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REGULAR ARTICLE

Hesitant fuzzy programming technique for multidimensional analysis of hesitant fuzzy preferences

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Abstract The linear programming technique for multidimensional analysis of preferences (LINMAP) is the most representative method for handling the multiple criteria decision making (MCDM) problems with respect to the preference information over alternatives. This paper utilizes the main structure of LINMAP to develop a novel hesitant fuzzy mathematical programming technique to handle MCDM problems within the decision environment of hesitant fuzzy elements (HFEs). Considering the hesitancy of the decision maker, both the pair-wise comparison preference information over alternatives and the evaluation information of alternatives with criteria are represented by the HFEs. Based on the incomplete pair-wise preference judgments over alternatives, we propose the concepts of the hesitant fuzzy consistency and inconsistency indices. Furthermore, we construct a hesitant fuzzy mathematical programming model to derive the weights of criteria and the positive-ideal solution. In this hesitant fuzzy programming model both the objective function and partial constraints' coefficients take the form of HFEs, and an effective approach based on the ranking method of HFEs is further developed to solve the new derived model. To address the incom-

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plete and inconsistent preference structures of criteria weights, we introduce several deviation variables and establish the bi-objective nonlinear programming model. At length, we employ a green supplier selection problem to illustrate the feasibility and applicability of the proposed technique and conduct a comparison analysis to validate its effectiveness.

Keywords Hesitant fuzzy information · Multiple criteria decision making · Inconsistency · Hesitant fuzzy programming model

1 Introduction

The linear programming technique for multidimensional analysis of preferences (LINMAP) was initially proposed by Srinivasan and Shocker (1973) for analyzing individual differences in preference judgments with regard to a set of alternatives. In this technique, the weights of decision criteria and the positive-ideal solution are unknown in advance. Thus, in the decision process the experts or decision makers (DMs) are required to provide not only the assessments of alternatives for criteria, but also the pair-wise comparison preference information over alternative. Based on the pair-wise preference information over alternatives, the linear programming optimal model is constructed to determine the weights of criteria and the positive-ideal solution. Furthermore, the best compromise alternative is generated by calculating the weighted Euclidean distance between each alternative and the positive-ideal solution.

The LINMAP is one of the most representative methods for handling the multiple criteria decision making (MCDM) problems with respect to the preference information over given alternatives (Hwang and Yoon 1981). Compared with most of the well-known MCDM methods (such as TOPSIS, ELECTRE) which only require the DM to provide the preference of alternatives with respect to criteria, the key feature of the LINMAP method is also to require the preference information on pair-wise comparisons of the alternatives. Different from the consistent methods based on pair-wise comparison matrix (Zhang et al. 2013; Zhu and Xu 2014) in which the pair-wise comparison information is complete, the LINMAP allows the pair-wise comparison preference information to be incomplete or even non-transitive.

In the classical LINMAP method, all decision data are crisp numbers. However, in the practical decision process it is more and more difficult for the DMs to utilize the crisp numbers to express their assessments because of a lack of knowledge or experience, intangible or non-monetary criteria, or a complex and uncertain environment. Instead, the DMs usually employ the fuzzy set (Zadeh 1965) or its extensions [such as interval numbers, triangular fuzzy number or trapezoid fuzzy number, intuitionistic fuzzy numbers (IFNs), and interval-valued IFNs, etc.] to express their assessments. To this end, some researchers have recently extended the classical LINMAP method into various fuzzy decision environments, such as the decision environment of triangular fuzzy number or trapezoid fuzzy number (Li and Yang 2004; Sadi-Nezhad and Akhtari 2008; Bereketli et al. 2011), the decision context of IFNs (Li et al. 2010), the decision environment of interval-valued IFNs (Chen 2013; Wang and Li 2012), the decision environment of interval type-2 trapezoid fuzzy number (Chen 2015) and the heterogeneous decision context involved multiple formats of information such as real numbers, interval numbers, fuzzy numbers and IFNs (Li and Wan 2013, 2014; Wan and Li 2013).

Although the usefulness and applicability of the LINMAP in the MCDM field have been intensively investigated, notably few attempts have been made to develop an appropriate LINMAP method to deal with the real-world MCDM problem in case of considering the hesitancy of DMs. As an important extension of fuzzy set, hesitant fuzzy set (HFS) which permits the membership degree of an element to a set to be represented as several possible values between 0 and 1, was recently proposed by Torra (2010) to describe the situations in which the DM or the decision organization hesitates among several values to assess an indicator, alternative, variable, etc. Pei and Yi (2015) investigated the properties of operations and algebraic structures of HFSs. HFSs show many advantages over traditional fuzzy sets and its other extensions, especially in multi-expert decision making with anonymity (Xia and Xu 2011). Basic elements in HFSs are HFEs (Xu and Xia 2011a). The HFEs are usually utilized to express assessments of alternatives when evaluating a practical MCDM problem in case of considering the DM's hesitancy. Applications of HFEs have been reported in many MCDM studies (Bedregal et al. 2014; Zhang and Xu 2014, 2015; Farhadinia 2013; Zhang 2013; Xu and Zhang 2013; Zhang and Wei 2013; Wang et al. 2014; etc.). A recent review of articles based on HFSs is presented in Rodríguez et al. (2014).

This paper leverages the classical LINMAP method to develop a novel hesitant fuzzy programming approach to solve the MCDM problem with incomplete weights in which the ratings of alternatives with each criterion are taken as HFEs and the incomplete judgments on pair-wise comparisons of alternatives with hesitant degrees are also represented by HFEs. Obviously, the proposed method combining the LINMAP with HFEs can not only extend the LINMAP, but also address the MCDM problem in case of considering the hesitancy of DMs. In the proposed method, hesitant fuzzy consistency and inconsistency indices are first defined on the basis of the incomplete preference judgments between alternatives. Then, a hesitant fuzzy programming model is constructed based on the idea that the hesitant fuzzy inconsistency index should be minimized and must be not larger than the hesitant fuzzy consistency index by some fixed HFEs. Furthermore, an effective approach is developed to solve the hesitant fuzzy programming model. In addition, we also establish the bi-objective nonlinear programming model to address the incomplete and inconsistent preference structures of criteria weights. At length, we employ a green supplier selection problem to illustrate the feasibility and applicability of the proposed technique and conduct a comparison analysis to validate its effectiveness.

The rest of this paper is organized as follows: Sect. 2 briefly reviews the concepts of HFEs, and meanwhile describes the hesitant fuzzy MCDM problem with incomplete weights. Section 3 proposes the hesitant fuzzy programming method. Section 4 illustrates the feasibility and applicability of the proposed method. Section 5 presents our conclusions.

2 Hesitant fuzzy MCDM problems with incomplete weights

The concept of HFEs is used extensively throughout this paper, and we first recall briefly the basic definitions of HFEs in this section. Then, we use HFEs to formulate a decision environment based on HFEs. Furthermore, we review briefly the structure of incomplete weight information.

2.1 Basic concept of HFEs

Definition 2.1 (Torra 2010) Let X be a reference set, an HFS A on X is defined in terms of a function $h_A(x)$ when applied to X returns a subset of [0, 1]. The mathematical form of the HFS A proposed by Xia and Xu (2011) was defined as follows:

$$A = \{\langle x, h_A(x)\rangle \,|x \in X\}, \tag{2.1}$$

where $h_A(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to A. They also called $h_A(x)$ an HFE denoted by h ($h = \{\gamma^\lambda|\lambda = 1, 2, \ldots, \#h\}$, where $\#h$ is the number of elements in h).

Remark 2.1 Bedregal et al. (2014) regarded the finite and nonempty HFSs as the typical HFSs, and the elements in typical HFSs as the typical HFEs. From now on, this paper will only focus on the typical HFEs. For convenience, we still employ the HFEs to denote the typical HFEs.

Definition 2.2 Given three HFEs represented by h, h_1 and h_2, respectively, and let $\alpha > 0$, then the operation laws of HFEs are defined as follows (Xia and Xu 2011):

$$\alpha h = \cup_{\gamma\in h}\{1 - (1 - \gamma)^\alpha\}, \quad h_1 \otimes h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\gamma_2\},$$
$$h^\alpha = \cup_{\gamma\in h}\{\gamma^\alpha\}, \quad h_1 \oplus h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}.$$

Following the definition of HFEs, we know that the number of values for different HFEs may be different and the values are usually out of order. In the real-world decision process, we should arrange them in any order for convenience. Without loss of generality, we assume that the values of an HFE h are arranged in an increasing order, and let γ^λ be the λth smallest value in h. Thus, for two HFEs h_1 and h_2, and let $\ell = \max\{\#h_1, \#h_2\}$, where $\#h_1$ and $\#h_2$ are, respectively, the numbers of values in the HFEs h_1 and h_2. In order to accurately calculate the distance between h_1 and h_2 with $\#h_1 \neq \#h_2$, we should extend the shorter one until both of them have the same length. Usually, the value which we need to add into the shorter HFE mainly depends on the DM's risk preference (Xu and Xia 2011a; Wang et al. 2014). To this end, Xu and Zhang (2013) developed a method of extension with a parameter η which can identify the adding value according to the DM's risk preference.

Definition 2.3 (Xu and Zhang 2013) For an HFE $h = \{\gamma^\lambda\,|\lambda = 1, 2, \ldots, \#h\}$, let $\gamma^- = \min\{\gamma|\,\gamma \in h\}$ and $\gamma^+ = \max\{\gamma|\,\gamma \in h\}$ be the minimum value and the maximum value in the HFE h, respectively; then $\bar{\gamma} = \eta\gamma^+ + (1 - \eta)\gamma^-$ is called an adding value, where $\eta(0 \leq \eta \leq 1)$ is the parameter determined by the DM according to his/her risk preference.

Apparently, the DMs can add different values to the shorter HFE according to their risk preference. If $\eta = 1$, then the adding value $\bar{\gamma} = \gamma^{+}$, which indicates that the DM is risk-seeking; if $\eta = 0$, then $\bar{\gamma} = \gamma^{-}$, which means that the DM is risk-averse; while if $\eta = \frac{1}{2}$, then $\bar{\gamma} = (\gamma^{+} + \gamma^{-})/2$, which reflects that the DM is risk-neutral. In this study, we assume that the DMs are all risk-averse (other situations can be studied similarly).

The hesitant fuzzy Hamming and Euclidean distances between HFEs developed by Xu and Xia (2011b) were introduced as below:

Definition 2.4 For two HFEs h_1 and h_2, and assume $\#h = \#h_1 = \#h_2$, then the hesitant fuzzy Hamming and Euclidean distances between them can be defined, respectively, as follows:

$$d_1(h_1, h_2) = \frac{1}{\#h} \sum_{\lambda=1}^{\#h} \left|\gamma_{h_1}^{\lambda} - \gamma_{h_2}^{\lambda}\right| \tag{2.2}$$

and

$$d_2(h_1, h_2) = \sqrt{\frac{1}{\#h} \sum_{\lambda=1}^{\#h} (\gamma_{h_1}^{\lambda} - \gamma_{h_2}^{\lambda})^2}. \tag{2.3}$$

To compare the magnitude of HFEs, Xia and Xu (2011) proposed a ranking approach as follows:

Definition 2.5 For an HFE $h = \{\gamma_h^{\lambda} | \lambda = 1, 2, \ldots, \#h\}$, $f(h) = \frac{1}{\#h} \sum_{\lambda=1}^{\#h} \gamma_h^{\lambda}$ is called the score function of h. Given two HFEs h_1 and h_2, we have: (1) if $f(h_1) > f(h_2)$, then $h_1 \succ h_2$; (2) if $f(h_1) = f(h_2)$, then $h_1 = h_2$; (3) if $f(h_1) < f(h_2)$, then $h_1 \prec h_2$.

In case of considering the DM's hesitancy, the HFE is an effective tool to express the evaluation values of alternatives when evaluating a real-life MCDM problem. In the following, a decision context based on HFEs is established for hesitant fuzzy MCDM problem with incomplete weights.

2.2 Decision environment based on HFEs

The MCDM is to identify the desirable compromise solution from the set of all feasible alternatives which are assessed based on a set of conflicting criteria. Let $\boldsymbol{A} = \{A_1, A_2, \ldots, A_m\} (m \geq 2)$ be a discrete set of m feasible alternatives, $\boldsymbol{C} = \{C_1, C_2, \ldots, C_n\}$ be a finite set of criteria. For convenience of description, let $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$. Consider an MCDM problem in which the evaluation information of alternative is expressed by HFEs. Thus, the ratings of the alternative $A_i \in \boldsymbol{A}(i \in M)$ with respect to the criteria $C_j \in \boldsymbol{C}(j \in N)$ can be represented by $h_{ij} = \{\gamma_{h_{ij}}^{1}, \gamma_{h_{ij}}^{2}, \ldots, \gamma_{h_{ij}}^{\#h_{ij}}\}$, and the MCDM problem with HFEs is expressed in the matrix format $H = (h_{ij})_{m \times n}$.

