(\tau^*(a^*)\big),Z^{*c}(a^*)\big),\quad \text{for } a^*\in E^*. \tag{2.9}$$

$Z^{*\mathbf{c}}$ is just the Hamiltonian section of $\widehat{Z}$ with respect to the canonical symplectic section Ω associated with E_Γ. In other words,

$$i_{Z^{*\mathbf{c}}}\Omega=d\widehat{Z}, \tag{2.10}$$

where d is the differential of the Lie algebroid $\tau^{\tau^*}:\mathcal{T}^{E_\Gamma}E^*_\Gamma\to E^*_\Gamma$ (for more details, see de León et al. 2005).

The Lie Algebroid $\widetilde{\tau}_\Gamma:\mathcal{T}^\Gamma\Gamma\to\Gamma$ Let $\mathcal{T}^\Gamma\Gamma$ be the Whitney sum $V\beta\oplus_\Gamma V\alpha$ of the vector bundles $V\beta\to\Gamma$ and $V\alpha\to\Gamma$, where $V\beta$ (resp., $V\alpha$) is the vertical bundle of β (resp., α). Then, the vector bundle $\widetilde{\tau}_\Gamma:\mathcal{T}^\Gamma\Gamma\equiv V\beta\oplus_\Gamma V\alpha\to\Gamma$ admits a Lie algebroid structure $([\![\cdot,\cdot]\!]^{\mathcal{T}^\Gamma\Gamma},\rho^{\mathcal{T}^\Gamma\Gamma})$. The anchor map $\rho^{\mathcal{T}^\Gamma\Gamma}$ is given by

$$\big(\rho^{\mathcal{T}^\Gamma\Gamma}\big)(X_g,Y_g)=X_g+Y_g$$

and the Lie bracket $[\![\cdot,\cdot]\!]^{\mathcal{T}^\Gamma\Gamma}$ on the space $\mathrm{Sec}(\widetilde{\tau}_\Gamma)$ is characterized for the following relation

$$[\![(\overrightarrow{X},\overleftarrow{Y}),(\overrightarrow{X'},\overleftarrow{Y'})]\!]^{\mathcal{T}^\Gamma\Gamma}=(-\overrightarrow{[\![X,X']\!]},\overleftarrow{[\![Y,Y']\!]}),$$

for $X,Y,X',Y'\in\mathrm{Sec}(\tau)$ (for more details, see Marrero et al. 2006).

On other hand, if X is a section of $\tau : E_\Gamma \to M$, one may define the sections $X^{(1,0)}$, $X^{(0,1)}$ (the β and α-lifts) and $X^{(1,1)}$ (the complete lift) of X to $\widetilde{\tau}_\Gamma : \mathcal{T}^\Gamma\Gamma \to \Gamma$ as follows:

$$X^{(1,0)}(g) = \big(\overrightarrow{X}(g), 0_g\big), \qquad X^{(0,1)}(g) = \big(0_g, \overleftarrow{X}(g)\big), \quad \text{and}$$
$$X^{(1,1)}(g) = \big(-\overrightarrow{X}(g), \overleftarrow{X}(g)\big).$$

We have that

$$[\![X^{(1,0)}, Y^{(1,0)}]\!]^{\mathcal{T}^\Gamma\Gamma} = -[\![X, Y]\!]^{(1,0)}, \qquad [\![X^{(0,1)}, Y^{(1,0)}]\!]^{\mathcal{T}^\Gamma\Gamma} = 0,$$
$$[\![X^{(0,1)}, Y^{(0,1)}]\!]^{\mathcal{T}^\Gamma\Gamma} = [\![X, Y]\!]^{(0,1)},$$

and, as a consequence,

$$[\![X^{(1,1)}, Y^{(1,0)}]\!]^{\mathcal{T}^\Gamma\Gamma} = [\![X, Y]\!]^{(1,0)}, \qquad [\![X^{(1,1)}, Y^{(0,1)}]\!]^{\mathcal{T}^\Gamma\Gamma} = [\![X, Y]\!]^{(0,1)},$$
$$[\![X^{(1,1)}, Y^{(1,1)}]\!]^{\mathcal{T}^\Gamma\Gamma} = [\![X, Y]\!]^{(1,1)}.$$

Now, if $g, h \in \Gamma$ one may introduce the linear monomorphisms ${}_h^{(1,0)} : (E_\Gamma)^*_{\alpha(h)} \to (\mathcal{T}_h^\Gamma\Gamma)^* \equiv V_h^*\beta \oplus V_h^*\alpha$ and ${}_g^{(0,1)} : (E_\Gamma)^*_{\beta(g)} \to (\mathcal{T}_g^\Gamma\Gamma)^* \equiv V_g^*\beta \oplus V_g^*\alpha$ given by

$$\gamma_h^{(1,0)}(X_h, Y_h) = \gamma\big(T_h(i \circ r_{h^{-1}})(X_h)\big), \tag{2.11}$$
$$\gamma_g^{(0,1)}(X_g, Y_g) = \gamma\big((T_g l_{g^{-1}})(Y_g)\big), \tag{2.12}$$

for $(X_g, Y_g) \in \mathcal{T}_g^\Gamma\Gamma$ and $(X_h, Y_h) \in \mathcal{T}_h^\Gamma\Gamma$.

Thus, if μ is a section of $\tau^* : E_\Gamma^* \to M$, one may define the corresponding lifts $\mu^{(1,0)}$ and $\mu^{(0,1)}$ as the sections of $\widetilde{\tau}_\Gamma^* : (\mathcal{T}^\Gamma\Gamma)^* \to \Gamma$ given by

$$\mu^{(1,0)}(h) = \mu_h^{(1,0)}, \quad \text{for } h \in \Gamma,$$
$$\mu^{(0,1)}(g) = \mu_g^{(0,1)}, \quad \text{for } g \in \Gamma.$$

Note that if $g \in \Gamma$ and $\{X_A\}$ (resp., $\{Y_B\}$) is a local basis of $\mathrm{Sec}(\tau)$ on an open subset U (resp., V) of M such that $\alpha(g) \in U$ (resp., $\beta(g) \in V$), then $\{X_A^{(1,0)}, Y_B^{(0,1)}\}$ is a local basis of $\mathrm{Sec}(\widetilde{\tau}_\Gamma)$ on the open subset $\alpha^{-1}(U) \cap \beta^{-1}(V)$. In addition, if $\{X^A\}$ (resp., $\{Y^B\}$) is the dual basis of $\{X_A\}$ (resp., $\{Y_B\}$), then $\{(X^A)^{(1,0)}, (Y^B)^{(0,1)}\}$ is the dual basis of $\{X_A^{(1,0)}, Y_B^{(0,1)}\}$.

2.3 Discrete Unconstrained Lagrangian Systems

In this section, we recall some results and constructions of our paper (Marrero et al. 2006) where we discussed the geometrical framework for discrete Mechanics on Lie groupoids, giving a unified point of view of several different sets of discrete equations such as the classical discrete Euler–Lagrange equations and the discrete Euler–Poincaré equations, and proposing new equations, such as the discrete Lagrange–Poincaré equations. We will extensively use these constructions in Sect. 3.

A *discrete unconstrained Lagrangian system on a Lie groupoid* consists of a Lie groupoid $\Gamma \rightrightarrows M$ (the *discrete space*) and a *discrete Lagrangian* $L_{\mathrm{d}} : \Gamma \to \mathbb{R}$.

2.3.1 Discrete Unconstrained Euler–Lagrange Equations

An *admissible sequence of order* N on the Lie groupoid Γ is an element $(g_1, \ldots, g_N)$ of $\Gamma^N \equiv \Gamma \times \cdots \times \Gamma$ such that $(g_k, g_{k+1}) \in \Gamma_2$, for $k = 1, \ldots, N-1$.

An admissible sequence $(g_1, \ldots, g_N)$ of order N is a solution of the *discrete unconstrained Euler–Lagrange equations* for L_{d} if

$$\sum_{k=1}^{N-1} \mathrm{d}^{\mathrm{o}}[L_{\mathrm{d}} \circ l_{g_k} + L_{\mathrm{d}} \circ r_{g_{k+1}} \circ i]\big(\epsilon(x_k)\big)_{|(E_\Gamma)_{x_k}} = 0,$$

where $\beta(g_k) = \alpha(g_{k+1}) = x_k$ and d^{o} is the standard differential on Γ, that is, the differential of the Lie algebroid $\tau_\Gamma : T\Gamma \to \Gamma$ (see Marrero et al. 2006).

The *discrete unconstrained Euler–Lagrange operator* $D_{\mathrm{DEL}} L_{\mathrm{d}} : \Gamma_2 \to E^*_\Gamma$ is given by

$$(D_{\mathrm{DEL}} L_{\mathrm{d}})(g, h) = \mathrm{d}^{\mathrm{o}}[L_{\mathrm{d}} \circ l_g + L_{\mathrm{d}} \circ r_h \circ i]\big(\epsilon(x)\big)_{|(E_\Gamma)_x},$$

for $(g, h) \in \Gamma_2$, with $\beta(g) = \alpha(h) = x \in M$ (see Marrero et al. 2006).

Thus, an admissible sequence $(g_1, \ldots, g_N)$ of order N is a solution of the discrete unconstrained Euler–Lagrange equations if and only if

$$(D_{\mathrm{DEL}} L_{\mathrm{d}})(g_k, g_{k+1}) = 0, \quad \text{for } k = 1, \ldots, N-1.$$

2.3.2 Discrete Poincaré–Cartan Sections

Consider the Lie algebroid $\tilde{\tau}_\Gamma : \mathcal{T}^\Gamma\Gamma \equiv V\beta \oplus_\Gamma V\alpha \to \Gamma$, and define the *Poincaré–Cartan 1-sections* $\Theta^-_{L_{\mathrm{d}}}, \Theta^+_{L_{\mathrm{d}}} \in \mathrm{Sec}((\tilde{\tau}_\Gamma)^*)$ as follows

$$\Theta^-_{L_{\mathrm{d}}}(g)(X_g, Y_g) = -X_g(L_{\mathrm{d}}), \qquad \Theta^+_{L_{\mathrm{d}}}(g)(X_g, Y_g) = Y_g(L_{\mathrm{d}}), \tag{2.13}$$

for each $g \in \Gamma$ and $(X_g, Y_g) \in \mathcal{T}^\Gamma_g\Gamma \equiv V_g\beta \oplus V_g\alpha$.

Since $\mathrm{d}L_{\mathrm{d}} = \Theta^+_{L_{\mathrm{d}}} - \Theta^-_{L_{\mathrm{d}}}$ and so, using $\mathrm{d}^2 = 0$, it follows that $\mathrm{d}\Theta^+_{L_{\mathrm{d}}} = \mathrm{d}\Theta^-_{L_{\mathrm{d}}}$. This means that there exists a unique 2-section $\Omega_{L_{\mathrm{d}}} = -\mathrm{d}\Theta^+_{L_{\mathrm{d}}} = -\mathrm{d}\Theta^-_{L_{\mathrm{d}}}$, which will be called the *Poincaré–Cartan* 2-section. This 2-section will be important to study the symplectic character of the discrete unconstrained Euler–Lagrange equations.

If g is an element of Γ such that $\alpha(g) = x$ and $\beta(g) = y$ and $\{X_A\}$ (resp., $\{Y_B\}$) is a local basis of $\mathrm{Sec}(\tau)$ on the open subset U (resp., V) of M, with $x \in U$ (resp., $y \in V$), then on $\alpha^{-1}(U) \cap \beta^{-1}(V)$ we have that

$$\begin{aligned} &\Theta^-_{L_{\mathrm{d}}} = -\overrightarrow{X_A}(L)\big(X^A\big)^{(1,0)}, \qquad \Theta^+_{L_{\mathrm{d}}} = \overleftarrow{Y_B}(L)\big(Y^B\big)^{(0,1)}, \\ &\Omega_{L_{\mathrm{d}}} = -\overrightarrow{X_A}\big(\overleftarrow{Y_B}(L_{\mathrm{d}})\big)\big(X^A\big)^{(1,0)} \wedge \big(Y^B\big)^{(0,1)}, \end{aligned} \tag{2.14}$$

where $\{X^A\}$ (resp., $\{Y^B\}$) is the dual basis of $\{X_A\}$ (resp., $\{Y_B\}$) (for more details, see Marrero et al. 2006).

2.3.3 Discrete Unconstrained Lagrangian Evolution Operator

Let $\Upsilon : \Gamma \to \Gamma$ be a smooth map such that:

- graph$(\Upsilon) \subseteq \Gamma_2$, that is, $(g, \Upsilon(g)) \in \Gamma_2$, for all $g \in \Gamma$ (Υ is a *second-order operator*) and
- $(g, \Upsilon(g))$ is a solution of the discrete unconstrained Euler–Lagrange equations, for all $g \in \Gamma$, that is

$$(D_{\mathrm{DEL}}L_{\mathrm{d}})\big(g, \Upsilon(g)\big) = 0, \quad \text{for all } g \in \Gamma.$$

In such a case

$$\overleftarrow{X}(g)(L_{\mathrm{d}}) - \overrightarrow{X}\big(\Upsilon(g)\big)(L_{\mathrm{d}}) = 0, \tag{2.15}$$

for every section X of $\tau : E_\Gamma \to M$ and every $g \in \Gamma$. The map $\Upsilon : \Gamma \to \Gamma$ is called a *discrete flow* or a *discrete unconstrained Lagrangian evolution operator for* L_{d}.

Now, let $\Upsilon : \Gamma \to \Gamma$ be a second-order operator. Then, the prolongation $\mathcal{T}\Upsilon : \mathcal{T}^\Gamma\Gamma \equiv V\beta \oplus_\Gamma V\alpha \to \mathcal{T}^\Gamma\Gamma \equiv V\beta \oplus_\Gamma V\alpha$ of Υ is the Lie algebroid morphism over $\Upsilon : \Gamma \to \Gamma$ defined as follows (see Marrero et al. 2006):

$$\begin{aligned}\mathcal{T}_g\Upsilon(X_g, Y_g) = \Big(\big(T_g(r_{g\Upsilon(g)} \circ i)\big)(Y_g), (T_g\Upsilon)(X_g)\\ + (T_g\Upsilon)(Y_g) - T_g(r_{g\Upsilon(g)} \circ i)(Y_g)\Big),\end{aligned} \tag{2.16}$$

for all $(X_g, Y_g) \in \mathcal{T}_g^\Gamma\Gamma \equiv V_g\beta \oplus V_g\alpha$. Moreover, from (2.4), (2.5) and (2.16), we obtain that

$$\mathcal{T}_g\Upsilon\big(\overrightarrow{X}(g), \overleftarrow{Y}(g)\big) = \big(-\overrightarrow{Y}\big(\Upsilon(g)\big), (T_g\Upsilon)\big(\overrightarrow{X}(g) + \overleftarrow{Y}(g)\big) + \overrightarrow{Y}\big(\Upsilon(g)\big)\big), \tag{2.17}$$

for all $X, Y \in \mathrm{Sec}(\tau)$.

Using (2.16), one may prove that (see Marrero et al. 2006):

(i) The map Υ is a discrete unconstrained Lagrangian evolution operator for L_{d} if and only if $(\mathcal{T}\Upsilon, \Upsilon)^*\Theta^-_{L_{\mathrm{d}}} = \Theta^+_{L_{\mathrm{d}}}$.
(ii) The map Υ is a discrete unconstrained Lagrangian evolution operator for L_{d} if and only if $(\mathcal{T}\Upsilon, \Upsilon)^*\Theta^-_{L_{\mathrm{d}}} - \Theta^-_{L_{\mathrm{d}}} = \mathrm{d}L_{\mathrm{d}}$.
(iii) If Υ is discrete unconstrained Lagrangian evolution operator, then

$$(\mathcal{T}\Upsilon, \Upsilon)^*\Omega_{L_{\mathrm{d}}} = \Omega_{L_{\mathrm{d}}}.$$

2.3.4 Discrete Unconstrained Legendre Transformations

Given a Lagrangian $L_{\mathrm{d}} : \Gamma \to \mathbb{R}$ we define the *discrete unconstrained Legendre transformations* $\mathbb{F}^-L_{\mathrm{d}} : \Gamma \to E_\Gamma^*$ and $\mathbb{F}^+L_{\mathrm{d}} : \Gamma \to E_\Gamma^*$ by (see Marrero et al. 2006)

$$\begin{aligned}\big(\mathbb{F}^-L_{\mathrm{d}}\big)(h)(v_{\epsilon(\alpha(h))}) &= -v_{\epsilon(\alpha(h))}(L_{\mathrm{d}} \circ r_h \circ i), \quad \text{for } v_{\epsilon(\alpha(h))} \in (E_\Gamma)_{\alpha(h)},\\ \big(\mathbb{F}^+L_{\mathrm{d}}\big)(g)(v_{\epsilon(\beta(g))}) &= v_{\epsilon(\beta(g))}(L_{\mathrm{d}} \circ l_g), \quad \text{for } v_{\epsilon(\beta(g))} \in (E_\Gamma)_{\beta(g)}.\end{aligned}$$

Now, we introduce the prolongations $\mathcal{T}^{\Gamma}\mathbb{F}^{-}L_{\mathrm{d}}:\mathcal{T}^{\Gamma}\Gamma\equiv V\beta\oplus_{\Gamma}V\alpha\to\mathcal{T}^{E_{\Gamma}}E_{\Gamma}^{*}$ and $\mathcal{T}^{\Gamma}\mathbb{F}^{+}L_{\mathrm{d}}:\mathcal{T}^{\Gamma}\Gamma\equiv V\beta\oplus_{\Gamma}V\alpha\to\mathcal{T}^{E_{\Gamma}}E_{\Gamma}^{*}$ as follows

$$\mathcal{T}_h^{\Gamma}\mathbb{F}^{-}L_{\mathrm{d}}(X_h,Y_h)=\big(T_h(i\circ r_{h^{-1}})(X_h),\big(T_h\mathbb{F}^{-}L_{\mathrm{d}}\big)(X_h)+\big(T_h\mathbb{F}^{-}L_{\mathrm{d}}\big)(Y_h)\big),\tag{2.18}$$

$$\mathcal{T}_g^{\Gamma}\mathbb{F}^{+}L_{\mathrm{d}}(X_g,Y_g)=\big((T_g l_{g^{-1}})(Y_g),\big(T_g\mathbb{F}^{+}L_{\mathrm{d}}\big)(X_g)+\big(T_g\mathbb{F}^{+}L_{\mathrm{d}}\big)(Y_g)\big),\tag{2.19}$$

for all $h,g\in\Gamma$ and $(X_h,Y_h)\in\mathcal{T}_h^{\Gamma}\Gamma\equiv V_h\beta\oplus V_h\alpha$ and $(X_g,Y_g)\in\mathcal{T}_g^{\Gamma}\Gamma\equiv V_g\beta\oplus V_g\alpha$ (see Marrero et al. 2006). We observe that the discrete Poincaré–Cartan 1-sections and 2-section are related to the canonical Liouville section of $(\mathcal{T}^{E_{\Gamma}}E_{\Gamma}^{*})^{*}\to E_{\Gamma}^{*}$ and the canonical symplectic section of $\wedge^2(\mathcal{T}^{E_{\Gamma}}E_{\Gamma}^{*})^{*}\to E_{\Gamma}^{*}$ by pull-back under the discrete unconstrained Legendre transformations, that is (see Marrero et al. 2006),

$$\big(\mathcal{T}^{\Gamma}\mathbb{F}^{-}L_{\mathrm{d}},\mathbb{F}^{-}L_{\mathrm{d}}\big)^{*}\Theta=\Theta_{L_{\mathrm{d}}}^{-},\qquad\big(\mathcal{T}^{\Gamma}\mathbb{F}^{+}L_{\mathrm{d}},\mathbb{F}^{+}L_{\mathrm{d}}\big)^{*}\Theta=\Theta_{L_{\mathrm{d}}}^{+},\tag{2.20}$$

$$\big(\mathcal{T}^{\Gamma}\mathbb{F}^{-}L_{\mathrm{d}},\mathbb{F}^{-}L_{\mathrm{d}}\big)^{*}\Omega=\Omega_{L_{\mathrm{d}}},\qquad\big(\mathcal{T}^{\Gamma}\mathbb{F}^{+}L_{\mathrm{d}},\mathbb{F}^{+}L_{\mathrm{d}}\big)^{*}\Omega=\Omega_{L_{\mathrm{d}}}.\tag{2.21}$$

2.3.5 Discrete Regular Lagrangians

A discrete Lagrangian $L_{\mathrm{d}}:\Gamma\to\mathbb{R}$ is said to be *regular* if the Poincaré–Cartan 2-section $\Omega_{L_{\mathrm{d}}}$ is nondegenerate on the Lie algebroid $\tilde{\tau}_{\Gamma}:\mathcal{T}^{\Gamma}\Gamma\equiv V\beta\oplus_{\Gamma}V\alpha\to\Gamma$ (see Marrero et al. 2006). In Marrero et al. (2006), we obtained some necessary and sufficient conditions for a discrete Lagrangian on a Lie groupoid Γ to be regular that we summarize as follows:

$$\begin{aligned}&L_{\mathrm{d}}\text{ is regular}\\ &\quad\Longleftrightarrow\quad\text{The Legendre transformation }\mathbb{F}^{+}L_{\mathrm{d}}\text{ is a local diffeomorphism}\\ &\quad\Longleftrightarrow\quad\text{The Legendre transformation }\mathbb{F}^{-}L_{\mathrm{d}}\text{ is a local diffeomorphism.}\end{aligned}$$

Locally, we deduce that L_{d} is regular if and only if for every $g\in\Gamma$ and every local basis $\{X_A\}$ (resp., $\{Y_B\}$) of $Sec(\tau)$ on an open subset U (resp., V) of M such that $\alpha(g)\in U$ (resp., $\beta(g)\in V$) we have that the matrix $(\overrightarrow{X}_A(\overleftarrow{Y}_B(L_{\mathrm{d}})))$ is regular on $\alpha^{-1}(U)\cap\beta^{-1}(V)$.

Now, let $L_{\mathrm{d}}:\Gamma\to\mathbb{R}$ be a discrete Lagrangian and g be a point of Γ. We define the $\mathbb{R}$-bilinear map $G_g^{L_{\mathrm{d}}}:(E_{\Gamma})_{\alpha(g)}\oplus(E_{\Gamma})_{\beta(g)}\to\mathbb{R}$ given by

$$G_g^{L_{\mathrm{d}}}(a,b)=\Omega_{L_{\mathrm{d}}}(g)\big(\big(-T_{\epsilon(\alpha(g))}(r_g\circ i)(a),0\big),\big(0,(T_{\epsilon(\beta(g))}l_g)(b)\big)\big).\tag{2.22}$$

Then, using (2.14), we have that

Proposition 2.1 *The discrete Lagrangian $L_{\mathrm{d}}:\Gamma\to\mathbb{R}$ is regular if and only if $G_g^{L_{\mathrm{d}}}$ is nondegenerate, for all $g\in\Gamma$, that is,*

$$G_g^{L_{\mathrm{d}}}(a,b)=0,\quad\textit{for all }b\in(E_{\Gamma})_{\beta(g)}\Rightarrow a=0$$

(*resp.*, $G_g^{L_{\mathrm{d}}}(a,b)=0$, *for all* $a\in(E_{\Gamma})_{\alpha(g)}\Rightarrow b=0$).

On the other hand, if $L_{\mathrm{d}} : \Gamma \to \mathbb{R}$ is a discrete Lagrangian on a Lie groupoid Γ, then we have that

$$\tau^* \circ \mathbb{F}^- L_{\mathrm{d}} = \alpha, \qquad \tau^* \circ \mathbb{F}^+ L_{\mathrm{d}} = \beta,$$

where $\tau^* : E_\Gamma^* \to M$ is the vector bundle projection. Using these facts, (2.18) and (2.19), we deduce the following result.

