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Multi-criteria outranking approach with hesitant fuzzy sets

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Abstract As a generalization of fuzzy sets, hesitant fuzzy sets constrain the membership degree of an element to be a set of possible values between zero and one. Those sets are considered useful in handling decision problems defined under uncertainties where decision-makers hesitate among several values before expressing their preferences. Motivated by the idea of traditional ELECTRE methods, the dominance relations and the opposition relations for hesitant fuzzy sets are introduced in this paper. In addition, several desirable properties are studied. Then, a novel outranking relation is developed, based on systematic comparison of assessments given to alternatives for each criterion. An outranking approach for multi-criteria decision-making problems with hesitant fuzzy sets, similar to ELECTRE III, is proposed for ranking alternatives. Finally, an example is given to verify the developed approach and demonstrate its validity and feasibility.

Keywords Multi-criteria decision-making · Hesitant fuzzy sets · Outranking relations · Dominance relations

1 Introduction

Since fuzzy set (FS) theory was first proposed by Zadeh (1965), it has been widely studied, developed and successfully applied in various fields, such as multi-criteria decision-making (MCDM) (Bellman and Zadeh 1990; Yager 1997), fuzzy logic and approximate reasoning (Zadeh 1975), pattern recognition (Pedrycz 1990). In real MCDM problems, the criteria weights and criteria values of alternatives may be inaccurate, uncertain or incomplete due to the fuzziness and uncertainty of decision-making

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problems. Some solutions can be derived from FSs, and especially fuzzy numbers can provide good solutions to such problems. In FSs, the universal membership degree of an element is a number, which ranges from 0 to 1 corresponding to the element in a universe. However, the membership degree in an FS is a single value, which reflects nothing about the lack of knowledge.

In practice, however, the information of alternatives corresponding to a fuzzy concept may be incomplete, i.e., the sum of the membership and non-membership degrees of an element in the universe corresponding to the fuzzy concept can be less than one. The FS theory fails in dealing with the insufficient understanding to the membership degree, while Atanassov's intuitionistic fuzzy sets (A-IFSs), an extension of Zadeh's FSs which were introduced by Atanassov (1986, 1999, 2000, 1989), successfully handled the problems by adding a non-membership degree. Therefore, A-IFSs were expected to be applicable to simulate human decision-making process and activities that require corresponding expertise and knowledge. MCDM problems with A-IFSs have received adequate attentions (Chen 2011; Chen and Yang 2012; Paternain et al. 2012; Xu 2011, 2012; Yue 2010; Wei 2010, 2011; Wei and Zhao 2012; Xu and Xia 2011a; Tan and Chen 2010; Akram and Dudek 2013; Zeng and Su 2011; Yu and Xu 2013; Pekala 2012; Mukherjee and Basu 2012; Xia et al. 2012). For example, Chen (2011) developed a TOPSIS-based non-linear programming methodology to handle MCDM problems with A-IFSs. Chen and Yang (2012) defined a new class of decision functions based on the weighted score function and the weighted accuracy function in the intuitionistic fuzzy setting. Paternain et al. (2012) presented a construction method for Atanassov's intuitionistic fuzzy preference relations. Several other methods based on a series of aggregation operators applied to MCDM problems with A-IFSs were put forward by Xu (2011), Wei and Zhao (2012), Wei (2010), Xu and Xia (2011a), Tan and Chen (2010), Zeng and Su (2011). As we know, A-IFSs can deal with fuzzy concepts "neither this nor that", but the membership and non-membership degrees of an element gathered are real numbers, respectively. In actual decision-making problems, however, the degrees in A-IFSs can be a set of real numbers instead of only one.

The purpose for introducing these sets is that determining the membership of an element into one single set is very difficult. In some circumstances, this difficulty is caused by a set of possible values. Now, we give an example to illustrate this problem. In the case where two experts discuss the membership of x into A, one may assign 0.3, and the other assigns 0.5. This situation can arise in an MCDM problem. Hesitant fuzzy sets (HFSs), another extension of traditional FSs, provide useful reference for our study of such situations. HFSs were first introduced by Torra (2010), Torra and Narukawa (2009), and they permit the membership degree of an element to be a set of multiple possible values between 0 and 1. HFSs are highly useful in handling the situations where people hesitate in expressing their preferences over objects. Xia and Xu (2011), Zhu et al. (2012), Wei (2012), Xia et al. (2013) and Zhang (2013) studied the aggregation operators of HFSs and applied them to MCDM problems. Farhadinia (2013a) defined a new score function to compare HFEs, which overcame the counterintuitive problem occurred in the method of Xia and Xu (2011). Verma and Sharma (2013) defined some new operations of HFSs. Yu et al. (2011) and Chen et al. (2013a) discussed the correlation coefficients of HFSs, together with their applications to clustering analysis. Xu and Xia (2011b) discussed the distance measures for HFSs.

They also (Xu and Xia 2011c) discussed the distance and correlation measures of hesitant fuzzy information. Based on the measures proposed by Xu and Xia (2011b), Peng et al. (2013) presented a generalized hesitant fuzzy synergetic weighted distance (GHFSWD) measure. Liao and Xu (2013) proposed a hesitant fuzzy VIKOR method based on some new defined hesitant fuzzy measures. Zhang and Wei (2013) presented both the E-VIKOR method and the TOPSIS method for MCDM problems with hesitant fuzzy information. Xu and Zhang (2013) developed an approach based on TOPSIS to deal with hesitant fuzzy MCDM problems with incomplete weights. Wei et al. (2013), Chen et al. (2013b) and Wei and Zhao (2013) developed some hesitant interval-valued fuzzy aggregation operators for MCDM problems, especially for the category with the criterion-related evaluations derived from hesitant interval-valued fuzzy information. Qian et al. (2013) put forward generalized HFSs and proved Zadeh's FSs, A-IFSs and HFSs were special cases of the new fuzzy sets. Farhadinia (2013b) introduced a mutual transformation of the entropy into the similarity measure for both HFSs and interval-valued HFSs. Yu and Zhang (2013) applied HFSs to personnel evaluation. However, some shortcomings of the existing methods for dealing with HFSs have emerged: (1) different operations of HFSs could produce different results. This leads to the difficulty puzzling decision-makers; (2) both distance measures and similarity measures are based on the assumption that both of the hesitant fuzzy elements (HFEs) in HFSs have the same length. If not, then the shorter one should be extended by adding some values in it until it has the same length of the longer one. In this case, different extension methods could produce different results; (3) the existing comparison methods have some counterintuitive problems to reflect the decision-makers' preference and they could lead to a reverse order in case of using different operations.