In the practical decision process the criteria set $\boldsymbol{C}$ can be divided into two sets, $\boldsymbol{C}_{\mathrm{I}}$ and $\boldsymbol{C}_{\mathrm{II}}$, where $\boldsymbol{C}_{\mathrm{I}}$ represents a collection of benefit criteria (the larger the better) and $\boldsymbol{C}_{\mathrm{II}}$ denotes a set of cost criteria (the smaller the better). Thus, $\boldsymbol{C}_{\mathrm{I}} \cap \boldsymbol{C}_{\mathrm{II}} = \emptyset$ and $\boldsymbol{C}_{\mathrm{I}} \cup \boldsymbol{C}_{\mathrm{II}} = \boldsymbol{C}$. Let $h_{ij}^{\mathrm{I}} = \{\gamma_{h_{ij}^{\mathrm{I}}}^{1}, \gamma_{h_{ij}^{I}}^{2}, \ldots, \gamma_{h_{ij}^{\mathrm{I}}}^{\#h_{ij}^{\mathrm{I}}}\}$ and $h_{ij}^{\mathrm{II}} = \{\gamma_{h_{ij}^{\mathrm{II}}}^{1}, \gamma_{h_{ij}^{\mathrm{II}}}^{2}, \ldots, \gamma_{h_{ij}^{\mathrm{II}}}^{\#h_{ij}^{\mathrm{II}}}\}$ represent the ratings of the alternative $\boldsymbol{A}_i \in A$ for the criteria $C_j \in \boldsymbol{C}_{\mathrm{I}}$ and $C_j \in \boldsymbol{C}_{\mathrm{II}}$, respectively. Apparently, the dimensions and measurements of criteria values are often different because the types of criteria are different. To eliminate the effect of different physical dimensions and measurements on the final decision results, in this study we transform the criteria values of the cost type into the criteria values of the benefit type by using the following equation:

$$h_{ij} = \begin{cases} h_{ij}^{\mathrm{I}} \ (= \{\gamma_{h_{ij}^{I}}^{1}, \gamma_{h_{ij}^{\mathrm{I}}}^{2}, \ldots, \gamma_{h_{ij}^{\mathrm{I}}}^{\#h_{ij}^{\mathrm{I}}}\}), & \text{for } C_j \in \boldsymbol{C}_{\mathrm{I}} \\ (h_{ij}^{\mathrm{II}})^c (= \{(1 - \gamma_{h_{ij}^{\mathrm{II}}}^{1}), (1 - \gamma_{h_{ij}^{\mathrm{II}}}^{2}), \ldots, (1 - \gamma_{h_{ij}^{\mathrm{II}}}^{\#h_{ij}^{\mathrm{II}}})\}), & \text{for } C_j \in \boldsymbol{C}_{\mathrm{II}} \end{cases} \quad (2.4)$$

As already mentioned in the introduction, the DM in many real-life decision situations may not only provide the ratings of alternatives with respect to each criterion, but also give the incomplete preference information on pair-wise comparisons of alternatives. In case of considering the DM's hesitancy, this study assumes that the preference information between alternatives is given by a set of ordered pairs $\tilde{\Omega} = \{(\xi, \zeta) | A_\xi \succeq_{\tilde{R}(\xi,\zeta)} A_\zeta\}$ with the hesitant fuzzy truth degrees $\tilde{R}(\xi, \zeta)(\xi, \zeta \in M)$, where $\tilde{R}(\xi, \zeta)$ is the preference information indicating the degree to which the DM prefers the alternative A_ξ to A_ζ. The $\tilde{R}(\xi, \zeta)$ is an HFE represented by $\tilde{R}(\xi, \zeta) = \{\gamma_{\tilde{R}(\xi,\zeta)}^{1}, \gamma_{\tilde{R}(\xi,\zeta)}^{2}, \ldots, \gamma_{\tilde{R}(\xi,\zeta)}^{\#\tilde{R}(\xi,\zeta)}\}$ for short, satisfying $0 \leq \gamma_{\tilde{R}(\xi,\zeta)}^{\lambda} \leq 1(\lambda = 1, 2, \ldots, \#\tilde{R}(\xi, \zeta))$.

Remark 2.2 It is easy to see that the pair-wise preference information provided by DMs is expressed by HFE, which is simply called hesitant fuzzy preference information originally developed by Zhu and Xu (2014). The cardinality $|\tilde{\Omega}|$ of $\tilde{\Omega}$, i.e., the number of alternative pairs in $\tilde{\Omega}$, is $\binom{m}{2} = \frac{1}{2}m(m-1)$ when the pair-wise preference information over alternatives is complete. In this study, the pair-wise preference information between alternatives given by the DM is allowed to be incomplete (i.e.,$|\tilde{\Omega}| < \binom{m}{2}$) and/or intransitive, which cannot be solved by the consistent method developed by Zhu and Xu (2014).

2.3 Incomplete weight information structure

In real-life decision process, the weights of criteria should be taken into account. Here we denote the criteria weighting vector by $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$, where w_j is the relative weight of the criterion C_j, satisfying the normalization condition: $\sum_{j=1}^{n} w_j = 1$ and $w_j \geq 0 (j \in N)$. In the classical LINMAP method, Srinivasan and Shocker (1973) pointed out that a solution is said to be trivial if $w_j = 0 (j \in N)$, and at the same time showed that the trivial solution does not convey any useful information so that the solution procedure should preclude it from being optimal. To avoid the trivial solution and without any loss of generality, we here stipulate $w_j \geq \varepsilon_w (j \in N)$, where ε_w is a sufficiently small positive constant and can be chosen by the DM in advance.

Let $\mathbf{\Delta_0}$ denote the set of the criteria weight information, and

$$\mathbf{\Delta_0} = \left\{ (w_1, w_2, \ldots, w_n) \middle| w_j \geq \varepsilon_w (j \in N), \sum\nolimits_{j=1}^{n} w_j = 1 \right\}. \tag{2.5}$$

Owing to the complexity and uncertainty of decision problems and the inherent subjective nature of human thinking, the information about criteria weights provided by the DM in many real decision situations is usually incomplete and has several different structure forms. Many studies (Chen 2014; Park and Kim 1997; Li and Wan 2013) have discussed the structure forms of criteria weights which are roughly divided into the following five forms:

(1) A weak ranking:

$$\mathbf{\Delta_1} = \left\{ (w_1, w_2, \ldots, w_n) \in \mathbf{\Delta_0} \middle| w_{j_1} \geq w_{j_2} \text{ for all } j_1 \in \Lambda_{(1)1} \text{ and } j_2 \in \Lambda_{(2)1} \right\}, \tag{2.6}$$

where $\Lambda_{(1)1}$ and $\Lambda_{(2)1}$ are two disjoint subsets of the subscript index set N of all criteria.

(2) A strict ranking:

$$\mathbf{\Delta_2} = \left\{ (w_1, w_2, \ldots, w_n) \in \mathbf{\Delta_0} \middle| \tau_{j_1 j_2}^{L} \leq w_{j_1} - w_{j_2} \leq \tau_{j_1 j_2}^{U} \right. \\ \left. \text{for all } j_1 \in \Lambda_{(1)2} \text{ and } j_2 \in \Lambda_{(2)2} \right\}, \tag{2.7}$$

where $\tau_{j_1 j_2}^{\mathrm{L}}$ and $\tau_{j_1 j_2}^{\mathrm{U}}$ are the constants that satisfy the condition: $0 < \tau_{j_1 j_2}^{\mathrm{L}} < \tau_{j_1 j_2}^{\mathrm{U}}$, $\Lambda_{(1)2}$ and $\Lambda_{(2)2}$ are two disjoint subsets of N.

(3) A ranking of differences:

$$\mathbf{\Delta_3} = \left\{ (w_1, w_2, \ldots, w_n) \in \mathbf{\Delta_0} \middle| w_{j_1} - w_{j_2} \geq w_{j_3} - w_{j_4} \text{ for all} \right. \\ \left. j_1 \in \Lambda_{(1)3}, j_2 \in \Lambda_{(2)3}, j_3 \in \Lambda_{(3)3} \text{ and } j_4 \in \Lambda_{(4)3} \right\} \tag{2.8}$$

where $\Lambda_{(1)3}$, $\Lambda_{(2)3}$, $\Lambda_{(3)3}$ and $\Lambda_{(4)3}$ are four disjoint subsets of N.

(4) A ranking with multiples:

$$\mathbf{\Delta_4} = \left\{ (w_1, w_2, \ldots, w_n) \in \mathbf{\Delta_0} \middle| w_{j_1} \geq \tau_{j_1 j_2} \cdot w_{j_2} \right. \\ \left. \text{for all } j_1 \in \Lambda_{(1)4} \text{ and } j_2 \in \Lambda_{(2)4} \right\}, \tag{2.9}$$

where $\tau_{j_1 j_2}$ is a constant that satisfies the condition: $\tau_{j_1 j_2} > 0$, $\Lambda_{(1)4}$ and $\Lambda_{(2)4}$ are two disjoint subsets of N.

(5) An interval form:

$$\mathbf{\Delta_5} = \left\{ (w_1, w_2, \ldots, w_n) \in \mathbf{\Delta_0} \middle| \tau_{j_1}^{\mathrm{L}} \leq w_{j_1} \leq \tau_{j_1}^{\mathrm{U}} \text{ for all } j_1 \in \Lambda_{(1)5} \right\} \tag{2.10}$$

where τ_{j1}^{L} and τ_{j1}^{U} are two constants that satisfy the condition: $0 < \tau_{j1}^{\mathrm{L}} < \tau_{j1}^{\mathrm{U}}$, $\Lambda_{(1)5}$ is a subset of N.

In general, the preference information structure of criteria importance may consist of several sets of the above basic sets or may contain all the five basic sets, which depends on the characteristic and need of the real-life decision problems. Let $\boldsymbol{\Delta}$ denote a set of the known information on the criteria weights, then we have

$$\boldsymbol{\Delta} = \boldsymbol{\Delta_1} \cup \boldsymbol{\Delta_2} \cup \boldsymbol{\Delta_3} \cup \boldsymbol{\Delta_4} \cup \boldsymbol{\Delta_5}. \tag{2.11}$$

3 Proposed hesitant fuzzy programming approach

In this section, we utilize the main structure of the LINMAP to develop a new hesitant fuzzy programming approach for handling aforementioned MCDM problems under hesitant fuzzy context. Meanwhile, to address the incomplete and inconsistent preference structures of criteria weights, we introduce several deviation variables and establish a bi-objective nonlinear programming model.

3.1 Hesitant fuzzy consistency and inconsistency indices

In hesitant fuzzy mathematical programming technique the DM for two given alternatives is presumed to prefer that alternative which is "closer" to the positive-ideal alternative. We here denote the hesitant fuzzy positive-ideal solution (HF-PIS) $\boldsymbol{A}^*$ by $\boldsymbol{A}^* = (h_1^*, h_2^*, \ldots, h_n^*)$ which is unknown in advance and needs to be determined, where $h_j^*(j \in N)$ is an HFE expressed as $h_j^* = \{\gamma_{h_j^*}^1, \gamma_{h_j^*}^2, \ldots, \gamma_{h_j^*}^{\#h_j^*}\}$ on the criterion C_j. All HFEs in the decision matrix have the same length (if not, then add some values into the shorter one until they have the same length according to Definition 2.3), thus we stipulate $\#h = \#h_{ij} = \#h_j^*(i \in M, j \in N)$ and $L = \{1, 2, \ldots, \#h\}$.

Then, the square of the hesitant fuzzy Euclidean distance between each alternative $A_i (i \in M)$ and the HF-PIS $\boldsymbol{A}^*$ is calculated by using Eq. (2.3) as:

$$S_i = \sum_{j=1}^{n} w_j d_2(h_{ij}, h_j^*)^2 = \sum_{j=1}^{n} w_j \left(\frac{1}{\#h} \sum_{\lambda=1}^{\#h} (\gamma_{h_{ij}}^{\lambda} - \gamma_{h_j^*}^{\lambda})^2 \right), \quad i \in M \tag{3.1}$$

In a manner similar to the classical LINMAP method, the proposed hesitant fuzzy mathematical programming technique requires the DM to provide not only the ratings of alternatives with respect to each criterion, but also the pair-wise comparison preference information over alternatives when evaluating the MCDM problems. Assume here that the DM provides the pair-wise comparison preference information over alternatives by a set of ordered pairs $\tilde{\Omega} = \{(\xi, \zeta) |\ A_\xi \succeq_{\tilde{R}(\xi,\zeta)} A_\zeta\}$ with the hesitant fuzzy truth degrees $\tilde{R}(\xi, \zeta)(\xi, \zeta \in M)$. By Eq. (2.3), the square of the distances between each pair alternatives $(\xi, \zeta) \in \tilde{\Omega}$ and the HF-PIS $\boldsymbol{A}^*$ can be obtained, respectively, as follows:

$$S_\xi = \sum_{j=1}^{n} w_j d_2(h_{\xi j}, h_j^*)^2 = \sum_{j=1}^{n} w_j \left(\frac{1}{\#h} \sum_{\lambda=1}^{\#h} (\gamma_{h_{\xi j}}^{\lambda} - \gamma_{h_j^*}^{\lambda})^2 \right), \tag{3.2}$$

and

$$S_\zeta = \sum_{j=1}^{n} w_j d_2(h_{\zeta j}, h_j^*)^2 = \sum_{j=1}^{n} w_j \left(\frac{1}{\#h} \sum_{\lambda=1}^{\#h} (\gamma_{h_{\zeta j}}^{\lambda} - \gamma_{h_j^*}^{\lambda})^2 \right). \tag{3.3}$$

For each pair of alternatives $(\xi, \zeta) \in \tilde{\Omega}$,

1. if $S_\xi > S_\zeta$, i.e., the alternative A_ξ is farther from the HF-PIS $\boldsymbol{A}^*$ than the alternative A_ζ, then the ranking order of the alternatives A_ξ and A_ζ determined by S_ξ and S_ζ based on $(\boldsymbol{w}, \boldsymbol{A}^*)$ is inconsistent with the ranking order obtained by the DM's preference relationship in $\tilde{\Omega}$.
2. if $S_\xi \leq S_\zeta$, namely, the alternative A_ξ is closer to the HF-PIS $\boldsymbol{A}^*$ than the alternative A_ζ, then the ranking order of the alternatives A_ξ and A_ζ determined by S_ξ and S_ζ based on $(\boldsymbol{w}, \boldsymbol{A}^*)$ is consistent with the ranking order obtained by the DM's preference relationship in $\tilde{\Omega}$.