Proposition 2.2 *Let $L_{\mathrm{d}} : \Gamma \to \mathbb{R}$ be a discrete Lagrangian function. Then, the following conditions are equivalent:*

(i) *L_{d} is regular.*
(ii) *The linear map $\mathcal{T}_h^\Gamma \mathbb{F}^- L_{\mathrm{d}} : V_h\beta \oplus V_h\alpha \to \mathcal{T}_{\mathbb{F}^- L_{\mathrm{d}}(h)}^{E_\Gamma} E_\Gamma^*$ is a linear isomorphism, for all $h \in \Gamma$.*
(iii) *The linear map $\mathcal{T}_g^\Gamma \mathbb{F}^+ L_{\mathrm{d}} : V_g\beta \oplus V_g\alpha \to \mathcal{T}_{\mathbb{F}^+ L_{\mathrm{d}}(g)}^{E_\Gamma} E_\Gamma{}^*$ is a linear isomorphism, for all $g \in \Gamma$.*

Finally, let $L_{\mathrm{d}} : \Gamma \to \mathbb{R}$ be a regular discrete Lagrangian function and $(g_0, h_0) \in \Gamma \times \Gamma$ be a solution of the discrete Euler–Lagrange equations for L_{d}. Then, one may prove (see Marrero et al. 2006) that there exist two open subsets U_0 and V_0 of Γ, with $g_0 \in U_0$ and $h_0 \in V_0$, and there exists a (local) discrete unconstrained Lagrangian evolution operator $\Upsilon_{L_{\mathrm{d}}} : U_0 \to V_0$ such that:

(i) $\Upsilon_{L_{\mathrm{d}}}(g_0) = h_0$.
(ii) $\Upsilon_{L_{\mathrm{d}}}$ is a diffeomorphism.
(iii) $\Upsilon_{L_{\mathrm{d}}}$ is unique, that is, if U_0' is an open subset of Γ, with $g_0 \in U_0'$, and $\Upsilon_{L_{\mathrm{d}}}' : U_0' \to \Gamma$ is a (local) discrete Lagrangian evolution operator, then

$$\Upsilon_{L_{\mathrm{d}}|U_0 \cap U_0'} = \Upsilon'_{L_{\mathrm{d}}|U_0 \cap U_0'}.$$

3 Discrete Nonholonomic (or Constrained) Lagrangian Systems on Lie Groupoids

In this section we find the discrete nonholonomic equations applying a discrete version of Hölder's principle (see Arnold 1978). We also discuss the existence of nonholonomic evolution operators giving characterizations which are the discrete versions of typical conditions for a continuous nonholonomic system (see Bates and Śniatycki 1992; Cortés et al. 2005; de León et al. 1997; de León and Martín de Diego 1996 and references therein). Finally we also study the reversibility of this evolution operator and the discrete nonholonomic momentum equation. With this framework, it would be possible to develop geometric integrators for the (reduced or not) equations of a continuous nonholonomic system.

3.1 Discrete Generalized Hölder's Principle

Let Γ be a Lie groupoid with structural maps

$$\alpha, \beta : \Gamma \to M, \qquad \epsilon : M \to \Gamma, \qquad i : \Gamma \to \Gamma, \qquad m : \Gamma_2 \to \Gamma.$$

Denote by $\tau : E_\Gamma \to M$ the Lie algebroid associated to Γ. Suppose that the rank of E_Γ is n and that the dimension of M is m.

A generalized discrete nonholonomic (or constrained) Lagrangian system on Γ is determined by:

- A *regular discrete Lagrangian* $L_{\mathrm{d}} : \Gamma \to \mathbb{R}$.
- A *constraint distribution*, $\mathcal{D}_{\mathrm{c}}$, which is a vector subbundle of the bundle $E_\Gamma \to M$ of admissible directions. We will denote by $\tau_{\mathcal{D}_{\mathrm{c}}} : \mathcal{D}_{\mathrm{c}} \to M$ the vector bundle projection and by $i_{\mathcal{D}_{\mathrm{c}}} : \mathcal{D}_{\mathrm{c}} \to E_\Gamma$ the canonical inclusion.
- A *discrete constraint embedded submanifold* $\mathcal{M}_{\mathrm{c}}$ of Γ, such that $\dim \mathcal{M}_{\mathrm{c}} = \dim \mathcal{D}_{\mathrm{c}} = m + r$, with $r \leq n$. We will denote by $i_{\mathcal{M}_{\mathrm{c}}} : \mathcal{M}_{\mathrm{c}} \to \Gamma$ the canonical inclusion.

Remark 3.1 Let $L_{\mathrm{d}} : \Gamma \to \mathbb{R}$ be a regular discrete Lagrangian on a Lie groupoid Γ and $\mathcal{M}_{\mathrm{c}}$ be a submanifold of Γ such that $\epsilon(M) \subseteq \mathcal{M}_{\mathrm{c}}$. Then, $\dim \mathcal{M}_{\mathrm{c}} = m + r$, with $0 \leq r \leq m$. Moreover, for every $x \in M$, we may introduce the subspace $\mathcal{D}_{\mathrm{c}}(x)$ of $E_\Gamma(x)$ given by

$$\mathcal{D}_{\mathrm{c}}(x) = T_{\epsilon(x)}\mathcal{M}_{\mathrm{c}} \cap E_\Gamma(x).$$

Since the linear map $T_{\epsilon(x)}\alpha : T_{\epsilon(x)}\mathcal{M}_{\mathrm{c}} \to T_x M$ is an epimorphism, we deduce that $\dim \mathcal{D}_{\mathrm{c}}(x) = r$. In fact, $\mathcal{D}_{\mathrm{c}} = \bigcup_{x \in M} \mathcal{D}_{\mathrm{c}}(x)$ is a vector subbundle of E_Γ (over M) of rank r. Thus, we may consider the discrete nonholonomic system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ on the Lie groupoid Γ.

For $g \in \Gamma$ fixed, we consider the following set of *admissible sequences* of order N:

$$\mathcal{C}_g^N = \big\{(g_1, \dots, g_N) \in \Gamma^N \bigm| (g_k, g_{k+1}) \in \Gamma_2,\\ \text{for } k = 1, \dots, N-1 \text{ and } g_1 \dots g_N = g\big\}.$$

Given a tangent vector at $(g_1, \dots, g_N)$ to the manifold $\mathcal{C}_g^N$, we may write it as the tangent vector at $t = 0$ of a curve in $\mathcal{C}_g^N$, $t \in (-\varepsilon, \varepsilon) \subseteq \mathbb{R} \to c(t)$ which passes through $(g_1, \dots, g_N)$ at $t = 0$. This type of curves is of the form

$$c(t) = \big(g_1 h_1(t), h_1^{-1}(t) g_2 h_2(t), \dots, h_{N-2}^{-1}(t) g_{N-1} h_{N-1}(t), h_{N-1}^{-1}(t) g_N\big),$$

where $h_k(t) \in \alpha^{-1}(\beta(g_k))$, for all t, and $h_k(0) = \epsilon(\beta(g_k))$ for $k = 1, \dots, N-1$.

Therefore, we may identify the tangent space to $\mathcal{C}_g^N$ at $(g_1, \dots, g_N)$ with

$$T_{(g_1, g_2, \dots, g_N)}\mathcal{C}_g^N \equiv \big\{(v_1, v_2, \dots, v_{N-1}) \bigm| v_k \in (E_\Gamma)_{x_k} \text{ and}\\ x_k = \beta(g_k), 1 \leq k \leq N-1\big\}.$$

Observe that each v_k is the tangent vector to the curve h_k at $t = 0$.

The curve c is called a *variation* of $(g_1, \dots, g_N)$ and $(v_1, v_2, \dots, v_{N-1})$ is called an *infinitesimal variation* of $(g_1, \dots, g_N)$.

Now, we define the *discrete action sum* associated to the discrete Lagrangian $L_\mathrm{d}:\Gamma\to\mathbb{R}$ as

$$\mathcal{S}L_\mathrm{d}: \qquad \mathcal{C}_g^N \longrightarrow \mathbb{R}$$
$$(g_1,\dots,g_N)\longmapsto \sum_{k=1}^N L_\mathrm{d}(g_k).$$

We define the *variation* $\delta\mathcal{S}L_\mathrm{d}: T_{(g_1,\dots,g_N)}\mathcal{C}_g^N\to\mathbb{R}$ as

$$\begin{aligned}
&\delta\mathcal{S}L_\mathrm{d}(v_1,\dots,v_{N-1})\\
&\quad=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{S}L_\mathrm{d}\big(c(t)\big)\\
&\quad=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big\{L_\mathrm{d}\big(g_1h_1(t)\big)+L_\mathrm{d}\big(h_1^{-1}(t)g_2h_2(t)\big)+\cdots\\
&\qquad+L_\mathrm{d}\big(h_{N-2}^{-1}(t)g_{N-1}h_{N-1}(t)\big)+L_\mathrm{d}\big(h_{N-1}^{-1}(t)g_N\big)\big\}\\
&\quad=\sum_{k=1}^{N-1}\big(\mathrm{d}^\mathrm{o}(L_\mathrm{d}\circ l_{g_k})\big(\epsilon(x_k)\big)(v_k)+\mathrm{d}^\mathrm{o}(L_\mathrm{d}\circ r_{g_{k+1}}\circ i)\big(\epsilon(x_k)\big)(v_k)\big),
\end{aligned}$$

where d^o is the standard differential on Γ, i.e., d^o is the differential of the Lie algebroid $\tau_\Gamma: T\Gamma\to\Gamma$. It is obvious from the last expression that the definition of variation $\delta\mathcal{S}L_\mathrm{d}$ does not depend on the choice of variations c of the sequence g whose infinitesimal variation is $(v_1,\dots,v_{N-1})$.

Next, we will introduce the subset $(\mathcal{V}_\mathrm{c})_g$ of $T_{(g_1,\dots,g_N)}\mathcal{C}_g^N$ defined by

$$(\mathcal{V}_\mathrm{c})_g=\big\{(v_1,\dots,v_{N-1})\in T_{(g_1,\dots,g_N)}\mathcal{C}_g^N\;\big|\;\forall k\in\{1,\dots,N-1\},\ v_k\in\mathcal{D}_\mathrm{c}\big\}.$$

Then, we will say that a sequence in $\mathcal{C}_g^N$ satisfying the constraints determined by $\mathcal{M}_\mathrm{c}$ is a *Hölder-critical point* of the discrete action sum $\mathcal{S}L_\mathrm{d}$ if the restriction of $\delta\mathcal{S}L_\mathrm{d}$ to $(\mathcal{V}_\mathrm{c})_g$ vanishes, i.e.,

$$\delta\mathcal{S}L_\mathrm{d}|_{(\mathcal{V}_\mathrm{c})_g}=0.$$

Definition 3.2 (Discrete Hölder's Principle) Given $g\in\Gamma$, a sequence $(g_1,\dots,g_N)\in\mathcal{C}_g^N$ such that $g_k\in\mathcal{M}_\mathrm{c}$, $1\le k\le N$, is a solution of the discrete nonholonomic Lagrangian system determined by $(L_\mathrm{d},\mathcal{M}_\mathrm{c},\mathcal{D}_\mathrm{c})$ if and only if $(g_1,\dots,g_N)$ is a Hölder-critical point of $\mathcal{S}L_\mathrm{d}$.

Remark 3.3 In the particular case when Γ is the pair groupoid $M\times M$, Definition 3.2 is the discrete version of Hölder's principle for an standard Lagrangian system subjected to nonholonomic constraints (see Arnold 1978). This kind of system is specified by a Lagrangian function $L: TM\to\mathbb{R}$ and a nonintegrable distribution $\mathcal{D}$ on M. Hölder's principle establishes that a curve $\sigma:[t_0,t_1]\to M$ satisfying the constraints ($\dot\sigma(t)\in\mathcal{D}_{c(t)}$) is a motion of the given nonholonomic Lagrangian system if and only if it is a critical point (in the sense of Hölder) of the action functional $\mathcal{S}(\sigma) = \int_{t_0}^{t_1} L(\sigma(t), \dot{\sigma}(t))\,\mathrm{d}t$, that is, $\mathrm{d}\mathcal{S}(\sigma)|_{\mathcal{V}_\sigma} = 0$, where $\mathcal{V}_\sigma = \{X \in T_\sigma C^2(t_0, t_1, q_0, q_1) \mid X(\sigma(t)) \in \mathcal{D}_{\sigma(t)}\ \forall t \in [t_0, t_1]\}$ and $C^2(t_0, t_1, q_0, q_1) = \{\sigma : [t_0, t_1] \to M \mid \sigma \in C^2, \sigma(t_0) = q_0, \sigma(t_1) = q_1\}$.

If $(g_1, \ldots, g_N) \in \mathcal{C}_g^N \cap (\mathcal{M}_{\mathrm{c}} \times \cdots \times \mathcal{M}_{\mathrm{c}})$, then $(g_1, \ldots, g_N)$ is a solution of the nonholonomic discrete Lagrangian system if and only if

$$\sum_{k=1}^{N-1} \big(\mathrm{d}^{\mathrm{o}}(L_{\mathrm{d}} \circ l_{g_k}) + \mathrm{d}^{\mathrm{o}}(L_{\mathrm{d}} \circ r_{g_{k+1}} \circ i)\big)\big(\epsilon(x_k)\big)_{|(\mathcal{D}_{\mathrm{c}})_{x_k}} = 0,$$

where $\beta(g_k) = \alpha(g_{k+1}) = x_k$. For $N = 2$, we obtain that $(g, h) \in \Gamma_2 \cap (\mathcal{M}_{\mathrm{c}} \times \mathcal{M}_{\mathrm{c}})$ (with $\beta(g) = \alpha(h) = x$) is a solution if

$$\mathrm{d}^{\mathrm{o}}(L_{\mathrm{d}} \circ l_g + L_{\mathrm{d}} \circ r_h \circ i)\big(\epsilon(x)\big)_{|(\mathcal{D}_{\mathrm{c}})_x} = 0.$$

These equations will be called the *discrete nonholonomic Euler–Lagrange equations for the system* $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$.

Let $(g_1, \ldots, g_N)$ be an element of $\mathcal{C}_g^N$. Suppose that $\beta(g_k) = \alpha(g_{k+1}) = x_k$, $1 \le k \le N-1$, and that $\{X_{Ak}\} = \{X_{ak}, X_{\alpha k}\}$ is a local adapted basis of $\mathrm{Sec}(\tau)$ on an open subset U_k of M, with $x_k \in U_k$. Here, $\{X_{ak}\}_{1 \le a \le r}$ is a local basis of $\mathrm{Sec}(\tau_{\mathcal{D}_{\mathrm{c}}})$ and, thus, $\{X^{\alpha k}\}_{r+1 \le \alpha \le n}$ is a local basis of the space of sections of the vector subbundle $\tau_{\mathcal{D}_{\mathrm{c}}^0} : \mathcal{D}_{\mathrm{c}}^0 \to M$, where $\mathcal{D}_{\mathrm{c}}^0$ is the annihilator of $\mathcal{D}_{\mathrm{c}}$ and $\{X^{ak}, X^{\alpha k}\}$ is the dual basis of $\{X_{ak}, X_{\alpha k}\}$. Then, the sequence $(g_1, \ldots, g_N)$ is a solution of the discrete nonholonomic equations if $(g_1, \ldots, g_N) \in \mathcal{M}_{\mathrm{c}} \times \cdots \times \mathcal{M}_{\mathrm{c}}$ and it satisfies the following closed system of difference equations

$$\begin{aligned} 0 &= \sum_{k=1}^{N-1} \big[\overleftarrow{X_{ak}}(g_k)(L_{\mathrm{d}}) - \overrightarrow{X_{ak}}(g_{k+1})(L_{\mathrm{d}})\big] \\ &= \sum_{k=1}^{N-1} \big[\big\langle \mathrm{d}L_{\mathrm{d}}, (X_{ak})^{(0,1)}\big\rangle(g_k) - \big\langle \mathrm{d}L_{\mathrm{d}}, (X_{ak})^{(1,0)}\big\rangle(g_{k+1})\big], \end{aligned}$$

for $1 \le a \le r$, d being the differential of the Lie algebroid $\pi^\tau : \mathcal{T}^\Gamma\Gamma \equiv V\beta \oplus_\Gamma V\alpha \to \Gamma$. For $N = 2$ we obtain that $(g, h) \in \Gamma_2 \cap (\mathcal{M}_{\mathrm{c}} \times \mathcal{M}_{\mathrm{c}})$ (with $\beta(g) = \alpha(h) = x$) is a solution if

$$\overleftarrow{X_a}(g)(L_{\mathrm{d}}) - \overrightarrow{X_a}(h)(L_{\mathrm{d}}) = 0,$$

where $\{X_a\}$ is a local basis of $\mathrm{Sec}(\tau_{\mathcal{D}_{\mathrm{c}}})$ on an open subset U of M such that $x \in U$.

Next, we describe an alternative version of these difference equations. First observe that using the Lagrange multipliers the discrete nonholonomic equations are rewritten as

$$\mathrm{d}^{\mathrm{o}}[L_{\mathrm{d}} \circ l_g + L_{\mathrm{d}} \circ r_h \circ i]\big(\epsilon(x)\big)(v) = \lambda_\alpha X^\alpha(x)(v),$$

for $v \in (E_\Gamma)_x$, with $(g, h) \in \Gamma_2 \cap (\mathcal{M}_{\mathrm{c}} \times \mathcal{M}_{\mathrm{c}})$ and $\beta(g) = \alpha(h) = x$. Here, $\{X^\alpha\}$ is a local basis of sections of the annihilator $\mathcal{D}_{\mathrm{c}}^0$.

Thus, the discrete nonholonomic equations are:

$$\overleftarrow{Y}(g)(L_{\rm d}) - \overrightarrow{Y}(h)(L_{\rm d}) = \lambda_\alpha\big(X^\alpha\big)(Y)|_{\beta(g)} \quad (g,h)\in\Gamma_2\cap(\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c}),$$

for all $Y\in \mathrm{Sec}(\tau)$ or, alternatively,

$$\big\langle \mathrm{d}L_{\rm d} - \lambda_\alpha(X^\alpha)^{(0,1)}, Y^{(0,1)}\big\rangle(g) - \big\langle \mathrm{d}L_{\rm d}, Y^{(1,0)}\big\rangle(h) = 0 \quad (g,h)\in\Gamma_2\cap(\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c}),$$

for all $Y\in \mathrm{Sec}(\tau)$.

On the other hand, we may define the *discrete nonholonomic Euler–Lagrange operator* $D_{\rm DEL}(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c}):\Gamma_2\cap(\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c})\to\mathcal{D}_{\rm c}^*$ as follows

$$D_{\rm DEL}(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})(g,h) = \mathrm{d}^{\rm o}[L_{\rm d}\circ l_g + L_{\rm d}\circ r_h\circ i]\big(\epsilon(x)\big)_{|(\mathcal{D}_{\rm c})_x},$$

for $(g,h)\in\Gamma_2\cap(\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c})$, with $\beta(g)=\alpha(h)=x\in M$.

Then, we may characterize the solutions of the discrete nonholonomic equations as the sequences $(g_1,\dots,g_N)$, with $(g_k,g_{k+1})\in\Gamma_2\cap(\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c})$, for each $k\in\{1,\dots,N-1\}$, and

$$D_{\rm DEL}(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})(g_k,g_{k+1}) = 0.$$

Remark 3.4

(i) The set $\Gamma_2\cap(\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c})$ is not, in general, a submanifold of $\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c}$.
(ii) Suppose that $\alpha_{\mathcal{M}_{\rm c}}:\mathcal{M}_{\rm c}\to M$ and $\beta_{\mathcal{M}_{\rm c}}:\mathcal{M}_{\rm c}\to M$ are the restrictions to $\mathcal{M}_{\rm c}$ of $\alpha:\Gamma\to M$ and $\beta:\Gamma\to M$, respectively. If $\alpha_{\mathcal{M}_{\rm c}}$ and $\beta_{\mathcal{M}_{\rm c}}$ are submersions, then $\Gamma_2\cap(\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c})$ is a submanifold of $\mathcal{M}_{\rm c}\times\mathcal{M}_{\rm c}$ of dimension $m+2r$.

3.2 Discrete Nonholonomic Legendre Transformations

Let $(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})$ be a discrete nonholonomic Lagrangian system. We define the *discrete nonholonomic Legendre transformations*

$$\mathbb{F}^-(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c}):\mathcal{M}_{\rm c}\to\mathcal{D}_{\rm c}^* \quad\text{and}\quad \mathbb{F}^+(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c}):\mathcal{M}_{\rm c}\to\mathcal{D}_{\rm c}^*$$

as follows:

$$\begin{aligned}&\mathbb{F}^-(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})(h)(v_{\epsilon(\alpha(h))}) = -v_{\epsilon(\alpha(h))}(L_{\rm d}\circ r_h\circ i),\\ &\quad\text{for } v_{\epsilon(\alpha(h))}\in\mathcal{D}_{\rm c}\big(\alpha(h)\big),\end{aligned}\tag{3.1}$$

$$\begin{aligned}&\mathbb{F}^+(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})(g)(v_{\epsilon(\beta(g))}) = v_{\epsilon(\beta(g))}(L_{\rm d}\circ l_g),\\ &\quad\text{for } v_{\epsilon(\beta(g))}\in\mathcal{D}_{\rm c}\big(\beta(g)\big).\end{aligned}\tag{3.2}$$

If $\mathbb{F}^-L_{\rm d}:\Gamma\to E_\Gamma^*$ and $\mathbb{F}^+L_{\rm d}:\Gamma\to E_\Gamma^*$ are the standard Legendre transformations associated with the Lagrangian function $L_{\rm d}$ and $i_{\mathcal{D}_{\rm c}}^*:E_\Gamma^*\to\mathcal{D}_{\rm c}^*$ is the dual map of the canonical inclusion $i_{\mathcal{D}_{\rm c}}:\mathcal{D}_{\rm c}\to E_\Gamma$, then

$$\begin{aligned}\mathbb{F}^-(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c}) &= i_{\mathcal{D}_{\rm c}}^*\circ\mathbb{F}^-L_{\rm d}\circ i_{\mathcal{M}_{\rm c}},\\ \mathbb{F}^+(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c}) &= i_{\mathcal{D}_{\rm c}}^*\circ\mathbb{F}^+L_{\rm d}\circ i_{\mathcal{M}_{\rm c}}.\end{aligned}\tag{3.3}$$

Remark 3.5

(i) Note that

$$\tau^*_{\mathcal{D}_c} \circ \mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c) = \alpha_{\mathcal{M}_c}, \qquad \tau^*_{\mathcal{D}_c} \circ \mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c) = \beta_{\mathcal{M}_c}. \tag{3.4}$$

(ii) If $D_{\mathrm{DEL}}(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is the discrete nonholonomic Euler–Lagrange operator, then

$$D_{\mathrm{DEL}}(L_d, \mathcal{M}_c, \mathcal{D}_c)(g,h) = \mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c)(g) - \mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c)(h),$$

for $(g,h) \in \Gamma_2 \cap (\mathcal{M}_c \times \mathcal{M}_c)$.

On the other hand, since by assumption $L_d : \Gamma \to \mathbb{R}$ is a regular discrete Lagrangian function, we have that the discrete Poincaré–Cartan 2-section Ω_{L_d} is symplectic on the Lie algebroid $\tilde{\tau}_\Gamma : \mathcal{T}^\Gamma\Gamma \to \Gamma$. Moreover, the regularity of L is equivalent to the fact that the Legendre transformations $\mathbb{F}^- L_d$ and $\mathbb{F}^+ L_d$ to be local diffeomorphisms (see Sect. 2.3.5).

Next, we will present necessary and sufficient conditions for the discrete nonholonomic Legendre transformations associated with the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ to be local diffeomorphisms.