All of those mentioned methods for solving MCDM problems with HFSs were in the category named function models. Several other methods were also listed and put into the category called relation models. Unlike function models, the latter ones performed their decision-making without a fusion method, but rather adopt outranking relations or priority functions to optimize, rank and classify alternatives in terms of priority among criteria. ELECTRE methods and PROMETHREE methods are the typical ones within that category. The prominent feature of a function model generally implicates a completely compensability hypothesis. In other words, no matter how worse an alternative A is, being compared to alternative B on a criterion, it can be compensated from other criteria, so as to make the overall consequence to be $A \succ B$. So the hypothesis causes the loss of partial information and the failure to reflect the demand of decision-makers in many situations. That deficiency can be removed using relation models. Non-compensation or conditional compensation principles are generally adopted by relation models. It means that if an alternative A is worse than the alternative B on a criterion and their difference exceeds a certain limitation, then decision-makers would ignore any case with $A \succ$ B, disregarding the significance of difference under any other criteria. Because of being more related to actual situation in solving decision-making problems, this hypothesis has been extensively appreciated in recent years. ELECTRE methods (Roy 1991), as representatives of relation models, are successfully and widely used in various fields due to their practical applicability, including biological engineering (Hanandeh and El-Zein 2010; Ermatita et al. 2011), energy sources (Cavallaro 2010; Haurant et al. 2011), environment (Kaya and Kahraman 2011; Achillas et al. 2010), economy (Bojković et al. 2010), value engineering (Marzouk 2011), medical science (Figueira et al. 2011), communication and transportation (Sawadogo and Anciaux 2011), and location selection problems (Devi and Yadav 2013; Ozcan et al. 2011). Among all ELECTRE methods, ELECTRE III, which is designed for ranking problems, has the ability to deal with imprecise, inaccurate and uncertain evaluations (Roy 1991).

While multiple previous studies about ELECTRE methods still focused on certain data, our research turns onto uncertain data, as an extension of ELECTRE III. Those uncertainties are expressed by HFSs. In this paper, an MCDM problem with HFSs is discussed as a central topic. Considering the importance of traditional ELECTRE methods playing in the ranking of alternatives, an outranking relation on HFSs with the properties is proposed. An outranking approach for MCDM problems with HFSs is proposed as a following-up. The main advantages of the proposed approach over those hesitant fuzzy operators are that it avoids the problems brought by the operations and comparison methods of HFSs, as well as it has the feature of non-compensatory and takes decision-makers' preferences into consideration, which are represented by choosing appropriate thresholds towards criteria.

The rest of this paper is organized to ensure the validity of the proposals. The detailed sequence is described as follows. In Sect. 2.1, we introduce the definitions of t-norm and t-conorm, and then we briefly review some basic concepts and operations of HFSs in Sect. 2.2. In Sect. 3, we define an outranking relation on HFSs, and some desirable properties are also studied in this section. Then, an outranking approach for MCDM problems with HFSs is developed in Sect. 4. Finally, an illustrative example is given to show the validity and feasibility of the proposed approach in Sect. 5, and the conclusions are in Sect. 6.

2 Preliminaries

2.1 t-norm and t-conorm

The t-norm and its dual t-conorm play an important role in the construction of averaging operators of HFSs (Achillas et al. 2010). Here, we introduce some basic concepts of them.

Definition 1 (Nguyen and Walker 1997; Klement and Mesiar 2005) A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is called t-norm if it satisfies the following conditions:

(1) $\forall x \in [0, 1], T(1, x) = x$;
(2) $\forall x, y \in [0, 1], T(x, y) = T(y, x)$;
(3) $\forall x, y, z \in [0, 1], T(x, T(y, z)) = T(T(x, y), z)$;
(4) If $x \leq x', y \leq y'$, then $T(x, y) \leq T(x', y')$.

Definition 2 (Nguyen and Walker 1997; Klement and Mesiar 2005) A function $S : [0, 1] \times [0, 1] \to [0, 1]$ is called t-conorm if it satisfies the following conditions:

(1) $\forall x \in [0, 1], S(0, x) = x$;
(2) $\forall x, y \in [0, 1], S(x, y) = S(y, x)$;

(3) $\forall x, y, z \in [0, 1]$, $S(x, S(y, z)) = S(S(x, y), z)$;
(4) $If\ x \leq x', y \leq y'$, then $S(x, y) \leq S(x', y')$.

Definition 3 (Nguyen and Walker 1997; Klement and Mesiar 2005) A t-norm function $T(x, y)$ is called Archimedean t-norm if it is continuous and $T(x, x) < x$ for all $x \in (0, 1)$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x, y \in (0, 1)$. A t-conorm function $S(x, y)$ is called Archimedean t-conorm if it is continuous and $S(x, x) > x$ for all $x \in (0, 1)$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x, y \in (0, 1)$.

It is well known (Klement and Mesiar 2005) that a strict Archimedean t-norm can be expressed via its additive generator k as $T(x, y) = k^{-1}(k(x) + k(y))$, and similarly, applied to its dual t-conorm $S(x, y) = l^{-1}(l(x) + l(y))$ with $l(t) = k(1 - t)$. We observe that an additive generator of a continuous Archimedean t-norm is a strictly decreasing function $k : [0, 1] \to [0, \infty)$.

There are some well-known Archimedean t-norms and t-conorms (Beliakov et al. 2007).

(1) Let $k(t) = -\log t, l(t) = -\log(1 - t), k^{-1}(t) = e^{-t}, h^{-1}(t) = 1 - e^{-t}$, and then Algebraic t-conorm and t-norm are obtained : $S(x, y) = 1 - (1 - x)(1 - y), T(x, y) = xy$.
(2) Let $k(t) = \log(\frac{2-t}{t}), l(t) = \log(\frac{2-(1-t)}{1-t}), k^{-1}(t) = \frac{2}{e^t+1}, h^{-1}(t) = 1 - \frac{2}{e^t+1}$, and then Einstein t-conorm and t-norm are obtained: $S(x, y) = \frac{x+y}{1+xy}, T(x, y) = \frac{xy}{1+(1-x)(1-y)}$.

2.2 Hesitant fuzzy sets

Definition 4 (Torra 2010) Let X be a universal set, and a hesitant fuzzy set (HFS) on X be in terms of a function that returns a subset of [0,1] when applied to X. It can be represented as follows:

$$E = \{\langle x, h_E(x)\rangle | x \in X\},$$

where $h_E(x)$ is a set of values within [0, 1], denoting the possible membership degrees of the element $x \in X$to the set E. $h_E(x)$ is a hesitant fuzzy element (HFE) (Xia and Xu 2011), and H is the set of all HFEs. Noteworthy, if Xcontains only one element, E is called a hesitant fuzzy number, briefly denoted by $E = \{h_E(x)\}$. The set of all hesitate fuzzy numbers is represented as HFNS.