Based on the above analysis, a hesitant fuzzy inconsistency index $(S_\zeta - S_\xi)^-$ is developed to measure the degree of inconsistency between the ranking order of the alternatives A_ξ and A_ζ in which one ranking order is determined by S_ξ and S_ζ based on $(\boldsymbol{w}, \boldsymbol{A}^*)$, and the other ranking order obtained by the DM's preference relationship in $\tilde{\Omega}$ as:

$$(S_\zeta - S_\xi)^- = \begin{cases} 0, & (S_\zeta \geq S_\xi) \\ \tilde{R}(\xi, \zeta) \times (S_\xi - S_\zeta), & (S_\zeta < S_\xi) \end{cases} \tag{3.4}$$

The hesitant fuzzy inconsistency index is also expressed in the following form:

$$(S_\zeta - S_\xi)^- = \tilde{R}(\xi, \zeta) \max\{0, (S_\xi - S_\zeta)\}. \tag{3.5}$$

The comprehensive hesitant fuzzy inconsistency index is defined as:

$$\tilde{B} = \sum_{(\xi,\zeta)\in\tilde{\Omega}} (S_\zeta - S_\xi)^- = \sum_{(\xi,\zeta)\in\tilde{\Omega}} \tilde{R}(\xi, \zeta) \max\{0, (S_\xi - S_\zeta)\}. \tag{3.6}$$

On the other hand, a hesitant fuzzy consistency index $(S_\zeta - S_\xi)^+$ is proposed as follows:

$$(S_\zeta - S_\xi)^+ = \begin{cases} \tilde{R}(\xi, \zeta) \times (S_\zeta - S_\xi), & (S_\zeta \geq S_\xi) \\ 0, & (S_\zeta < S_\xi) \end{cases} \tag{3.7}$$

which is utilized to measure the degree of consistency between the ranking order of the alternatives A_ξ and A_ζ in which one ranking order is determined by S_ξ and S_ζ based on $(\boldsymbol{w}, \boldsymbol{A}^*)$, and the other ranking order obtained by the DM's preference relationship in $\tilde{\Omega}$.

Obviously, the hesitant fuzzy consistency index in Eq. (3.7) can be also rewritten as:

$$(S_\zeta - S_\xi)^+ = \tilde{R}(\xi, \zeta) \max\{0, (S_\zeta - S_\xi)\}. \tag{3.8}$$

And the comprehensive hesitant fuzzy consistency index is obtained as below:

$$\tilde{G} = \sum_{(\xi,\zeta)\in\tilde{\Omega}} (S_\zeta - S_\xi)^+ = \sum_{(\xi,\zeta)\in\tilde{\Omega}} \tilde{R}(\xi,\zeta)\max\{0, (S_\zeta - S_\xi)\}. \tag{3.9}$$

3.2 Optimization model

According to the definitions of hesitant fuzzy consistency and inconsistency indices, it is easy to know that in the practical decision process the smaller the hesitant fuzzy inconsistency index is, the better the final decision result is. Meanwhile, the hesitant fuzzy inconsistency index $\tilde{B}$ should be no bigger than the hesitant fuzzy consistency index $\tilde{G}$. We assume the hesitant fuzzy consistency index $\tilde{G}$ is not smaller than the hesitant fuzzy inconsistency index $\tilde{B}$ by ε_h. Here, ε_h is an arbitrary HFE given by the DM in advance.

Correspondingly, we follow the above rule to construct an optimal model in order to determine the weighting vector and the HF-PIS as follows:

$$\begin{aligned}&\min\ \{\tilde{B}\}\\ &\text{s.t.}\ \begin{cases}\tilde{G} - \tilde{B} \geq \varepsilon_h\\ \boldsymbol{w} \in \boldsymbol{\Delta}\end{cases}\end{aligned} \tag{3.10}$$

where $\boldsymbol{\Delta}$ is the preference structure of criteria importance introduced in Sect. 2.3.

Using Eqs. (3.6) and (3.9), it can be easily derived that

$$\begin{aligned}\tilde{G} - \tilde{B} &= \sum_{(\xi,\zeta)\in\tilde{\Omega}} \{(S_\zeta - S_\xi)^+ - (S_\zeta - S_\xi)^-\}\\ &= \sum_{(\xi,\zeta)\in\tilde{\Omega}} (\tilde{R}(\xi,\zeta) \times (S_\zeta - S_\xi)).\end{aligned} \tag{3.11}$$

Combining with Eqs. (3.2) and (3.3), the above Eq. (3.11) can be rewritten as follows:

$$\begin{aligned}\tilde{G} - \tilde{B} = \sum_{(\xi,\zeta)\in\tilde{\Omega}} \Bigg\{\tilde{R}(\xi,\zeta) \times \Bigg\{\sum_{j=1}^{n} w_j\left(\frac{1}{\#h}\sum_{\lambda=1}^{\#h}((\gamma^\lambda_{h_{\zeta j}})^2 - (\gamma^\lambda_{h_{\xi j}})^2)\right)\\ -2\sum_{j=1}^{n} w_j\left(\frac{1}{\#h}\sum_{\lambda=1}^{\#h}\gamma^\lambda_{h_j^*}(\gamma^\lambda_{h_{\xi j}} - \gamma^\lambda_{h_{\zeta j}})\right)\Bigg\}\Bigg\}.\end{aligned} \tag{3.12}$$

Using Eqs. (3.6) and (3.11), the mathematical programming model (3.10) can be rewritten as:

$$
\begin{aligned}
&\min \left\{ \sum_{(\xi,\zeta)\in\tilde{\Omega}} \{\tilde{R}(\xi,\zeta) \times \max\{0, (S_\xi - S_\zeta)\}\} \right\} \\
&\text{s.t.} \begin{cases} \sum_{(\xi,\zeta)\in\tilde{\Omega}} \{\tilde{R}(\xi,\zeta) \times (S_\zeta - S_\xi)\} \geq \varepsilon_h \\ \boldsymbol{w} \in \boldsymbol{\Delta} \end{cases}
\end{aligned} \tag{3.13}
$$

For each pair of alternatives $(\xi, \zeta) \in \tilde{\Omega}$, let $z_{\xi\zeta} = \max\{0, (S_\xi - S_\zeta)\}$, then $z_{\xi\zeta} \geq S_\xi - S_\zeta$, i.e., $z_{\xi\zeta} + S_\zeta - S_\xi \geq 0$ and $z_{\xi\zeta} \geq 0$. Thus, the mathematical programming model (3.13) can be converted into the following model:

$$
\begin{aligned}
&\min \left\{ \sum_{(\xi,\zeta)\in\tilde{\Omega}} z_{\xi\zeta} \tilde{R}(\xi,\zeta) \right\} \\
&\text{s.t.} \begin{cases} \sum_{(\xi,\zeta)\in\tilde{\Omega}} (\tilde{R}(\xi,\zeta) \times (S_\zeta - S_\xi)) \geq \varepsilon_h \\ z_{\xi\zeta} - S_\xi + S_\zeta \geq 0 \quad ((\xi,\zeta) \in \tilde{\Omega}) \\ z_{\xi\zeta} \geq 0 \quad ((\xi,\zeta) \in \tilde{\Omega}) \\ \boldsymbol{w} \in \boldsymbol{\Delta} \end{cases}
\end{aligned} \tag{3.14}
$$

It is easy to notice that in the model (3.14) the objective function is an HFE and the right and left coefficients of the first constraint condition are also HFEs. We call the model (3.14) a hesitant fuzzy mathematical programming model. To our best knowledge, there is no method for solving such a kind of hesitant fuzzy programming models. Therefore, we next develop an effective method for solving the model (3.14).

Let $g_{\zeta\xi} = S_\zeta - S_\xi$ and $\vartheta_j^\lambda = w_j \gamma_{h_j^*}^\lambda$, since $0 \leq \gamma_{h_j^*}^1 \leq \cdots \leq \gamma_{h_j^*}^{\#h} \leq 1 (j \in N)$ and $0 \leq w_j \leq 1 (j \in N)$, thus we obtain $0 \leq \vartheta_j^1 \leq \cdots \leq \vartheta_j^{\#h} \leq w_j (j \in N)$. By Eqs. (3.2) and (3.3), we have

$$
\begin{aligned}
g_{\zeta\xi} &= \sum_{j=1}^{n} w_j \left(\frac{1}{\#h} \sum_{\lambda=1}^{\#h} ((\gamma_{h_{\zeta j}}^\lambda)^2 - (\gamma_{h_{\xi j}}^\lambda)^2) \right) - 2 \sum_{j=1}^{n} w_j \left(\frac{1}{\#h} \sum_{\lambda=1}^{\#h} \gamma_{h_j^*}^\lambda (\gamma_{h_{\zeta j}}^\lambda - \gamma_{h_{\xi j}}^\lambda) \right) \\
&= \sum_{j=1}^{n} w_j \left(\frac{1}{\#h} \sum_{\lambda=1}^{\#h} ((\gamma_{h_{\zeta j}}^\lambda)^2 - (\gamma_{h_{\xi j}}^\lambda)^2) \right) - 2 \sum_{j=1}^{n} \left(\frac{1}{\#h} \sum_{\lambda=1}^{\#h} \vartheta_j^\lambda (\gamma_{h_{\zeta j}}^\lambda - \gamma_{h_{\xi j}}^\lambda) \right)
\end{aligned} \tag{3.15}
$$

Consequently, the model (3.14) is further transformed into the following hesitant fuzzy mathematical programming model:

$$\min \left\{ \sum_{(\xi,\zeta)\in\tilde{\Omega}} \tilde{R}(\xi,\zeta) \times z_{\xi\zeta} \right\}$$
$$\text{s.t.} \begin{cases} \sum_{(\xi,\zeta)\in\tilde{\Omega}} (g_{\zeta\xi} \times \tilde{R}(\xi,\zeta)) \geq \varepsilon_h \\ z_{\xi\zeta} + g_{\zeta\xi} \geq 0 \quad ((\xi,\zeta) \in \tilde{\Omega}) \\ z_{\xi\zeta} \geq 0 \quad ((\xi,\zeta) \in \tilde{\Omega}) \\ 0 \leq \vartheta_j^1 \leq \cdots \leq \vartheta_j^{\#h} \leq w_j \; (j \in N) \\ \boldsymbol{w} \in \boldsymbol{\Delta} \end{cases} \tag{3.16}$$

It is easy to see that in the model (3.16) there exist $(|\tilde{\Omega}| + n + nl)$ variables that need to be determined, including $|\tilde{\Omega}|$ variables $z_{\xi\zeta}((\xi,\zeta) \in \tilde{\Omega})$, n weights of criteria $w_j(j \in N)$, and $n \bullet \#h$ variables of $\vartheta_j^\lambda(j \in N, \lambda \in L)$; and at the same time we have $(|\tilde{\Omega}| + 2nl - n + 1)$ inequalities (excluding the non-negative constraints for the variables and the incomplete weighed information $\boldsymbol{\Delta}$). In order to determine objectively these variables, the number $(|\tilde{\Omega}| + 2nl - n + 1)$ of inequalities should not be very small. In general, the larger the number $(|\tilde{\Omega}|)$ is the more precise and reliable the obtained results $(\boldsymbol{w}, \boldsymbol{A}^*)$ are.

Then, according to the ranking idea of HFEs introduced in Sect. 2, the model (3.16) can be transformed into the following programming model (3.17) as:

$$\min \left\{ \sum_{(\xi,\zeta)\in\tilde{\Omega}} z_{\xi\zeta} f(\tilde{R}(\xi,\zeta)) \right\}$$
$$\text{s.t.} \begin{cases} \sum_{(\xi,\zeta)\in\tilde{\Omega}} g_{\zeta\xi} f(\tilde{R}(\xi,\zeta)) \geq f(\varepsilon_h) \\ z_{\xi\zeta} + g_{\zeta\xi} \geq 0 \quad ((\xi,\zeta) \in \tilde{\Omega}) \\ z_{\xi\zeta} \geq 0 \quad ((\xi,\zeta) \in \tilde{\Omega}) \\ 0 \leq \vartheta_j^1 \leq \cdots \leq \vartheta_j^{\#h} \leq w_j \quad (j \in N) \\ \boldsymbol{w} \in \boldsymbol{\Delta} \end{cases} \tag{3.17}$$

where $f(\tilde{R}(\xi,\zeta))$ and $f(\varepsilon_h)$ are the score functions of $\tilde{R}(\xi,\zeta)$ and ε_h, respectively.

It is easily observed that the model (3.17) is a crisp linear programming model which can be solved by using the Simplex method and needs very low time cost relative to the nonlinear programming model. Then, the solutions of the model (3.17), i.e., the criteria weighting vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ and $\vartheta_j^\lambda(j \in N,\ \lambda \in L)$ can be easily obtained. Based on the derived HF-PIS and weights of criteria, the distances between the alternatives and the HF-PIS are calculated and the best compromise alternative that has the shortest distance to the HF-PIS is obtained.

3.3 Issues that involve inconsistent preference structure of criteria weights

In some practical decision problems, the weight information of criteria provided by the DM may be inconsistent when using the five basic ranking forms (i.e., Eqs. 2.6–2.10) to express them, especially in some complex and uncertain environments. These inconsistent opinions on the importance of criteria may result in no feasible solutions that satisfy all conditions in $\mathbf{\Delta}$. To this end, inspired the idea of Chen (2014) some non-negative deviation variables are presented to relax the conditions in $\mathbf{\Delta}$, and a bi-objective nonlinear programming model is formulated to address the problem with inconsistent weight information.