Let F be the vector subbundle (over Γ) of $\tilde{\tau}_\Gamma : \mathcal{T}^\Gamma\Gamma \to \Gamma$ whose fiber at the point $h \in \Gamma$ is

$$F_h = \bigl\{\gamma_h^{(1,0)} \bigm| \gamma \in \mathcal{D}_c\bigl(\alpha(h)\bigr)^0\bigr\}^0 \subseteq \mathcal{T}^\Gamma_h\Gamma.$$

In other words,

$$F_h^0 = \bigl\{\gamma_h^{(1,0)} \bigm| \gamma \in \mathcal{D}_c\bigl(\alpha(h)\bigr)^0\bigr\}.$$

Note that the rank of F is $n + r$.

We also consider the vector subbundle $\bar{F}$ (over Γ) of $\tilde{\tau}_\Gamma : \mathcal{T}^\Gamma\Gamma \to \Gamma$ of rank $n + r$ whose fiber at the point $g \in \Gamma$ is

$$\bar{F}_g = \bigl\{\gamma_g^{(0,1)} \bigm| \gamma \in \mathcal{D}_c\bigl(\beta(g)\bigr)^0\bigr\}^0 \subseteq \mathcal{T}^\Gamma_g\Gamma.$$

Now, let $\rho^{\mathcal{T}^\Gamma\Gamma} : \mathcal{T}^\Gamma\Gamma \to T\Gamma$ be the anchor map of the Lie algebroid $\tilde{\tau}_\Gamma : \mathcal{T}^\Gamma\Gamma \to \Gamma$. Then, we will denote by $\mathcal{H}_h$ the subspace of $\mathcal{T}^\Gamma_h\Gamma$ given by

$$\mathcal{H}_h = \bigl(\rho^{\mathcal{T}^\Gamma\Gamma}\bigr)^{-1}(T_h\mathcal{M}_c) \cap F_h, \quad \text{for } h \in \mathcal{M}_c.$$

In a similar way, for every $g \in \mathcal{M}_c$ we will introduce the subspace $\bar{\mathcal{H}}_g$ of $\mathcal{T}^\Gamma_g\Gamma$ defined by

$$\bar{\mathcal{H}}_g = \bigl(\rho^{\mathcal{T}^\Gamma\Gamma}\bigr)^{-1}(T_g\mathcal{M}_c) \cap \bar{F}_g.$$

On the other hand, let h be a point of $\mathcal{M}_c$ and $G_h^{L_d} : (E_\Gamma)_{\alpha(h)} \oplus (E_\Gamma)_{\beta(h)} \to \mathbb{R}$ be the $\mathbb{R}$-bilinear map given by (2.22). We will denote by $(\overleftarrow{E}_\Gamma)_h^{\mathcal{M}_c}$ the subspace of $(E_\Gamma)_{\beta(h)}$ defined by

$$\bigl(\overleftarrow{E}_\Gamma\bigr)_h^{\mathcal{M}_c} = \bigl\{b \in (E_\Gamma)_{\beta(h)} \bigm| (T_{\epsilon(\beta(h))}l_h)(b) \in T_h\mathcal{M}_c\bigr\}$$

and by $G_h^{L_d c}: (\mathcal{D}_c)_{\alpha(h)} \times (\overleftarrow{E}_\Gamma)_h^{\mathcal{M}_c} \to \mathbb{R}$ the restriction to $(\mathcal{D}_c)_{\alpha(h)} \times (\overleftarrow{E}_\Gamma)_h^{\mathcal{M}_c}$ of the $\mathbb{R}$-bilinear map $G_h^{L_d}$.

In a similar way, if g is a point of Γ we will consider the subspace $(\overrightarrow{E}_\Gamma)_g^{\mathcal{M}_c}$ of $(E_\Gamma)_{\alpha(g)}$ defined by

$$\left(\overrightarrow{E}_\Gamma\right)_g^{\mathcal{M}_c} = \left\{a \in (E_\Gamma)_{\alpha(g)} \mid \left(T_{\epsilon(\alpha(g))}(r_g \circ i)\right)(a) \in T_g\mathcal{M}_c\right\}$$

and the restriction $\bar{G}_g^{L_d c}: (\overrightarrow{E}_\Gamma)_g^{\mathcal{M}_c} \times (\mathcal{D}_c)_{\beta(g)} \to \mathbb{R}$ of $G_g^{L_d}$ to the space $(\overrightarrow{E}_\Gamma)_g^{\mathcal{M}_c} \times (\mathcal{D}_c)_{\beta(g)}$.

Then, we have the following result.

Theorem 3.6 *If $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is a discrete nonholonomic Lagrangian system, the following conditions are equivalent*:

(i) *The discrete nonholonomic Legendre transformation $\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c)$ (resp., $\mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c)$) is a local diffeomorphism.*

(ii) *For every $h \in \mathcal{M}_c$ (resp., $g \in \mathcal{M}_c$)*

$$\left(\rho^{\mathcal{T}^\Gamma\Gamma}\right)^{-1}(T_h\mathcal{M}_c) \cap F_h^\perp = \{0\}$$

(resp., $(\rho^{\mathcal{T}^\Gamma\Gamma})^{-1}(T_g\mathcal{M}_c) \cap \bar{F}_g^\perp = \{0\}$).

(iii) *For every $h \in \mathcal{M}_c$ (resp., $g \in \mathcal{M}_c$) the dimension of the vector subspace $\mathcal{H}_h$ (resp., $\bar{\mathcal{H}}_g$) is $2r$ and the restriction to the vector subbundle $\mathcal{H}$ (resp., $\bar{\mathcal{H}}$) of the Poincaré–Cartan 2-section Ω_{L_d} is nondegenerate.*

(iv) *For every $h \in \mathcal{M}_c$ (resp., $g \in \mathcal{M}_c$)*

$$\left\{b \in \left(\overleftarrow{E}_\Gamma\right)_h^{\mathcal{M}_c} \mid G_h^{L_d c}(a, b) = 0,\ \forall a \in (\mathcal{D}_c)_{\alpha(h)}\right\} = \{0\}$$

(resp., $\{a \in (\overrightarrow{E}_\Gamma)_g^{\mathcal{M}_c} \mid G_g^{L_d c}(a, b) = 0,\ \forall b \in (\mathcal{D}_c)_{\beta(g)}\} = \{0\}$).

Proof The proof of this theorem may be found in the Appendix of this paper. □

3.3 Nonholonomic Evolution Operators and Regular Discrete Nonholonomic Lagrangian Systems

First of all, we introduce the definition of a nonholonomic evolution operator.

Definition 3.7 Let $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system and $\Upsilon_{nh}: \mathcal{M}_c \to \mathcal{M}_c$ be a differentiable map. Υ_{nh} is said to be a discrete nonholonomic evolution operator for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ if:

(i) graph$(\Upsilon_{nh}) \subseteq \Gamma_2$, that is, $(g, \Upsilon_{nh}(g)) \in \Gamma_2$, for all $g \in \mathcal{M}_c$ and

(ii) $(g, \Upsilon_{nh}(g))$ is a solution of the discrete nonholonomic equations, for all $g \in \mathcal{M}_c$, that is,

$$d^o(L_d \circ l_g + L_d \circ r_{\Upsilon_{nh}(g)} \circ i)\left(\epsilon\left(\beta(g)\right)\right)_{|\mathcal{D}_c(\beta(g))} = 0, \quad \text{for all } g \in \mathcal{M}_c.$$

Remark 3.8 If $\Upsilon_{\mathrm{nh}} : \mathcal{M}_{\mathrm{c}} \to \mathcal{M}_{\mathrm{c}}$ is a differentiable map then, from (3.1), (3.2) and (3.4), we deduce that Υ_{nh} is a discrete nonholonomic evolution operator for $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ if and only if

$$\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) \circ \Upsilon_{\mathrm{nh}} = \mathbb{F}^{+}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}).$$

Now, we will introduce the notion of a regular discrete nonholonomic Lagrangian system.

Definition 3.9 A discrete nonholonomic Lagrangian system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is said to be regular if the discrete nonholonomic Legendre transformations $\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ and $\mathbb{F}^{+}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ are local diffeomorphims.

From Theorem 3.6, we deduce Corollary 3.10.

Corollary 3.10 *Let $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ be a discrete nonholonomic Lagrangian system. Then, the following conditions are equivalent*:

(i) *The system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is regular.*
(ii) *The following relations hold*

$$\left(\rho^{\mathcal{T}^{\Gamma}\Gamma}\right)^{-1}(T_h\mathcal{M}_{\mathrm{c}}) \cap F_h^{\perp} = \{0\}, \quad \textit{for all } h \in \mathcal{M}_{\mathrm{c}},$$

$$\left(\rho^{\mathcal{T}^{\Gamma}\Gamma}\right)^{-1}(T_g\mathcal{M}_{\mathrm{c}}) \cap \bar{F}_g^{\perp} = \{0\}, \quad \textit{for all } g \in \mathcal{M}_{\mathrm{c}}.$$

(iii) *$\mathcal{H}$ and $\bar{\mathcal{H}}$ are symplectic subbundles of rank $2r$ of the symplectic vector bundle $(\mathcal{T}^{\Gamma}_{\mathcal{M}_{\mathrm{c}}}\Gamma, \Omega_{L_{\mathrm{d}}})$.*
(iv) *If g and h are points of $\mathcal{M}_{\mathrm{c}}$, then the $\mathbb{R}$-bilinear maps $G_h^{L_{\mathrm{d}}c}$ and $\bar{G}_g^{L_{\mathrm{d}}c}$ are right and left nondegenerate, respectively.*

The map $G_h^{L_{\mathrm{d}}c}$ (resp., $\bar{G}_g^{L_{\mathrm{d}}c}$) is right nondegenerate (resp., left nondegenerate) if

$$G_h^{L_{\mathrm{d}}c}(a, b) = 0, \quad \forall a \in (\mathcal{D}_{\mathrm{c}})_{\alpha(h)} \quad \Rightarrow \quad b = 0$$

(resp., $\bar{G}_g^{L_{\mathrm{d}}c}(a, b) = 0, \forall b \in (\mathcal{D}_{\mathrm{c}})_{\beta(g)} \Rightarrow a = 0$).

Remark 3.11 Corollary 3.10 may be considered as the discrete version of some results obtained by several authors (see Bates and Śniatycki 1992; Cortés et al. 2005; de León et al. 1997; de León and Martín de Diego 1996) about the characterization of the regularity of continuous nonholonomic Lagrangian systems.

Every solution of the discrete nonholonomic equations for a regular discrete nonholonomic Lagrangian system determines a unique local discrete nonholonomic evolution operator. More precisely, we may prove the following result:

Theorem 3.12 *Let $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ be a regular discrete nonholonomic Lagrangian system and $(g_0, h_0) \in \mathcal{M}_{\mathrm{c}} \times \mathcal{M}_{\mathrm{c}}$ be a solution of the discrete nonholonomic equations for $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$. Then, there exist two open subsets U_0 and V_0 of Γ, with $g_0 \in U_0$ and $h_0 \in V_0$, and there exists a local discrete nonholonomic evolution operator $\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})} : U_0 \cap \mathcal{M}_{\mathrm{c}} \to V_0 \cap \mathcal{M}_{\mathrm{c}}$ such that*:

(i) $\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}(g_0) = h_0$.
(ii) $\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}$ *is a diffeomorphism.*
(iii) $\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}$ *is unique, that is, if U_0' is an open subset of Γ, with $g_0 \in U_0'$, and $\Upsilon_{\mathrm{nh}} : U_0' \cap \mathcal{M}_{\mathrm{c}} \to \mathcal{M}_{\mathrm{c}}$ is a (local) discrete nonholonomic evolution operator, then*

$$\big(\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}\big)_{|U_0 \cap U_0' \cap \mathcal{M}_{\mathrm{c}}} = (\Upsilon_{\mathrm{nh}})_{|U_0 \cap U_0' \cap \mathcal{M}_{\mathrm{c}}}.$$

Proof From Remark 3.5, we deduce that

$$\mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})(g_0) = \mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})(h_0) = \mu_0 \in \mathcal{D}_{\mathrm{c}}^*.$$

Thus, we can choose two open subsets U_0 and V_0 of Γ, with $g_0 \in U_0$ and $h_0 \in V_0$, and an open subset W_0 of E_Γ^* such that $\mu_0 \in W_0$ and

$$\begin{aligned} &\mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) : U_0 \cap \mathcal{M}_{\mathrm{c}} \to W_0 \cap \mathcal{D}_{\mathrm{c}}^*, \\ &\mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) : V_0 \cap \mathcal{M}_{\mathrm{c}} \to W_0 \cap \mathcal{D}_{\mathrm{c}}^* \end{aligned}$$

are diffeomorphisms. Therefore, from Remark 3.8, we obtain that

$$\begin{aligned} \Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})} = {}& \big(\mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})^{-1} \circ \mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})\big)_{|U_0 \cap \mathcal{M}_{\mathrm{c}}} : \\ & U_0 \cap \mathcal{M}_{\mathrm{c}} \to V_0 \cap \mathcal{M}_{\mathrm{c}} \end{aligned}$$

is a (local) discrete nonholonomic evolution operator. Moreover, it is clear that $\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}(g_0) = h_0$ and it follows that $\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}$ is a diffeomorphism.

Finally, if U_0' is an open subset of Γ, with $g_0 \in U_0'$, and $\Upsilon_{\mathrm{nh}} : U_0' \cap \mathcal{M}_{\mathrm{c}} \to \mathcal{M}_{\mathrm{c}}$ is another (local) discrete nonholonomic evolution operator, then $(\Upsilon_{\mathrm{nh}})_{|U_0 \cap U_0' \cap \mathcal{M}_{\mathrm{c}}}$ is also a (local) discrete nonholonomic evolution operator. Consequently, from Remark 3.8, we conclude that

$$\begin{aligned} (\Upsilon_{\mathrm{nh}})_{|U_0 \cap U_0' \cap \mathcal{M}_{\mathrm{c}}} &= \big[\mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})^{-1} \circ \mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})\big]_{|U_0 \cap U_0' \cap \mathcal{M}_{\mathrm{c}}} \\ &= \big(\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}\big)_{|U_0 \cap U_0' \cap \mathcal{M}_{\mathrm{c}}}. \end{aligned}$$

□

3.4 Reversible Discrete Nonholonomic Lagrangian Systems

Let $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ be a discrete nonholonomic Lagrangian system on a Lie groupoid $\Gamma \rightrightarrows M$.

Following the terminology used in McLachlan and Perlmutter (2006) for the particular case when Γ is the pair groupoid $M \times M$, we will introduce the following definition

Definition 3.13 The discrete nonholonomic Lagrangian system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is said to be reversible if

$$L_{\mathrm{d}} \circ i = L_{\mathrm{d}}, \quad i(\mathcal{M}_{\mathrm{c}}) = \mathcal{M}_{\mathrm{c}},$$

$i : \Gamma \to \Gamma$ being the inversion of the Lie groupoid Γ.

For a reversible discrete nonholonomic Lagrangian system we have the following result:

Proposition 3.14 *Let $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ be a reversible nonholonomic Lagrangian system on a Lie groupoid Γ. Then, the following conditions are equivalent:*

(i) *The discrete nonholonomic Legendre transformation $\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is a local diffeomorphism.*
(ii) *The discrete nonholonomic Legendre transformation $\mathbb{F}^{+}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is a local diffeomorphism.*

Proof If $h \in \mathcal{M}_{\mathrm{c}}$ then, using (3.1) and the fact that $L_{\mathrm{d}} \circ i = L_{\mathrm{d}}$, it follows that

$$\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})(h)(v_{\epsilon(\alpha(h))}) = -v_{\epsilon(\alpha(h))}\big(L_{\mathrm{d}} \circ l_h^{-1}\big)$$

for $v_{\epsilon(\alpha(h))} \in (\mathcal{D}_{\mathrm{c}})_{\alpha(h)}$. Thus, from (3.2), we obtain that

$$\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})(h)(v_{\epsilon(\alpha(h))}) = -\mathbb{F}^{+}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})\big(h^{-1}\big)(v_{\epsilon(\beta(h^{-1}))}).$$

This implies that

$$\mathbb{F}^{+}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) = -\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) \circ i.$$

Therefore, since the inversion is a diffeomorphism (in fact, we have that $i^2 = id$), we deduce the result. □

Using Theorem 3.6, Definition 3.9 and Proposition 3.14, we may obtain necessary and sufficient conditions for a reversible nonholonomic Lagrangian system on a Lie groupoid to be regular.

Next, we will prove that a reversible nonholonomic Lagrangian system is dynamically reversible.

Proposition 3.15 *Let $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ be a reversible nonholonomic Lagrangian system on a Lie groupoid Γ and (g, h) be a solution of the discrete nonholonomic Euler–Lagrange equations for $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$. Then (h^{-1}, g^{-1}) is also a solution of these equations. In particular, if the system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is regular and $\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}$ is the (local) discrete nonholonomic evolution operator for $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$, then $\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}$ is reversible, that is,*

$$\Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})} \circ i \circ \Upsilon_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})} = i.$$

Proof Using that $i(\mathcal{M}_c) = \mathcal{M}_c$, we deduce that

$$\big(h^{-1}, g^{-1}\big) \in \Gamma_2 \cap (\mathcal{M}_c \times \mathcal{M}_c).$$

Now, suppose that $\beta(g) = \alpha(h) = x$ and that $v \in (\mathcal{D}_c)_x$. Then, since $L_d \circ i = L_d$, it follows that

$$\begin{aligned}
d^o[L_d \circ l_{h^{-1}} + L_d \circ r_{g^{-1}} \circ i]\big(\varepsilon(x)\big)(v) &= v(L_d \circ i \circ r_h \circ i) + v(L_d \circ i \circ l_g) \\
&= v(L_d \circ l_g) + v(L_d \circ r_h \circ i) \\
&= 0.
\end{aligned}$$

Thus, we conclude that (h^{-1}, g^{-1}) is a solution of the discrete nonholonomic Euler–Lagrange equations for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$.

If the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is regular and $g \in \mathcal{M}_c$, we have that $(g, \Upsilon_{nh}^{(L_d,\mathcal{M},\mathcal{D}_c)}(g))$ is a solution of the discrete nonholonomic Euler–Lagrange equations for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$. Therefore, $(i(\Upsilon_{nh}^{(L_d,\mathcal{M},\mathcal{D}_c)}(g)), i(g))$ is also a solution of the dynamical equations which implies that

$$\Upsilon_{nh}^{(L_d,\mathcal{M},\mathcal{D}_c)}\big(i\big(\Upsilon_{nh}^{(L_d,\mathcal{M},\mathcal{D}_c)}(g)\big)\big) = i(g). \qquad \square$$

Remark 3.16 Proposition 3.15 was proved in McLachlan and Perlmutter (2006) for the particular case when Γ is the pair groupoid.

3.5 Lie Groupoid Morphisms and Reduction

Let (Φ, Φ_0) be a Lie groupoid morphism between the Lie groupoids $\Gamma \rightrightarrows M$ and $\Gamma' \rightrightarrows M'$.

Denote by $(E(\Phi), \Phi_0)$ the corresponding morphism between the Lie algebroids E_Γ and $E_{\Gamma'}$ of Γ and Γ', respectively (see Sect. 2.2).

If $L_d : \Gamma \to \mathbb{R}$ and $L'_d : \Gamma' \to \mathbb{R}$ are discrete Lagrangians on Γ and Γ' such that

$$L_d = L'_d \circ \Phi$$

then, from Theorem 4.6 in Marrero et al. (2006), we have that

$$(D_{DEL} L_d)(g, h)(v) = \big(D_{DEL} L'_d\big)\big(\Phi(g), \Phi(h)\big)\big(E_x(\Phi)(v)\big)$$

for $(g, h) \in \Gamma_2$ and $v \in (E_\Gamma)_x$, where $x = \beta(g) = \alpha(h) \in M$.

Using this fact, we deduce the following result:

Corollary 3.17 *Let (Φ, Φ_0) be a Lie groupoid morphism between the Lie groupoids $\Gamma \rightrightarrows M$ and $\Gamma' \rightrightarrows M'$. Suppose that $L'_d : \Gamma' \to \mathbb{R}$ is a discrete Lagrangian on Γ', that $(L_d = L'_d \circ \Phi, \mathcal{M}_c, \mathcal{D}_c)$ is a discrete nonholonomic Lagrangian system on Γ and that $(g, h) \in \Gamma_2 \cap (\mathcal{M}_c \times \mathcal{M}_c)$. Then*:

(i) *The pair (g, h) is a solution of the discrete nonholonomic problem $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ if and only if $(D_{DEL} L'_d)(\Phi(g), \Phi(h))$ vanishes over the set $(E_{\beta(g)}\Phi)((\mathcal{D}_c)_{\beta(g)})$.*

(ii) *If* $(L'_{\mathrm{d}}, \mathcal{M}'_{\mathrm{c}}, \mathcal{D}'_{\mathrm{c}})$ *is a discrete nonholonomic Lagrangian system on* Γ' *such that* $(\Phi(g), \Phi(h)) \in \mathcal{M}'_{\mathrm{c}} \times \mathcal{M}'_{\mathrm{c}}$ *and* $(E_{\beta(g)}(\Phi))((\mathcal{D}_{\mathrm{c}})_{\beta(g)}) = (\mathcal{D}'_{\mathrm{c}})_{\Phi_0(\beta(g))}$, *then* (g, h) *is a solution for the discrete nonholonomic problem* $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ *if and only if* $(\Phi(g), \Phi(h))$ *is a solution for the discrete nonholonomic problem* $(L'_{\mathrm{d}}, \mathcal{M}'_{\mathrm{c}}, \mathcal{D}'_{\mathrm{c}})$.

3.6 Discrete Nonholonomic Hamiltonian Evolution Operator

Let $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ be a regular discrete nonholonomic system. Assume, without the loss of generality, that the discrete nonholonomic Legendre transformations $\mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) : \mathcal{M}_{\mathrm{c}} \to \mathcal{D}^*_{\mathrm{c}}$ and $\mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) : \mathcal{M}_{\mathrm{c}} \to \mathcal{D}^*_{\mathrm{c}}$ are global diffeomorphisms. Then, $\gamma_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})} = \mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})^{-1} \circ \mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is the discrete nonholonomic evolution operator and one may define the *discrete nonholonomic Hamiltonian evolution operator*, $\tilde{\gamma}_{\mathrm{nh}} : \mathcal{D}^*_{\mathrm{c}} \to \mathcal{D}^*_{\mathrm{c}}$, by

$$\tilde{\gamma}_{\mathrm{nh}} = \mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) \circ \gamma_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})} \circ \mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})^{-1}. \tag{3.5}$$

From Remark 3.8, we have

$$\begin{aligned}\tilde{\gamma}_{\mathrm{nh}} &= \mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) \circ \gamma_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})} \circ \mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})^{-1} \\ &= \mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}) \circ \mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})^{-1}.\end{aligned}$$

The following commutative diagram illustrates the situation

$$\begin{tikzcd}
& \mathcal{M}_{\mathrm{c}} \arrow[rr, "\gamma_{\mathrm{nh}}^{(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})}"] \arrow[dl, "\mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})"'] \arrow[dr, "\mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})"] & & \mathcal{M}_{\mathrm{c}} \arrow[dl, "\mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})"'] \arrow[dr, "\mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})"] & \\
\mathcal{D}^*_{\mathrm{c}} \arrow[rr, "\tilde{\gamma}_{\mathrm{nh}}"] & & \mathcal{D}^*_{\mathrm{c}} \arrow[rr, "\tilde{\gamma}_{\mathrm{nh}}"] & & \mathcal{D}^*_{\mathrm{c}}
\end{tikzcd}$$

Remark 3.18 The discrete nonholonomic evolution operator is an application from $\mathcal{D}^*_{\mathrm{c}}$ to itself. It is remarkable that $\mathcal{D}^*_{\mathrm{c}}$ is also the appropriate nonholonomic momentum space for a continuous nonholonomic system defined by a Lagrangian $L : E_\Gamma \to \mathbb{R}$ and the constraint distribution $\mathcal{D}_{\mathrm{c}}$. Therefore, in the regular case, the solution of the continuous nonholonomic Lagrangian system also determines a flow from $\mathcal{D}^*_{\mathrm{c}}$ to itself. We consider that this would be a good starting point to compare the discrete and continuous dynamics and eventually to establish a backward error analysis for nonholonomic systems.