Torra (2010) defined some operations on HFEs, and Xia and Xu (2011) and Xia (2012) further defined some operations on HFEs and score functions.

Definition 5 (Xia 2012) Let $h_1, h_2, h \in$ HFNS, $\lambda \geq 0$. The four operations are defined as follows:

(1) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{l^{-1}(l(\gamma_1) + l(\gamma_2))\}$;
(2) $\lambda h = \cup_{\gamma \in h}\{l^{-1}(\lambda l(\gamma))\}$;

(3) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{k^{-1}(k(\gamma_1) + k(\gamma_2))\}$;
(4) $h^\lambda = \cup_{\gamma \in h}\{k^{-1}(\lambda k(\gamma))\}$,

where $l(t) = k(1-t)$, and $k : [0, 1] \to [0, \infty)$ is a strictly decreasing function.
Let $k(t) = -\log t$, $l(t) = -\log(1-t)$, and then

(1) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
(2) $\lambda h = \cup_{\gamma \in h}\{1 - (1-\gamma)^\lambda\}$;
(3) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$;
(4) $h^\lambda = \cup_{\gamma \in h}\{\gamma^\lambda\}$.

All those four operations were once introduced by Xia and Xu (2011).
Let $k(t) = \log(\frac{2-t}{t})$, $l(t) = \log(\frac{2-(1-t)}{1-t})$, and then

(5) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2}\}$;
(6) $\lambda h = \cup_{\gamma \in h}\{1 - \frac{2}{(\frac{1+\gamma}{1-\gamma})^\lambda + 1}\}$;
(7) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\frac{\gamma_1\gamma_2}{2-\gamma_1-\gamma_2+\gamma_1\gamma_2}\}$;
(8) $h^\lambda = \cup_{\gamma \in h}\{\frac{2}{(\frac{2-\gamma}{\gamma})^\lambda + 1}\}$.

Example 1 Let $h_1 = \{0.1, 0.3\}$, $h_2 = \{0.2, 0.3, 0.5\}$.

(1) Let $k(t) = -\log t$, $l(t) = -\log(1-t)$, and then $h^1_\oplus = h_1 \oplus h_2 = \{0.28, 0.37, 0.64, 0.44, 0.51, 0.65\}$, and $h^1_\otimes = h_1 \otimes h_2 = \{0.02, 0.03, 0.05, 0.06, 0.09, 0.15\}$.
(2) Let $k(t) = \log(\frac{2-t}{t})$, $l(t) = \log(\frac{2-(1-t)}{1-t})$, and then

$$h^2_\oplus = h_1 \oplus h_2 = \{0.2941, 0.3884, 0.5714, 0.4717, 0.5505, 0.6957\},$$

and $h^2_\otimes = h_1 \otimes h_2 = \{0.1163, 0.0184, 0.0345, 0.0385, 0.0604, 0.1111\}$.

Apparently, $h^1_\oplus \neq h^2_\oplus$, $h^1_\otimes \neq h^2_\otimes$. Therefore, the results varied with the application of different operations.

Definition 6 (Xia and Xu 2011) Let $h \in H$, and $s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma$ be the score function of h, where $\#h$ is the number of elements in h.

For two HFEs h_1 and h_2,
if $s(h_1) > s(h_2)$, then $h_1 > h_2$;
if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

The impropriety of using the definition in the comparison of two HFEs, is illustrated in the following example.

Example 2 Based on Example 1, we have $s(h^1_\oplus) = 0.4817$, $s(h^2_\oplus) = 0.4953$, $s(h^1_\otimes) = 0.0667$, and $s(h^2_\otimes) = 0.0458$. According to Definition 6, the following are true: $h^1_\oplus < h^2_\oplus$ and $h^1_\otimes > h^2_\otimes$. Therefore, arithmetic average operator and geometric average operator based on different operations could lead to different ranking results. In addition, the comparison method in Definition 6 can result in the counterintuitive problem. For example, let $h_1 = \{0.5\}$, $h_2 = \{0.1, 0.9\}$ and $h_3 = \{0.1, 0.5, 0.9\}$

be three HFEs. Clearly, $h_1 \neq h_2 \neq h_3$. But by applying Definition 6, we get $s(h_1) = s(h_2) = s(h_3)$, then $h_1 = h_2 = h_3$, which is contradictory to our intuition.

To overcome the counterintuitive problem, Farhadinia (2013a) defined a new score function.

Definition 7 (Farhadinia 2013a) Let $h = \cup_{\gamma \in h}\{\gamma\} = \{\gamma_j | j = 1, 2, \ldots, l(h)\}$ be an HFE, where $l(h)$ is the number of elements in h. Then, the score function of h is defined as

$$S(h) = \frac{\sum_{j=1}^{l(h)} \delta(j)\gamma_j}{\sum_{j=1}^{l(h)} \delta(j)},$$

where $\{\delta(j) | j = 1, 2, \ldots, l(h)\}$ is a positive-valued monotonic increasing sequence of index j.

In Farhadinia (2013a), the new score function proposed to compare HFEs was defined, and it performed better than that of Xia and Xu (2011). The new score function overcame the counterintuitive problem occurred in the method of Xia and Xu (2011). However, the new score function was defined on the assumption that the values in HFEs are arranged in an ascending order and if two HFEs do not have the same length, then the shorter one should be extended by adding the greatest number in it until both of the HFEs have the same length. In this extension method, the decision-makers are assumed to be optimistic in decision-making process, so they employed the maximum. This method did not take into account the case that pessimists prefer the minimum. When optimists and pessimists are bound in one group, these methods will no longer be applicable. So this extension method cannot comprehensively reflect the decision-makers' risky tendency in making decisions.

Furthermore, Torra and Narukawa (2009) and Torra (2010) proposed an aggregation principle for HFEs.

Definition 8 (Torra 2010; Torra and Narukawa 2009) Let $E = \{h_1, h_2, \ldots, h_n\}$ be a set of HFEs, ϑ be a function on E, $\vartheta : [0, 1]^n \to [0, 1]$, and then

$$\vartheta_E = \bigcup_{\gamma \in h_1 \times h_2 \times \cdots \times h_n} \{\vartheta(\gamma)\}.$$

3 Outranking relations on HFSs

In ELECTRE III method, for the jth criterion being considered, the concordance index and the discordance index are constructed with three associated thresholds: the preference threshold p_j, the indifference threshold q_j, and the veto threshold v_j. Among those three thresholds, p_j is used to justify the preference in favor of either of the actions, q_j stands for being compatible with indifference between two actions, and v_j is assigned to introduce discordance into the outranking relations. Note that, in this paper, we only consider the simple case where the thresholds p_j, q_j and v_j

are constants under each criterion. This simplification of using constant thresholds aids the illustration of ELECTRE III method and our approach. Actually, they can be generalized to functions varying with the value of the criteria $g_j(a)$; that is, the case of variable thresholds $p_j(g_j(a))$, $q_j(g_j(a))$ and $v_j(g_j(a))$ (Roy 1991).