For convenience, we denote $\mathbf{\Delta}$ as follows:

$$\begin{aligned}
\mathbf{\Delta} = \{&(w_1, w_2, \ldots, w_n) \in \mathbf{\Delta_0} | w_{j_1} \geq w_{j_2} \text{ for all } j_1 \in \Lambda_{(1)1} \text{ and } j_2 \in \Lambda_{(2)1}\\
&w_{j_1} - w_{j_2} \geq \tau^L_{j_1 j_2}, w_{j_1} - w_{j_2} \leq \tau^U_{j_1 j_2} \text{ for all } j_1 \in \Lambda_{(1)2} \text{ and } j_2 \in \Lambda_{(2)2}\\
&w_{j_1} - w_{j_2} - w_{j_3} + w_{j_4} \geq 0 \text{ for all } j_1 \in \Lambda_{(1)3}, j_2 \in \Lambda_{(2)3}, j_3 \in \Lambda_{(3)3}\\
&\text{ and } j_4 \in \Lambda_{(4)3}\\
&w_{j_1}/w_{j_2} \geq \tau_{j_1 j_2} \text{ for all } j_1 \in \Lambda_{(1)4} \text{ and } j_2 \in \Lambda_{(2)4}\\
&w_{j_1} \geq \tau^L_{j_1},\ w_{j_1} \leq \tau^U_{j_1} \text{ for all } j_1 \in \Lambda_{(1)5}\},
\end{aligned}$$

where $j_1 \neq j_2 \neq j_3 \neq j_4$.

Now, we introduce several non-negative deviation variables to relax the conditions in $\mathbf{\Delta}$ if the $\mathbf{\Delta}$ includes the inconsistent weights information of criteria. This relaxed result denoted by $\mathbf{\Delta}^\dagger$ which is presented as follows:

$$\begin{aligned}
\mathbf{\Delta}^\dagger = \{&(w_1, w_2, \ldots, w_n) \in \mathbf{\Delta_0} | w_{j_1} + \varpi^-_{1 j_1 j_2} \geq w_{j_2} \text{ for all } j_1 \in \Lambda_{(1)1} \text{ and } j_2 \in \Lambda_{(2)1}\\
&w_{j_1} - w_{j_2} + \varpi^-_{2 j_1 j_2} \geq \tau^{\mathrm{L}}_{j_1 j_2}, w_{j_1} - w_{j_2} - \varpi^+_{2 j_1 j_2} \leq \tau^{\mathrm{U}}_{j_1 j_2} \text{ for all } j_1 \in \Lambda_{(1)2}\\
&\text{ and } j_2 \in \Lambda_{(2)2}\\
&w_{j_1} - w_{j_2} - w_{j_3} + w_{j_4} + \varpi^-_{3 j_1 j_2} \geq 0 \text{ for all } j_1 \in \Lambda_{(1)3}, j_2 \in \Lambda_{(2)3}, j_3 \in \Lambda_{(3)3}\\
&\text{ and } j_4 \in \Lambda_{(4)3}\\
&w_{j_1}/w_{j_2} + \varpi^-_{4 j_1 j_2} \geq \tau_{j_1 j_2} \text{ for all } j_1 \in \Lambda_{(1)4} \text{ and } j_2 \in \Lambda_{(2)4}\\
&w_{j_1} + \varpi^-_{5 j_1} \geq \tau^L_{j_1},\ w_{j_1} - \varpi^+_{5 j_1} \leq \tau^U_{j_1} \text{ for all } j_1 \in \Lambda_{(1)5}\}.
\end{aligned} \tag{3.18}$$

It is noted that all the deviation values $\varpi^-_{1 j_1 j_2}, \varpi^-_{2 j_1 j_2}, \varpi^+_{2 j_1 j_2}, \varpi^-_{3 j_1 j_2 j_3 j_4}, \varpi^-_{4 j_1 j_2}$, $\varpi^-_{5 j_1}, \varpi^+_{5 j_1} (j_1, j_2, j_3, j_4 \in N)$ are non-negative real numbers and are also unknown in advance. To handle the inconsistent weight information, a bi-objective nonlinear programming model is constructed as below:

$$\min \left\{ \sum_{(\xi,\zeta)\in\tilde{\Omega}} z_{\xi\zeta} f(\tilde{R}(\xi,\zeta)) \right\}$$
$$\min \left\{ \sum_{j_1,j_2,j_3,j_4\in N} (\varpi^-_{1j_1j_2} + \varpi^-_{2j_1j_2} + \varpi^+_{2j_1j_2} + \varpi^-_{3j_1j_2j_3j_4} + \varpi^-_{4j_1j_2} + \varpi^-_{5j_1j_2} + \varpi^+_{5j_1j_2}) \right\}$$
$$\text{s.t.} \begin{cases} \sum_{(\xi,\zeta)\in\tilde{\Omega}} (g_{\zeta\xi} \times f(\tilde{R}(\xi,\zeta))) \geq f(\varepsilon_h) \\ z_{\xi\zeta} + g_{\zeta\xi} \geq 0 \quad ((\xi,\zeta)\in\tilde{\Omega}) \\ z_{\xi\zeta} \geq 0 \quad ((\xi,\zeta)\in\tilde{\Omega}) \\ 0 \leq \vartheta^1_j \leq \cdots \leq \vartheta^{\#h}_j \leq w_j \quad (j\in N) \\ (w_1, w_2, \ldots, w_n) \in \mathbf{\Delta}^\dagger \\ \varpi^-_{1j_1j_2} \geq 0,\ j_1\in\Lambda_{(1)1} \text{ and } j_2\in\Lambda_{(2)1} \\ \varpi^-_{2j_1j_2} \geq 0,\ \varpi^+_{2j_1j_2} \geq 0,\ j_1\in\Lambda_{(1)2} \text{ and } j_2\in\Lambda_{(2)2} \\ \varpi^-_{3j_1j_2j_3j_4} \geq 0,\ j_1\in\Lambda_{(1)3},\ j_2\in\Lambda_{(2)3},\ j_3\in\Lambda_{(3)3} \text{ and } j_4\in\Lambda_{(4)3} \\ \varpi^-_{4j_1j_2} \geq 0,\ j_1\in\Lambda_{(1)4} \text{ and } j_2\in\Lambda_{(2)4} \\ \varpi^-_{5j_1} \geq 0,\ \varpi^+_{5j_1} \geq 0,\ j_1\in\Lambda_{(1)5} \end{cases} \tag{3.19}$$

By using the min–max operator (Chen 2014), the above bi-objective nonlinear programming model is converted into the single-objective programming model as follows:

$$\min \{\chi\}$$
$$\text{s.t.} \begin{cases} \sum_{(\xi,\zeta)\in\tilde{\Omega}} z_{\xi\zeta} f(\tilde{R}(\xi,\zeta)) \leq \chi \\ \sum_{j_1,j_2,j_3,j_4\in N} (\varpi^-_{1j_1j_2} + \varpi^-_{2j_1j_2} + \varpi^+_{2j_1j_2} + \varpi^-_{3j_1j_2j_3j_4} + \varpi^-_{4j_1j_2} + \varpi^-_{5j_1j_2} + \varpi^+_{5j_1j_2}) \leq \chi \\ \sum_{(\xi,\zeta)\in\tilde{\Omega}} (g_{\zeta\xi} \times f(\tilde{R}(\xi,\zeta))) \geq f(\varepsilon_h) \\ z_{\xi\zeta} + g_{\zeta\xi} \geq 0 \quad ((\xi,\zeta)\in\tilde{\Omega}) \\ z_{\xi\zeta} \geq 0 \quad ((\xi,\zeta)\in\tilde{\Omega}) \\ 0 \leq \vartheta^1_j \leq \cdots \leq \vartheta^{\#h}_j \leq w_j \quad (j\in N) \\ (w_1, w_2, \ldots, w_n) \in \mathbf{\Delta}^\dagger \\ \varpi^-_{1j_1j_2} \geq 0,\ j_1\in\Lambda_{(1)1} \text{ and } j_2\in\Lambda_{(2)1} \\ \varpi^-_{2j_1j_2} \geq 0,\ \varpi^+_{2j_1j_2} \geq 0,\ j_1\in\Lambda_{(1)2} \text{ and } j_2\in\Lambda_{(2)2} \\ \varpi^-_{3j_1j_2j_3j_4} \geq 0,\ j_1\in\Lambda_{(1)3},\ j_2\in\Lambda_{(2)3},\ j_3\in\Lambda_{(3)3} \text{ and } j_4\in\Lambda_{(4)3} \\ \varpi^-_{4j_1j_2} \geq 0,\ j_1\in\Lambda_{(1)4} \text{ and } j_2\in\Lambda_{(2)4} \\ \varpi^-_{5j_1} \geq 0,\ \varpi^+_{5j_1} \geq 0,\quad j_1\in\Lambda_{(1)5} \end{cases} \tag{3.20}$$

Apparently, the programming model (3.20) is a crisp linear programming which can be easily solved by using LINGO 11.0. By solving the above model, the optimal weighting vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$, the optimal deviation values $\varpi^-_{1j_1j_2}, \varpi^-_{2j_1j_2}, \varpi^+_{2j_1j_2}, \varpi^-_{3j_1j_2j_3j_4}, \varpi^-_{4j_1j_2}, \varpi^-_{5j_1}, \varpi^+_{5j_1} (j_1, j_2, j_3, j_4 \in N)$ and the HF-PIS $\boldsymbol{A}^* = (h^*_1, h^*_2, \ldots, h^*_n)$ are obtained, respectively. Furthermore, based on the derived HF-PIS and optimal weights, the distances of the alternatives to the HF-PIS are calculated and thus the best compromise alternative that has the shortest distance to the HF-PIS is obtained.

Table 1 The algorithm of the proposed decision method

Step 1	Identify the evaluation criteria and the incomplete weight information structure
Step 2	Express the pair-wise comparison preference information over alternatives with hesitant fuzzy truth degrees represented by $\tilde{\Omega} = \left\{(\xi, \zeta) \middle| A_\xi \succeq_{\tilde{R}(\xi,\zeta)} A_\zeta \right\}$
Step 3	Construct the hesitant fuzzy decision matrix H and obtain the normalized decision matrix H^N by Eq. (2.4)
Step 4	Calculate the hesitant fuzzy consistency and inconsistency indices by Eqs. (3.6) and (3.9), respectively
Step 5	If the weight information of criteria is incomplete and consistent, we construct the hesitant fuzzy programming model according to the model (3.16), and solve it by transforming the derived model into a linear programming model in the sense of model (3.17); if the weight information is incomplete and inconsistent, we establish a bi-objective programming model according to model (3.19), and solve it by transforming this model into a linear programming model in the sense of model (3.20)
Step 6	Get the optimal weight vector $\boldsymbol{w}$ and the HF-PIS $\boldsymbol{A}^*$ through solving model (3.17) or (3.20) by using LINGO 11.0
Step 7	Calculate the relative distances $S_i(i \in M)$ of the alternatives $A_i(i \in M)$ from the HF-PIS $\boldsymbol{A}^*$ using Eq. (2.3)
Step 8	The ranking order of alternatives is generated according to the increasing order of the distances $S_i(i \in M)$ and the best alternative from the alternative set $\boldsymbol{A}$ is determined

3.4 The algorithm of the proposed decision method

Now, we present a practical algorithm of the proposed approach for solving the aforementioned MCDM problem, which can be summarized as in Table 1:

4 Case illustration and discussions

In this section, an MCDM problem involved with the supplier selection problem is presented to demonstrate the applicability and the implementation process of the proposed method.

4.1 Description of the supplier selection problem

With the increase of public awareness of the need to protect the environment, it is urgent for businesses to introduce and promote business practices that help ease the negative impacts of their actions on the environment (Wang and Chan 2013). In the automobile manufacturing industries, the manufacturers want to improve their environmental management practices, not only internally, but also with their suppliers. To this end, the automobile manufacturing company plans to find some environmentally and economically powerful suppliers as strategic partners, with whom the company intends to build long-term collaborative relationships. There are five qualified suppliers which are named, for our purposes, as A_1, A_2, A_3, A_4 and A_5. A decision organization including three experts (from the purchasing department, management department, environment department, respectively) is invited to evaluate these five suppliers and help the company choose an optimal supplier as its strategic partner. The supplier selection criteria have been determined by the decision organization as follows: (1) C_1 is the delivery capability; (2) C_2 is the environmental performance; (3) C_3 is the cost of product; (4) C_4 is the quality of product. It is noted that the criterion C_3 is the cost criterion and others are the benefit criteria. The preference structure of criteria importance is also given as follows:

$$\mathbf{\Delta} = \left\{ (w_1, w_2, w_3, w_4) \in \mathbf{\Delta_0} \left| \begin{array}{l} w_4 \geq w_1, w_2 \geq 2w_1, \ 0.15 \leq w_4 \leq 0.5, \\ 0.05 \leq w_2 - w_3 \leq 0.3, w_4 - w_3 \geq w_2 - w_1 \end{array} \right. \right\}.$$

To get more reasonable evaluation results, in the real-world decision process the experts are required to give their evaluations anonymously. For the green supplier selection problem, the original assessments of suppliers on each criterion provided anonymously by the three experts are listed in Table 2. Although all of the experts provide their evaluation values of alternatives under each criterion, some of these values may be repeated. Considering the decision information provided anonymously by experts, we only collect all of the possible values for an alternative under a criterion, and each value provided only means that it is a possible value, while the times that the values repeated are negligible. Obviously, the HFE is just a tool to deal with such cases, and all possible evaluations for an alternative under each criterion can be considered as an HFE (Xu and Xia 2011a). The collective opinions of the original assessments of suppliers with respect to criteria provided by the decision organization are taken as HFEs, listed in Table 2. For the element {0.4, 0.5, 0.7} in Table 2, it means that the decision organization has hesitancy among 0.4, 0.5 and 0.7 when providing the assessment of the alternative A_1 with respect to C_1, and the others have the similar meanings.