3.7 The Discrete Nonholonomic Momentum Map

Let $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ be a regular discrete nonholonomic Lagrangian system on a Lie groupoid $\Gamma \rightrightarrows M$ and $\tau : E_\Gamma \to M$ be the Lie algebroid of Γ.

Suppose that $\mathfrak{g}$ is a Lie algebra and that $\Psi : \mathfrak{g} \to \mathrm{Sec}(\tau)$ is a $\mathbb{R}$-linear map. Then, for each $x \in M$, we consider the vector subspace $\mathfrak{g}^x$ of $\mathfrak{g}$ given by

$$\mathfrak{g}^x = \{\xi \in \mathfrak{g} \mid \Psi(\xi)(x) \in (\mathcal{D}_{\mathrm{c}})_x\}$$

and the disjoint union of these vector spaces

$$\mathfrak{g}^{\mathcal{D}_{\mathrm{c}}} = \bigcup_{x \in M} \mathfrak{g}^x.$$

We will denote by $(\mathfrak{g}^{\mathcal{D}_{\mathrm{c}}})^*$ the disjoint union of the dual spaces, that is,

$$(\mathfrak{g}^{\mathcal{D}_{\mathrm{c}}})^* = \bigcup_{x \in M} (\mathfrak{g}^x)^*.$$

Next, we define the *discrete nonholonomic momentum map* $J^{\mathrm{nh}} : \Gamma \to (\mathfrak{g}^{\mathcal{D}_{\mathrm{c}}})^*$ as follows: $J^{\mathrm{nh}}(g) \in (\mathfrak{g}^{\beta(g)})^*$ and

$$J^{\mathrm{nh}}(g)(\xi) = \Theta^+_{L_{\mathrm{d}}}\big(\Psi(\xi)^{(1,1)}\big)(g) = \overleftarrow{\Psi(\xi)}(g)(L_{\mathrm{d}}), \quad \text{for } g \in \Gamma \text{ and } \xi \in \mathfrak{g}^{\beta(g)}.$$

If $\widetilde{\xi} : M \to \mathfrak{g}$ is a smooth map such that $\widetilde{\xi}(x) \in \mathfrak{g}^x$, for all $x \in M$, then we may consider the smooth function $J^{\mathrm{nh}}_{\widetilde{\xi}} : \Gamma \to \mathbb{R}$ defined by

$$J^{\mathrm{nh}}_{\widetilde{\xi}}(g) = J^{\mathrm{nh}}(g)\big(\widetilde{\xi}\big(\beta(g)\big)\big), \quad \forall g \in \Gamma.$$

Definition 3.19 The Lagrangian L_{d} is said to be $\mathfrak{g}$-invariant with respect Ψ if

$$\Psi(\xi)^{(1,1)}(L_{\mathrm{d}}) = \overleftarrow{\Psi(\xi)}(L_{\mathrm{d}}) - \overrightarrow{\Psi(\xi)}(L_{\mathrm{d}}) = 0, \quad \forall \xi \in \mathfrak{g}.$$

Now, we will prove the following result

Theorem 3.20 *Let $\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}} : \mathcal{M}_{\mathrm{c}} \to \mathcal{M}_{\mathrm{c}}$ be the local discrete nonholonomic evolution operator for the regular system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$. If L_{d} is $\mathfrak{g}$-invariant with respect to $\Psi : \mathfrak{g} \to \mathrm{Sec}(\tau)$ and $\widetilde{\xi} : M \to \mathfrak{g}$ is a smooth map such that $\widetilde{\xi}(x) \in \mathfrak{g}^x$, for all $x \in M$, then*

$$\begin{aligned} &J^{\mathrm{nh}}_{\widetilde{\xi}}\big(\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}(g)\big) - J^{\mathrm{nh}}_{\widetilde{\xi}}(g) \\ &\quad = \overleftarrow{\Psi\big(\widetilde{\xi}\big(\beta\big(\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}(g)\big)\big) - \widetilde{\xi}\big(\beta(g)\big)\big)}\big(\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}(g)\big)(L_{\mathrm{d}}) \end{aligned} \tag{3.6}$$

for $g \in \mathcal{M}_{\mathrm{c}}$.

Proof Using that the Lagrangian L_{d} is $\mathfrak{g}$-invariant with respect to Ψ, we have that

$$\begin{aligned} &\overrightarrow{\Psi\big(\tilde{\xi}\big(\alpha\big(\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}(g)\big)\big)\big)}\big(\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}(g)\big)(L_{\mathrm{d}}) \\ &\quad = \overleftarrow{\Psi\big(\tilde{\xi}\big(\alpha\big(\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}(g)\big)\big)\big)}\big(\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}(g)\big)(L_{\mathrm{d}}). \end{aligned} \tag{3.7}$$

Also, since $(g, \Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g))$ is a solution of the discrete nonholonomic equations:

$$\overleftarrow{\Psi\big(\tilde{\xi}\big(\beta(g)\big)\big)}(g)(L_{\rm d}) = \overrightarrow{\Psi\big(\tilde{\xi}\big(\alpha\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)\big)\big)}\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)(L_{\rm d}). \tag{3.8}$$

Thus, from (3.7) and (3.8), we find that

$$\overleftarrow{\Psi\big(\tilde{\xi}\big(\beta(g)\big)\big)}(g)(L_{\rm d}) = \overleftarrow{\Psi\big(\tilde{\xi}\big(\beta(g)\big)\big)}\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)(L_{\rm d}).$$

Therefore,

$$\begin{aligned} J_{\tilde{\xi}}^{\rm nh}\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big) - J_{\tilde{\xi}}^{\rm nh}(g) &= \overleftarrow{\Psi\big(\tilde{\xi}\big(\beta\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)\big)\big)}\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)(L_{\rm d}) \\ &\quad - \overleftarrow{\Psi\big(\tilde{\xi}\big(\beta(g)\big)\big)}(g)(L_{\rm d}) \\ &= \overleftarrow{\Psi\big(\tilde{\xi}\big(\beta\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)\big)\big)}\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)(L_{\rm d}) \\ &\quad - \overleftarrow{\Psi\big(\tilde{\xi}\big(\beta(g)\big)\big)}\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)(L_{\rm d}) \\ &= \overleftarrow{\Psi\big(\tilde{\xi}\big(\beta\big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)\big) - \tilde{\xi}\big(\beta(g)\big)\big)} \\ &\quad \times \big(\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}(g)\big)(L_{\rm d}). \qquad \square \end{aligned}$$

Remark 3.21 Theorem 3.20 may be considered as the discrete version of a result which was proved in Cortés et al. (2005) (see also Bloch et al. 1996; Cantrijn et al. 1998, 1999) for a continuous Lagrangian system on a Lie algebroid which is invariant under the action of a symmetry Lie group.

Equation (3.6) will be called *the discrete nonholonomic momentum equation*. Theorem 3.20 suggests we introduce the following definition.

Definition 3.22 An element $\xi \in \mathfrak{g}$ is said to be a *horizontal symmetry* for the discrete nonholonomic system $(L_{\rm d}, \mathcal{M}_{\rm c}, \mathcal{D}_{\rm c})$ and the map $\Psi : \mathfrak{g} \to {\rm Sec}(\tau)$ if

$$\Psi(\xi)(x) \in (\mathfrak{D}_{\rm c})_x, \quad \text{for all } x \in M.$$

Now, from Theorem 3.20, we conclude that:

Corollary 3.23 *If $L_{\rm d}$ is $\mathfrak{g}$-invariant with respect to Ψ and $\xi \in \mathfrak{g}$ is a horizontal symmetry for $(L_{\rm d}, \mathcal{M}_{\rm c}, \mathcal{D}_{\rm c})$ and $\Psi : \mathfrak{g} \to {\rm Sec}(\tau)$, then $J_{\tilde{\xi}}^{\rm nh} : \Gamma \to \mathbb{R}$ is a constant of the motion for $\Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})}$, that is,*

$$J_{\tilde{\xi}}^{\rm nh} \circ \Upsilon_{\rm nh}^{(L_{\rm d},\mathcal{M}_{\rm c},\mathcal{D}_{\rm c})} = J_{\tilde{\xi}}^{\rm nh}.$$

4 Examples

4.1 Discrete Holonomic Lagrangian Systems on a Lie Groupoid

Let us examine the case when the system is subjected to holonomic constraints.

Let $L_{\mathrm{d}}:\Gamma\to\mathbb{R}$ be a discrete Lagrangian on a Lie groupoid $\Gamma \rightrightarrows M$. Suppose that $\mathcal{M}_{\mathrm{c}}\subseteq\Gamma$ is a Lie subgroupoid of Γ over $M'\subseteq M$, that is, $\mathcal{M}_{\mathrm{c}}$ is a Lie groupoid over M' with structural maps

$$\begin{aligned} &\alpha_{|\mathcal{M}_{\mathrm{c}}}:\mathcal{M}_{\mathrm{c}}\to M', \qquad &&\beta_{|\mathcal{M}_{\mathrm{c}}}:\mathcal{M}_{\mathrm{c}}\to M',\\ &\epsilon_{|M'}:M'\to\mathcal{M}_{\mathrm{c}}, \qquad &&i_{|\mathcal{M}_{\mathrm{c}}}:\mathcal{M}_{\mathrm{c}}\to\mathcal{M}_{\mathrm{c}}, \end{aligned}$$

the canonical inclusions $i_{\mathcal{M}_{\mathrm{c}}}:\mathcal{M}_{\mathrm{c}}\to\Gamma$ and $i_{M'}:M'\to M$ are injective immersions and the pair $(i_{\mathcal{M}_{\mathrm{c}}}, i_{M'})$ is a Lie groupoid morphism. We may assume, without the loss of generality, that $M'=M$ (in other cases we will replace the Lie groupoid Γ by the Lie subgroupoid Γ' over M' defined by $\Gamma'=\alpha^{-1}(M')\cap\beta^{-1}(M')$).

Then, if $L_{\mathcal{M}_{\mathrm{c}}}=L_{\mathrm{d}}\circ i_{\mathcal{M}_{\mathrm{c}}}$ and $\tau_{\mathcal{M}_{\mathrm{c}}}:E_{\mathcal{M}_{\mathrm{c}}}\to M$ is the Lie algebroid of $\mathcal{M}_{\mathrm{c}}$, we have that the discrete (unconstrained) Euler–Lagrange equations for the Lagrangian function $L_{\mathcal{M}_{\mathrm{c}}}$ are

$$\overleftarrow{X}(g)(L_{\mathcal{M}_{\mathrm{c}}})-\overrightarrow{X}(h)(L_{\mathcal{M}_{\mathrm{c}}})=0 \quad (g,h)\in(\mathcal{M}_{\mathrm{c}})_2, \tag{4.1}$$

for $X\in\mathrm{Sec}(\tau_{\mathcal{M}_{\mathrm{c}}})$.

We are interested in writing these equations in terms of the Lagrangian L_{d} defined on the Lie groupoid Γ. From Corollary 4.7 (iii) in Marrero et al. (2006), we deduce that $(g,h)\in(\mathcal{M}_{\mathrm{c}})_2$ is a solution of (4.1) if and only if $D_{\mathrm{DEL}}L_{\mathrm{d}}(g,h)$ vanishes over $\mathrm{Im}(E_{\beta(g)}(i_{\mathcal{M}_{\mathrm{c}}}))$. Here, $E(i_{\mathcal{M}_{\mathrm{c}}}):E_{\mathcal{M}_{\mathrm{c}}}\to E_\Gamma$ is the Lie algebroid morphism induced between $E_{\mathcal{M}_{\mathrm{c}}}$ and E_Γ by the Lie groupoid morphism $(i_{\mathcal{M}_{\mathrm{c}}}, id)$. Therefore, we may consider the discrete holonomic system as the discrete nonholonomic system $(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})$, where $\mathcal{D}_{\mathrm{c}}=(E(i_{\mathcal{M}_{\mathrm{c}}}))(E_{\mathcal{M}_{\mathrm{c}}})\cong E_{\mathcal{M}_{\mathrm{c}}}$.

In this particular case, when the subgroupoid $\mathcal{M}_{\mathrm{c}}$ is determined by the vanishing set of $n-r$ independent real C^∞-functions $\phi^\gamma:\Gamma\to\mathbb{R}$

$$\mathcal{M}_{\mathrm{c}}=\left\{g\in\Gamma \;\middle|\; \phi^\gamma(g)=0,\ \text{for all } \gamma\right\},$$

then the discrete holonomic equations are equivalent to

$$\overleftarrow{Y}(g)(L_{\mathrm{d}})-\overrightarrow{Y}(h)(L_{\mathrm{d}})=\lambda_\gamma \mathrm{d}^{\mathrm{o}}\phi^\gamma\big(\epsilon\big(\beta(g)\big)\big)\big(Y\big(\beta(g)\big)\big), \quad \phi^\gamma(g)=\phi^\gamma(h)=0,$$

for all $Y\in\mathrm{Sec}(\tau)$, where d^{o} is the standard differential on Γ. This algorithm is a generalization of the Shake algorithm for holonomic systems (see Cortés and Martínez 2001; Leimkuhler and Reich 2004; Marsden and West 2001; McLachlan and Perlmutter 2006 for similar results on the pair groupoid $Q\times Q$).

4.2 Discrete Nonholonomic Lagrangian Systems on the Pair Groupoid

Let $(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})$ be a discrete nonholonomic Lagrangian system on the pair groupoid $Q\times Q\rightrightarrows Q$ and suppose that (q_0,q_1) is a point of $\mathcal{M}_{\mathrm{c}}$. Then, using the results of

Sect. 3.1, we deduce that $((q_0, q_1), (q_1, q_2)) \in (Q \times Q)_2$ is a solution of the discrete nonholonomic Euler–Lagrange equations for $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ if and only if

$$\begin{aligned} &\big(D_2 L_{\mathrm{d}}(q_0, q_1) + D_1 L_{\mathrm{d}}(q_1, q_2)\big)_{|\mathcal{D}_{\mathrm{c}}(q_1)} = 0,\\ &(q_1, q_2) \in \mathcal{M}_{\mathrm{c}}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} &D_2 L_{\mathrm{d}}(q_0, q_1) + D_1 L_{\mathrm{d}}(q_1, q_2) = \sum_{j=1}^{n-r} \lambda_j A^j(q_1),\\ &(q_1, q_2) \in \mathcal{M}_{\mathrm{c}}, \end{aligned}$$

where λ_j are the Lagrange multipliers and $\{A^j\}$ is a local basis of the annihilator $\mathcal{D}_{\mathrm{c}}^0$. These equations were considered in Cortés and Martínez (2001) and McLachlan and Perlmutter (2006).

Note that if $(q_1, q_2) \in \Gamma = Q \times Q$ then, in this particular case, $G^{L_{\mathrm{d}}}_{(q_1,q_2)} : T_{q_1}Q \times T_{q_2}Q \to \mathbb{R}$ is just the $\mathbb{R}$-bilinear map $(D_2 D_1 L_{\mathrm{d}})(q_1, q_2)$.

On the other hand, if $(q_1, q_2) \in \mathcal{M}_{\mathrm{c}}$ we have that

$$\begin{aligned} \overleftarrow{(TQ)}^{\mathcal{M}_{\mathrm{c}}}_{(q_1,q_2)} &= \big\{ v_{q_2} \in T_{q_2}Q \;\big|\; (0, v_{q_2}) \in T_{(q_1,q_2)}\mathcal{M}_{\mathrm{c}} \big\},\\ \overrightarrow{(TQ)}^{\mathcal{M}_{\mathrm{c}}}_{(q_1,q_2)} &= \big\{ v_{q_1} \in T_{q_1}Q \;\big|\; (v_{q_1}, 0) \in T_{(q_1,q_2)}\mathcal{M}_{\mathrm{c}} \big\}. \end{aligned}$$

Thus, the system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is regular if and only if for every $(q_1, q_2) \in \mathcal{M}_{\mathrm{c}}$ the following conditions hold:

$$\left.\begin{aligned} &\text{If } v_{q_1} \in \overrightarrow{(TQ)}^{\mathcal{M}_{\mathrm{c}}}_{(q_1,q_2)} \text{ and}\\ &\langle D_2 D_1 L_{\mathrm{d}}(q_1, q_2) v_{q_1}, v_{q_2}\rangle = 0, \quad \forall v_{q_2} \in \mathcal{D}_{\mathrm{c}}(q_2) \end{aligned}\right\} \implies v_{q_1} = 0,$$

and

$$\left.\begin{aligned} &\text{if } v_{q_2} \in \overleftarrow{(TQ)}^{\mathcal{M}_{\mathrm{c}}}_{(q_1,q_2)} \text{ and}\\ &\langle D_2 D_1 L_{\mathrm{d}}(q_1, q_2) v_{q_1}, v_{q_2}\rangle = 0, \quad \forall v_{q_1} \in \mathcal{D}_{\mathrm{c}}(q_1) \end{aligned}\right\} \implies v_{q_2} = 0.$$

The first condition was obtained in McLachlan and Perlmutter (2006) in order to guarantee the existence of a unique local nonholonomic evolution operator $\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}$ for the system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$. However, in order to assure that $\Upsilon^{(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})}_{\mathrm{nh}}$ is a (local) diffeomorphism one must assume that the second condition also holds.

Example 4.1 (Discrete Nonholonomically Constrained Particle) Consider the discrete nonholonomic system determined by:

(a) A discrete Lagrangian $L_{\mathrm{d}} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$:

$$L_{\mathrm{d}}(x_0, y_0, z_0, x_1, y_1, z_1) = \frac{1}{2}\left[\left(\frac{x_1 - x_0}{h}\right)^2 + \left(\frac{y_1 - y_0}{h}\right)^2 + \left(\frac{z_1 - z_0}{h}\right)^2\right].$$

(b) A constraint distribution of $Q=\mathbb{R}^3$,

$$\mathcal{D}_c = \operatorname{span}\left\{X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, X_2 = \frac{\partial}{\partial y}\right\}.$$

(c) A discrete constraint submanifold $\mathcal{M}_c$ of $\mathbb{R}^3\times\mathbb{R}^3$ determined by the constraint

$$\phi(x_0,y_0,z_0,x_1,y_1,z_1) = \frac{z_1-z_0}{h} - \left(\frac{y_1+y_0}{2}\right)\left(\frac{x_1-x_0}{h}\right).$$

The system $(L_d,\mathcal{M}_c,\mathcal{D}_c)$ is a discretization of a classical continuous nonholonomic system: the nonholonomic free particle (for a discussion on this continuous system see, for instance, Bloch et al. 1996; Cortés 2002). Note that if $E_{(\mathbb{R}^3\times\mathbb{R}^3)}\cong T\mathbb{R}^3$ is the Lie algebroid of the pair groupoid $\mathbb{R}^3\times\mathbb{R}^3 \rightrightarrows \mathbb{R}^3$, then

$$T_{(x_1,y_1,z_1,x_1,y_1,z_1)}\mathcal{M}_c \cap E_{(\mathbb{R}^3\times\mathbb{R}^3)}(x_1,y_1,z_1) = \mathcal{D}_c(x_1,y_1,z_1).$$

Since

$$\overleftarrow{X_1} = \frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z_1}, \qquad \overrightarrow{X_1} = -\frac{\partial}{\partial x_0} - y_0\frac{\partial}{\partial z_0}, \qquad \overleftarrow{X_2} = \frac{\partial}{\partial y_1}, \qquad \overrightarrow{X_2} = -\frac{\partial}{\partial y_0},$$

then, the discrete nonholonomic equations are

$$\left(\frac{x_2-2x_1+x_0}{h^2}\right) + y_1\left(\frac{z_2-2z_1+z_0}{h^2}\right) = 0, \tag{4.2}$$

$$\frac{y_2-2y_1+y_0}{h^2} = 0, \tag{4.3}$$

which together with the constraint equation determine a well-posed system of difference equations.

We have that

$$\begin{aligned}
D_2D_1L_d &= -\frac{1}{h}\{\mathrm{d}x_0\wedge\mathrm{d}x_1 + \mathrm{d}y_0\wedge\mathrm{d}y_1 + \mathrm{d}z_0\wedge\mathrm{d}z_1\}\\
(\overrightarrow{T\mathbb{R}^3})^{\mathcal{M}_c}_{(x_0,y_0,z_0,x_1,y_1,z_1)} &= \left\{a_0\frac{\partial}{\partial x_0} + b_0\frac{\partial}{\partial y_0} + c_0\frac{\partial}{\partial z_0} \in T_{(x_0,y_0,z_0)}\mathbb{R}^3 / \right.\\
&\qquad \left. c_0 = \frac{1}{2}\big(a_0(y_1+y_0) - b_0(x_1-x_0)\big)\right\}.\\
(\overleftarrow{T\mathbb{R}^3})^{\mathcal{M}_c}_{(x_0,y_0,z_0,x_1,y_1,z_1)} &= \left\{a_1\frac{\partial}{\partial x_1} + b_1\frac{\partial}{\partial y_1} + c_1\frac{\partial}{\partial z_1} \in T_{(x_1,y_1,z_1)}\mathbb{R}^3 / \right.\\
&\qquad \left. c_1 = \frac{1}{2}\big(a_1(y_1+y_0) + b_1(x_1-x_0)\big)\right\}.
\end{aligned}$$

Thus, if we consider the open subset of $\mathcal{M}_c$ defined by

$$\left\{(x_0,y_0,z_0,x_1,y_1,z_1)\in\mathcal{M}_c \,\middle|\, 2+y_1^2+y_1y_0\neq 0, 2+y_0^2+y_0y_1\neq 0\right\},$$

then in this subset the discrete nonholonomic system is regular.

Let $\Psi : \mathfrak{g} = \mathbb{R}^2 \to \mathfrak{X}(\mathbb{R}^3)$ given by $\Psi(a,b) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial z}$. Then $\mathfrak{g}^{\mathcal{D}_c} = \operatorname{span}\{\Psi(\tilde{\xi}) = X_1\}$, where $\tilde{\xi} : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $\tilde{\xi}(x,y,z) = (1,y)$. Moreover, the Lagrangian L_d is $\mathfrak{g}$-invariant with respect to Ψ. Therefore,

$$\begin{aligned}J^{\mathrm{nh}}_{\tilde{\xi}}(x_1,y_1,z_1,x_2,y_2,z_2) - J^{\mathrm{nh}}_{\tilde{\xi}}(x_0,y_0,z_0,x_1,y_1,z_1)\\ = \overleftarrow{\Psi(0,y_2-y_1)}(x_1,y_1,z_1,x_2,y_2,z_2)(L_d),\end{aligned}$$

that is,

$$\left(\frac{x_2-x_1}{h^2} + y_2\frac{z_2-z_1}{h^2}\right) - \left(\frac{x_1-x_0}{h^2} + y_1\frac{z_1-z_0}{h^2}\right) = (y_2-y_1)\left(\frac{z_2-z_1}{h^2}\right).$$

This equation is precisely (4.2).

4.3 Discrete Nonholonomic Lagrangian Systems on a Lie Group

Let G be a Lie group. G is a Lie groupoid over a single point and the Lie algebra $\mathfrak{g}$ of G is just the Lie algebroid associated with G.