Definition 9 (Roy 1991) Let G be the criteria set $G = \{g_1, \ldots, g_j, \ldots, g_m\}$, B be the set of alternatives or actions $B = \{a_1, \ldots, a_i, \ldots, a_n\}$. Two thresholds under the criterion g_j have been specified to construct the fuzzy concordance index: the indifference threshold q_j and the preference threshold $p_j (0 \leq q_j \leq p_j)$. Let a_1 and a_2 be two alternatives, where $a_1, a_2 \in B$, and then the relations can be defined as follows.

(1) If $g_j(a_1) - g_j(a_2) \geq p_j$, then a_1 is strictly preferred to a_2, denoted by $P(a_1, a_2)$.
(2) If $q_j < g_j(a_1) - g_j(a_2) < p_j$, then a_1 is weakly preferred to a_2, denoted by $W(a_1, a_2)$.
(3) If $\left|g_j(a_1) - g_j(a_2)\right| \leq q_j$, then a_1 is indifferent to a_2, denoted by $I(a_1, a_2)$.

The concordance index for the single criterion is defined as follows.

(1) If $g_j(a_1) + q_j \geq g_j(a_2)$, then $c_j(a_1, a_2) = 1$.
(2) If $g_j(a_1) + q_j < g_j(a_2) < g_j(a_1) + p_j$, then $c_j(a_1, a_2) = \frac{g_j(a_1) - g_j(a_2) + p_j}{p_j - q_j}$.
(3) If $g_j(a_1) + p_j \leq g_j(a_2)$, then $c_j(a_1, a_2) = 0$.

Definition 10 (Roy 1991) A veto threshold $v_j (\geq p_j)$ is introduced based on Definition 9. Then, the discordance index $d(a_1, a_2)$ is defined as follows.

(1) If $g_j(a_2) - g_j(a_1) \leq p_j$, then $d_j(a_1, a_2) = 0$.
(2) If $p_j < g_j(a_2) - g_j(a_1) < v_j$, then $d_j(a_1, a_2) = \frac{g_j(a_2) - g_j(a_1) - p_j}{v_j - p_j}$.
(3) If $g_j(a_2) - g_j(a_1) \geq v_j$, then $d_j(a_1, a_2) = 1$.

It should be mentioned that if there is a criterion for which the alternative a_2 performs better than the alternative a_1 by at least the veto threshold even if other criteria favor the outranking of a_2 by a_1, then any outranking of a_2 by a_1 indicated by the concordance index can be overruled.

Following ELECTRE III, we define the outranking relations, a concordance index and a discordance index for HFNs.

Definition 11 Let $h_1, h_2 \in$ HFNS, p,$q (0 \leq q \leq p)$ be two thresholds, then we define the concordance index for HFNs as follows:

$$r_{p,q}(h_1, h_2) = \max_{\gamma_1 \in h_1} \min_{\gamma_2 \in h_2} c_{p,q}(\gamma_1, \gamma_2).$$

In fact, the maxmin operator used in Definition 11 is an extension of the traditional concordance index by integrating ELECTRE III method and hesitant fuzzy information. Especially, if h_1 is indifferent to h_2, then the concordance index should be equal to one, which is still in accordance with the traditional concordance index. Therefore, the concordance index of ELECTRE III combined with HFNs in HFSs makes it more suitable for dealing with the uncertain data represented by HFSs, unlike the traditional ELECTRE III method which always deals with certain data. It is not difficult to find that if both h_1 and h_2 are reduced to the single values, $r_{p,q}(h_1, h_2)$ will turn into a concordance index which has been introduced in Definition 9.

According to Definition 11, we could easily obtain the following property.

Property 1 Let $h_1, h_2 \in$ HFNS, p and $q(0 \leq q \leq p)$ be two thresholds, and then

$$0 \leq r_{p,q}(h_1, h_2) \leq 1.$$

Definition 12 The strict dominance relation, the weak dominance relation and the indifference relation of HFNs can be defined as follows.

(1) If $r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) = 1$ (which is equivalent to $r_{p,q}(h_2, h_1) = 0$ and $r_{p,q}(h_1, h_2) = 1$), then h_1 strongly dominates h_2 (h_2 is strongly dominated by h_1), denoted by $h_1 >_S h_2$.
(2) If $r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) = 0$, then h_1 is indifferent to h_2, denoted by $h_1 \sim h_2$.
(3) If $0 < r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) < 1$, then h_1 weakly dominates h_2 (h_2 is weakly dominated by h_1), denoted by $h_1 >_W h_2$.
(4) If $0 < r_{p,q}(h_2, h_1) - r_{p,q}(h_1, h_2) < 1$, then h_2 weakly dominates h_1 (h_1 is weakly dominated by h_2), denoted by $h_2 >_W h_1$.

Example 3 Let $p = 0.2, q = 0.1$.

(1) If $h_1 = \{0.5, 0.7\}$, $h_2 = \{0.1, 0.2\}$,then $r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) = 1$. So $h_1 >_S h_2$.
(2) If$h_1 = \{0.4, 0.5\}$, $h_1 = \{0.2, 0.6\}$, then $r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) = 0$. So $h_1 \sim h_2$.
(3) If $h_1 = \{0.4, 0.5\}$, $h_2 = \{0.3, 0.35\}$, then $r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) = 0.5$. So $h_1 >_W h_2$.

Property 2 Let $h_1, h_2 \in$ HFNS, p and $q(0 \leq q \leq p)$ be two thresholds, and then $h_1 >_S h_2$ if and only if $\min\{\gamma_1 | \gamma_1 \in h_1\} - \max\{\gamma_2 | \gamma_2 \in h_2\} \geq p$.