Moreover, the decision organization also provides the HFEs of ordered pairs for the preferences over the alternatives as follows:

$$\tilde{\Omega} = \left\{ \begin{array}{l} \langle (1,2), \tilde{R}(1,2) \rangle, \langle (2,3), \tilde{R}(2,3) \rangle, \langle (2,4), \tilde{R}(2,4) \rangle, \langle (2,5), \tilde{R}(2,5) \rangle, \\ \langle (3,1), \tilde{R}(3,1) \rangle, \langle (3,4), \tilde{R}(3,4) \rangle, \langle (4,5), \tilde{R}(4,5) \rangle \end{array} \right\}$$

where the corresponding hesitant fuzzy truth degrees are listed as follows:

$$\begin{array}{l} \tilde{R}(1,2) = \{0.5, 0.6, 0.7\}, \quad \tilde{R}(2,3) = \{0.6, 0.65, 0.7\}, \\ \tilde{R}(2,4) = \{0.8, 0.85, 0.9\}, \quad \tilde{R}(2,5) = \{0.5, 0.7\}, \\ \tilde{R}(3,1) = \{0.4, 0.5, 0.6\}, \quad \tilde{R}(3,4) = \{0.6, 0.7, 0.95\}, \quad \tilde{R}(4,5) = \{0.7, 0.9\}. \end{array}$$

and $\tilde{R}(1,2) = \{0.5, 0.6, 0.7\}$ means that the decision organization has hesitancy among the values 0.5, 0.6 and 0.7 when providing the degrees to which the alternative A_1 is superior to A_2, and the others have the similar meanings.

Table 2 Ratings of the green suppliers under various criteria provided by DMs

Suppliers	CriterionC_1				Criterion C_2			
	DM1	DM2	DM3	Collective opinion	DM1	DM2	DM3	Collective opinion
A_1	0.5	0.4	0.7	{0.4,0.5,0.7}	0.7	0.7	0.7	{0.7}
A_2	0.5	0.6	0.4	{0.4,0.5,0.6}	0.8	0.7	0.9	{0.7,0.8,0.9}
A_3	0.4	0.4	0.4	{0.4}	0.6	0.9	0.6	{0.6,0.9}
A_4	0.4	0.6	0.3	{0.3, 0.4,0.6}	0.4	0.6	0.5	{0.4,0.5,0.6}
A_5	0.7	0.8	0.6	{0.6,0.7,0.8 }	0.5	0.3	0.4	{0.3,0.4,0.5}
Suppliers	Criterion C_3				Criterion C_4			
	DM1	DM2	DM3	Collective opinion	DM1	DM2	DM3	Collective opinion
A_1	0.5	0.4	0.55	{0.4,0.5,0.55}	0.9	0.8	0.85	{0.8,0.85,0.9}
A_2	0.55	0.4	0.45	{0.4,0.45,0.55}	0.6	0.7	0.7	{0.6,0.7}
A_3	0.3	0.45	0.55	{0.3,0.45,0.55}	0.6	0.8	0.7	{0.6,0.7,0.8}
A_4	0.1	0.1	0.1	{0.1}	0.3	0.25	0.15	{0.15,0.25,0.3}
A_5	0.5	0.6	0.5	{0.5,0.6}	0.7	0.45	0.4	{0.4, 0.45, 0.7}

Table 3 The normalized decision matrix provided by the decision organization

	C_1	C_2	C_3	C_4
A_1	{0.4, 0.5, 0.7}	{0.7, 0.7, 0.7}	{0.45, 0.5, 0.6}	{0.8, 0.85, 0.9}
A_2	{0.4, 0.5, 0.6}	{0.7, 0.8, 0.9}	{0.45, 0.55, 0.6}	{0.6, 0.6, 0.7}
A_3	{0.4, 0.4, 0.4}	{0.6, 0.6, 0.9}	{0.45, 0.55, 0.7}	{0.6, 0.7, 0.8}
A_4	{0.3, 0.4, 0.6}	{0.4, 0.5, 0.6}	{0.9, 0.9, 0.9}	{0.15, 0.25, 0.3}
A_5	{0.6, 0.7, 0.8 }	{0.3, 0.4, 0.5}	{0.4, 0.4, 0.5}	{0.4, 0.45, 0.75}

4.2 Illustration of the proposed approach

In the following, we employ the proposed method to solve the above green supplier evaluation problem. Firstly, we normalize the hesitant fuzzy decision data in Table 2, and the normalized results are listed in Table 3.

Taking the $\varepsilon_h = \{0.01\}$ and $\varepsilon_w = 0.01$, we utilize the model (3.16) to construct the hesitant fuzzy programming model (5.1) which is displayed in Appendix. Then, the derived model (5.1) is converted into the crisp linear programming model based on the ranking approach of HFEs (i.e., using the model (3.17)). By solving the corresponding linear programming model using LINGO 11.0, the components of the optimal solution can be obtained as below:

$$\boldsymbol{w} = (w_1, w_2, w_3, w_4)^{\mathrm{T}} = (0.1217, 0.2434, 0.1934, 0.4415)^{\mathrm{T}},$$
$$\vartheta_1^1 = 0,\ \vartheta_1^2 = 0.0017,\ \vartheta_1^3 = 0.1217,\ \vartheta_2^1 = 0.1921,\ \vartheta_2^2 = 0.2434,\ \vartheta_2^3 = 0.2434,$$
$$\vartheta_3^1 = 0.1594,\ \vartheta_3^2 = 0.1594,\ \vartheta_3^3 = 0.1594,\ \vartheta_4^1 = 0,\ \vartheta_4^2 = 0.3824,\ \vartheta_4^3 = 0.3824.$$

Using the equation $\gamma_{h_j^*}^{\lambda} = \vartheta_j^{\lambda}/w_j$, we can further obtain the HF-PIS $\boldsymbol{A}^*$ as:

$$\boldsymbol{A}^* = (h_1^*, h_2^*, h_3^*, h_4^*) = (\{0, 0.014, 1\}, \{0.7892, 1, 1\}, \{0.8242, 0.8242, 0.8242\}, \{0, 0.8661, 0.8661\}).$$

The distances d_i^* between the alternatives $A_i\,(i = 1, 2, 3, 4, 5)$ and the HF-PIS $\boldsymbol{A}^*$ are calculated by utilizing Eq. (2.3) as below:

$$d_1^* = 0.1484, \quad d_2^* = 0.1162, \quad d_3^* = 0.1164, \quad d_4^* = 0.1692, \quad d_5^* = 0.1852.$$

By comparing the derived distances, we obtain the ranking order of all alternatives as: $A_5 \prec A_4 \prec A_1 \prec A_3 \prec A_2$*, thus the best optimal alternative is the green supplier* A_2.

4.3 Sensitivity analysis

In this section, we conduct a sensitivity analysis by modifying the parameter ε_h. In general, we may increase or decrease the values of the parameter ε_h, and recalculate the optimal weighting vector $\boldsymbol{w}$, the HF-PIS $\boldsymbol{A}^*$ and the ranking orders of alternatives. By comparing the decision results with different values of ε_h, we can make a conclusion whether the decision results are sensitive to the values of the parameter ε_h.

As far as this green supplier selection example is concerned, we change the value of the parameter ε_h from $\{0.0001\}$ to $\{0.1\}$, and obtain the change results of the optimal weighting vector $\boldsymbol{w}$, the HF-PIS $\boldsymbol{A}^*$ and the ranking order of alternatives, listed in Table 4.

From the sensitivity analysis results presented in Table 4, it is noticed that both the optimal weighting vector $\boldsymbol{w}$ and the HF-PIS $\boldsymbol{A}^*$ are not sensitive to the HFE ε_h, and in spite of the alteration in the value of ε_h, the obtained ranking orders of alternatives are usually consistent. In general, the HFE ε_h is given by the DM a priori and ε_h should not be too big which usually takes the value less than {0.1}.

4.4 Discussion of inconsistent weights of criteria

As mentioned previously, the DM might express the inconsistent opinions on the criteria importance when he/she employs the five basic ranking forms introduced in Sect. 2.3. In the above green supplier selection problem, let $\boldsymbol{\Delta} = \boldsymbol{\Delta_1} \cup \boldsymbol{\Delta_2} \cup \boldsymbol{\Delta_3} \cup \boldsymbol{\Delta_4} \cup \boldsymbol{\Delta_5}$ and the five basic ranking forms be expressed as follows, respectively:

$$\begin{aligned}
\boldsymbol{\Delta_1} &= \{(w_1, w_2, w_3, w_4) \in \boldsymbol{\Delta_0} \,|w_4 \geq w_1\},\\
\boldsymbol{\Delta_2} &= \{(w_1, w_2, w_3, w_4) \in \boldsymbol{\Delta_0} \,|0.05 \leq w_2 - w_3 \leq 0.3\},\\
\boldsymbol{\Delta_3} &= \{(w_1, w_2, w_3, w_4) \in \boldsymbol{\Delta_0} \,|w_4 - w_3 \geq w_2 - w_1\},\\
\boldsymbol{\Delta_4} &= \{(w_1, w_2, w_3, w_4) \in \boldsymbol{\Delta_0} \,|w_2 \geq 2w_1\},\\
\boldsymbol{\Delta_5} &= \{(w_1, w_2, w_3, w_4) \in \boldsymbol{\Delta_0} \,|0.15 \leq w_4 \leq 0.5\}.
\end{aligned}$$

Now, we assume that a new condition $w_3 \geq w_2$ is added to the set $\boldsymbol{\Delta_1}$, then the set $\boldsymbol{\Delta}_1$ is updated as follows:

$$\boldsymbol{\Delta_1}^{(\text{new})} = \{(w_1, w_2, w_3, w_4) \in \boldsymbol{\Delta_0} \,|w_4 \geq w_1, w_3 \geq w_2\}$$

and $\boldsymbol{\Delta}$ is updated as follows:

$$\boldsymbol{\Delta}^{(new)} = \left\{(w_1, w_2, w_3, w_4) \in \boldsymbol{\Delta_0} \,\middle|\, \begin{array}{l} w_4 \geq w_1, \quad w_3 \geq w_2,\ 0.05 \leq w_2 - w_3 \leq 0.3, \\ w_4 - w_3 \geq w_2 - w_1,\ w_2 \geq 2w_1, \quad 0.15 \leq w_4 \leq 0.5 \end{array}\right\}.$$

It is easy to see that the condition of $w_3 \geq w_2$ in $\boldsymbol{\Delta}_1^{(\text{new})}$ is conflict with the condition of $0.05 \leq w_2 - w_3 \leq 0.3$ in $\boldsymbol{\Delta_2}$. In other words, the weights in $\boldsymbol{\Delta}^{(\text{new})}$ exist partially inconsistent which result in no feasible solutions that satisfy all conditions in $\boldsymbol{\Delta}^{(\text{new})}$. Thus, we introduce several deviation variables to relax the conditions in $\boldsymbol{\Delta}^{(\text{new})}$ into $\boldsymbol{\Delta}^{\dagger(\text{new})}$, as follows:

Table 4 The sensitivity analysis results with different values of ε_h

ε_h	$\boldsymbol{w} = (w_1, w_2, w_3, w_4)$	$\boldsymbol{A}^* = (h_1^*, h_2^*, h_3^*, h_4^*)$	The ranking results of alternatives
$\{0.0001\}$	$(0.1155, 0.231, 0.181, 0.4725)$	$(\{0, 0.014, 1\}, \{0.7892, 1, 1\}, \{0.8242, 0.8242, 0.8242\}, \{0, 0.8661, 0.8661\})$	$A_5 \prec A_4 \prec A_1 \prec A_3 \prec A_2$
$\{0.001\}$	$(0.1155, 0.231, 0.181, 0.4725)$	$(\{0, 0.014, 1\}, \{0.7892, 1, 1\}, \{0.8242, 0.8242, 0.8242\}, \{0, 0.8661, 0.8661\})$	$A_5 \prec A_4 \prec A_1 \prec A_3 \prec A_2$
$\{0.01\}$	$(0.1217, 0.2434, 0.1934, 0.4415)$	$(\{0, 0.014, 1\}, \{0.7892, 1, 1\}, \{0.8242, 0.8242, 0.8242\}, \{0, 0.8661, 0.8661\})$	$A_5 \prec A_4 \prec A_1 \prec A_3 \prec A_2$
$\{0.02\}$	$(0.1241, 0.2481, 0.1981, 0.4297)$	$(\{0, 0.2119, 1\}, \{0.8759, 0.8759, 1\}, \{0.8445, 0.8445, 0.8445\}, \{0, 0.8797, 0.8797\})$	$A_5 \prec A_4 \prec A_1 \prec A_3 \prec A_2$
$\{0.05\}$	$(0.1216, 0.2432, 0.1932, 0.442)$	$(\{0, 0.6686, 1\}, \{0.757, 0.757, 0.9996\}, \{1, 1, 1\}, \{0.2269, 0.8649, 0.8649\})$	$A_5 \prec A_4 \prec A_1 \prec A_3 \prec A_2$
$\{0.08\}$	$(0.1142, 0.2284, 0.1784, 0.479)$	$(\{0, 0.3835, 0.9378\}, \{1, 1, 1\}, \{1, 1, 1\}, \{0.0635, 0.9344, 0.9344\})$	$A_5 \prec A_4 \prec A_1 \prec A_3 \prec A_2$
$\{0.1\}$	$(0.1156, 0.2312, 0.1812, 0.472)$	$(\{0, 0.4721, 0.7852\}, \{1, 1, 1\}, \{1, 1, 1\}, \{0.1297, 0.9252, 0.9252\})$	$A_5 \prec A_4 \prec A_1 \prec A_3 \prec A_2$

$$\mathbf{\Delta}^{\dagger(\text{new})} = \left\{ (w_1, w_2, \ldots, w_n) \in \mathbf{\Delta}_0 \left| \begin{array}{l} w_4 + \varpi_{141}^- \geq w_1, \quad w_3 + \varpi_{132}^- \geq w_2, \quad w_2 - w_3 \ + \varpi_{223}^- \geq 0.05, \\ w_4 + \varpi_{54}^- \geq 0.15, \quad w_2 - w_3 - \varpi_{223}^+ \leq 0.3, \\ w_4 - w_3 - w_2 + w_1 + \varpi_{34321}^- \geq 0 \, , \\ w_2/w_1 + \varpi_{421}^- \geq 2, \quad w_4 - \varpi_{54}^+ \leq 0.5 \end{array} \right. \right\}$$

where all deviation variables $\varpi_{141}^-, \varpi_{132}^-, \varpi_{223}^-, \varpi_{223}^+, \varpi_{34321}^-, \varpi_{421}^-, \varpi_{54}^+ and\ \varpi_{54}^-$ are non-negative real numbers.