If $g, h \in G$, $v_h \in T_hG$ and $\alpha_h \in T^*_hG$ we will use the following notation:

$$\begin{aligned} gv_h &= (T_hl_g)(v_h) \in T_{gh}G, & v_hg &= (T_hr_g)(v_h) \in T_{hg}G,\\ g\alpha_h &= \big(T^*_{gh}l_{g^{-1}}\big)(\alpha_h) \in T^*_{gh}G, & \alpha_hg &= \big(T^*_{hg}r_{g^{-1}}\big)(\alpha_h) \in T^*_{hg}G.\end{aligned}$$

Now, let $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system on the Lie group G, that is, $L_d : G \to \mathbb{R}$ is a discrete Lagrangian, $\mathcal{M}_c$ is a submanifold of G and $\mathcal{D}_c$ is a vector subspace of $\mathfrak{g}$.

If $g_1 \in \mathcal{M}_c$, then $(g_1, g_2) \in G \times G$ is a solution of the discrete nonholonomic Euler–Lagrange equations for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ if and only if

$$\begin{aligned} &g_1^{-1}\mathrm{d}L_d(g_1) - \mathrm{d}L_d(g_2)g_2^{-1} = \sum_{j=1}^{n-r}\lambda_j\mu^j,\\ &g_k \in \mathcal{M}_c, \quad k = 1,2,\end{aligned} \tag{4.4}$$

where λ_j are the Lagrange multipliers and $\{\mu^j\}$ is a basis of the annihilator $\mathcal{D}_c^0$ of $\mathcal{D}_c$. These equations were obtained in McLachlan and Perlmutter (2006) (see Theorem 3 in McLachlan and Perlmutter 2006).

Taking $p_k = \mathrm{d}L_d(g_k)g_k^{-1}$, $k = 1,2$ then

$$\begin{aligned} &p_2 - \mathrm{Ad}^*_{g_1} p_1 = -\sum_{j=1}^{n-r}\lambda^j\mu_j,\\ &g_k \in \mathcal{M}_c, \quad k = 1,2,\end{aligned} \tag{4.5}$$

where $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of G on $\mathfrak{g}$. These equations were obtained in Fedorov and Zenkov (2005a) and called *discrete Euler–Poincaré–Suslov equations*.

On the other hand, from (2.14), we have that

$$\Omega_{L_{\mathrm{d}}}\big((\overrightarrow{\eta},\overleftarrow{\mu}),(\overrightarrow{\eta}',\overleftarrow{\mu}')\big)=\overrightarrow{\eta}'\big(\overleftarrow{\mu}(L_{\mathrm{d}})\big)-\overrightarrow{\eta}\big(\overleftarrow{\mu}'(L_{\mathrm{d}})\big).$$

Thus, if $g\in G$ then, using (2.22), it follows that the $\mathbb{R}$-bilinear map $G_g^{L_{\mathrm{d}}}:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ is given by

$$G_g^{L_{\mathrm{d}}}(\xi,\eta)=-\overleftarrow{\eta}(g)\big(\overrightarrow{\xi}(L_{\mathrm{d}})\big).$$

Therefore, the system $(L_{\mathrm{d}},\mathcal{M}_{\mathrm{c}},\mathcal{D}_{\mathrm{c}})$ is regular if and only if for every $g\in\mathcal{M}_{\mathrm{c}}$ the following conditions hold:

$$\begin{aligned}&\eta\in\mathfrak{g}/\overleftarrow{\eta}(g)\in T_g\mathcal{M}_{\mathrm{c}}\quad\text{and}\quad\overleftarrow{\eta}(g)\big(\overrightarrow{\xi}(L_{\mathrm{d}})\big)=0,\quad\forall\xi\in\mathcal{D}_{\mathrm{c}}\quad\Longrightarrow\quad\eta=0,\\&\xi\in\mathfrak{g}/\overrightarrow{\xi}(g)\in T_g\mathcal{M}_{\mathrm{c}}\quad\text{and}\quad\overleftarrow{\eta}(g)\big(\overrightarrow{\xi}(L_{\mathrm{d}})\big)=0,\quad\forall\eta\in\mathcal{D}_{\mathrm{c}}\quad\Longrightarrow\quad\xi=0.\end{aligned}$$

We illustrate this situation with two simple examples previously considered in Fedorov and Zenkov (2005a).

4.3.1 The Discrete Suslov System

(See Fedorov and Zenkov (2005a).) The Suslov system studies the motion of a rigid body suspended at its center of mass under the action of the following nonholonomic constraint: the body angular velocity is orthogonal to some fixed direction.

The configuration space is $G=\mathrm{SO}(3)$ and the elements of the Lie algebra $\mathfrak{so}(3)$ may be identified with $\mathbb{R}^3$ and represented by coordinates $(\omega_x,\omega_y,\omega_z)$. Without loss of generality, let us choose as fixed direction the third vector of the body frame $\bar{e}_1,\bar{e}_2,\bar{e}_3$. Then, the nonholonomic constraint is $\omega_z=0$.

The discretization of this system is modelled by considering the discrete Lagrangian $L_{\mathrm{d}}:\mathrm{SO}(3)\to\mathbb{R}$ defined by $L_{\mathrm{d}}(\Omega)=\frac{1}{2}\operatorname{Tr}(\Omega J)$, where J represents the mass matrix (a symmetric positive-definite matrix with components $(J_{ij})_{1\le i,j\le 3}$).

The constraint submanifold $\mathcal{M}_{\mathrm{c}}$ is determined by the constraint $\operatorname{Tr}(\Omega E_3)=0$ (see Fedorov and Zenkov 2005a), where

$$E_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\qquad E_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\qquad E_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$$

is the standard basis of $\mathfrak{so}(3)$, the Lie algebra of SO(3).

The vector subspace $\mathcal{D}_{\mathrm{c}}=\operatorname{span}\{E_1,E_2\}$. Therefore, $\mathcal{D}_{\mathrm{c}}^0=\operatorname{span}\{E^3\}$. Moreover, the exponential map of SO(3) is a diffeomorphism from an open subset of $\mathcal{D}_{\mathrm{c}}$ (which contains the zero vector) to an open subset of $\mathcal{M}_{\mathrm{c}}$ (which contains the identity element I). In particular, $T_I\mathcal{M}_{\mathrm{c}}=\mathcal{D}_{\mathrm{c}}$.

On the other hand, the *discrete Euler–Poincaré–Suslov equations* are given by

$$\overleftarrow{E_i}(\Omega_1)(L_{\mathrm{d}})-\overrightarrow{E_i}(\Omega_2)(L_{\mathrm{d}})=0,\qquad\operatorname{Tr}(\Omega_i E_3)=0,\quad i\in\{1,2\}.$$

After some straightforward operations, we deduce that the above equations are equivalent to

$$\mathrm{Tr}\big((E_i\Omega_2 - \Omega_1 E_i)J\big) = 0, \qquad \mathrm{Tr}(\Omega_i E_3) = 0, \quad i \in \{1, 2\}$$

or, considering the components $\Omega_k = (\Omega_{ij}^{(k)})$ of the elements of SO(3), we have that:

$$\begin{pmatrix} J_{23}\Omega_{33}^{(1)} - J_{33}\Omega_{32}^{(1)} + J_{22}\Omega_{23}^{(1)} \\ -J_{23}\Omega_{22}^{(1)} + J_{12}\Omega_{13}^{(1)} - J_{13}\Omega_{12}^{(1)} \end{pmatrix} = \begin{pmatrix} -J_{23}\Omega_{33}^{(2)} - J_{22}\Omega_{32}^{(2)} - J_{12}\Omega_{31}^{(2)} \\ +J_{33}\Omega_{23}^{(2)} + J_{23}\Omega_{22}^{(2)} + J_{13}\Omega_{21}^{(2)} \end{pmatrix}$$

$$\begin{pmatrix} -J_{13}\Omega_{33}^{(1)} + J_{33}\Omega_{31}^{(1)} - J_{12}\Omega_{23}^{(1)} \\ +J_{23}\Omega_{21}^{(1)} - J_{11}\Omega_{13}^{(1)} + J_{13}\Omega_{11}^{(1)} \end{pmatrix} = \begin{pmatrix} J_{13}\Omega_{33}^{(2)} + J_{12}\Omega_{32}^{(2)} + J_{11}\Omega_{31}^{(2)} \\ -J_{33}\Omega_{13}^{(2)} - J_{23}\Omega_{12}^{(2)} - J_{13}\Omega_{11}^{(2)} \end{pmatrix}$$

$$\Omega_{12}^{(1)} = \Omega_{21}^{(1)}, \qquad \Omega_{12}^{(2)} = \Omega_{21}^{(2)}.$$

Moreover, since the discrete Lagrangian verifies that

$$L_{\mathrm{d}}(\Omega) = \frac{1}{2}\mathrm{Tr}(\Omega J) = \frac{1}{2}\mathrm{Tr}\big(\Omega^t J\big) = L_{\mathrm{d}}\big(\Omega^{-1}\big)$$

and also the constraint satisfies $\mathrm{Tr}(\Omega E_3) = -\mathrm{Tr}(\Omega^{-1}E_3)$, then this discretization of the Suslov system is reversible. The regularity condition in $\Omega \in \mathrm{SO}(3)$ is, in this particular case,

$$\begin{aligned} &\eta \in \mathfrak{so}(3) / \mathrm{Tr}(E_1\Omega\eta J) = 0, \qquad \mathrm{Tr}(E_2\Omega\eta J) = 0 \quad \text{and} \\ &\mathrm{Tr}(\Omega\eta E_3) = 0 \quad \Longrightarrow \quad \eta = 0. \end{aligned}$$

It is easy to show that the system is regular in a neighborhood of the identity I.

4.3.2 The Discrete Chaplygin Sleigh

(See Fedorov (2007), Fedorov and Zenkov (2005a).) The Chaplygin sleigh system describes the motion of a rigid body sliding on a horizontal plane. The body is supported at three points, two of which slide freely without friction while the third is a knife edge, a constraint that allows no motion orthogonal to this edge (see Neimark and Fufaev 1972).

The configuration space of this system is the group SE(2) of Euclidean motions of $\mathbb{R}^2$. An element $\Omega \in \mathrm{SE}(2)$ is represented by a matrix

$$\Omega = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } \theta, x, y \in \mathbb{R}.$$

Thus, (θ, x, y) are local coordinates on SE(2).

A basis of the Lie algebra $\mathfrak{se}(2) \cong \mathbb{R}^3$ of SE(2) is given by

$$e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and we have that

$$[e, e_1] = e_2, \qquad [e, e_2] = -e_1, \qquad [e_1, e_2] = 0.$$

An element $\xi \in \mathfrak{se}(2)$ is of the form

$$\xi = \omega\, e + v_1\, e_1 + v_2\, e_2,$$

and the exponential map $\exp : \mathfrak{se}(2) \cong \mathbb{R}^3 \to \mathrm{SE}(2)$ of SE(2) is given by

$$\exp(\omega, v_1, v_2) = \left(\omega, v_1 \frac{\sin\omega}{\omega} + v_2\left(\frac{\cos\omega - 1}{\omega}\right), -v_1\left(\frac{\cos\omega - 1}{\omega}\right) + v_2 \frac{\sin\omega}{\omega}\right),$$

if $\omega \neq 0$,

and

$$\exp(0, v_1, v_2) = (0, v_1, v_2).$$

Note that the restriction of this map to the open subset $U =]-\pi, \pi[\times\mathbb{R}^2 \subseteq \mathbb{R}^3 \cong \mathfrak{se}(2)$ is a diffeomorphism onto the open subset $\exp(U)$ of SE(2).

A discretization of the Chaplygin sleigh may be constructed as follows:

– The discrete Lagrangian $L_{\mathrm{d}} : \mathrm{SE}(2) \to \mathbb{R}$ is given by

$$L_{\mathrm{d}}(\Omega) = \frac{1}{2}\operatorname{Tr}\bigl(\Omega \mathbb{J} \Omega^{\mathrm{T}}\bigr) - \operatorname{Tr}(\Omega \mathbb{J}),$$

where $\mathbb{J}$ is the matrix:

$$\mathbb{J} = \begin{pmatrix} (J/2) + ma^2 & mab & ma \\ mab & (J/2) + mb^2 & mb \\ ma & mb & m \end{pmatrix}$$

(see Fedorov and Zenkov 2005a).

– The vector subspace $\mathcal{D}_{\mathrm{c}}$ of $\mathfrak{se}(2)$ is

$$\mathcal{D}_{\mathrm{c}} = \operatorname{span}\{e, e_1\} = \bigl\{(\omega, v_1, v_2) \in \mathfrak{se}(2) \bigm| v_2 = 0\bigr\}.$$

– The constraint submanifold $\mathcal{M}_{\mathrm{c}}$ of SE(2) is

$$\mathcal{M}_{\mathrm{c}} = \exp(U \cap \mathcal{D}_{\mathrm{c}}). \tag{4.6}$$

Thus, we have that

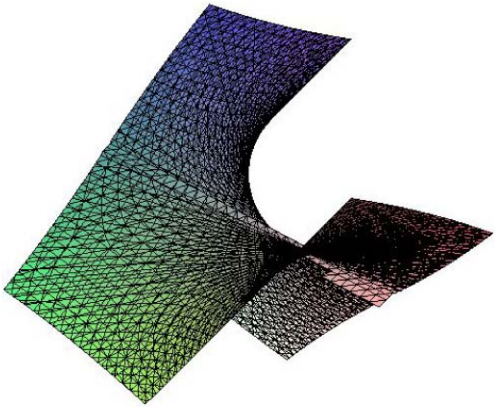
$$\begin{aligned}\mathcal{M}_{\mathrm{c}} = {} & \bigl\{(\theta, x, y) \in \mathrm{SE}(2) \bigm| -\pi < \theta < \pi, \theta \neq 0, (1 - \cos\theta)x - y\sin\theta = 0\bigr\} \\ & \cup \bigl\{(0, x, 0) \in \mathrm{SE}(2) \bigm| x \in \mathbb{R}\bigr\}.\end{aligned}$$

From (4.6) it follows that $I \in \mathcal{M}_{\mathrm{c}}$ and $T_I\mathcal{M}_{\mathrm{c}} = \mathcal{D}_{\mathrm{c}}$. In fact, one may prove that

$$T_{(0,x,0)}\mathcal{M}_{\mathrm{c}} = \operatorname{span}\left\{\frac{\partial}{\partial\theta}_{|(0,x,0)} + \frac{x}{2}\frac{\partial}{\partial y}_{|(0,x,0)}, \frac{\partial}{\partial x}_{|(0,x,0)}\right\},$$

for $x \in \mathbb{R}$ (see Fig. 1).

Fig. 1 Submanifold $\mathcal{M}_{\mathrm{c}}$



Now, the discrete Euler–Poincaré–Suslov equations are:

$$\begin{aligned}&\overleftarrow{e}(\theta_1, x_1, y_1)(L_{\mathrm{d}}) - \overrightarrow{e}(\theta_2, x_2, y_2)(L_{\mathrm{d}}) = 0,\\&\overleftarrow{e_1}(\theta_1, x_1, y_1)(L_{\mathrm{d}}) - \overrightarrow{e_1}(\theta_2, x_2, y_2)(L_{\mathrm{d}}) = 0,\end{aligned}$$

and the condition $(\theta_k, x_k, y_k) \in \mathcal{M}_{\mathrm{c}}$, with $k \in \{1, 2\}$. We rewrite these equations as the following system of difference equations:

$$\begin{pmatrix}-am\cos\theta_1 - bm\sin\theta_1 + am\\ +mx_1\cos\theta_1 + my_1\sin\theta_1\end{pmatrix} = \begin{pmatrix}mx_2 + am\cos\theta_2\\ -bm\sin\theta_2 - am\end{pmatrix}$$

$$\begin{pmatrix}am\, y_1\cos\theta_1 - am\, x_1\sin\theta_1 - bm\, x_1\cos\theta_1\\ -bm\, y_1\sin\theta_1 + (a^2m + b^2m + J)\sin\theta_1\end{pmatrix} = \begin{pmatrix}am\, y_2 - bm\, x_2\\ +(a^2m + b^2m + J)\sin\theta_2\end{pmatrix}$$

together with the condition

$$(\theta_k, x_k, y_k) \in \mathcal{M}_{\mathrm{c}}, \quad k \in \{1, 2\}.$$

On the other hand, one may prove that the discrete nonholonomic Lagrangian system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D})$ is reversible.

Finally, consider a point $(0, x, 0) \in \mathcal{M}_{\mathrm{c}}$ and an element $\eta \equiv (\omega, v_1, v_2) \in \mathfrak{se}(2)$ such that

$$\begin{aligned}&\overleftarrow{\eta}(0, x, 0) \in T_{(0,x,0)}\mathcal{M}_{\mathrm{c}}, \qquad \overleftarrow{\eta}(0, x, 0)\big(\overrightarrow{e}(L_{\mathrm{d}})\big) = 0,\\&\overleftarrow{\eta}(0, x, 0)\big(\overrightarrow{e_1}(L_{\mathrm{d}})\big) = 0.\end{aligned}$$

Then, if we assume that $a^2m + J + am\frac{x}{2} \neq 0$, it follows that $\eta = 0$.

Thus, the discrete nonholonomic Lagrangian system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is regular in a neighborhood of the identity I.

4.4 Discrete Nonholonomic Lagrangian Systems on an Action Lie Groupoid

Let H be a Lie group with identity element $\mathfrak{e}$ and $\cdot : M \times H \to M$, $(x, h) \in M \times H \mapsto xh \in M$, a right action of H on M. Thus, we may consider the action Lie groupoid

$\Gamma = M \times H$ over M with structural maps given by

$$\begin{aligned} &\tilde{\alpha}(x,h) = x, \qquad \tilde{\beta}(x,h) = xh, \qquad \tilde{\epsilon}(x) = (x,\mathfrak{e}), \\ &\tilde{m}\big((x,h),(xh,h')\big) = \big(x,hh'\big), \qquad \tilde{i}(x,h) = \big(xh,h^{-1}\big). \end{aligned} \tag{4.7}$$

Now, let $\mathfrak{h} = T_{\mathfrak{e}}H$ be the Lie algebra of H and $\Phi : \mathfrak{h} \to \mathfrak{X}(M)$ the map given by

$$\Phi(\eta) = \eta_M, \quad \text{for } \eta \in \mathfrak{h},$$

where η_M is the infinitesimal generator of the action $\cdot : M \times H \to M$ corresponding to η. Then, Φ is a Lie algebra morphism and the corresponding action Lie algebroid $pr_1 : M \times \mathfrak{h} \to M$ is just the Lie algebroid of $\Gamma = M \times H$.

We have that $Sec(pr_1) \cong \{\tilde{\eta} : M \to \mathfrak{h} \mid \tilde{\eta} \text{ is smooth}\}$ and that the Lie algebroid structure $([\![\cdot,\cdot]\!]_\Phi, \rho_\Phi)$ on $pr_1 : M \times H \to M$ is defined by

$$\begin{aligned} &[\![\tilde{\eta},\tilde{\mu}]\!]_\Phi(x) = \big[\tilde{\eta}(x),\tilde{\mu}(x)\big] + \big(\tilde{\eta}(x)\big)_M(x)(\tilde{\mu}) - \big(\tilde{\mu}(x)\big)_M(x)(\tilde{\eta}), \\ &\rho_\Phi(\tilde{\eta})(x) = \big(\tilde{\eta}(x)\big)_M(x), \end{aligned}$$

for $\tilde{\eta},\tilde{\mu} \in Sec(pr_1)$ and $x \in M$. Here, $[\cdot,\cdot]$ denotes the Lie bracket of $\mathfrak{h}$.

If $(x,h) \in \Gamma = M \times H$, then the left-translation $l_{(x,h)} : \tilde{\alpha}^{-1}(xh) \to \tilde{\alpha}^{-1}(x)$ and the right-translation $r_{(x,h)} : \tilde{\beta}^{-1}(x) \to \tilde{\beta}^{-1}(xh)$ are given

$$l_{(x,h)}\big(xh,h'\big) = \big(x,hh'\big), \qquad r_{(x,h)}\big(x\big(h'\big)^{-1},h'\big) = \big(x\big(h'\big)^{-1},h'h\big). \tag{4.8}$$

Now, if $\eta \in \mathfrak{h}$, then η defines a constant section $C_\eta : M \to \mathfrak{h}$ of $pr_1 : M \times \mathfrak{h} \to M$ and, using (2.4), (2.5), (4.7) and (4.8), we have that the left-invariant and the right-invariant vector fields $\overleftarrow{C}_\eta$ and $\overrightarrow{C}_\eta$, respectively, on $M \times H$ are defined by

$$\overrightarrow{C}_\eta(x,h) = \big(-\eta_M(x), \overrightarrow{\eta}(h)\big), \qquad \overleftarrow{C}_\eta(x,h) = \big(0_x, \overleftarrow{\eta}(h)\big), \tag{4.9}$$

for $(x,h) \in \Gamma = M \times H$.

Note that if $\{\eta_i\}$ is a basis of $\mathfrak{h}$ then $\{C_{\eta_i}\}$ is a global basis of $Sec(pr_1)$.

On the other hand, we will denote by $\exp_\Gamma : E_\Gamma = M \times \mathfrak{h} \to \Gamma = M \times H$ the map given by

$$\exp_\Gamma(x,\eta) = \big(x, \exp_H(\eta)\big), \quad \text{for } (x,\eta) \in E_\Gamma = M \times \mathfrak{h},$$

where $\exp_H : \mathfrak{h} \to H$ is the exponential map of the Lie group H. Note that if $\Phi_{(x,e)} : \mathbb{R} \to \Gamma = M \times H$ is the integral curve of the left-invariant vector field $\overleftarrow{C}_\eta$ on $\Gamma = M \times H$ such that $\Phi_{(x,e)}(0) = (x,e)$, then (see (4.9))

$$\exp_\Gamma(x,\eta) = \Phi_{(x,e)}(1).$$

Next, suppose that $L_d : \Gamma = M \times H \to \mathbb{R}$ is a Lagrangian function, that $\mathcal{D}_c$ is a constraint distribution such that $\{X^\alpha\}$ is a local basis of sections of the annihilator $\mathcal{D}_c^0$, and $\mathcal{M}_c \subseteq \Gamma$ is the discrete constraint submanifold.

For every $h \in H$ (resp., $x \in M$) we will denote by L_h (resp., L_x) the real function on M (resp., on H) given by $L_h(y) = L_{\mathrm{d}}(y, h)$ (resp., $L_x(h') = L_{\mathrm{d}}(x, h')$). A composable pair $((x, h_k), (xh_k, h_{k+1})) \in \Gamma_2 \cap (\mathcal{M}_{\mathrm{c}} \times \mathcal{M}_{\mathrm{c}})$ is a solution of the discrete nonholonomic Euler–Lagrange equations for the system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ if

$$\overleftarrow{C}_\eta(x, h_k)(L_{\mathrm{d}}) - \overrightarrow{C}_\eta(xh_k, h_{k+1})(L_{\mathrm{d}}) = \lambda_\alpha X^\alpha(xh_k)(\eta), \quad \text{for all } \eta \in \mathfrak{h},$$

or, in other terms (see (4.9))

$$\big\{(T_{\mathfrak{e}} l_{h_k})(\eta)\big\}(L_x) - \big\{(T_{\mathfrak{e}} r_{h_{k+1}})(\eta)\big\}(L_{xh_k}) + \eta_M(xh_k)(L_{h_{k+1}}) = \lambda_\alpha X^\alpha(xh_k)(\eta),$$

for all $\eta \in \mathfrak{h}$.