Proof (1) Necessity: $h_1 >_S h_2 \Rightarrow \min\{r | r \in h_1\} - \max\{r | r \in h_2\} \geq p$.
According to Definition 12, if $h_1 >_S h_2$, then $r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) = 1$. Since $0 \leq r_{p,q}(h_1, h_2) \leq 1$ and $0 \leq r_{p,q}(h_2, h_1) \leq 1$, $r_{p,q}(h_2, h_1) = 0$. Then we get $\max\limits_{\gamma_2 \in h_2} \min\limits_{\gamma_1 \in h_1} c_{p,q}(\gamma_2, \gamma_1) = 0$. As we know from Definition 9 that $c_{p,q}(\gamma_2, \gamma_1) \in [0, 1]$, so $c_{p,q}(\gamma_2, \gamma_1) = 0$. Hence, $\gamma_1 - \gamma_2 \geq p$, for any $\gamma_1 \in h_1$, $\gamma_2 \in h_2$. Therefore, $\min\{\gamma_1 | \gamma_1 \in h_1\} - \max\{\gamma_2 | \gamma_2 \in h_2\} \geq p$ is certainly validated.
(2) Sufficiency: $\min\{\gamma_1 | \gamma_1 \in h_1\} - \max\{\gamma_2 | \gamma_2 \in h_2\} \geq p \Rightarrow h_1 >_S h_2$.

Because $\min\{\gamma_1 | \gamma_1 \in h_1\} - \max\{\gamma_2 | \gamma_2 \in h_2\} \geq p$, we have $\gamma_1 - \gamma_2 \geq p$ for any $\gamma_1 \in h_1$, $\gamma_2 \in h_2$. From Definition 9, $c_{p,q}(\gamma_2, \gamma_1) = 0$, $c_{p,q}(\gamma_1, \gamma_2) = 1$, and then $\max\limits_{\gamma_1 \in h_1} \min\limits_{\gamma_2 \in h_2} c_{p,q}(\gamma_1, \gamma_2) = 1$, which indicates $r_{p,q}(h_1, h_2) = 1$. Therefore, from Definition 12, we get $h_1 >_S h_2$. □

Property 3 Let $h_1, h_2, h_3 \in$ HFNS, and p and $q(0 \leq q \leq p)$ be two thresholds. If $h_1 >_S h_2$, $h_2 >_S h_3$, then there is $h_1 >_S h_3$.

Proof According to Property 2, if $h_1 >_S h_2$, then $\min\{\gamma_1|\gamma_1 \in h_1\} - \max\{\gamma_2|\gamma_2 \in h_2\} \geq p$.

If $h_2 >_S h_3$, then $\min\{\gamma_2|\gamma_2 \in h_2\} - \max\{\gamma_3|\gamma_3 \in h_3\} \geq p$. So

$$\left.\begin{array}{l}\min\{\gamma_1|\gamma_1 \in h_1\} - \max\{\gamma_2|\gamma_2 \in h_2\} \geq p \\ \min\{\gamma_2|\gamma_2 \in h_2\} - \max\{\gamma_3|\gamma_3 \in h_3\} \geq p\end{array}\right\}$$
$$\Rightarrow \min\{\gamma_1|\gamma_1 \in h_1\} - \max\{\gamma_3|\gamma_3 \in h_3\} \geq 2p \geq p.$$

Therefore, $h_1 >_S h_3$. □

Property 4 Let $h_1, h_2, h \in$ HFNS, and p and $q(0 \leq q \leq p)$ be two thresholds.

(1) The strong dominance relations are categorized as:
  ① irreflexivity: $\forall h \in$ HFNS, $h \not>_S h$;
  ② asymmetry: $\forall h_1, h_2 \in$ HFNS, $h_1 >_S h_2 \nRightarrow h_2 >_S h_1$;
  ③ transitivity: $\forall h, h_1, h_2 \in$ HFNS, $h >_S h_1, h_1 >_S h_2 \Rightarrow h >_S h_2$.
(2) The weak dominance relations are categorized as:
  ④ irreflexivity: $\forall h \in$ HFNS, $h \not>_W h$;
  ⑤ asymmetry: $\forall h_1, h_2 \in$ HFNS, $h_1 >_W h_2 \nRightarrow h_2 >_W h_1$;
  ⑥ non-transitivity: $\exists h, h_1, h_2 \in$ HFNS, such that $h >_W h_1, h_1 >_W h_2 \nRightarrow h >_W h_2$.
(3) The indifference relations are categorized as:
  ⑦ reflexivity: $\forall h \in$ HFNS, $h \sim h$;
  ⑧ symmetry: $\forall h_1, h_2 \in$ HFNS, $h_1 \sim h_2 \Rightarrow h_2 \sim h_1$;
  ⑨ non-transitivity: $\exists h, h_1, h_2 \in$ HFNS, such that $h \sim h_1, h_1 \sim h_2 \nRightarrow h \sim h_2$.

According Definitions 11 and 12, and Properties 2 and 3, it is clear to prove that ①–⑤, ⑦ and ⑧ hold. Therefore, only ⑥ and ⑨ need to be proven.

Example 4 We exemplify those false arguments on ⑥ and ⑨.

(1) Let $h = \{0.5, 0.6\}$, $h_1 = \{0.3, 0.4\}$, $h_2 = \{0.1, 0.2\}$, $p = 0.25$, $q = 0.15$, and then $r_{p,q}(h, h_1) - r_{p,q}(h_1, h) = 0.5$, $r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) = 0.5$, $r_{p,q}(h, h_2) - r_{p,q}(h_2, h) = 1$.So $h >_W h_1, h_1 >_W h_2$, but $h >_S h_2$.
(2) Let $h = \{0.2, 0.5\}$, $h_1 = \{0.3, 0.6\}$, $h_2 = \{0.4, 0.7\}$, $p = 0.2$, $q = 0.1$, and then $r_{p,q}(h, h_1) - r_{p,q}(h_1, h) = 1 - 1 = 0$, $r_{p,q}(h_1, h_2) - r_{p,q}(h_2, h_1) = 1 - 1 = 0$, $r_{p,q}(h_2, h) - r_{p,q}(h, h_2) = 1$. So $h \sim h_1, h_1 \sim h_2$, but $h_2 >_S h$.

Similar to dominance relations, we define the strong opposition relation, the weak opposition relation and the indifferent opposition relation.

Definition 13 Let $h_1, h_2 \in$ HFNS, p and $v(p \leq v)$ be two thresholds, and then we define the discordance index for HFNs as follows:

$$t_{p,v}(h_1, h_2) = \max_{\gamma_1 \in h_1} \min_{\gamma_2 \in h_2} d_{p,v}(\gamma_1, \gamma_2).$$

Similarly, the maxmin operator used in Definition 13 is an extension of the traditional discordance index by integrating ELECTRE III method and hesitant fuzzy information.

Especially, if h_1 is indifferent to h_2, then the discordance index should be equal to zero, which is still in accordance with the traditional discordance index. The conclusions are easy to drawn that when both h_1 and h_2 are reduced to the single values, $t_{p,v}(h_1, h_2)$ turns into a discordance index as introduced in Definition 10.

According to Definition 13, it is easy to get the following property.

Property 5 Let $h_1, h_2 \in$ HFNS, p and $v(p \le v)$ be two thresholds, and then

$$0 \le t_{p,v}(h_1, h_2) \le 1.$$

Definition 14 The strong opposition relation, weak opposition relation and indifferent opposition relation for HFNs are defined as follows.