To handle the inconsistent weight information, according to the model (3.19) proposed in Sect. 3.3, we construct a bi-objective nonlinear programming model (5.2) which is displayed in Appendix. Based on the model (3.20) proposed in Sect. 3.3, the model (5.2) can be converted into the following optimal model (5.3) which is also displayed in Appendix.

By solving the model (5.3) using the Simplex method, the components of the optimal solution can be obtained. By combining with the equation $\gamma_{h_j^*}^{\lambda} = \vartheta_j^{\lambda}/w_j$, we can obtain the optimal weight vector $\boldsymbol{w}$ and the HF-PIS $\boldsymbol{A}^*$ as follows:

$$\boldsymbol{w} = (w_1, w_2, w_3, w_4)^{\mathrm{T}} = (0.133, 0.267, 0.233, 0.367)^{\mathrm{T}}, \ \chi = 0.05,$$
$$\boldsymbol{A}^* = (h_1^*, h_2^*, h_3^*, h_4^*) = (\{0, 1, 1\}, \{0, 1, 1\}, \{0.7712, 0.7712, 0.7712\}, \{0.0825, 1, 1\}),$$
$$\varpi_{141}^- = \varpi_{223}^+ = \varpi_{34321}^- = \varpi_{421}^- = \varpi_{54}^+ = \varpi_{54}^- = 0, \ \varpi_{132}^- = 0.0335, \ \varpi_{223}^- = 0.0165.$$

Then, the distances $d_i^*(i = 1, 2, 3, 4, 5)$ of the alternatives $A_i(i = 1, 2, 3, 4, 5)$ to the HF-PIS $\boldsymbol{A}^*$ are calculated by using Eq. (2.3) as below:

$$d_1^* = 0.1658, \ d_2^* = 0.1468, \ d_3^* = 0.1471, \ d_4^* = 0.2109, \ d_5^* = 0.1681.$$

By comparing these relative distances, we can obtain the ranking orders of the alternatives as: $A_4 \prec A_5 \prec A_1 \prec A_3 \prec A_2$. It is easy to see that the ranking order of alternatives determined based on inconsistent weights is remarkably different from that based on the consistent weights. The main reason is that the weight distribution among these four criteria under the inconsistent weight information is different from those using the consistent information.

4.5 Comparison analysis of the obtained results

Recently, Wan and Li (2013) extended the classical LINMAP method for solving the heterogeneous MCDM problems which involve IFNs, trapezoidal fuzzy numbers, intervals and real numbers, and at the same time the DM's preference is given through the pair-wise comparisons of alternatives which are represented as IFNs. In this section, we conduct a comparison with the Wan and Li's method. We use the HFEs' envelopes, i.e., intuitionistic fuzzy data, instead of the hesitant fuzzy data in the above green supplier selection problem, and the hesitant fuzzy data of the above example (i.e., Table 3) is converted into intuitionistic fuzzy data as depicted in Table 5.

Because the above decision problem considers the pair-wise comparisons between alternatives with the hesitant fuzzy truth degree $\tilde{R}(\xi, \zeta)$, to make this comparison more

Table 5 Intuitionistic fuzzy normalized decision matrix

	C_1	C_2	C_3	C_4
A_1	(0.4, 0.3)	(0.7, 0.3)	(0.45, 0.4)	(0.8, 0.1)
A_2	(0.4, 0.4)	(0.7, 0.1)	(0.45, 0.4)	(0.6, 0.3)
A_3	(0.4, 0.6)	(0.6, 0.1)	(0.45, 0.3)	(0.6, 0.2)
A_4	(0.3, 0.4)	(0.4, 0.4)	(0.9, 0.1)	(0.15, 0.7)
A_5	(0.6, 0.2)	(0.3, 0.5)	(0.4, 0.5)	(0.4, 0.25)

fair we also assume that the $\tilde{R}(\xi, \zeta)$ reduces to the HFEs' envelopes, i.e., intuitionistic fuzzy preferences. Thereby, the hesitant fuzzy truth degrees are also transformed into the corresponding intuitionistic fuzzy truth degrees as follows:

$$\tilde{R}(1,2) = (0.5, 0.3),\ \tilde{R}(2,3) = (0.6, 0.3),$$
$$\tilde{R}(2,4) = (0.8, 0.1),\ \tilde{R}(2,5) = (0.5, 0.3),$$
$$\tilde{R}(3,1) = (0.4, 0.4),\quad \tilde{R}(3,4) = (0.6, 0.05),\ \tilde{R}(4,5) = (0.7, 0.1),$$

where $\tilde{R}(1,2) = (0.5, 0.3)$ means that the intensity of which the alternative A_1 is superior to A_2 is 0.5, and the intensity of which the alternative A_1 is inferior to A_2 is 0.3; and the others have the similar meanings.

Then, we utilize the Wan and Li's method to construct the corresponding fuzzy mathematical programming model (5.4) which is displayed in Appendix. Solving the model (5.4) by the method introduced in subsection 4.3 in Li and Wan (2013), the optimal solutions (taking $\varepsilon = (0.01, 0.99)$ and $\omega = 0.5$) are obtained as follows:

$$w_1 = 0.1375,\ \ w_2 = 0.275,\ \ w_3 = 0.225,\ \ w_4 = 0.3625,\ \ u_1 = 0,\ \ u_2 = 0.275,$$
$$u_3 = 0.1811,\ \ u_4 = 0.1343,\ \ v_1 = 0.1357,\ \ v_2 = 0.0,\ \ v_3 = 0.0,\ \ v_4 = 0.1222.$$
$$\boldsymbol{A}^+ = (\langle 0, 1\rangle, \langle 1, 0\rangle, \langle 0.6751, 0\rangle, \langle 0.4651, 0.5349\rangle).$$

Therefore, the squares of the distances of the alternatives $A_i (i = 1, 2, 3, 4, 5)$ from the intuitionistic fuzzy PIS $\boldsymbol{A}^+$ can be calculated as below:

$$d_5^* = 0.2588 > d_1^* = 0.1601 > d_4^* = 0.1598 > d_2^* = 0.1124 > d_3^* = 0.1051.$$

Comparing these distances, the ranking orders of the alternatives $A_i (i = 1, 2, 3, 4, 5)$ for the decision organization are generated as:

$$A_5 \prec A_1 \prec A_4 \prec A_2 \prec A_3.$$

Thus, the best alternative is the green supplier A_3.

To provide a better view of the comparison results, we put the results of the ranking of alternatives obtained by the proposed method and Wan and Li's method into Fig. 1.

From Fig. 1, we know that the ranking order of the alternatives obtained by Wan and Li's method is different from that obtained by the method proposed in this paper. The

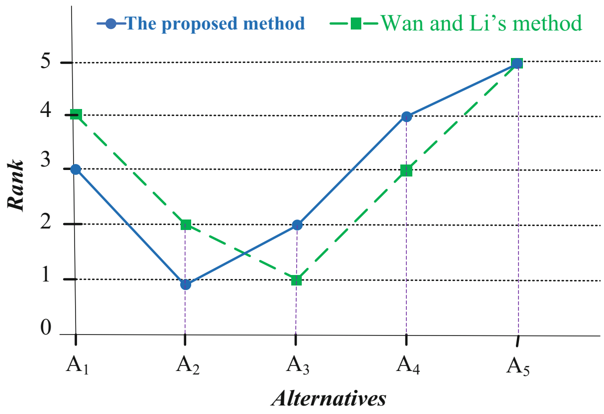

Fig. 1 The pictorial representation of the rankings of alternatives

main reason is that in our proposed method the decision information is expressed by HFEs, while in Wan and Li's method the decision data is represented by IFNs. Using the Wan and Li's method to handle the HFEs decision data, it needs to transform HFEs into IFNs, which gives rise to a difference in the accuracy of data in the two types and further has an effect on the final decision results. Additionally, in the proposed method we have constructed a new hesitant fuzzy mathematical programming model (i.e., the model (3.16)) where both the objective function and the constraints of the model contain the HFEs, to estimate the HF-PIS and the weight vector of criteria. Based on the ranking method of HFEs, we have technically developed an effective method without adding any new parameter for solving this kind of model with HFEs; whereas, in the Wan and Li's method the constructed model (i.e., the model (18) in Li and Wan 2013) is the fuzzy mathematical programming model with IFNs, and the derived model is solved by the weighted average method with a new weighted parameter $\omega \in (0, 1)$. Apparently, compared with Wan and Li's method, our proposed method is capable of better modeling the real-world MCDM problems, especially dealing with the MCDM problems in case of considering the hesitancy of the DM.

On the other hand, it is easily observed that the proposed method is based on the idea that the most preferred alternative is the solution with shortest distance to the hesitant fuzzy positive-ideal solution, which is similar to the TOPSIS method. Nevertheless, the TOPSIS method and the proposed method require different types of preference information and decision conditions. In the TOPSIS approach, the weights of criteria and the positive-ideal solution are required to be known in advance. While in the proposed method, the weights of criteria and the positive-ideal solution are unknown, but the incomplete and/or intransitive pair-wise comparison preference information over alternatives is required to be provided by the DM in advance. Based on the incomplete preference information on paired comparison of alternatives, an optimal model in the proposed method is constructed to determine the weights of criteria and the positive-ideal solution. Apparently, compared with the TOPSIS method, the proposed approach does not require experts to provide the weights of criteria and the positive-ideal solution in advance, but constructs the optimal model to objectively determine the weights and the positive-ideal solution, which avoids the subjective randomness of selecting the weights of criteria and the positive-ideal solution.

5 Conclusions and future research directions

The theory of HFEs has turned out to be an efficient tool in quantifying the ambiguous and vague nature of subjective judgments in the real-world decision process. This paper has developed a hesitant fuzzy programming approach for handling the MCDM problems with incomplete weights in which the ratings of alternatives with each criterion are taken as HFEs and the incomplete judgments on pair-wise comparisons of alternatives with hesitant degrees are also represented by HFEs. Compared with Wan and Li's method, our proposed method is capable of better modeling the real-world MCDM problems, especially dealing with the MCDM problems in case of considering the hesitancy of the DM. This proposed approach makes several contributions to the literature and practices. First, the concept of hesitant fuzzy programming model in which both the objective function and some constraints' coefficients take the form of HFEs has been proposed. Second, an effective approach has been presented to solve the derived model. Third, a bi-objective programming model has been constructed to address the issues of incomplete and inconsistent weights of the criteria. Finally, the real-life green supplier selection problem is introduced to illustrate the feasibility and applicability of the proposed method.

In future studies we will develop the corresponding decision support systems based on the proposed method to solve the real-world decision problems in case of considering the DM's hesitation. We will also focus on some additional experimental studies with different sizes of randomly generated problems and discuss how the components of the optimal weight vector are obtained. Furthermore, the potential of combining the proposed hesitant fuzzy programming method with other useful MCDM techniques within the environment of HFEs will also be taken into consideration in the future.

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Appendix

$$
\min \begin{Bmatrix} \{0.5, 0.6, 0.7\} \otimes z_{12} + \{0.6, 0.65, 0.7\} \otimes z_{23} + \{0.8, 0.85, 0.9\} \otimes z_{24} + \{0.5, 0.7\} \otimes z_{25} \\ + \{0.4, 0.5, 0.6\} \otimes z_{31} + \{0.6, 0.7, 0.95\} \otimes z_{34} + \{0.7, 0.9\} \otimes z_{45} \end{Bmatrix}
$$