4.4.1 The Discrete Veselova System

As a concrete example of a nonholonomic system on an action Lie groupoid we consider a discretization of the Veselova system (see Veselov and Veselova 1989). In the continuous theory (Cortés et al. 2005), the configuration manifold is the action Lie algebroid $pr_1 : S^2 \times \mathfrak{so}(3) \to S^2$ with Lagrangian

$$L_{\mathrm{c}}(\gamma, \omega) = \frac{1}{2}\omega \cdot I\omega - mgl\gamma \cdot \mathrm{e},$$

where S^2 is the unit sphere in $\mathbb{R}^3$, $\omega \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ is the angular velocity, γ is the direction opposite to the gravity and e is a unit vector in the direction from the fixed point to the center of mass, all them expressed in a frame fixed to the body. The constants m, g and l are respectively the mass of the body, the strength of the gravitational acceleration and the distance from the fixed point to the center of mass. The matrix I is the inertia tensor of the body. Moreover, the constraint subbundle $\mathcal{D}_{\mathrm{c}} \to S^2$ is given by

$$\gamma \in S^2 \mapsto \mathcal{D}_{\mathrm{c}}(\gamma) = \big\{\omega \in \mathbb{R}^3 \simeq \mathfrak{so}(3) \;\big|\; \gamma \cdot \omega = 0\big\}.$$

Note that the section $\phi : S^2 \to S^2 \times \mathfrak{so}(3)^*$, $(x, y, z) \mapsto ((x, y, z), xe^1 + ye^2 + ze^3)$, where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$ and $\{e^1, e^2, e^3\}$ is the dual basis, is a global basis for $\mathcal{D}_{\mathrm{c}}^0$.

If $\omega \in \mathfrak{so}(3)$ and $\widehat{\omega}$ is the skew-symmetric matrix of order 3 such that $\widehat{\omega}v = \omega \times v$ then the Lagrangian function L_{c} may be expressed as follows

$$L_{\mathrm{c}}(\gamma, \omega) = \frac{1}{2}\operatorname{Tr}\big(\widehat{\omega}\mathbb{I}\widehat{\omega}^{\mathrm{T}}\big) - mgl\gamma \cdot \mathrm{e},$$

where $\mathbb{I} = \frac{1}{2}\operatorname{Tr}(I)I_{3\times 3} - I$. Here, $I_{3\times 3}$ is the identity matrix. Thus, we may define a discrete Lagrangian $L_{\mathrm{d}} : \Gamma = S^2 \times \mathrm{SO}(3) \to \mathbb{R}$ for the system by (see Marrero et al. 2006)

$$L_{\mathrm{d}}(\gamma, \Omega) = -\frac{1}{h}\operatorname{Tr}(\mathbb{I}\Omega) - hmgl\gamma \cdot \mathrm{e}.$$

On the other hand, we consider the open subset of SO(3)

$$V = \{ \Omega \in \mathrm{SO}(3) \mid \operatorname{Tr} \Omega \neq \pm 1 \}$$

and the real function $\psi : S^2 \times V \to \mathbb{R}$ given by

$$\psi(\gamma, \Omega) = \gamma \cdot \big(\widehat{\Omega - \Omega^{\mathrm{T}}} \big).$$

One may check that the critical points of ψ are

$$C_\psi = \{ (\gamma, \Omega) \in S^2 \times V \mid \Omega\gamma - \gamma = 0 \}.$$

Thus, the subset $\mathcal{M}_{\mathrm{c}}$ of $\Gamma = S^2 \times \mathrm{SO}(3)$ defined by

$$\mathcal{M}_{\mathrm{c}} = \{ (\gamma, \Omega) \in (S^2 \times V) - C_\psi \mid \gamma \cdot \big(\widehat{\Omega - \Omega^{\mathrm{T}}} \big) = 0 \},$$

is a submanifold of Γ of codimension one. $\mathcal{M}_{\mathrm{c}}$ is the discrete constraint submanifold.

We have that the map $\exp_\Gamma : S^2 \times \mathfrak{so}(3) \to S^2 \times \mathrm{SO}(3)$ is a diffeomorphism from an open subset of $\mathcal{D}_{\mathrm{c}}$, which contains the zero section, to an open subset of $\mathcal{M}_{\mathrm{c}}$, which contains the subset of Γ given by

$$\tilde{\epsilon}(S^2) = \{ (\gamma, e) \in S^2 \times \mathrm{SO}(3) \}.$$

So, it follows that

$$(\mathcal{D}_{\mathrm{c}})(\gamma) = T_{(\gamma, e)} \mathcal{M}_{\mathrm{c}} \cap E_\Gamma(\gamma), \quad \text{for } \gamma \in S^2.$$

Following the computations of Marrero et al. (2006) we get the nonholonomic discrete Euler–Lagrange equations

$$M_{k+1} - \Omega_k^{\mathrm{T}} M_k \Omega_k + mglh^2 \big(\widehat{\gamma_{k+1} \times \mathrm{e}} \big) = \lambda \widehat{\gamma_{k+1}},$$

$$\gamma_k \big(\widehat{\Omega_k - \Omega_k^{\mathrm{T}}} \big) = 0, \quad \gamma_{k+1} \big(\widehat{\Omega_{k+1} - \Omega_{k+1}^{\mathrm{T}}} \big) = 0,$$

for $((\gamma_k, \Omega_k), (\gamma_k \Omega_k, \Omega_{k+1})) \in \Gamma_2$, where $M = \Omega \mathbb{I} - \mathbb{I} \Omega^{\mathrm{T}}$. Therefore, in terms of the axial vector Π in $\mathbb{R}^3$ defined by $\hat{\Pi} = M$, we can write the equations in the form

$$\Pi_{k+1} = \Omega_k^{\mathrm{T}} \Pi_k - mglh^2 \gamma_{k+1} \times \mathrm{e} + \lambda \gamma_{k+1},$$

$$\gamma_k \big(\widehat{\Omega_k - \Omega_k^{\mathrm{T}}} \big) = 0, \quad \gamma_{k+1} \big(\widehat{\Omega_{k+1} - \Omega_{k+1}^{\mathrm{T}}} \big) = 0.$$

Note that, using the expression of an arbitrary element of SO(3) in terms of the Euler angles (see Chap. 15 in Marsden and Ratiu 1999), we deduce that the discrete constraint submanifold $\mathcal{M}_{\mathrm{c}}$ is reversible, that is, $i(\mathcal{M}_{\mathrm{c}}) = \mathcal{M}_{\mathrm{c}}$. However, the discrete nonholonomic Lagrangian system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is not reversible. In fact, it is easy to prove that $L_{\mathrm{d}} \circ i \neq L_{\mathrm{d}}$.

On the other hand, if $\gamma \in S^2$ and $\xi, \eta \in \mathbb{R}^3 \cong \mathfrak{so}(3)$, then it follows that

$$\overrightarrow{C}_\xi(\gamma, I_3)\big(\overleftarrow{C}_\eta(L_{\mathrm{d}}) \big) = -\xi \cdot I\eta.$$

Consequently, the nonholonomic system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is regular in a neighborhood (in $\mathcal{M}_{\mathrm{c}}$) of the submanifold $\tilde{\epsilon}(S^2)$.

4.5 Discrete Nonholonomic Lagrangian Systems on an Atiyah Lie Groupoid

Let $p : Q \to M = Q/G$ be a principal G-bundle and choose a local trivialization $G \times U$, where U is an open subset of M. Then, one may identify the open subset $(p^{-1}(U) \times p^{-1}(U))/G \simeq ((G \times U) \times (G \times U))/G$ of the Atiyah groupoid $(Q \times Q)/G$ with the product manifold $(U \times U) \times G$. Indeed, it is easy to prove that the map

$$\begin{aligned} \big((G \times U) \times (G \times U)\big)/G &\to (U \times U) \times G, \\ \big[\big((g,x),(g',y)\big)\big] &\to \big((x,y), g^{-1}g'\big), \end{aligned}$$

is bijective. Thus, the restriction to $((G \times U) \times (G \times U))/G$ of the Lie groupoid structure on $(Q \times Q)/G$ induces a Lie groupoid structure in $(U \times U) \times G$ with source, target and identity section given by

$$\begin{aligned} &\alpha : (U \times U) \times G \to U; \quad \big((x,y),g\big) \to x, \\ &\beta : (U \times U) \times G \to U; \quad \big((x,y),g\big) \to y, \\ &\epsilon : U \to (U \times U) \times G; \quad x \to \big((x,x),\mathfrak{e}\big), \end{aligned}$$

and with multiplication $m : ((U \times U) \times G)_2 \to (U \times U) \times G$ and inversion $i : (U \times U) \times G \to (U \times U) \times G$ defined by

$$\begin{aligned} m\big(\big((x,y),g\big),\big((y,z),h\big)\big) &= \big((x,z),gh\big), \\ i\big((x,y),g\big) &= \big((y,x),g^{-1}\big). \end{aligned} \tag{4.10}$$

The Lie algebroid $A((U \times U) \times G)$ may be identified with the vector bundle $TU \times \mathfrak{g} \to U$. Thus, the fiber over the point $x \in U$ is the vector space $T_xU \times \mathfrak{g}$. Therefore, a section of $A((U \times U) \times G)$ is a pair $(X, \tilde{\xi})$, where X is a vector field on U and $\tilde{\xi}$ is a map from U on $\mathfrak{g}$. The space $\mathrm{Sec}(A((U \times U) \times G))$ is generated by sections of the form $(X, 0)$ and $(0, C_\xi)$, with $X \in \mathfrak{X}(U)$, $\xi \in \mathfrak{g}$ and $C_\xi : U \to \mathfrak{g}$ being the constant map $C_\xi(x) = \xi$, for all $x \in U$ (see Marrero et al. 2006 for more details).

Now, suppose that $L_{\mathrm{d}} : (U \times U) \times G \to \mathbb{R}$ is a Lagrangian function, $\mathcal{D}_{\mathrm{c}}$ a vector subbundle of $TU \times \mathfrak{g}$ and $\mathcal{M}_{\mathrm{c}}$ a constraint submanifold on $(U \times U) \times G$. Take a basis of sections $\{Y^\alpha\}$ of the annihilator $\mathcal{D}_{\mathrm{c}}^o$. Then, the discrete nonholonomic equations are

$$\overleftarrow{(X_\alpha, \tilde{\eta}_\alpha)}\big((x,y),g_k\big)(L_{\mathrm{d}}) - \overrightarrow{(X_\alpha, \tilde{\eta}_\alpha)}\big((y,z),g_{k+1}\big)(L_{\mathrm{d}}) = 0,$$

with $(X_\alpha, \tilde{\eta}_\alpha) : U \to TU \times \mathfrak{g}$ a basis of the space $\mathrm{Sec}(\tau_{\mathcal{D}_{\mathrm{c}}})$ and $(((x,y),g_k),((y,z),g_{k+1})) \in (\mathcal{M}_{\mathrm{c}} \times \mathcal{M}_{\mathrm{c}}) \cap ((U \times U) \times G)_2$. The above equations may be also written as

$$\begin{aligned} \overleftarrow{(X,0)}\big((x,y),g_k\big)(L_{\mathrm{d}}) - \overrightarrow{(X,0)}\big((y,z),g_{k+1}\big)(L_{\mathrm{d}}) &= \lambda_\alpha Y^\alpha(y)\big(X(y)\big), \\ \overleftarrow{(0,C_\xi)}\big((x,y),g_k\big)(L_{\mathrm{d}}) - \overrightarrow{(0,C_\xi)}\big((y,z),g_{k+1}\big)(L_{\mathrm{d}}) &= \lambda_\alpha Y^\alpha(y)\big(C_\xi(y)\big), \end{aligned}$$

with $X \in \mathfrak{X}(U)$, $\xi \in \mathfrak{g}$ and $(((x,y),g_k),((y,z),g_{k+1})) \in (\mathcal{M}_c \times \mathcal{M}_c) \cap ((U \times U) \times G)_2$. An equivalent expression of these equations is

$$\begin{aligned} &D_2 L_d\big((x,y),g_k\big) + D_1 L_d\big((y,z),g_{k+1}\big) = \lambda_\alpha \mu^\alpha(y), \\ &p_{k+1}(y,z) = Ad^*_{g_k} p_k(x,y) - \lambda_\alpha \tilde{\eta}^\alpha(y), \end{aligned} \tag{4.11}$$

where $p_k(\bar{x},\bar{y}) = \mathrm{d}(r^*_{g_k} L_{(\bar{x},\bar{y},)})(\mathfrak{e})$ for $(\bar{x},\bar{y}) \in U \times U$ and we write $Y^\alpha \equiv (\mu^\alpha, \tilde{\eta}^\alpha)$, μ^α being a 1-form on U and $\tilde{\eta}^\alpha : U \to \mathfrak{g}^*$ a smooth map.

4.5.1 A Discretization of the Equations of Motion of a Rolling Ball Without Sliding on a Rotating Table with Constant Angular Velocity

A (homogeneous) sphere of radius $r > 0$, mass m and inertia about any axis I rolls without sliding on a horizontal table which rotates with constant angular velocity Ω about a vertical axis through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere (see Neimark and Fufaev 1972).

The configuration space for the continuous system is $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$ and we shall use the notation $(x, y; R)$ to represent a typical point in Q. Then, the nonholonomic constraints are

$$\dot{x} + \frac{r}{2}\mathrm{Tr}\big(\dot{R}R^{\mathrm{T}}E_2\big) = -\Omega y, \qquad \dot{y} - \frac{r}{2}\mathrm{Tr}\big(\dot{R}R^{\mathrm{T}}E_1\big) = \Omega x,$$

where $\{E_1, E_2, E_3\}$ is the standard basis of $\mathfrak{so}(3)$.

The matrix $\dot{R}R^{\mathrm{T}}$ is skew symmetric, therefore we may write

$$\dot{R}R^{\mathrm{T}} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix},$$

where (w_1, w_2, w_3) represents the angular velocity vector of the sphere measured with respect to the inertial frame. Then, we may rewrite the constraints in the usual form:

$$\dot{x} - r w_2 = -\Omega y, \qquad \dot{y} + r w_1 = \Omega x.$$

The Lagrangian for the rolling ball is:

$$\begin{aligned} L_c\big(x, y; R, \dot{x}, \dot{y}; \dot{R}\big) &= \frac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big) + \frac{1}{4}I\,\mathrm{Tr}\big(\dot{R}R^{\mathrm{T}}\big(\dot{R}R^{\mathrm{T}}\big)^{\mathrm{T}}\big) \\ &= \frac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big) + \frac{1}{2}I\big(\omega_1^2 + \omega_2^2 + \omega_3^2\big). \end{aligned}$$

Moreover, it is clear that $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$ is the total space of a trivial principal SO(3)-bundle over $\mathbb{R}^2$ and the bundle projection $\phi : Q \to M = \mathbb{R}^2$ is just the canonical projection on the first factor. Therefore, we may consider the corresponding Atiyah algebroid $E' = TQ/\mathrm{SO}(3)$ over $M = \mathbb{R}^2$. We will identify the tangent bundle to SO(3) with $\mathfrak{so}(3) \times \mathrm{SO}(3)$ by using right translation.

Under this identification between $T(\mathrm{SO}(3))$ and $\mathfrak{so}(3)\times\mathrm{SO}(3)$ the tangent action of SO(3) on $T(\mathrm{SO}(3))\cong\mathfrak{so}(3)\times\mathrm{SO}(3)$ is the trivial action

$$\big(\mathfrak{so}(3)\times\mathrm{SO}(3)\big)\times\mathrm{SO}(3)\to\mathfrak{so}(3)\times\mathrm{SO}(3)\qquad\big((\omega,R),S\big)\mapsto(\omega,RS).\quad(4.12)$$

Thus, the Atiyah algebroid $TQ/\mathrm{SO}(3)$ is isomorphic to the product manifold $T\mathbb{R}^2\times\mathfrak{so}(3)$ and the vector bundle projection is $\tau_{\mathbb{R}^2}\circ pr_1$, where $pr_1: T\mathbb{R}^2\times\mathfrak{so}(3)\to T\mathbb{R}^2$ and $\tau_{\mathbb{R}^2}: T\mathbb{R}^2\to\mathbb{R}^2$ are the canonical projections.

A section of $E'=TQ/\mathrm{SO}(3)\cong T\mathbb{R}^2\times\mathfrak{so}(3)\to\mathbb{R}^2$ is a pair (X,u), where X is a vector field on $\mathbb{R}^2$ and $u:\mathbb{R}^2\to\mathfrak{so}(3)$ is a smooth map. Therefore, a global basis of sections of $T\mathbb{R}^2\times\mathfrak{so}(3)\to\mathbb{R}^2$ is

$$s_1'=\left(\frac{\partial}{\partial x},0\right),\qquad s_2'=\left(\frac{\partial}{\partial y},0\right),\qquad s_3'=(0,E_1),$$
$$s_4'=(0,E_2),\qquad s_5'=(0,E_3).$$

The anchor map $\rho': E'=TQ/\mathrm{SO}(3)\cong T\mathbb{R}^2\times\mathfrak{so}(3)\to T\mathbb{R}^2$ is the projection over the first factor and if $[\![\cdot,\cdot]\!]'$ is the Lie bracket on the space $\mathrm{Sec}(E'=TQ/\mathrm{SO}(3))$ then the only nonzero fundamental Lie brackets are

$$[\![s_3',s_4']\!]'=s_5',\qquad [\![s_4',s_5']\!]'=s_3',\qquad [\![s_5',s_3']\!]'=s_4'.$$

Moreover, the Lagrangian function $L_c=T$ and the constraint functions are SO(3)-invariant. Consequently, L_c induces a Lagrangian function L_c' on $E'=TQ/\mathrm{SO}(3)$

$$L_c'(x,y,\dot{x},\dot{y};\omega)=\frac{1}{2}m\big(\dot{x}^2+\dot{y}^2\big)+\frac{1}{4}I\,\mathrm{Tr}\big(\omega\omega^{\mathrm{T}}\big)=\frac{1}{2}m\big(\dot{x}^2+\dot{y}^2\big)-\frac{1}{4}I\,\mathrm{Tr}\big(\omega^2\big),$$

where $(x,y,\dot{x},\dot{y})$ are the standard coordinates on $T\mathbb{R}^2$ and $\omega\in\mathfrak{so}(3)$. The constraint functions defined on $E'=TQ/\mathrm{SO}(3)$ are

$$\dot{x}+\frac{r}{2}\mathrm{Tr}(\omega E_2)=-\Omega y,\qquad \dot{y}-\frac{r}{2}\mathrm{Tr}(\omega E_1)=\Omega x.\quad(4.13)$$

We have a nonholonomic system on the Atiyah algebroid $E'=TQ/\mathrm{SO}(3)\cong T\mathbb{R}^2\times\mathfrak{so}(3)$. This kind of system was recently analyzed by J. Cortés et al. (2005) (in particular, this example was carefully studied).

Equations (4.13) define an affine subbundle of the vector bundle $E'\cong T\mathbb{R}^2\times\mathfrak{so}(3)\to\mathbb{R}^2$ which is modeled over the vector subbundle $\mathcal{D}_c'$ generated by the sections

$$\mathcal{D}_c'=\big\{s_5',rs_1'+s_4',rs_2'-s_3'\big\}.$$

Our objective is to discretize this example directly on the Atiyah algebroid. The Atiyah groupoid is now identified to $\mathbb{R}^2\times\mathbb{R}^2\times\mathrm{SO}(3)\rightrightarrows\mathbb{R}^2$. We may construct the discrete Lagrangian by

$$L_d'(x_0,y_0,x_1,y_1;W_1)=L_c'\left(x_0,y_0,\frac{x_1-x_0}{h},\frac{y_1-y_0}{h};(\log W_1)/h\right),$$

where $\log : SO(3) \to \mathfrak{so}(3)$ is the (local)-inverse of the exponential map $\exp : \mathfrak{so}(3) \to SO(3)$. For simplicity instead of this procedure we use the following approximation

$$\log W_1/h \approx \frac{W_1 - I_{3\times 3}}{h},$$

where $I_{3\times 3}$ is the identity matrix.

Then

$$\begin{aligned}L'_{\mathrm{d}}(x_0, y_0, x_1, y_1; W_1) &= L'_{\mathrm{c}}\left(x_0, y_0, \frac{x_1 - x_0}{h}, \frac{y_1 - y_0}{h}; \frac{W_1 - I_{3\times 3}}{h}\right) \\ &= \frac{1}{2}m\left[\left(\frac{x_1 - x_0}{h}\right)^2 + \left(\frac{y_1 - y_0}{h}\right)^2\right] + \frac{I}{(2h)^2}\operatorname{Tr}(I_{3\times 3} - W_1).\end{aligned}$$

Eliminating constants, we may consider as discrete Lagrangian

$$L'_{\mathrm{d}} = \frac{1}{2}m\left[\left(\frac{x_1 - x_0}{h}\right)^2 + \left(\frac{y_1 - y_0}{h}\right)^2\right] - \frac{I}{2h^2}\operatorname{Tr}(W_1).$$

The *discrete constraint submanifold* $\mathcal{M}'_{\mathrm{c}}$ of $\mathbb{R}^2 \times \mathbb{R}^2 \times SO(3)$ is determined by the constraints

$$\begin{aligned}\frac{x_1 - x_0}{h} + \frac{r}{2h}\operatorname{Tr}(W_1 E_2) &= -\Omega\frac{y_1 + y_0}{2}, \\ \frac{y_1 - y_0}{h} - \frac{r}{2h}\operatorname{Tr}(W_1 E_1) &= \Omega\frac{x_1 + x_0}{2}.\end{aligned}$$

We have that the system $(L'_{\mathrm{d}}, \mathcal{M}'_{\mathrm{c}}, \mathcal{D}'_{\mathrm{c}})$ is not reversible. Note that the Lagrangian function L'_{d} is reversible. However, the constraint submanifold $\mathcal{M}'_{\mathrm{c}}$ is not reversible.

The discrete nonholonomic Euler–Lagrange equations for the system $(L'_{\mathrm{d}}, \mathcal{M}'_{\mathrm{c}}, \mathcal{D}'_{\mathrm{c}})$ are

$$\begin{aligned}\overleftarrow{s'_5}(x_0, y_0, x_1, y_1; W_1)\big(L'_{\mathrm{d}}\big) - \overrightarrow{s'_5}(x_1, y_1, x_2, y_2; W_2)\big(L'_{\mathrm{d}}\big) &= 0, \\ \overleftarrow{\big(rs'_1 + s'_4\big)}(x_0, y_0, x_1, y_1; W_1)\big(L'_{\mathrm{d}}\big) - \overrightarrow{\big(rs'_1 + s'_4\big)}(x_1, y_1, x_2, y_2; W_2)\big(L'_{\mathrm{d}}\big) &= 0, \\ \overleftarrow{\big(rs'_2 - s'_3\big)}(x_0, y_0, x_1, y_1; W_1)\big(L'_{\mathrm{d}}\big) - \overrightarrow{\big(rs'_2 - s'_3\big)}(x_1, y_1, x_2, y_2; W_2)\big(L'_{\mathrm{d}}\big) &= 0\end{aligned}$$

with the constraints defining $\mathcal{M}_{\mathrm{c}}$.