(1) If $t_{p,v}(h_1, h_2) = 1$, then h_1 strongly opposes $h_2(h_2$ is strongly opposed by $h_1)$, denoted by $h_1 >_{SO} h_2$.
(2) If $t_{p,v}(h_1, h_2) - t_{p,v}(h_2, h_1) = 0$, then h_1 is indifferently opposed to h_2, denoted by $h_1 \sim_O h_2$.
(3) If $0 < t_{p,v}(h_1, h_2) - t_{p,v}(h_2, h_1) < 1$, then h_1 weakly opposes h_2 (h_2 is weakly opposed by h_1), denoted by $h_1 >_{WO} h_2$.
(4) If $0 < t_{p,v}(h_2, h_1) - t_{p,v}(h_1, h_2) < 1$, then h_2 weakly opposes h_1 (h_1 is weakly opposed by h_2), denoted by $h_2 >_{WO} h_1$.

Example 5 Let $p = 0.2$, $v = 0.3$.

(1) If $h_1 = \{0.1, 0.2\}$, $h_2 = \{0.5, 0.7\}$, then $t_{p,v}(h_1, h_2) = 1$. So $h_1 >_{SO} h_2$.
(2) If $h_1 = \{0.2, 0.5\}$, $h_2 = \{0.1, 0.6\}$, then $t_{p,v}(h_1, h_2) - t_{p,v}(h_2, h_1) = 0$. So $h_1 \sim_O h_2$.
(3) If $h_1 = \{0.2, 0.5\}$, $h_2 = \{0.45, 0.75\}$, then $t_{p,v}(h_1, h_2) - t_{p,v}(h_2, h_1) = 0.5$. So $h_1 >_{WO} h_2$.
According to Definitions 10, 13 and 14, similar to Properties 2, 3 and 4, the following properties are true.

Property 6 Let $h_1, h_2 \in$ HFNS, p and $v(p < v)$ be two thresholds, and then $h_1 >_{SO} h_2$, if and only if $\min\{r|r \in h_2\} - \max\{r|r \in h_1\} \ge v$.

Property 7 Let $h_1, h_2, h_3 \in$ HFNS, and p and $v(p < v)$ be two thresholds. If $h_1 >_{SO} h_2$ and $h_2 >_{SO} h_3$, then $h_1 >_{SO} h_3$.

Property 8 Let $h_1, h_2, h \in$ HFNS, p and $v(p < v)$ be two thresholds, then

(1) The strict opposition relations are categorized into:
①irreflexivity: $\forall h \in$ HFNS, $h \not>_{SO} h$;
②asymmetry: $\forall h_1, h_2 \in$ HFNS, $h_1 >_{SO} h_2 \not\Rightarrow h_2 >_{SO} h_1$;
③transitivity: $\forall h, h_1, h_2 \in$ HFNS, $h >_{SO} h_1, h_1 >_{SO} h_2 \Rightarrow h >_{SO} h_2$.
(2) The weak opposition relations are also categorized into:
④irreflexivity: $\forall h \in$ HFNS, $h \not>_{WO} h$;
⑤asymmetry: $\forall h_1, h_2 \in$ HFNS, $h_1 >_{WO} h_2 \not\Rightarrow h_2 >_{WO} h_1$;
⑥non-transitivity: $\exists h, h_1, h_2 \in$ HFNS, such that $h >_{WO} h_1, h_1 >_{WO} h_2 \not\Rightarrow h >_{WO} h_2$.

(3) The indifferent opposition relations are categorized into:
⑦ reflexivity: $\forall h \in$ HFNS, $h \sim_O h$;
⑧ symmetry: $\forall h_1, h_2 \in$ HFNS, $h_1 \sim_O h_2 \Rightarrow h_2 \sim_O h_1$;
⑨ non-transitivity: $\exists h, h_1, h_2 \in$ HFNS, such that $h \sim_O h_1, h_1 \sim_O h_2 \not\Rightarrow h \sim_O h_2$.

Example 6 We exemplify the false arguments on ⑥ and ⑨.

(1) Let $h = \{0.1, 0.4\}, h_1 = \{0.3, 0.5\}, h_2 = \{0.5, 0.6\}, p = 0.15, v = 0.25$, and then $t_{p,v}(h, h_1) - t_{p,v}(h_1, h) = 0.5$, $t_{p,v}(h_1, h_2) - t_{p,v}(h_2, h_1)$=0.5, $t_{p,v}(h, h_2) = 1$. So $h_1 >_{WO} h_2, h >_{WO} h_1, h >_{SO} h_2$.
(2) Let $h = \{0.1, 0.4\}$, $h_1 = \{0.2, 0.5\}$, $h_2 = \{0.3, 0.6\}$, $p = 0.1$, $v = 0.2$, and then $t_{p,v}(h, h_1) - t_{p,v}(h_1, h)$=0, $t_{p,v}(h_1, h_2) - t_{p,v}(h_2, h_1) = 0$, $t_{p,v}(h, h_2) - t_{p,v}(h_2, h) = 1$. So $h \sim_O h_1, h_1 \sim_O h_2, h >_{SO} h_2$.

4 Outranking approach of MCDM with HFNs

A total number of $n$alternatives are contained in an MCDM ranking or selection problem with hesitant fuzzy information. They are denoted by $A = \{a_1, a_2, \ldots, a_n\}$, where each alternative is assessed by means of m criteria, denoted by $C = \{c_1, c_2, \ldots, c_m\}$. a_{ij} is the value of the alternative a_i for the criterion c_j, represented by HFNs. Decision-makers are requested to provide their preferences anonymously so as to protect their privacies to obtain a more reasonable result. The weight of the criterion c_j is ω_j, where $j = 1, 2, \ldots, m$, $\sum_{j=1}^{m} \omega_j = 1$.

Our method is an integration of HFSs and ELECTRE III to solve MCDM problems mentioned above. We set the thresholds q_j, p_j and v_j $(0 \leq q_j \leq p_j \leq v_j)$ associate with the criterion c_j.

The set of subscripts that for all criteria meeting the constraint $a_{ik} >_Z a_{sk}$ or $a_{ik} >_W a_{sk}$ is

$$O(a_i, a_s) = \{k | 1 \leq k \leq m, a_{ik} >_Z a_{sk}\, or\, a_{sk} >_W a_{ik}\} \quad (i = 1, 2, \ldots, n; s = 1, 2, \ldots, n), \tag{1}$$

where $Z = \{S, W, I\}$. $a_{ik} >_S a_{sk}$ means a_{ik} strongly dominates a_{sk}. $a_{ik} >_I a_{sk}$ means a_{ik} is indifferent to a_{sk}. $a_{ik} >_W a_{sk}$ means a_{ik} weekly dominates a_{sk}.