$$
\text{s.t.} \begin{cases}
-0.00667w_1 + 0.157w_2 + 0.0175w_3 - 0.321w_4 + 0.0667\vartheta_1^1 - 0.2\vartheta_1^2 - 0.0667\vartheta_2^2 \\
+0.0333\vartheta_2^3 + 0.133\vartheta_4^1 + 0.167\vartheta_4^2 + 0.133\vartheta_4^3 + z_{12} \geq 0 \\
-0.133w_1 - 0.137w_2 + 0.0433w_3 + 0.0933w_4 + 0.133\vartheta_1^1 + 0.133\vartheta_1^2 + 0.133\vartheta_2^2 \\
+0.0667\vartheta_2^3 - 0.0667\vartheta_3^1 - 0.0667\vartheta_4^1 - 0.0667\vartheta_4^2 + z_{23} \geq 0 \\
-0.09w_1 - 0.39w_2 + 0.522w_3 - 0.345w_4 + 0.333\vartheta_1^2 + 0.0667\vartheta_1^3 + 0.2\vartheta_2^2 + 0.2\vartheta_2^3 \\
-0.2\vartheta_3^1 - 0.233\vartheta_3^2 - 0.3\vartheta_3^3 + 0.267\vartheta_4^1 + 0.233\vartheta_4^2 + 0.3\vartheta_4^3 + z_{24} \geq 0 \\
0.203w_1 - 0.48w_2 - 0.0983w_3 - 0.095w_4 - 0.133\vartheta_1^1 + 0.2\vartheta_1^2 - 0.133\vartheta_1^3 + 0.267\vartheta_2^2 \\
+0.267\vartheta_2^3 + 0.0667\vartheta_3^1 + 0.1\vartheta_3^2 + 0.0333\vartheta_3^3 - 0.0333\vartheta_4^1 + 0.1\vartheta_4^2 + 0.133\vartheta_4^3 + z_{25} \geq 0 \\
0.14w_1 - 0.02w_2 - 0.608w_3 + 0.288w_4 - 0.2\vartheta_1^1 - 0.0667\vartheta_1^2 - 0.0667\vartheta_2^2 - 0.0667\vartheta_2^3 \\
+0.0667\vartheta_3^1 + 0.0333\vartheta_3^2 - 0.0667\vartheta_4^1 - 0.1\vartheta_4^2 - 0.133\vartheta_4^3 + z_{31} \geq 0 \\
0.0433w_1 - 0.253w_2 + 0.478w_3 - 0.438w_4 - 0.133\vartheta_1^1 + 0.2\vartheta_1^2 + 0.0667\vartheta_1^3 - 0.0667\vartheta_2^2 \\
+0.133\vartheta_2^3 - 0.133\vartheta_3^1 - 0.233\vartheta_3^2 - 0.3\vartheta_3^3 + 0.333\vartheta_4^1 + 0.3\vartheta_4^2 + 0.3\vartheta_4^3 + z_{34} \geq 0 \\
0.293w_1 - 0.09w_2 - 0.63w_3 + 0.25w_4 - 0.133\vartheta_1^1 - 0.133\vartheta_1^2 + 0.2\vartheta_1^3 + 0.0667\vartheta_2^2 \\
+0.0667\vartheta_2^3 + 0.267\vartheta_3^1 + 0.333\vartheta_3^2 + 0.0333\vartheta_3^3 - 0.3\vartheta_4^1 - 0.133\vartheta_4^2 - 0.167\vartheta_4^3 + z_{45} \geq 0 \\
\{0.5, 0.6, 0.7\} \otimes (-0.00667w_1 + 0.157w_2 + 0.0175w_3 - 0.321w_4 + 0.0667\vartheta_1^1 \\
-0.2\vartheta_1^2 - 0.0667\vartheta_2^2 + 0.0333\vartheta_2^3 + 0.133\vartheta_4^1 + 0.167\vartheta_4^2 + 0.133\vartheta_4^3) \\
+ \{0.6, 0.65, 0.7\} \otimes (-0.133w_1 - 0.137w_2 + 0.0433w_3 + 0.0933w_4 + 0.133\vartheta_1^1 \\
+0.133\vartheta_1^2 + 0.133\vartheta_2^2 + 0.0667\vartheta_2^3 - 0.0667\vartheta_3^1 - 0.0667\vartheta_4^1 - 0.0667\vartheta_4^2) \\
+ \{0.8, 0.85, 0.9\} \otimes (-0.09w_1 - 0.39w_2 + 0.522w_3 - 0.345w_4 + 0.333\vartheta_1^2 + 0.0667\vartheta_1^3 \\
+0.2\vartheta_2^2 + 0.2\vartheta_2^3 - 0.2\vartheta_3^1 - 0.233\vartheta_3^2 - 0.3\vartheta_3^3 + 0.267\vartheta_4^1 + 0.233\vartheta_4^2 + 0.3\vartheta_4^3) \\
+ \{0.5, 0.7\} \otimes (0.203w_1 - 0.48w_2 - 0.0983w_3 - 0.095w_4 - 0.133\vartheta_1^1 + 0.2\vartheta_1^2 - 0.133\vartheta_1^3 \\
+0.267\vartheta_2^2 + 0.267\vartheta_2^3 + 0.0667\vartheta_3^1 + 0.1\vartheta_3^2 + 0.0333\vartheta_3^3 - 0.0333\vartheta_4^1 + 0.1\vartheta_4^2 + 0.133\vartheta_4^3) \\
+ \{0.4, 0.5, 0.6\} \otimes (0.14w_1 - 0.02w_2 - 0.608w_3 + 0.288w_4 - 0.2\vartheta_1^1 - 0.0667\vartheta_1^2 \\
-0.0667\vartheta_2^2 - 0.0667\vartheta_2^3 + 0.0667\vartheta_3^1 + 0.0333\vartheta_3^2 - 0.0667\vartheta_4^1 - 0.1\vartheta_4^2 - 0.133\vartheta_4^3) \\
+ \{0.6, 0.7, 0.95\} \otimes (0.0433w_1 - 0.253w_2 + 0.478w_3 - 0.438w_4 - 0.133\vartheta_1^1 + 0.2\vartheta_1^2 + 0.0667\vartheta_1^3 \\
-0.0667\vartheta_2^2 + 0.133\vartheta_2^3 - 0.133\vartheta_3^1 - 0.233\vartheta_3^2 - 0.3\vartheta_3^3 + 0.333\vartheta_4^1 + 0.3\vartheta_4^2 + 0.3\vartheta_4^3) \\
+ \{0.7, 0.9\} \otimes (0.293w_1 - 0.09w_2 - 0.63w_3 + 0.25w_4 - 0.133\vartheta_1^1 - 0.133\vartheta_1^2 + 0.2\vartheta_1^3 + 0.0667\vartheta_2^2 \\
+0.0667\vartheta_2^3 + 0.267\vartheta_3^1 + 0.333\vartheta_3^2 + 0.0333\vartheta_3^3 - 0.3\vartheta_4^1 - 0.133\vartheta_4^2 - 0.167\vartheta_4^3) \succeq \{0.01\} \\
w_4 \geq w_1,\ 0.05 \leq w_2 - w_2 \leq 0.3,\ \ w_4 - w_3 \geq w_2 - w_1,\ w_2 \geq 2w_1,\ 0.15 \leq w_4 \leq 0.5 \\
w_1 + w_2 + w_3 + w_4 = 1,\ 0.01 \leq w_j \leq 1 \quad (j = 1, 2, 3, 4) \\
z_{12}, z_{23}, z_{24}, z_{31}, z_{34}, z_{25}, z_{45} \geq 0 \\
0 \leq \vartheta_j^1 \leq \vartheta_j^2 \leq \vartheta_j^3 \leq w_j \ (j = 1, 2, 3, 4)
\end{cases} \tag{5.1}
$$

$$
\begin{aligned}
&\min \left\{ \begin{array}{l} \{0.5, 0.6, 0.7\} \otimes z_{12} + \{0.6, 0.65, 0.7\} \otimes z_{23} + \{0.8, 0.85, 0.9\} \otimes z_{24} + \{0.5, 0.7\} \otimes z_{25} \\ + \{0.4, 0.5, 0.6\} \otimes z_{31} + \{0.6, 0.7, 0.95\} \otimes z_{34} + \{0.7, 0.9\} \otimes z_{45} \end{array} \right\} \\
&\min \left\{ \varpi_{141}^- + \varpi_{132}^- + \varpi_{223}^- + \varpi_{223}^+ + \varpi_{34321}^- + \varpi_{421}^- + \varpi_{54}^+ + \varpi_{54}^- \right\} \\
&\text{s.t.} \left\{ \begin{array}{l}
-0.00667 w_1 + 0.157 w_2 + 0.0175 w_3 - 0.321 w_4 + 0.0667 \vartheta_1^1 - 0.2 \vartheta_1^2 - 0.0667 \vartheta_2^2 \\
+0.0333 \vartheta_2^3 + 0.133 \vartheta_4^1 + 0.167 \vartheta_4^2 + 0.133 \vartheta_4^3 + z_{12} \geq 0 \\
-0.133 w_1 - 0.137 w_2 + 0.0433 w_3 + 0.0933 w_4 + 0.133 \vartheta_1^1 + 0.133 \vartheta_1^2 + 0.133 \vartheta_2^2 + 0.0667 \vartheta_2^3 \\
-0.0667 \vartheta_3^1 - 0.0667 \vartheta_4^1 - 0.0667 \vartheta_4^2 + z_{23} \geq 0 \\
-0.09 w_1 - 0.39 w_2 + 0.522 w_3 - 0.345 w_4 + 0.333 \vartheta_1^2 + 0.0667 \vartheta_1^3 + 0.2 \vartheta_2^2 + 0.2 \vartheta_2^3 \\
-0.2 \vartheta_3^1 - 0.233 \vartheta_3^2 - 0.3 \vartheta_3^3 + 0.267 \vartheta_4^1 + 0.233 \vartheta_4^2 + 0.3 \vartheta_4^3 + z_{24} \geq 0 \\
0.203 w_1 - 0.48 w_2 - 0.0983 w_3 - 0.095 w_4 - 0.133 \vartheta_1^1 + 0.2 \vartheta_1^2 - 0.133 \vartheta_1^3 + 0.267 \vartheta_2^2 + 0.267 \vartheta_2^3 \\
+0.0667 \vartheta_3^1 + 0.1 \vartheta_3^2 + 0.0333 \vartheta_3^3 - 0.0333 \vartheta_4^1 + 0.1 \vartheta_4^2 + 0.133 \vartheta_4^3 + z_{25} \geq 0 \\
0.14 w_1 - 0.02 w_2 - 0.608 w_3 + 0.288 w_4 - 0.2 \vartheta_1^1 - 0.0667 \vartheta_1^2 - 0.0667 \vartheta_2^2 - 0.0667 \vartheta_2^3 \\
+0.0667 \vartheta_3^1 + 0.0333 \vartheta_3^2 - 0.0667 \vartheta_4^1 - 0.1 \vartheta_4^2 - 0.133 \vartheta_4^3 + z_{31} \geq 0 \\
0.0433 w_1 - 0.253 w_2 + 0.478 w_3 - 0.438 w_4 - 0.133 \vartheta_1^1 + 0.2 \vartheta_1^2 + 0.0667 \vartheta_1^3 - 0.0667 \vartheta_2^2 \\
+0.133 \vartheta_2^3 - 0.133 \vartheta_3^1 - 0.233 \vartheta_3^2 - 0.3 \vartheta_3^3 + 0.333 \vartheta_4^1 + 0.3 \vartheta_4^2 + 0.3 \vartheta_4^3 + z_{34} \geq 0 \\
0.293 w_1 - 0.09 w_2 - 0.63 w_3 + 0.25 w_4 - 0.133 \vartheta_1^1 - 0.133 \vartheta_1^2 + 0.2 \vartheta_1^3 + 0.0667 \vartheta_2^2 + 0.0667 \vartheta_2^3 \\
+0.267 \vartheta_3^1 + 0.333 \vartheta_3^2 + 0.0333 \vartheta_3^3 - 0.3 \vartheta_4^1 - 0.133 \vartheta_4^2 - 0.167 \vartheta_4^3 + z_{45} \geq 0 \\
\{0.5, 0.6, 0.7\} \otimes (-0.00667 w_1 + 0.157 w_2 + 0.0175 w_3 - 0.321 w_4 + 0.0667 \vartheta_1^1 - 0.2 \vartheta_1^2 \\
-0.0667 \vartheta_2^2 + 0.0333 \vartheta_2^3 + 0.133 \vartheta_4^1 + 0.167 \vartheta_4^2 + 0.133 \vartheta_4^3) \\
+ \{0.6, 0.65, 0.7\} \otimes (-0.133 w_1 - 0.137 w_2 + 0.0433 w_3 + 0.0933 w_4 + 0.133 \vartheta_1^1 \\
+0.133 \vartheta_1^2 + 0.133 \vartheta_2^2 + 0.0667 \vartheta_2^3 - 0.0667 \vartheta_3^1 - 0.0667 \vartheta_4^1 - 0.0667 \vartheta_4^2) \\
+ \{0.8, 0.85, 0.9\} \otimes (-0.09 w_1 - 0.39 w_2 + 0.522 w_3 - 0.345 w_4 + 0.333 \vartheta_1^2 + 0.0667 \vartheta_1^3 \\
+0.2 \vartheta_2^2 + 0.2 \vartheta_2^3 - 0.2 \vartheta_3^1 - 0.233 \vartheta_3^2 - 0.3 \vartheta_3^3 + 0.267 \vartheta_4^1 + 0.233 \vartheta_4^2 + 0.3 \vartheta_4^3) \\
+ \{0.5, 0.7\} \otimes (0.203 w_1 - 0.48 w_2 - 0.0983 w_3 - 0.095 w_4 - 0.133 \vartheta_1^1 + 0.2 \vartheta_1^2 - 0.133 \vartheta_1^3 \\
+0.267 \vartheta_2^2 + 0.267 \vartheta_2^3 + 0.0667 \vartheta_3^1 + 0.1 \vartheta_3^2 + 0.0333 \vartheta_3^3 - 0.0333 \vartheta_4^1 + 0.1 \vartheta_4^2 + 0.133 \vartheta_4^3) \\
+ \{0.4, 0.5, 0.6\} \otimes (0.14 w_1 - 0.02 w_2 - 0.608 w_3 + 0.288 w_4 - 0.2 \vartheta_1^1 - 0.0667 \vartheta_1^2 \\
-0.0667 \vartheta_2^2 - 0.0667 \vartheta_2^3 + 0.0667 \vartheta_3^1 + 0.0333 \vartheta_3^2 - 0.0667 \vartheta_4^1 - 0.1 \vartheta_4^2 - 0.133 \vartheta_4^3) \\
+ \{0.6, 0.7, 0.95\} \otimes (0.0433 w_1 - 0.253 w_2 + 0.478 w_3 - 0.438 w_4 - 0.133 \vartheta_1^1 + 0.2 \vartheta_1^2 + 0.0667 \vartheta_1^3 \\
-0.0667 \vartheta_2^2 + 0.133 \vartheta_2^3 - 0.133 \vartheta_3^1 - 0.233 \vartheta_3^2 - 0.3 \vartheta_3^3 + 0.333 \vartheta_4^1 + 0.3 \vartheta_4^2 + 0.3 \vartheta_4^3) \\
+ \{0.7, 0.9\} \otimes (0.293 w_1 - 0.09 w_2 - 0.63 w_3 + 0.25 w_4 - 0.133 \vartheta_1^1 - 0.133 \vartheta_1^2 + 0.2 \vartheta_1^3 + 0.0667 \vartheta_2^2 \\
+0.0667 \vartheta_2^3 + 0.267 \vartheta_3^1 + 0.333 \vartheta_3^2 + 0.0333 \vartheta_3^3 - 0.3 \vartheta_4^1 - 0.133 \vartheta_4^2 - 0.167 \vartheta_4^3) \succeq \{0.01\} \\
z_{12}, z_{23}, z_{24}, z_{31}, z_{34}, z_{25}, z_{45} \geq 0 \\
0 \leq \vartheta_j^1 \leq \vartheta_j^2 \leq \vartheta_j^3 \leq w_j \ (j = 1, 2, 3, 4) \\
w_4 + \varpi_{141}^- \geq w_1, w_3 + \varpi_{132}^- \geq w_2, w_2 - w_3 \ + \varpi_{223}^- \geq 0.05, w_2 - w_3 - \varpi_{223}^+ \leq 0.3, \\
w_4 - w_3 - w_2 + w_1 + \varpi_{34321}^- \geq 0 \ , w_2/w_1 + \varpi_{421}^- \geq 2, w_4 - \varpi_{54}^+ \leq 0.5, \ w_4 + \varpi_{54}^- \geq 0.15 \\
w_1 + w_2 + w_3 + w_4 = 1, \ \ 0.01 \leq w_j \leq 1 \ \ (j = 1, 2, 3, 4) \\
\varpi_{141}^-, \varpi_{132}^-, \varpi_{223}^-, \varpi_{223}^+, \varpi_{34321}^-, \varpi_{421}^-, \varpi_{54}^+, \varpi_{54}^- \geq 0
\end{array} \right.
\end{aligned}
\tag{5.2}
$$