On the other hand, the vector fields $\overleftarrow{s}'_5$, $\overrightarrow{s}'_5$, $\overleftarrow{rs'_1 + s'_4}$, $\overrightarrow{rs'_1 + s'_4}$, $\overleftarrow{rs'_2 - s'_3}$ and $\overrightarrow{rs'_2 - s'_3}$ on $(\mathbb{R}^2 \times \mathbb{R}^2) \times SO(3)$ are given by

$$\begin{aligned}&\overleftarrow{s}'_5 = \big((0,0), \overleftarrow{E}_3\big), &&\overrightarrow{s}'_5 = \big((0,0), \overrightarrow{E}_3\big), \\ &\overleftarrow{rs'_1 + s'_4} = \left(\left(0, r\frac{\partial}{\partial x}\right), \overleftarrow{E}_2\right), &&\overrightarrow{rs'_1 + s'_4} = \left(\left(-r\frac{\partial}{\partial x}, 0\right), \overrightarrow{E}_2\right),\end{aligned}$$

$$\overleftarrow{rs_2'-s_3'} = \left(\left(0, r\frac{\partial}{\partial y}\right), -\overleftarrow{E}_1\right), \qquad \overrightarrow{rs_2'-s_3'} = \left(\left(-r\frac{\partial}{\partial y}, 0\right), -\overrightarrow{E}_1\right),$$

where $\overleftarrow{E}_i$ (resp., $\overrightarrow{E}_i$) is the left-invariant (resp., right-invariant) vector field on SO(3) induced by $E_i \in \mathfrak{so}(3)$, for $i \in \{1, 2, 3\}$. Thus, we deduce the following system of equations:

$$\begin{aligned}
\operatorname{Tr}\big((W_1 - W_2)E_3\big) &= 0,\\
rm\left(\frac{x_2 - 2x_1 + x_0}{h^2}\right) + \frac{I}{2h^2}\operatorname{Tr}\big((W_1 - W_2)E_2\big) &= 0,\\
rm\left(\frac{y_2 - 2y_1 + y_0}{h^2}\right) - \frac{I}{2h^2}\operatorname{Tr}\big((W_1 - W_2)E_1\big) &= 0,\\
\frac{x_2 - x_1}{h} + \frac{r}{2h}\operatorname{Tr}(W_2E_2) + \Omega\frac{y_2 + y_1}{2} &= 0,\\
\frac{y_2 - y_1}{h} - \frac{r}{2h}\operatorname{Tr}(W_2E_1) - \Omega\frac{x_2 + x_1}{2} &= 0,
\end{aligned}$$

where $(x_0, x_1, y_0, y_1; W_1)$ are known. Simplifying we obtain the following system of equations:

$$\frac{x_2 - 2x_1 + x_0}{h^2} + \frac{I\Omega}{I + mr^2}\frac{y_2 - y_0}{2h} = 0, \tag{4.14}$$

$$\frac{y_2 - 2y_1 + y_0}{h^2} - \frac{I\Omega}{I + mr^2}\frac{x_2 - x_0}{2h} = 0, \tag{4.15}$$

$$\operatorname{Tr}\big((W_1 - W_2)E_3\big) = 0, \tag{4.16}$$

$$\frac{x_2 - x_1}{h} + \frac{r}{2h}\operatorname{Tr}(W_2E_2) + \Omega\frac{y_2 + y_1}{2} = 0, \tag{4.17}$$

$$\frac{y_2 - y_1}{h} - \frac{r}{2h}\operatorname{Tr}(W_2E_1) - \Omega\frac{x_2 + x_1}{2} = 0. \tag{4.18}$$

Now, consider the open subset U of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathrm{SO}(3)$

$$U = \left(\mathbb{R}^2 \times \mathbb{R}^2\right) \times \left\{W \in \mathrm{SO}(3) \;\middle|\; W - \operatorname{Tr}(W)I_{3\times 3} \text{ is regular}\right\}.$$

Then, using Corollary 3.10 (iv), we deduce that the discrete nonholonomic Lagrangian system $(L_d', \mathcal{M}_c', \mathcal{D}_c')$ is regular in the open subset U' of $\mathcal{M}_c'$ given by $U' = U \cap \mathcal{M}_c'$.

If we denote by $u_k = (x_{k+1} - x_k)/h$ and $v_k = (y_{k+1} - y_k)/h$, $k \in \mathbb{N}$, then from (4.14) and (4.15) we deduce that

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix} = A\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \frac{1}{4 + \alpha^2h^2}\begin{pmatrix} 4 - \alpha^2h^2 & -4\alpha h \\ 4\alpha h & 4 - \alpha^2h^2 \end{pmatrix}\begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

or in other terms

$$x_{k+2} = \frac{8x_{k+1} + (\alpha^2h^2 - 4)x_k - 4\alpha h(y_{k+1} - y_k)}{\alpha^2h^2 + 4}$$

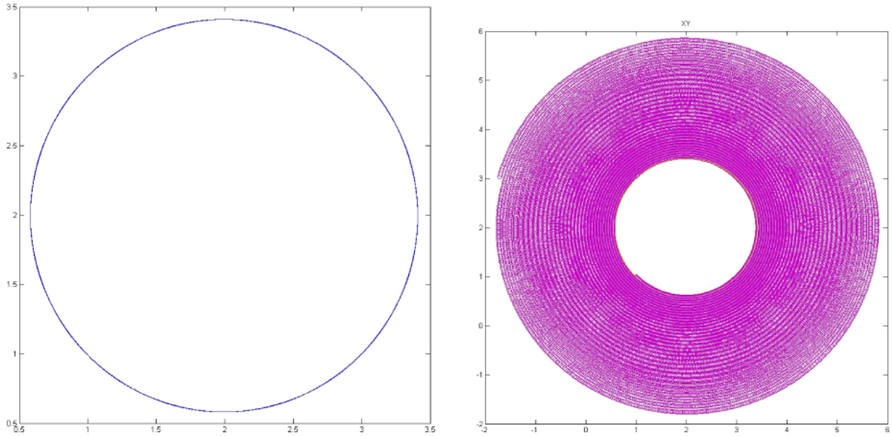


Fig. 2 Orbits for the discrete nonholonomic equations of motion (*left*) and a standard numerical method (*right*) (initial conditions $x_0 = 0.99$, $y_0 = 1$, $x_1 = 1$, $y_1 = 0.99$ and $h = 0.01$ after 20,000 steps)

$$y_{k+2} = \frac{8y_{k+1} + (\alpha^2h^2 - 4)y_k + 4\alpha(x_{k+1} - x_k)}{\alpha^2h^2 + 4},$$

where $\alpha = \frac{I\Omega}{I+mr^2}$. Since $A \in \mathrm{SO}(2)$, the discrete nonholonomic model predicts that the point of contact of the ball will sweep out a circle on the table in agreement with the continuous model. Figure 2 shows the excellent behavior of the proposed numerical method.

4.6 Discrete Chaplygin Systems

Now we present the theory for a particular (but typical) example of discrete nonholonomic systems: *discrete Chaplygin systems*. These kinds of systems were considered in the case of the pair groupoid in Cortés and Martínez (2001).

For any groupoid $\Gamma \rightrightarrows M$, the map $\chi : \Gamma \to M \times M$, $g \mapsto (\alpha(g), \beta(g))$ is a morphism over M from Γ to the pair groupoid $M \times M$ (usually called the *anchor* of Γ). The induced morphism of Lie algebroids is precisely the anchor $\rho : E_\Gamma \to TM$ of E_Γ (the Lie algebroid of Γ).

Definition 4.2 A *discrete Chaplygin system* on the groupoid Γ is a discrete nonholonomic problem $(L_\mathrm{d}, \mathcal{M}_\mathrm{c}, \mathcal{D}_\mathrm{c})$ such that

- $(L_\mathrm{d}, \mathcal{M}_\mathrm{c}, \mathcal{D}_\mathrm{c})$ is a regular discrete nonholonomic Lagrangian system.
- $\chi_{\mathcal{M}_\mathrm{c}} = \chi \circ i_{\mathcal{M}_\mathrm{c}} : \mathcal{M}_\mathrm{c} \to M \times M$ is a diffeomorphism.
- $\rho \circ i_{\mathcal{D}_\mathrm{c}} : \mathcal{D}_\mathrm{c} \to TM$ is an isomorphism of vector bundles.

Denote by $\tilde{L}_\mathrm{d} : M \times M \to \mathbb{R}$ the discrete Lagrangian defined by $\tilde{L}_\mathrm{d} = L_\mathrm{d} \circ i_{\mathcal{M}_\mathrm{c}} \circ (\chi_{\mathcal{M}_\mathrm{c}})^{-1}$.

In the following, we want to express the dynamics on $M \times M$, by finding relations between the dynamics defined by the nonholonomic system on Γ and $M \times M$.

From our hypothesis, for any vector field $Y \in \mathfrak{X}(M)$ there exists a unique section $X \in \mathrm{Sec}(\tau_{\mathcal{D}_c})$ such that $\rho \circ i_{\mathcal{D}_c} \circ X = Y$.

Now, using (2.4), (2.5) and (2.6), it follows that

$$T_g\alpha\big(\overrightarrow{X}(g)\big) = -Y\big(\alpha(g)\big) \quad \text{and} \quad T_g\beta\big(\overleftarrow{X}(g)\big) = Y\big(\beta(g)\big)$$

with some abuse of notation. In other words,

$$\mathcal{T}_g\chi\big(X^{(1,0)}(g)\big) = Y^{(1,0)}\big(\alpha(g),\beta(g)\big) \quad \text{and} \quad \mathcal{T}_g\chi\big(X^{(0,1)}(g)\big) = Y^{(0,1)}\big(\alpha(g),\beta(g)\big)$$

for $g \in \mathcal{M}_c$, where $\mathcal{T}\chi : \mathcal{T}^\Gamma\Gamma \cong V\beta \oplus_\Gamma V\alpha \to \mathcal{T}^{M\times M}(M\times M) \cong T(M\times M)$ is the prolongation of the morphism χ given by

$$(\mathcal{T}_g\chi)(X_g, Y_g) = \big((T_g\alpha)(X_g), (T_g\beta)(Y_g)\big),$$

for $g \in \Gamma$ and $(X_g, Y_g) \in \mathcal{T}_g^\Gamma\Gamma \cong V_g\beta \oplus V_g\alpha$.

Since $\chi_{\mathcal{M}_c}$ is a diffeomorphism, there exists a unique $X'_g \in T_g\mathcal{M}_c$ (resp., $\bar{X}'_g \in T_g\mathcal{M}_c$) such that

$$(\mathcal{T}_g\chi_{\mathcal{M}_c})\big(X'_g\big) = Y^{(1,0)}\big(\alpha(g),\beta(g)\big) = \big(-Y\big(\alpha(g)\big), 0_{\beta(g)}\big)$$

(resp., $(\mathcal{T}_g\chi_{\mathcal{M}_c})(\bar{X}'_g) = Y^{(0,1)}(\alpha(g),\beta(g)) = (0_{\alpha(g)}, Y(\beta(g))))$ for all $g \in \mathcal{M}_c$.

Thus,

$$\begin{aligned} X'_g &\in T_g\mathcal{M}_c \cap V_g\beta, & \overrightarrow{X}(g) - X'_g &= Z'_g \in V_g\alpha \cap V_g\beta,\\ \bar{X}'_g &\in T_g\mathcal{M}_c \cap V_g\alpha, & \overleftarrow{X}(g) - \bar{X}'_g &= \bar{Z}'_g \in V_g\alpha \cap V_g\beta, \end{aligned}$$

for all $g \in \mathcal{M}_c$.

Now, if $(g,h) \in \Gamma_2 \cap (\mathcal{M}_c \times \mathcal{M}_c)$, then

$$\begin{aligned} \overleftarrow{X}(g)(L_d) - \overrightarrow{X}(h)(L_d) &= \bar{X}'_g(L_d) + \bar{Z}'_g(L_d) - X'_h(L_d) - Z'_h(L_d)\\ &= \overleftarrow{Y}\big(\alpha(g),\beta(g)\big)(\tilde{L}_d) - \overrightarrow{Y}\big(\alpha(h),\beta(h)\big)(\tilde{L}_d)\\ &\quad + \bar{Z}'_g(L_d) - Z'_h(L_d). \end{aligned}$$

Therefore, if we use the following notation

$$\begin{aligned} &\big(\alpha(g),\beta(g)\big) = (x,y), & &\big(\alpha(h),\beta(h)\big) = (y,z)\\ &F_Y^+(x,y) = -\bar{Z}'_{\chi_{\mathcal{M}_c}^{-1}(x,y)}(L_d), & &F_Y^-(y,z) = Z'_{\chi_{\mathcal{M}_c}^{-1}(y,z)}(L_d), \end{aligned}$$

then

$$\overleftarrow{X}(g)(L_d) - \overrightarrow{X}(h)(L_d) = \overleftarrow{Y}(x,y)(\tilde{L}_d) - \overrightarrow{Y}(y,z)(\tilde{L}_d) - F_Y^+(x,y) + F_Y^-(y,z).$$

In conclusion, we have proved that (g,h) is a solution of the discrete nonholonomic Euler–Lagrange equations for the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ if and only if $((x,y),(y,z))$

is a solution of the reduced equations

$$\overleftarrow{Y}(x,y)(\tilde{L}_{\mathrm{d}}) - \overrightarrow{Y}(y,z)(\tilde{L}_{\mathrm{d}}) = F_Y^+(x,y) - F_Y^-(y,z), \quad Y \in \mathfrak{X}(M).$$

Note that the above equations are the standard forced discrete Euler–Lagrange equations (see Marsden and West 2001).

4.6.1 The Discrete Two-Wheeled Planar Mobile Robot

We now consider a discrete version of the two-wheeled planar mobile robot (Cortés 2002; Cortés et al. 2005) (see also Kobilarov and Sukhatme 2007). The position and orientation of the robot is determined, with respect a fixed Cartesian reference, by an element $\Omega = (\theta, x, y) \in \mathrm{SE}(2)$, that is, a matrix

$$\Omega = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the different positions of the two wheels are described by elements $(\phi, \psi) \in \mathbb{T}^2$. Therefore, the configuration space is $\mathrm{SE}(2) \times \mathbb{T}^2$. The system is subjected to three nonholonomic constraints: one constraint induced by the condition of no lateral sliding of the robot and the other two by the rolling conditions of both wheels.

It is well known that this system is SE(2)-invariant and that the system may be described as a nonholonomic system on the Lie algebroid $\mathfrak{se}(2) \times T\mathbb{T}^2 \to \mathbb{T}^2$ (see Cortés et al. 2005). In this case, the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}\left(J\omega^2 + m\left(v^1\right)^2 + m\left(v^2\right)^2 + 2m_0 l\omega v^2 + J_2\dot{\phi}^2 + J_2\dot{\psi}^2\right) \\ &= \frac{1}{2}\operatorname{Tr}\left(\xi \mathbb{J} \xi^{\mathrm{T}}\right) + \frac{J_2}{2}\dot{\phi}^2 + \frac{J_2}{2}\dot{\psi}^2, \end{aligned}$$

where

$$\xi = \omega e + v^1 e_1 + v^2 e_2 = \begin{pmatrix} 0 & -\omega & v^1 \\ \omega & 0 & v^2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{J} = \begin{pmatrix} J/2 & 0 & m_0 l \\ 0 & J/2 & 0 \\ m_0 l & 0 & m \end{pmatrix}.$$

Here, $m = m_0 + 2m_1$, where m_0 is the mass of the robot without the two wheels, m_1 the mass of each wheel, J its the moment of inertia with respect to the vertical axis, J_2 the axial moments of inertia of the wheels and l the distance between the center of mass of the robot and the intersection point of the horizontal symmetry axis of the robot and the horizontal line connecting the centers of the two wheels.

The nonholonomic constraints are

$$\begin{aligned} v^1 + \frac{R}{2}\dot{\phi} + \frac{R}{2}\dot{\psi} &= 0, \\ v^2 &= 0, \\ \omega + \frac{R}{2c}\dot{\phi} - \frac{R}{2c}\dot{\psi} &= 0, \end{aligned} \tag{4.19}$$

determining a submanifold $\mathcal{M}$ of $\mathfrak{se}(2)\times T\mathbb{T}^2$, where R is the radius of the two wheels and $2c$ the lateral length of the robot.

In order to discretize the above nonholonomic system, we consider the Atiyah groupoid $\Gamma=\mathrm{SE}(2)\times(\mathbb{T}^2\times\mathbb{T}^2)\rightrightarrows\mathbb{T}^2$. The Lie algebroid of $\mathrm{SE}(2)\times(\mathbb{T}^2\times\mathbb{T}^2)\rightrightarrows\mathbb{T}^2$ is $T\mathbb{T}^2\times\mathfrak{se}(2)\to\mathbb{T}^2$. Then:

– The discrete Lagrangian $L_{\mathrm{d}}:\mathrm{SE}(2)\times(\mathbb{T}^2\times\mathbb{T}^2)\to\mathbb{R}$ is given by

$$\begin{aligned}L_{\mathrm{d}}(\Omega_k,\phi_k,\psi_k,\phi_{k+1},\psi_{k+1})&=\frac{1}{2h^2}\operatorname{Tr}\big((\Omega_k-I_{3\times3})\mathbb{J}(\Omega_k-I_{3\times3})^{\mathrm{T}}\big)\\&\quad+\frac{J_1}{2}\frac{(\Delta\phi_k)^2}{h^2}+\frac{J_1}{2}\frac{(\Delta\psi_k)^2}{h^2},\end{aligned}$$

where $I_{3\times3}$ is the identity matrix, $\Delta\phi_k=\phi_{k+1}-\phi_k$, $\Delta\psi_k=\psi_{k+1}-\psi_k$ and

$$\Omega_k=\begin{pmatrix}\cos\theta_k & -\sin\theta_k & x_k\\ \sin\theta_k & \cos\theta_k & y_k\\ 0 & 0 & 1\end{pmatrix}.$$

We obtain that

$$\begin{aligned}L_{\mathrm{d}}&=\frac{1}{2h^2}\big(mx_k^2+my_k^2-2lm_0x_k(1-\cos\theta_k)\\&\quad+2J(1-\cos\theta_k)+2lm_0y_k\sin\theta_k\big)+\frac{1}{2}J_1\frac{(\Delta\phi_k)^2}{h^2}+\frac{1}{2}J_1\frac{(\Delta\psi_k)^2}{h^2}.\end{aligned}$$

– The constraint vector subbundle of $\mathfrak{se}(2)\times T\mathbb{T}^2$ is generated by the sections

$$\left\{s_1=\frac{R}{2}e_1+\frac{R}{2c}e-\frac{\partial}{\partial\phi},\ s_2=\frac{R}{2}e_1-\frac{R}{2c}e-\frac{\partial}{\partial\psi}\right\}.$$

– The continuous constraints of the two-wheeled planar robot are written in matrix form (see 4.19):

$$\xi=\begin{pmatrix}0 & -\omega & v^1\\ \omega & 0 & v^2\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & \frac{R}{2c}\dot\phi-\frac{R}{2c}\dot\psi & -\frac{R}{2}\dot\phi-\frac{R}{2}\dot\psi\\ -\frac{R}{2c}\dot\phi+\frac{R}{2c}\dot\psi & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

We discretize the previous constraints using the exponential on SE(2) (see Sect. 4.3.2) and discretizing the velocities on the right-hand side

$$\Omega_k=\begin{pmatrix}\cos(\frac{R}{2c}\Delta\phi_k-\frac{R}{2c}\Delta\psi_k) & \sin(\frac{R}{2c}\Delta\phi_k-\frac{R}{2c}\Delta\psi_k) & -c\frac{\Delta\phi_k+\Delta\psi_k}{\Delta\phi_k-\Delta\psi_k}\times\sin(\frac{R}{2c}\Delta\phi_k-\frac{R}{2c}\Delta\psi_k)\\ -\sin(\frac{R}{2c}\Delta\phi_k-\frac{R}{2c}\Delta\psi_k) & \cos(\frac{R}{2c}\Delta\phi_k-\frac{R}{2c}\Delta\psi_k) & c\frac{\Delta\phi_k+\Delta\psi_k}{\Delta\phi_k-\Delta\psi_k}\times(1-\cos(\frac{R}{2c}\Delta\phi_k-\frac{R}{2c}\Delta\psi_k))\\ 0 & 0 & 1\end{pmatrix}$$

if $\Delta\phi_k \neq \Delta\psi_k$, and

$$\Omega_k = \begin{pmatrix} 1 & 0 & -R\Delta\phi_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

if $\Delta\phi_k = \Delta\psi_k$.

Therefore, the constraint submanifold $\mathcal{M}_{\mathrm{c}}$ is defined as

$$\theta_k = -\frac{R}{2c}\Delta\phi_k + \frac{R}{2c}\Delta\psi_k, \tag{4.20}$$

$$x_k = -c\frac{\Delta\phi_k + \Delta\psi_k}{\Delta\phi_k - \Delta\psi_k}\sin\left(\frac{R}{2c}\Delta\phi_k - \frac{R}{2c}\Delta\psi_k\right), \tag{4.21}$$

$$y_k = c\frac{\Delta\phi_k + \Delta\psi_k}{\Delta\phi_k - \Delta\psi_k}\left(1 - \cos\left(\frac{R}{2c}\Delta\phi_k - \frac{R}{2c}\Delta\psi_k\right)\right) \tag{4.22}$$

if $\Delta\phi_k \neq \Delta\psi_k$ and $\theta_k = 0$, $x_k = -R\Delta\phi_k$ and $y_k = 0$ if $\Delta\phi_k = \Delta\psi_k$.

We have that the discrete nonholonomic system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is reversible. Moreover, if $\epsilon_\Gamma : \mathbb{T}^2 \to \mathrm{SE}(2) \times (\mathbb{T}^2 \times \mathbb{T}^2)$ is the identity section of the Lie groupoid $\Gamma = \mathrm{SE}(2) \times (\mathbb{T}^2 \times \mathbb{T}^2)$, then it is clear that

$$\epsilon_\Gamma\left(\mathbb{T}^2\right) = \{I_{3\times 3}\} \times \Delta_{\mathbb{T}^2\times\mathbb{T}^2} \subseteq \mathcal{M}_{\mathrm{c}}.$$

Here, $\Delta_{\mathbb{T}^2\times\mathbb{T}^2}$ is the diagonal in $\mathbb{T}^2 \times \mathbb{T}^2$. In addition, the system $(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is regular in a neighborhood U of the submanifold $\epsilon_\Gamma(\mathbb{T}^2) = \{I_{3\times 3}\} \times \Delta_{\mathbb{T}^2\times\mathbb{T}^2}$ in $\mathcal{M}_{\mathrm{c}}$. Note that

$$T_{(I_{3\times 3},\phi_1,\psi_1,\phi_1,\psi_1)}\mathcal{M}_{\mathrm{c}} \cap E_\Gamma(\phi_1, \psi_1) = \mathcal{D}_{\mathrm{c}}(\phi_1, \psi_1),$$

for $(\phi_1, \psi_1) \in \mathbb{T}^2$, where $E_\Gamma = \mathfrak{se}(2) \times T\mathbb{T}^2$ is the Lie algebroid of the Lie groupoid $\Gamma = \mathrm{SE}(2) \times (\mathbb{T}^2 \times \mathbb{T}^2)$.

On the other hand, it is easy to show that the system $(L_{\mathrm{d}}, U, \mathcal{D}_{\mathrm{c}})$ is a discrete Chaplygin system.

