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REGULAR ARTICLE

Line planning in public transportation: models and methods

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Abstract The problem of defining suitable lines in a public transportation system (bus, railway, tram, or underground) is an important real-world problem that has also been well researched in theory. Driven by applications, it often lacks a clear description, but is rather stated in an informal way. This leads to a variety of different published line planning models. In this paper, we introduce some of the basic line planning models, identify their characteristics, and review literature on models, mathematical approaches, and algorithms for line planning. Moreover, we point out related topics as well as current and future directions of research.

Keywords Line planning · Public transportation · Mathematical programming

1 The line planning problem in public transportation

Given the increasing demand for mobility, an efficient organization of public (passenger) transportation becomes more and more important. This is reflected not only in practice but also by an increasing number of research papers dealing with the optimization of public transport. The goal of the optimization process is on the one hand, to offer a high quality of service for the passengers while, on the other hand, the costs for setting up and running the transit system should be small.

As noted by many authors (see e.g., Ceder and Wilson 1986; Liebchen and Möhring 2007; Desaulniers and Hickman 2007), the planning process in public transportation consists of several consecutive planning phases. As shown in Fig. 1, the process starts

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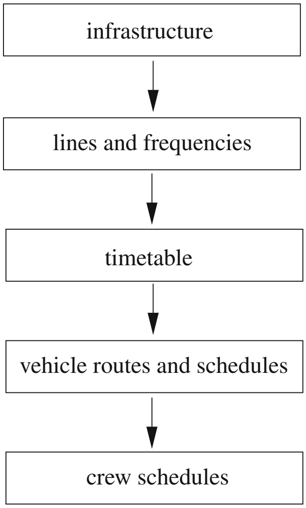

Fig. 1 The planning steps in public transportation in their classical order

with network design, usually followed by line planning, timetabling, and vehicle and crew scheduling.

In this paper, we review the line planning process. We assume that the infrastructure is already given and represented as a *public transportation network (PTN)*. In particular, we assume the stations to be fixed and a set of possible edges to be given. These can be streets (in bus transportation) or the track system (in rail, tram, or underground transportation). We are looking for a set of *lines* (a formal definition will be given in the next section) in the PTN along which service will be offered. Line planning hence includes to determine the number and the routes of the lines. It also includes the determination of the *frequencies* of the lines, i.e., how often service should be offered. A line plan together with the frequencies of the lines is called a *line concept*. The *line planning problem* is to find a line concept

- which is feasible in the sense that it can be operated,
- which ensures that public transport is convenient for the passengers, and
- whose costs are small.

We remark that lines together with a rough time frame have been also denoted as *(periodic) service intensions* (see Caimi et al. 2009) as the line concept contains the customer-relevant information about the public transport system.

Stating the line planning problem as above leaves most modeling aspects open. In particular, there are many different ways to model the costs of a line concept, and there are even more possibilities to evaluate its quality for the passengers. Moreover, different constraints (ensuring feasibility) may be considered. This may explain why there exist so many different *models* for the line planning problem.

The remainder of the paper is structured as follows. In the next section, we introduce the basic notation and discuss characteristics of line planning models. We also present some “basic” and some common line planning models. In Sect. 3, we review literature on these line planning models including modeling aspects and solution approaches.

Finally, recent directions of research are sketched in Sect. 4, and suggestions for future research are identified in Sect. 5.

2 Models for the line planning problem

2.1 Modeling the line planning problem

In this section, we will identify the basic ingredients of the line planning models treated in the literature. In order to describe them properly, we need some notation.

Notation 1 *Let $PTN = (V, E)$ be a public transportation network with its stops or stations V and its direct connections E.*

The PTN describes the underlying street or track network which we assume to be given and fixed. In this section, we assume that all lines will be operated by homogeneous vehicles, i.e., we only consider one single transport mode. This is certainly not a realistic assumption, but nevertheless makes algorithmically sense since different modes of transport are often considered one after another. In Sect. 3, we will mention generalizations of these assumptions made in the literature.

Notation 2 *A* **line** *l is a path in the public transportation network PTN.*
The **frequency** *f_l of a line l says how often service is offered along line l within a given time period I (e.g., an hour, a day).*
A **line concept** *$(\mathcal{L}, f)$ is a set of lines $\mathcal{L}$ together with their frequencies f_l for all $l \in \mathcal{L}$.*

Note that the frequencies determine how often a line is served within the period I, but leave it open if this happens in regular intervals (as done in a cyclic timetable) or in an aperiodic way. It is important to choose the interval I appropriate for the goals one has in mind: If a special line concept for the morning peak should be determined then I should refer to the time span of the morning peak; if the line concept should be the same for all periods of the day then I can be chosen as, e.g., a month.

Also there may be several restrictions on the lines, e.g., it is sometimes required that lines should be symmetric (i.e., operated in both directions).

As mentioned before, there are two conflicting goals when determining a line concept. On the one hand, costs should be small; on the other hand, the line concept should be as good as possible for the passengers. Consequently, there are cost- and passenger-oriented components in all models for the line planning problem. We will call a model *passenger-oriented* if its objective is to maximize the quality of the line concept (costs may be restricted in the constraints). Analogously, a model in which the goal is to minimize the costs while the constraints ensure a minimum level of quality for the passengers will be called *cost-oriented*.

In the following we discuss how cost functions and service quality can be modeled and evaluated. We will also point out other typical constraints in line planning models.