Using the weight vector ω associated with criteria, we define the comprehensive concordance index $C(a_i, a_s)$ as follows:

$$C(a_i, a_s) = \sum_{k \in O(a_i, a_s)} \omega_k r_{p_k, q_k}(a_{ik}, a_{sk}). \tag{2}$$

Here $C(a_i, a_s)$ ranges from 0 to 1, a value of 0 indicates that alternative a_i is worse than alternative a_s.

Then, the concordance matrix C can be fabricated as

$$C = \begin{pmatrix} - & c_{12} & c_{13} & \cdots & c_{1(n-1)} & c_{1n} \\ c_{21} & - & c_{23} & \cdots & c_{2(n-1)} & c_{2n} \\ \cdots & \cdots & - & \cdots & \cdots & \cdots \\ c_{(n-1)1} & c_{(n-1)2} & c_{(n-1)3} & \cdots & - & c_{(n-1)n} \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{n(n-1)} & \end{pmatrix}. \tag{3}$$

The credibility index of outranking relations is defined as follows:

$$\sigma(a_i, a_s) = \begin{cases} C(a_i, a_s), & \text{If } F = \phi \\ C(a_i, a_s) \prod\limits_{j \in F} \frac{1 - t_{p_j, v_j}(a_{ij}, a_{sj})}{1 - C(a_i, a_s)}, & \text{If } F \neq \phi \end{cases}, \tag{4}$$

where $F = \{j : t_{p_j, v_j}(a_{ij}, a_{sj}) > C(a_i, a_s)\}$.

The ranking index of the alternatives is defined as follows:

$$\delta(a_i) = \sum_{s=1}^{n} \sigma(a_i, a_s) - \sum_{s=1}^{n} \sigma(a_s, a_i). \tag{5}$$

Under such a correlation, the larger $\delta(a_i)$ is, the better the alternative a_i is.

It is now feasible to develop a new approach for MCDM problems mentioned above.

Step 1. Determine the thresholds.
The thresholds p_k, q_k and v_k, which are associated with the criterion c_k and satisfy $0 \le q_k \le p_k \le v_k$, are set by decision-maker(s).
Step 2. Calculate the concordance index to get the concordance matrix.
According to Definitions 9 and 11, $\mathrm{r}_{p_k, q_k}(a_{ik}, a_{jk})(i = 1, 2, \ldots, n, j = 1, 2, \ldots, n)$ are calculated for each $c_k (k = 1, 2, \ldots, m)$, so as the comprehensive concordance index with ω_k according to (2). Then, the concordance matrix is obtained with consistence to (3).
Step 3. Calculate the credibility index.
According to Definitions 10 and 13, the detailed values of $t_{p_k, v_k}(a_{ik}, a_{jk})(i = 1, 2, \ldots, n, j = 1, 2, \ldots, n)$ can be calculated. According to the results of Step 2, it is necessary to find all the values of k which meet $t_{p_k, v_k}(a_{ik}, a_{jk}) > C(a_i, a_j)$. Thus, the credibility index of outranking relations is calculated according to (4).
Step 4. Rank all alternatives.
Using ranking index expressed by (5), $\delta(a_i)$ is obtained, and the ranking of all alternatives is thus followed.

5 Illustrative example

In this section, an example is adapted from Wei (2012) for further illustration. The school of management in a Chinese university is planning to introduce some outstanding teachers from overseas for strengthening academic capability and enhancing their

Table 1 Hesitant fuzzy decision matrix

	c_1	c_2	c_3	c_4
a_1	{0.4, 0.5, 0.7}	{0.5, 0.8}	{0.6, 0.7, 0.9}	{0.5, 0.6}
a_2	{0.6, 0.7, 0.8}	{0.5, 0.6}	{0.4, 0.6, 0.7}	{0.4, 0.5}
a_3	{0.6, 0.8}	{0.2, 0.3, 0.5}	{0.4, 0.6}	{0.5, 0.7}
a_4	{0.5, 0.6, 0.7}	{0.4, 0.5}	{0.8, 0.9}	{0.3, 0.4, 0.5}
a_5	{0.6, 0.7}	{0.5, 0.7}	{0.7, 0.8}	{0.2, 0.3, 0.4}

teaching quality. The project has attracted great attention over the campus. The university's president, the dean of management school and the human resource officer set up the panel of decision-makers, responding the major responsibility in the introduction. They make strict evaluation of five alternatives, denoted by a_1, a_2, a_3, a_4, a_5, according to the following four criteria: morality, research capability, teaching skills and education backgrounds, denoted by c_1, c_2, c_3, c_4 with the weight vector ω =(0.45, 0.25, 0.2, 0.1). The evaluation of these five candidates $a_i (i = 1, 2, 3, 4, 5)$ is performed with HFNs by three decision-makers under $c_k (k = 1, 2, 3, 4)$. The hesitant fuzzy decision matrix $(a_{ij})_{5\times4}$ is constructed and shown in Table 1.

The procedures of obtaining the optimal alternative, using the developed method, are shown as follows.

Step 1. Determine the thresholds.
For simplicity of calculation, we set $q_k = q = 0.05$, $p_k = p = 0.25$, $v_k = v = 0.3$ for all criteria $c_k (k = 1,2,3,4)$.
Step 2. Calculate the concordance matrix.
Then, the values of $r_{p_k,q_k}(a_{ik}, a_{jk}) (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ are calculated for each criterion. The comprehensive concordance index is calculated based on

$$C(a_i, a_j) = \sum_{k\in O(a_i,a_j)} \omega_k r_{p_k,q_k}(a_{ik}, a_{jk}).$$

So, the concordance matrix is:

$$C = \begin{pmatrix} - & 0.8875 & 0.8625 & 1 & 1 \\ 0.6375 & - & 0.925 & 0.85 & 0.8875 \\ 0.55 & 0.8875 & - & 0.8 & 0.6625 \\ 0.725 & 0.825 & 0.8125 & - & 0.8125 \\ 0.8125 & 0.8625 & 0.7875 & 0.925 & - \end{pmatrix}.$$

Step 3. Calculate the credibility index.
Calculate $t_{p_k,v_k}(a_{ik}, a_{jk}) (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$. According to the results of Step 2, a filter of all the values of k to meet $t_{p_k,v_k}(a_{ik}, a_{jk}) > C(a_i, a_j)$ is needed. The calculation of the credibility index of outranking relations according to (4) can be obtained as follows.