The reduced Lagrangian on $\mathbb{T}^2 \times \mathbb{T}^2$ is

$$\tilde{L}_{\mathrm{d}} = \begin{cases} \frac{1}{h^2}(mc^2(\frac{\Delta\phi_k+\Delta\psi_k}{\Delta\phi_k-\Delta\psi_k})^2(1-\cos(\frac{R}{2c}\Delta\phi_k - \frac{R}{2c}\Delta\psi_k)) & \\ \quad + J(1-\cos(\frac{R}{2c}\Delta\phi_k - \frac{R}{2c}\Delta\psi_k))) & \\ \quad + \frac{1}{2}J_1\frac{(\Delta\phi_k)^2}{h^2} + \frac{1}{2}J_1\frac{(\Delta\psi_k)^2}{h^2}, & \text{if } \Delta\phi_k \neq \Delta\psi_k, \\ (J_1 + \frac{mR^2}{2})\frac{(\Delta\phi_k)^2}{h^2}, & \text{if } \Delta\phi_k = \Delta\psi_k. \end{cases}$$

The discrete nonholonomic equations are

$$\overleftarrow{s_1}|_{(\Omega_1,\phi_1,\psi_1,\phi_2,\psi_2)}(L_{\mathrm{d}}) - \overrightarrow{s_1}|_{(\Omega_2,\phi_2,\psi_2,\phi_3,\psi_3)}(L_{\mathrm{d}}) = 0,$$
$$\overleftarrow{s_2}|_{(\Omega_1,\phi_1,\psi_1,\phi_2,\psi_2)}(L_{\mathrm{d}}) - \overrightarrow{s_2}|_{(\Omega_2\phi_2,\psi_2,\phi_3,\psi_3)}(L_{\mathrm{d}}) = 0.$$

These equations in coordinates are

$$
\begin{aligned}
2J_1(\phi_3 - 2\phi_2 + \phi_1) = {} & lRm_0(\cos\theta_2 + \cos\theta_1) + \frac{JR}{c}(\sin\theta_2 - \sin\theta_1) \\
& - \frac{R\cos\theta_1}{c}(lm_0 y_1 + cmx_1) + \frac{R\sin\theta_1}{c}(lm_0 x_1 - cmy_1) \\
& + \frac{R}{c}\big(cmx_2 + lm_0(y_2 - 2c)\big),
\end{aligned} \tag{4.23}
$$

$$
\begin{aligned}
2J_1(\psi_3 - 2\psi_2 + \psi_1) = {} & lRm_0(\cos\theta_2 + \cos\theta_1) - \frac{JR}{c}(\sin\theta_2 - \sin\theta_1) \\
& + \frac{R\cos\theta_1}{c}(lm_0 y_1 - cmx_1) - \frac{R\sin\theta_1}{c}(lm_0 x_1 + cmy_1) \\
& + \frac{R}{c}\big(cmx_2 - lm_0(y_2 + 2c)\big).
\end{aligned} \tag{4.24}
$$

Substituting constraints (4.20), (4.21) and (4.22) in (4.23) and (4.24) we obtain a set of equations of the type $0 = f_1(\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3)$ and $0 = g_1(\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3)$ which are the reduced equations of the Chaplygin system.

5 Conclusions and Future Work

In this paper we have elucidated the geometrical framework for nonholonomic discrete Mechanics on Lie groupoids. We have developed discrete nonholonomic equations on Lie groupoids that are general enough to produce practical integrators for continuous nonholonomic systems (reduced or not). The geometric properties related with these equations have been completely studied and the applicability of these developments has been stated in several interesting examples.

Of course, much work remains to be done to clarify the nature of discrete nonholonomic mechanics. Many of this future work was stated in McLachlan and Perlmutter (2006) and, in particular, we emphasize the need for the following:

- A complete backward error analysis which explains the very good energy behavior showed in examples or the preservation of a discrete energy (see Fedorov and Zenkov 2005a).
- Related to the previous question, the construction of a discrete exact model for a continuous nonholonomic system (see Iglesias et al. 2007; Marsden and West 2001; McLachlan and Perlmutter 2006).
- To study discrete nonholonomic systems that preserve a volume form on the constraint surface mimicking the continuous case (see, for instance, Fedorov and Jovanovic 2004; Zenkov and Bloch 2003 for this last case).
- To analyze the discrete Hamiltonian framework and the construction of integrators depending on different discretizations.
- And the construction of a discrete nonholonomic connection in the case of Atiyah groupoids (see Leok 2004; Marrero et al. 2006).

Related with some of the previous questions, in the conclusions of the paper by R. McLachlan and M. Perlmutter (2006), the authors raised the question of the possibility of the definition of generalized constraint forces dependent on all the points q_{k-1}, q_k and q_{k+1} (instead of just q_k) for the case of the pair groupoid. We think that the discrete nonholonomic Euler–Lagrange equations can be generalized to consider this case of general constraint forces that, moreover, are closest to the continuous model (see de León et al. 2004; McLachlan and Perlmutter 2006).

Appendix

We will use the same notation as in Sect. 3.2.

Lemma 6.1 *F (resp., $\bar{F}$) is a coisotropic vector subbundle of the symplectic vector bundle $(\mathcal{T}^\Gamma\Gamma, \Omega_{L_\mathrm{d}})$, that is,*

$$F_h^\perp \subseteq F_h, \quad \textit{for every } h \in \Gamma$$

(resp., $\bar{F}_g^\perp \subseteq \bar{F}_g$, for every $g \in \Gamma$), where $F_h^\perp$ (resp., $\bar{F}_g^\perp$) is the symplectic orthogonal of F_h (resp., $\bar{F}_g$) in the symplectic vector space $(\mathcal{T}_h^\Gamma\Gamma, \Omega_{L_\mathrm{d}}(h))$ (resp., $(\mathcal{T}_g^\Gamma\Gamma, \Omega_{L_\mathrm{d}}(g))$).

Proof If $h \in \Gamma$ we have that

$$F_h^\perp = \flat_{\Omega_{L_\mathrm{d}}(h)}^{-1}\big(F_h^0\big),$$

$\flat_{\Omega_{L_\mathrm{d}}(h)} : \mathcal{T}_h^\Gamma\Gamma \to (\mathcal{T}_h^\Gamma\Gamma)^*$ being the canonical isomorphism induced by the symplectic form $\Omega_{L_\mathrm{d}}(h)$. Thus, using (2.14), we deduce that

$$F_h^\perp = \big\{\flat_{\Omega_{L_\mathrm{d}}(h)}^{-1}\big(\gamma_h^{(1,0)}\big) \mid \gamma \in \mathcal{D}_\mathrm{c}\big(\alpha(h)\big)^0\big\} \subseteq \{0\} \oplus V_h\alpha \subseteq F_h.$$

The coisotropic character of $\bar{F}_g$ is proved in a similar way. □

We also have the following result

Lemma 6.2 *Let $\mathcal{T}^\Gamma\mathbb{F}^-L_\mathrm{d} : \mathcal{T}^\Gamma\Gamma \to \mathcal{T}^{E_\Gamma}E_\Gamma^*$ (resp., $\mathcal{T}^\Gamma\mathbb{F}^+L_\mathrm{d} : \mathcal{T}^\Gamma\Gamma \to \mathcal{T}^{E_\Gamma}E_\Gamma^*$) be the prolongation of the Legendre transformation $\mathbb{F}^-L_\mathrm{d} : \Gamma \to E_\Gamma^*$ (resp., $\mathbb{F}^+L_\mathrm{d} : \Gamma \to E_\Gamma^*$). Then,*

$$\begin{aligned}\big(\mathcal{T}_h^\Gamma\mathbb{F}^-L_\mathrm{d}\big)(F_h) &= \mathcal{T}_{\mathbb{F}^-L_\mathrm{d}(h)}^{\mathcal{D}_\mathrm{c}}E_\Gamma^* \\ &= \big\{(v_{\alpha(h)}, X_{\mathbb{F}^-L_\mathrm{d}(h)}) \in \mathcal{T}_{\mathbb{F}^-L_\mathrm{d}(h)}^{E_\Gamma}E_\Gamma^* \mid v_{\alpha(h)} \in \mathcal{D}_\mathrm{c}\big(\alpha(h)\big)\big\},\end{aligned}$$

for $h \in \mathcal{M}_\mathrm{c}$ (resp.,

$$
\begin{aligned}
\big(\mathcal{T}_g^{\Gamma}\mathbb{F}^{+}L_{\mathrm{d}}\big)(\bar{F}_g) &= \mathcal{T}^{\mathcal{D}_{\mathrm{c}}}_{\mathbb{F}^{+}L_{\mathrm{d}}(g)}E^*_\Gamma \\
&= \big\{(v_{\beta(g)}, X_{\mathbb{F}^{+}L_{\mathrm{d}}(g)}) \in \mathcal{T}^{E_\Gamma}_{\mathbb{F}^{+}L_{\mathrm{d}}(g)}E^*_\Gamma \,\big|\, v_{\beta(g)} \in \mathcal{D}_{\mathrm{c}}\big(\beta(g)\big)\big\},
\end{aligned}
$$

for $g \in \mathcal{M}_{\mathrm{c}}$).

Proof It follows using (2.11), (2.18) (resp., (2.12), (2.19)) and Proposition 2.2. Note that $\dim \mathcal{T}^{\mathcal{D}_{\mathrm{c}}}_{\mathbb{F}^{-}L_{\mathrm{d}}(h)}E^*_\Gamma = \dim F_h$ (resp., $\dim \mathcal{T}^{\mathcal{D}_{\mathrm{c}}}_{\mathbb{F}^{+}L_{\mathrm{d}}(g)}E^*_\Gamma = \dim \bar{F}_g$). □

Now, we may prove Theorem 3.6.

Proof of Theorem 3.6 (i) ⇒ (ii) If $h \in \mathcal{M}_{\mathrm{c}}$ and $(X_h, Y_h) \in (\rho^{\mathcal{T}^\Gamma\Gamma})^{-1}(T_h\mathcal{M}_{\mathrm{c}}) \cap F_h^{\perp}$ then, using the fact that $F_h^{\perp} \subseteq \{0\} \oplus V_h\alpha$ (see the proof of Lemma 6.1), we have that $X_h = 0$. Therefore,

$$Y_h \in V_h\alpha \cap T_h\mathcal{M}_{\mathrm{c}}. \tag{6.1}$$

Next, we will see that

$$\big(T_h\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})\big)(Y_h) = 0. \tag{6.2}$$

From (3.4) and (6.1), it follows that $(T_h\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}))(Y_h)$ is vertical with respect to the projection $\tau^*_{\mathcal{D}_{\mathrm{c}}} : \mathcal{D}^*_{\mathrm{c}} \to M$.

Thus, it is sufficient to prove that

$$\Big(\big(T_h\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})\big)(Y_h)\Big)\big(\hat{Z}\big) = 0, \quad \text{for all } Z \in \mathrm{Sec}(\tau_{\mathcal{D}_{\mathrm{c}}}).$$

Here, $\hat{Z} : \mathcal{D}^*_{\mathrm{c}} \to \mathbb{R}$ is the linear function on $\mathcal{D}^*_{\mathrm{c}}$ induced by the section $Z : M \to \mathcal{D}_{\mathrm{c}}$.

Now, using (3.3), we deduce that

$$\Big(\big(T_h\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})\big)(Y_h)\Big)\big(\hat{Z}\big) = \mathrm{d}\big(\hat{Z} \circ i^*_{\mathcal{D}_{\mathrm{c}}}\big)\big(\big(\mathbb{F}^{-}L_{\mathrm{d}}\big)(h)\big)\big(0, \big(T_h\mathbb{F}^{-}L_{\mathrm{d}}\big)(Y_h)\big),$$

where d is the differential of the Lie algebroid $\tau^{\tau^*} : \mathcal{T}^{E_\Gamma}E^*_\Gamma \to E^*_\Gamma$.

Consequently, if $Z^{*\mathbf{c}} : E^*_\Gamma \to \mathcal{T}^{E_\Gamma}E^*_\Gamma$ is the complete lift of $Z \in \mathrm{Sec}(\tau)$, we have that (see (2.10)),

$$
\begin{aligned}
&\Big(\big(T_h\mathbb{F}^{-}(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})\big)(Y_h)\Big)\big(\hat{Z}\big) \\
&\quad = \Omega\big(\mathbb{F}^{-}L_{\mathrm{d}}(h)\big)\big(Z^{*\mathbf{c}}\big(\mathbb{F}^{-}L_{\mathrm{d}}(h)\big), \big(0, \big(T_h\mathbb{F}^{-}L_{\mathrm{d}}\big)(Y_h)\big)\big),
\end{aligned} \tag{6.3}
$$

Ω being the canonical symplectic section associated with the Lie algebroid E_Γ.

On the other hand, since $Z \in \mathrm{Sec}(\tau_{\mathcal{D}_{\mathrm{c}}})$, it follows that $Z^{*\mathbf{c}}(\mathbb{F}^{-}L_{\mathrm{d}}(h))$ is in $\mathcal{T}^{\mathcal{D}_{\mathrm{c}}}_{\mathbb{F}^{-}L_{\mathrm{d}}(h)}E^*_\Gamma$ and, from Lemma 6.2, we conclude that there exists $(X'_h, Y'_h) \in F_h$ such that

$$\big(\mathcal{T}_h^{\Gamma}\mathbb{F}^{-}L_{\mathrm{d}}\big)\big(X'_h, Y'_h\big) = Z^{*\mathbf{c}}\big(\big(\mathbb{F}^{-}L_{\mathrm{d}}\big)(h)\big). \tag{6.4}$$

Moreover, using (2.18), we obtain that

$$\big(\mathcal{T}_h^{\Gamma}\mathbb{F}^{-}L_{\mathrm{d}}\big)(0, Y_h) = \big(0, \big(T_h\mathbb{F}^{-}L_{\mathrm{d}}\big)(Y_h)\big). \tag{6.5}$$

Thus, from (2.21), (6.3), (6.4) and (6.5), we deduce that

$$\big((T_h\mathbb{F}^-(L_\mathrm{d},\mathcal{M},\mathcal{D}_\mathrm{c}))(Y_h)\big)\big(\widehat{Z}\big)=-\Omega_{L_\mathrm{d}}(h)\big((0,Y_h),\big(X'_h,Y'_h\big)\big).$$

Therefore, since $(0,Y_h)\in F_h^{\perp}$, it follows that (6.2) holds, which implies that $Y_h=0$.

This proves that $(\rho^{\mathcal{T}^\Gamma\Gamma})^{-1}(T_h\mathcal{M}_\mathrm{c})\cap F_h^{\perp}=\{0\}$.

If $\mathbb{F}^+(L_\mathrm{d},\mathcal{M}_\mathrm{c},\mathcal{D}_\mathrm{c})$ is a local diffeomorphism then, proceeding as above, we have that $(\rho^{\mathcal{T}^\Gamma\Gamma})^{-1}(T_g\mathcal{M}_\mathrm{c})\cap\bar{F}_g^{\perp}=\{0\}$, for all $g\in\mathcal{M}_\mathrm{c}$.

(ii) $\Rightarrow$ (iii) Assume that $h\in\mathcal{M}_\mathrm{c}$ and that

$$\big(\rho^{\mathcal{T}^\Gamma\Gamma}\big)^{-1}(T_h\mathcal{M}_\mathrm{c})\cap F_h^{\perp}=\{0\}. \tag{6.6}$$

Let U be an open subset of Γ, with $h\in U$, and $\{\phi^\gamma\}_{\gamma=1,\dots,n-r}$ a set of independent real C^∞-functions on U such that

$$\mathcal{M}_\mathrm{c}\cap U=\big\{h'\in U\;\big|\;\phi^\gamma\big(h'\big)=0,\ \text{for all }\gamma\big\}.$$

If d is the differential of the Lie algebroid $\tilde{\tau}_\Gamma:\mathcal{T}^\Gamma\Gamma\to\Gamma$, then it is easy to prove that

$$\big(\rho^{\mathcal{T}^\Gamma\Gamma}\big)^{-1}(T_h\mathcal{M}_\mathrm{c})=\big\langle\big\{\mathrm{d}\phi^\gamma(h)\big\}\big\rangle^0.$$

Thus,

$$\dim\big(\big(\rho^{\mathcal{T}^\Gamma\Gamma}\big)^{-1}(T_h\mathcal{M}_\mathrm{c})\big)\geq n+r. \tag{6.7}$$

On the other hand, $\dim F_h^{\perp}=n-r$. Therefore, from (6.6) and (6.7), we obtain that

$$\dim\big(\big(\rho^{\mathcal{T}^\Gamma\Gamma}\big)^{-1}(T_h\mathcal{M}_\mathrm{c})\big)=n+r,$$

and

$$\mathcal{T}_h^\Gamma\Gamma=\big(\rho^{\mathcal{T}^\Gamma\Gamma}\big)^{-1}(T_h\mathcal{M}_\mathrm{c})\oplus F_h^{\perp}.$$

Consequently, using Lemma 6.1, we deduce that

$$F_h=\mathcal{H}_h\oplus F_h^{\perp}. \tag{6.8}$$

This implies that $\dim\mathcal{H}_h=2r$. Moreover, from (6.8), we also get that

$$\mathcal{H}_h\cap\mathcal{H}_h^{\perp}\subseteq\mathcal{H}_h\cap F_h^{\perp}$$

and, since $\mathcal{H}_h\cap F_h^{\perp}=(\rho^{\mathcal{T}^\Gamma\Gamma})^{-1}(T_h\mathcal{M}_\mathrm{c})\cap F_h^{\perp}$ (see Lemma 6.1), it follows that $\mathcal{H}_h\cap\mathcal{H}_h^{\perp}=\{0\}$.

Thus, we have proved that $\mathcal{H}_h$ is a symplectic subspace of the symplectic vector space $(\mathcal{T}_h^\Gamma\Gamma,\Omega_{L_\mathrm{d}}(h))$.

If $(\rho^{\mathcal{T}^\Gamma\Gamma})^{-1}(T_g\mathcal{M}_\mathrm{c})\cap\bar{F}_g^{\perp}=\{0\}$, for all $g\in\mathcal{M}_\mathrm{c}$ then, proceeding as above, we obtain that $\bar{\mathcal{H}}_g$ is a symplectic subspace of the symplectic vector space $(\mathcal{T}_g^\Gamma\Gamma,\Omega_{L_\mathrm{d}}(g))$, for all $g\in\mathcal{M}_\mathrm{c}$.

(iii) $\Rightarrow$ (iv) Assume that $h \in \mathcal{M}_c$, that $\mathcal{H}_h$ is a symplectic subspace of dimension $2r$ of the symplectic vector space $(\mathcal{T}_h^\Gamma \Gamma, \Omega_{L_d}(h))$ and that $b \in (\overleftarrow{E}_\Gamma)_h^{\mathcal{M}_c}$ satisfies the following condition

$$G_h^{L_d}(a, b) = 0, \quad \forall a \in (\mathcal{D}_c)_{\alpha(h)}.$$

Then, $Y_h = (T_{\epsilon(\beta(h))} l_h)(b) \in T_h\mathcal{M}_c \cap V_h\alpha$ and $(0, Y_h) \in (\rho^{\mathcal{T}^\Gamma \Gamma})^{-1}(T_h\mathcal{M}_c)$.

Moreover, if $(X'_h, Y'_h) \in F_h$, we have that

$$X'_h = -\big(T_{\epsilon(\alpha(h))}(r_h \circ i)\big)(a), \quad \text{with } a \in (\mathcal{D}_c)_{\alpha(h)}.$$

Thus, from (2.14) and (2.22), we deduce that

$$\Omega_{L_d}(h)\big((X'_h, Y'_h), (0, Y_h)\big) = \Omega_{L_d}(h)\big((X'_h, 0), (0, Y_h)\big) = G_h^{L_d}(a, b) = 0.$$

Therefore,

$$(0, Y_h) \in \big(\rho^{\mathcal{T}^\Gamma \Gamma}\big)^{-1}(T_h\mathcal{M}_c) \cap F_h^\perp$$

which, using Lemma 6.1 and the fact that $\mathcal{H}_h \subseteq F_h$, implies that $(0, Y_h) \in \mathcal{H}_h \cap \mathcal{H}_h^\perp = \{0\}$. Consequently, $b = 0$.

If $\bar{\mathcal{H}}_g$ is a symplectic subspace of dimension $2r$ of the symplectic vector space $(\mathcal{T}_g^\Gamma \Gamma, \Omega_{L_d}(g))$, for all $g \in \mathcal{M}_c$ then, proceeding as above, we obtain that

$$\big\{a \in \big(\overrightarrow{E}_\Gamma\big)_g^{\mathcal{M}_c} \mid G_g^{L_d c}(a, b) = 0, \text{ for all } b \in (\mathcal{D}_c)_{\beta(g)}\big\} = \{0\}.$$

(iv) $\Rightarrow$ (i) Suppose that $h \in \mathcal{M}_c$, that

$$\big\{b \in \big(\overleftarrow{E}_\Gamma\big)_h^{\mathcal{M}_c} \mid G_h^{L_d}(a, b) = 0, \ \forall a \in (\mathcal{D}_c)_{\alpha(h)}\big\} = \{0\}$$

and that Y_h is a tangent vector to $\mathcal{M}_c$ at h such that

$$\big(T_h\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c)\big)(Y_h) = 0. \tag{6.9}$$

Then, from (3.4), it follows that $Y_h \in V_h\alpha$. Thus,

$$Y_h = \big(T_{\epsilon(\alpha(h))} l_h\big)(b), \quad \text{with } b \in \big(\overleftarrow{E}_\Gamma\big)_h^{\mathcal{M}_c}.$$

Now, using (2.21) and (2.22), we have that

$$G_h^{L_d}(a, b) = \Omega\big(\mathbb{F}^- L_d(h)\big)\big(\big(0, \big(T_h\mathbb{F}^- L_d\big)(Y_h)\big), \big(a, Y_{\mathbb{F}^- L_d(h)}\big)\big)$$

for $a \in (\mathcal{D}_c)_{\alpha(h)}$, with $Y_{\mathbb{F}^- L_d(h)} \in T_{\mathbb{F}^- L_d(h)}^{E_\Gamma} E_\Gamma^*$ and $(a, Y_{\mathbb{F}^- L_d(h)}) \in \mathcal{T}_{\mathbb{F}^- L_d(h)}^{E_\Gamma} E_\Gamma^*$.

Next, we take a section $Z \in Sec(\tau_{\mathcal{D}_c})$ such that $Z(\alpha(h)) = a$. Then (see (2.9)),

$$(a, Y_{\mathbb{F}^- L_d(h)}) = Z^{*\mathbf{c}}\big(\mathbb{F}^- L_d(h)\big) + \big(0, Y'_{\mathbb{F}^- L_d(h)}\big),$$

where $Y'_{\mathbb{F}^- L_d(h)} \in T_{\mathbb{F}^- L_d(h)} E_\Gamma^*$ and $Y'_{\mathbb{F}^- L_d(h)}$ is vertical with respect to the projection $\tau^* : E_\Gamma^* \to M$.

Therefore, since

$$\Omega\big(\mathbb{F}^- L_{\mathrm{d}}(h)\big)\big(\big(0, \big(T_h\mathbb{F}^- L_{\mathrm{d}}\big)(Y_h)\big), \big(0, Y'_{\mathbb{F}^- L_{\mathrm{d}}(h)}\big)\big) = 0,$$

(see (3.7) in de León et al. 2005), we have that

$$G_h^{L_{\mathrm{d}}}(a,b) = -\Omega\big(\mathbb{F}^- L_{\mathrm{d}}(h)\big)\big(Z^{*\mathbf{c}}\big(\mathbb{F}^- L_{\mathrm{d}}(h)\big), \big(0, \big(T_h\mathbb{F}^- L_{\mathrm{d}}\big)(Y_h)\big)\big)$$

and consequently, from (2.10), (3.3) and (6.9), it follows that

$$\begin{aligned} G_h^{L_{\mathrm{d}}}(a,b) &= -\mathrm{d}\big(\hat{Z} \circ i^*_{\mathcal{D}_{\mathrm{c}}}\big)\big(\mathbb{F}^- L_{\mathrm{d}}(h)\big)\big(0, \big(T_h\mathbb{F}^- L_{\mathrm{d}}\big)(Y_h)\big) \\ &= -\big(T_h\mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})\big)(Y_h)(\hat{Z}) = 0. \end{aligned}$$

This implies that $b = 0$ and $Y_h = 0$. Thus, we have proved that $\mathbb{F}^-(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is a local diffeomorphism.

If

$$\big\{a \in \big(\overrightarrow{E}_\Gamma\big)_g^{\mathcal{M}_{\mathrm{c}}} / G_g^{L_{\mathrm{d}}c}(a,b) = 0, \ \forall b \in (\mathcal{D}_{\mathrm{c}})_{\beta(g)}\big\} = \{0\},$$

for all $g \in \mathcal{M}_{\mathrm{c}}$ then, proceeding as above, we obtain that $\mathbb{F}^+(L_{\mathrm{d}}, \mathcal{M}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}})$ is a local diffeomorphism. □

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