$$
\min \{\chi\}
$$

$$
\text{s.t.} \begin{cases}
0.6z_{12} + 0.65z_{23} + 0.85z_{24} + 0.6z_{25} + 0.5z_{31} + 0.75z_{34} + 0.8z_{45} \le \chi \\
\varpi_{141}^{-} + \varpi_{132}^{-} + \varpi_{223}^{-} + \varpi_{223}^{+} + \varpi_{34321}^{-} + \varpi_{421}^{-} + \varpi_{54}^{+} + \varpi_{54}^{-} \le \chi \\
-0.00667w_1 + 0.157w_2 + 0.0175w_3 - 0.321w_4 + 0.0667\vartheta_1^1 - 0.2\vartheta_1^2 - 0.0667\vartheta_2^2 \\
+0.0333\vartheta_2^3 + 0.133\vartheta_4^1 + 0.167\vartheta_4^2 + 0.133\vartheta_4^3 + z_{12} \ge 0 \\
-0.133w_1 - 0.137w_2 + 0.0433w_3 + 0.0933w_4 + 0.133\vartheta_1^1 + 0.133\vartheta_1^2 + 0.133\vartheta_2^2 + 0.0667\vartheta_2^3 \\
-0.0667\vartheta_3^1 - 0.0667\vartheta_4^1 - 0.0667\vartheta_4^2 + z_{23} \ge 0 \\
-0.09w_1 - 0.39w_2 + 0.522w_3 - 0.345w_4 + 0.333\vartheta_1^2 + 0.0667\vartheta_1^3 + 0.2\vartheta_2^2 + 0.2\vartheta_2^3 \\
-0.2\vartheta_3^1 - 0.233\vartheta_3^2 - 0.3\vartheta_3^3 + 0.267\vartheta_4^1 + 0.233\vartheta_4^2 + 0.3\vartheta_4^3 + z_{24} \ge 0 \\
0.203w_1 - 0.48w_2 - 0.0983w_3 - 0.095w_4 - 0.133\vartheta_1^1 + 0.2\vartheta_1^2 - 0.133\vartheta_1^3 + 0.267\vartheta_2^2 + 0.267\vartheta_2^3 \\
+0.0667\vartheta_3^1 + 0.1\vartheta_3^2 + 0.0333\vartheta_3^3 - 0.0333\vartheta_4^1 + 0.1\vartheta_4^2 + 0.133\vartheta_4^3 + z_{25} \ge 0 \\
0.14w_1 - 0.02w_2 - 0.608w_3 + 0.288w_4 - 0.2\vartheta_1^1 - 0.0667\vartheta_1^2 - 0.0667\vartheta_2^2 - 0.0667\vartheta_2^3 \\
+0.0667\vartheta_3^1 + 0.0333\vartheta_3^2 - 0.0667\vartheta_4^1 - 0.1\vartheta_4^2 - 0.133\vartheta_4^3 + z_{31} \ge 0 \\
0.0433w_1 - 0.253w_2 + 0.478w_3 - 0.438w_4 - 0.133\vartheta_1^1 + 0.2\vartheta_1^2 + 0.0667\vartheta_1^3 - 0.0667\vartheta_2^2 \\
+0.133\vartheta_2^3 - 0.133\vartheta_3^1 - 0.233\vartheta_3^2 - 0.3\vartheta_3^3 + 0.333\vartheta_4^1 + 0.3\vartheta_4^2 + 0.3\vartheta_4^3 + z_{34} \ge 0 \\
0.293w_1 - 0.09w_2 - 0.63w_3 + 0.25w_4 - 0.133\vartheta_1^1 - 0.133\vartheta_1^2 + 0.2\vartheta_1^3 + 0.0667\vartheta_2^2 + 0.0667\vartheta_2^3 \\
+0.267\vartheta_3^1 + 0.333\vartheta_3^2 + 0.0333\vartheta_3^3 - 0.3\vartheta_4^1 - 0.133\vartheta_4^2 - 0.167\vartheta_4^3 + z_{45} \ge 0 \\
0.6 \times (-0.00667w_1 + 0.157w_2 + 0.0175w_3 - 0.321w_4 + 0.0667\vartheta_1^1 - 0.2\vartheta_1^2 \\
-0.0667\vartheta_2^2 + 0.0333\vartheta_2^3 + 0.133\vartheta_4^1 + 0.167\vartheta_4^2 + 0.133\vartheta_4^3) \\
+0.65 \times (-0.133w_1 - 0.137w_2 + 0.0433w_3 + 0.0933w_4 + 0.133\vartheta_1^1 \\
+0.133\vartheta_1^2 + 0.133\vartheta_2^2 + 0.0667\vartheta_2^3 - 0.0667\vartheta_3^1 - 0.0667\vartheta_4^1 - 0.0667\vartheta_4^2) \\
+0.85 \times (-0.09w_1 - 0.39w_2 + 0.522w_3 - 0.345w_4 + 0.333\vartheta_1^2 + 0.0667\vartheta_1^3 \\
+0.2\vartheta_2^2 + 0.2\vartheta_2^3 - 0.2\vartheta_3^1 - 0.233\vartheta_3^2 - 0.3\vartheta_3^3 + 0.267\vartheta_4^1 + 0.233\vartheta_4^2 + 0.3\vartheta_4^3) \\
+0.6 \times (0.203w_1 - 0.48w_2 - 0.0983w_3 - 0.095w_4 - 0.133\vartheta_1^1 + 0.2\vartheta_1^2 - 0.133\vartheta_1^3 \\
+0.267\vartheta_2^2 + 0.267\vartheta_2^3 + 0.0667\vartheta_3^1 + 0.1\vartheta_3^2 + 0.0333\vartheta_3^3 - 0.0333\vartheta_4^1 + 0.1\vartheta_4^2 + 0.133\vartheta_4^3) \\
+0.5 \times (0.14w_1 - 0.02w_2 - 0.608w_3 + 0.288w_4 - 0.2\vartheta_1^1 - 0.0667\vartheta_1^2 \\
-0.0667\vartheta_2^2 - 0.0667\vartheta_2^3 + 0.0667\vartheta_3^1 + 0.0333\vartheta_3^2 - 0.0667\vartheta_4^1 - 0.1\vartheta_4^2 - 0.133\vartheta_4^3) \\
+0.75 \times (0.0433w_1 - 0.253w_2 + 0.478w_3 - 0.438w_4 - 0.133\vartheta_1^1 + 0.2\vartheta_1^2 + 0.0667\vartheta_1^3 \\
-0.0667\vartheta_2^2 + 0.133\vartheta_2^3 - 0.133\vartheta_3^1 - 0.233\vartheta_3^2 - 0.3\vartheta_3^3 + 0.333\vartheta_4^1 + 0.3\vartheta_4^2 + 0.3\vartheta_4^3) \\
+0.8 \times (0.293w_1 - 0.09w_2 - 0.63w_3 + 0.25w_4 - 0.133\vartheta_1^1 - 0.133\vartheta_1^2 + 0.2\vartheta_1^3 + 0.0667\vartheta_2^2 \\
+0.0667\vartheta_2^3 + 0.267\vartheta_3^1 + 0.333\vartheta_3^2 + 0.0333\vartheta_3^3 - 0.3\vartheta_4^1 - 0.133\vartheta_4^2 - 0.167\vartheta_4^3) \ge 0.01 \\
z_{12}, z_{23}, z_{24}, z_{31}, z_{34}, z_{25}, z_{45} \ge 0 \\
w_4 + \varpi_{141}^{-} \ge w_1, w_3 + \varpi_{132}^{-} \ge w_2, w_2 - w_3 + \varpi_{223}^{-} \ge 0.05, w_2 - w_3 - \varpi_{223}^{+} \le 0.3, \\
w_4 - w_3 - w_2 + w_1 + \varpi_{34321}^{-} \ge 0, w_2/w_1 + \varpi_{421}^{-} \ge 2, w_4 - \varpi_{54}^{+} \le 0.5, \ w_4 + \varpi_{54}^{-} \ge 0.15 \\
w_1 + w_2 + w_3 + w_4 = 1, \ \ 0.01 \le w_j \le 1 \ \ (j = 1, 2, 3, 4) \\
0 \le \vartheta_j^1 \le \vartheta_j^2 \le \vartheta_j^3 \le w_j \ (j = 1, 2, 3, 4) \\
\varpi_{141}^{-}, \varpi_{132}^{-}, \varpi_{223}^{-}, \varpi_{223}^{+}, \varpi_{34321}^{-}, \varpi_{421}^{-}, \varpi_{54}^{+}, \varpi_{54}^{-} \ge 0
\end{cases} \tag{5.3}
$$

$$
\min\ \left\{\langle 0.5,0.3\rangle z_{12}+\langle 0.6,0.3\rangle z_{23}+\langle 0.8,01\rangle z_{24}+\langle 0.5,0.3\rangle z_{25}+\langle 0.4,0.4\rangle z_{31}+\langle 0.6,0.05\rangle z_{34}+\langle 0.7,0.1\rangle z_{45}\right\}
$$

$$
\text{s.t.}\ \begin{cases}
0.2u_2-0.1u_1+0.6u_4-0.2v_1+0.4v_2-0.2v_4+0.13w_1-0.26w_2-0.1w_4+z_{12}\geq 0\\
0.2u_2-0.2u_1+0.1u_3+0.5u_4-0.4v_1+0.1v_2+0.2v_3+0.2v_4+0.28w_1-0.17w_2-0.14w_3-0.13w_4+z_{23}\geq 0\\
0.2u_1+0.3u_2-0.6u_3+0.2u_4+0.1v_1-0.3v_2+0.15v_3-0.35v_4-0.14w_1-0.19w_2-0.3675w_3-0.02w_4+z_{24}\geq 0\\
0.4u_2-0.2u_1+0.05u_4+0.2v_1-0.4v_2-0.15v_3+0.3v_4-0.36w_1-0.48w_2+0.0725w_3-0.595w_4+z_{25}\geq 0\\
0.3u_1-0.4u_2-0.1u_3+0.5u_4+0.6v_1-0.5v_2-0.2v_3-0.48w_1+0.36w_2+0.13w_3+0.22w_4+z_{31}\geq 0\\
0.4u_1+0.1u_2-0.7u_3+0.1u_4+0.5v_1-0.4v_2-0.05v_3-0.55v_4-0.48w_1+0.07w_2+0.4825w_3+0.105w_4+z_{34}\geq 0\\
0.4u_2-0.2u_1+0.05u_4+0.2v_1-0.4v_2-0.15v_3+0.3v_4+0.04w_1-0.08w_2+0.0725w_3-0.395w_4+z_{45}\geq 0\\
\langle 0.5,0.3\rangle\,(0.2u_2-0.1u_1+0.6u_4-0.2v_1+0.4v_2-0.2v_4+0.13w_1-0.26w_2-0.1w_4)+\\
\langle 0.6,0.3\rangle\,(0.2u_2-0.2u_1+0.1u_3+0.5u_4-0.4v_1+0.1v_2+0.2v_3+0.2v_4+0.28w_1-0.17w_2-0.14w_3-0.13w_4)+\\
\langle 0.8,01\rangle\,(0.2u_1+0.3u_2-0.6u_3+0.2u_4+0.1v_1-0.3v_2+0.15v_3-0.35v_4-0.14w_1-0.19w_2-0.3675w_3-0.02w_4)+\\
\langle 0.5,0.3\rangle\,(0.4u_2-0.2u_1+0.05u_4+0.2v_1-0.4v_2-0.15v_3+0.3v_4-0.36w_1-0.48w_2+0.0725w_3-0.595w_4)+\\
\langle 0.4,0.4\rangle\,(0.3u_1-0.4u_2-0.1u_3+0.5u_4+0.6v_1-0.5v_2-0.2v_3-0.48w_1+0.36w_2+0.13w_3+0.22w_4)+\\
\langle 0.6,0.05\rangle\,(0.4u_1+0.1u_2-0.7u_3+0.1u_4+0.5v_1-0.4v_2-0.05v_3-0.55v_4-0.48w_1+0.07w_2+0.4825w_3+\\
0.105w_4)+\langle 0.7,0.1\rangle\,(0.4u_2-0.2u_1+0.05u_4+0.2v_1-0.4v_2-0.15v_3+\ 0.3v_4+\\
0.04w_1-0.08w_2+0.0725w_3-0.395w_4)\succeq\langle 0.01,0.99\rangle\\
w_2\geq 2w_1,\ 0.05\leq w_2-w_2\leq 0.3,\ \ 0.15\leq w_4\leq 0.5\\
w_1+w_2+w_3+w_4=1,\ 0.05\leq w_j\leq 1\quad (j=1,2,3,4)\\
z_{12},z_{23},z_{24},z_{31},z_{34},z_{25},z_{45}\geq 0\\
u_j,v_j\geq 0,\ u_j+v_j\leq w_j\quad (j=1,2,3,4)
\end{cases}
\tag{5.4}
$$

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