Since $r_{p,q}(a_{11}, a_{21}) = 0.75$, and $r_{p,q}(a_{21}, a_{11}) = 1$, then $r_{p,q}(a_{21}, a_{11}) - r_{p,q}(a_{11}, a_{21}) = 0.25$. So $a_{21} >_W a_{11}$.

Similarly, we can get $r_{p,q}(a_{12}, a_{22}) = 1$, $r_{p,q}(a_{13}, a_{23}) = 1$, and $r_{p,q}(a_{14}, a_{24}) = 1$. According to the weight vector ω =(0.45,0.25,0.2,0.1) and (1), there is $C(a_1, a_2) = 0.75 \times 0.45 + 1 \times 0.25 + 1 \times 0.2 + 1 \times 0.1 = 0.8875$.

And because $t_{p,v}(a_{11}, a_{21}) = 0$, $t_{p,v}(a_{12}, a_{22}) = 0$, $t_{p,v}(a_{13}, a_{23}) = 0$, and $t_{p,v}(a_{14}, a_{24}) = 0$, we get $F = \phi$.

From (4), we can get $\sigma(a_1, a_2) = 0.8875$. In a similar way, we can get other values of $\sigma(a_i, a_j)(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ as follows:

$\sigma(a_1, a_3) = 0.8625$, $\sigma(a_1, a_4) = 1$, $\sigma(a_1, a_5) = 1$, $\sigma(a_2, a_1) = 0.6375$, $\sigma(a_2, a_3) = 0.925$, $\sigma(a_2, a_4) = 0$, $\sigma(a_2, a_5) = 0$, $\sigma(a_3, a_1) = 0$, $\sigma(a_3, a_2) = 0$, $\sigma(a_3, a_4) = 0$, $\sigma(a_3, a_5) = 0$, $\sigma(a_4, a_1) = 0.725$, $\sigma(a_4, a_2) = 0.825$, $\sigma(a_4, a_3) = 0.8125$, $\sigma(a_4, a_5) = 0.8125$, $\sigma(a_5, a_1) = 0$, $\sigma(a_5, a_2) = 0.8625$, $\sigma(a_5, a_3) = 0$, and $\sigma(a_5, a_4) = 0.925$.

Step 4. Rank all alternatives.

Using (5), we calculate the values of $\delta(a_i)(i = 1, 2, 3, 4, 5)$. They are

$\delta(a_1) = 2.3875$, $\delta(a_2) = -1.0125$, $\delta(a_3) = -2.6$, $\delta(a_4) = 1.25$, and $\delta(a_5) = -0.025$. Thus, the ranking of the alternatives are: $a_1 \succ a_4 \succ a_5 \succ a_2 \succ a_3$, and the most desirable alternative is a_1.

The results based on the HFPWA operator and the HFPWG operator which were presented by Wei (2012) are $a_5 \succ a_2 \succ a_1 \succ a_4 \succ a_3$ and $a_2 \succ a_5 \succ a_1 \succ a_4 \succ a_3$, respectively. Based on the HFPWA operator of Wei (2012), we apply the Conditions (5)–(8) in Definition 5 to calculate the example, then the result is $a_5 \succ a_2 \succ a_1 \succ a_4 \succ a_3$. Although the final results related to the same illustrative example, based on the HFPWA operator, are the same, we do have provided some other scenarios such as Examples 1 and 2 to justify that different results could be obtained in case that different operations are applied.

Farhadinia (2013a) presented a new score function for ranking HFEs and applied the same HFPWA operator of Wei (2012) to the same illustrative example. The final result was $a_5 \succ a_1 \succ a_2 \succ a_4 \succ a_3$. These methods were based on the same operations but not the comparison methods. Although they were based on the same example, these methods produced different ranking results. So the final result is not robust through different approaches. As we have mentioned in Examples 1 and 2 that different operations could produce different ranking results, to avoid such problems caused by different operations and comparison methods, we have proposed a method that was not based on any operations or comparison methods.

The previous stated ranking result is different from those obtained from either Wei's method or Farhadinia's method. The reasons were concluded but not limited to the pairwise comparison that our approach based upon, among the sets of possible values of the pair of HFNs. It is an extension of the traditional ELECTRE III method, and has the non-compensatory as its distinguished feature to other MCDM methods. It indicates that good scores on other criteria may not, in particular, be used to compensate a very bad score on a criterion. As the results yielded from different operations and comparison methods vary a lot, it is difficult for decision-makers to choose the best one. Such difficulty has been illustrated with two methods improved by Wei (2012) and Farhadinia (2013a), which are based on the same example and Examples 1 and 2. The main advantages of the proposed approach over those hesitant fuzzy operators are not only overcoming the problems caused using operations and comparison methods, but also the feature of non-compensatory and the consideration of decision-makers' preferences, which are represented by choosing the appropriate thresholds towards the criteria. And it is easy to verify that the proposed approach has the ability to deal with the case where some values may repeat more than once in an HFS. Moreover, this integration makes the new method more appropriate for MCDM problems.

6 Conclusion

The HFS theory is a powerful tool to deal with MCDM problems, when decision-makers have hesitancy in providing their preferences over objects. Unlike A-IFSs, HFSs expand the membership degree of an element from the single values to a set of possible values. In this paper, the theories about HFSs are reviewed at first. Then, the definition of dominance relation and opposition relation for HFSs is given, as motivated by the idea of traditional ELECTRE methods. Some properties are then described in details. Based on the previous definitions of outranking relations, this paper contains an outranking approach for MCDM problems with HFNs. The prominent advantages of the proposed approach are both the elimination of shortcomings of regular operations and comparison methods, and the feature of non-compensatory which distinguished itself from many other MCDM methods. Such approach also takes decision-makers' preferences into consideration, being represented by choosing the appropriate thresholds towards the criteria. It can be extended to the case where some values may repeat more than once in an HFS. This method is useful when the decision-makers deal with multi-criteria ranking problem with hesitant fuzzy information.

In this paper, the thresholds p, q and v are considered as constants to each criterion. In some other scenarios, they can be generalized to the functions varying with the value of the criteria, or to say, in the case of variable thresholds. The weight vector, in the paper, is given with certain information, but in some cases, decision-makers could give the imprecise weight information, especially in the form of HFSs. Assessing the weights by HFSs is a critical issue which remains to the further research. In addition, only the case that the membership degree of HFSs is a real number has been studied in the paper. It is considerable and meaningful to extend them into interval numbers or other uncertain forms in further research. Moreover, it is also a critical issue that the evaluations given by some decision-makers are partially or totally concordant. HFSs, as the rising theory, remain many theoretical problems needed to be solved as soon as possible, and hence in the future, a more comprehensive and useful method should be put forward to deal with the more imprecise, inaccurate and uncertain evaluations given by decision-makers.

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