Modeling the costs

In order to determine the costs of a line concept, data about the expenses for vehicles, drivers, etc. need to be known. Often they are aggregated to some fixed cost per line and some variable cost which depends on the frequency. The most common approximation of the costs of a line concept $(\mathcal{L}, f)$ is

$$c(\mathcal{L}, f) = \sum_{l \in \mathcal{L}} \text{cost}_l f_l. \tag{1}$$

where cost_l describes the costs for operating one vehicle along line l during the planning period I.

It depends on the length of l, the time needed for a complete trip along line l, and on the costs per kilometer and per minute driving, but is independent of the frequency f_l. Many details about the calculation of more realistic costs may be considered with respect to the specific transportation mode such as, e.g., the number of cars of a train in railway transportation (see Claessens et al. 1998). We remark that the final costs depend not only on the lines but also on the vehicle and crew schedules, but these cannot be taken into account since they are not known in the line planning stage.

In order to keep the costs in a passenger-oriented model small enough $c(\mathcal{L}, f)$ can be bounded by a budget constraint

$$\text{(BUD)} \quad \sum_{l \in \mathcal{L}} \text{cost}_l f_l \leq B \tag{2}$$

for some given budget $B \geq 0$.

Modeling the quality for the passengers

Also for dealing with the passengers' data, it is helpful to assume one single transit mode served by a homogeneous fleet. If this is not the case, the first step suggested by many authors is the *system split procedure* (Oltrogge 1994; Bouma and Oltrogge 1994) which distributes the passengers to the different modes of transport and hence results in one problem for each transport mode. Note that this cannot always be done, e.g., if a new rapid transit line within an existing transport system is to be introduced, or if the different transport modes cannot be separately investigated. Such generalizations will be pointed out in Sect. 3.

After the system split, we are left with a (single-mode) line planning problem for each type of transportation. The following two types of data about the passengers may be considered.

Notation 3 $(W_{uv})_{u,v \in V}$ *denotes the* **origin–destination matrix (OD-matrix)***, i.e., W_{uv} is the number of passengers who want to travel from an origin (station) $u \in V$ to a destination (station) $v \in V$ within the planning period I. If $W_{uv} > 0$ we say that (u, v) is an* **OD-pair**.
The **traffic load** *w_e is the number of passengers traveling along edge $e \in E$ within the planning period I.*

In order to evaluate the quality of a system for the passengers, the most common objective functions are based on the following notation.

Notation 4 *A* **direct traveler** *is a passenger who does not have to transfer between different lines on his way between his origin* u *and his destination* v, $u, v \in V$.
The **riding time** *of a passenger is defined as the time the passenger sits in a bus/train on his way between his origin and his destination. It neglects the time needed for transfers.*

The **traveling time** *of a passenger consists of the riding time and a penalty (time) for every transfer.*

The exact traveling time can only be calculated if the timetable is known (which is not the case at the stage of line planning). Using a penalty for each transfer is hence an approximation of the real transfer time.

Common objective functions are to maximize the number of direct travelers, to minimize the sum of all riding times of all passengers, or to minimize the sum of all traveling times of all passengers.

Let us now consider constraints which can be used in a cost-oriented model to ensure a minimum level of quality for the passengers. The most simple ones are based on the traffic loads w_e which are often assumed to be known beforehand. Given a line concept $(\mathcal{L}, f)$ the capacity constraints

$$\text{(CAP)} \quad \sum_{l \in \mathcal{L}: e \in l} \text{cap}_l f_l \geq w_e \quad \text{for all } e \in E \tag{3}$$

ensure that w_e passengers can be transported along edge e, where cap_l denotes the capacity of a vehicle operating line l, i.e., the number of seats in the bus, tram, underground, or train, respectively. Note that models using (CAP) assume the capacities to be fixed and known in advance.

In some papers, in particular, when the goal is to maximize the number of direct travelers, a modified capacity constraint (CAP') is used which requires enough capacity on every edge for each single line instead of looking at the sum of lines.

An even simpler constraint which guarantees a minimum level of service is to look at the *edge frequencies*, i.e., the number of vehicles running along the edges. A line concept $(\mathcal{L}, f)$ satisfies the *lower edge frequency requirements* if

$$\text{(LEF)} \quad \sum_{l \in \mathcal{L}: e \in l} f_l \geq f_e^{\min} \quad \text{for all } e \in E. \tag{4}$$

The parameter $f_e^{\min}$ is often defined as the minimal number of vehicles needed to transport all passengers: If the capacity cap is the same for all vehicles, and w_e is an approximation of the number of passengers traveling along edge e the lower frequency requirements with $f_e^{\min} := \lceil \frac{w_e}{\text{cap}} \rceil$ are equivalent to the capacity constraints (CAP).

If a set of OD-pairs is given, a minimal requirement is that all passengers can travel between their origins and their destinations, no matter how long the journey takes and how many transfers are necessary, i.e., every OD-pair should be connected by the lines in the line concept. This condition is denoted as (CON).

Apart from the different objectives and constraints, there is one more crucial difference in passenger-oriented line planning models: Up to the beginning of this century, mathematical line planning models used a given OD-matrix only to distribute the passengers on paths in the transportation network *before* the lines were known. This first step is known as *traffic assignment* and generates the traffic loads w_e by routing passengers through the PTN along suitable paths. Many procedures for traffic assignment are known. These include shortest path approaches or approaches in which one assumes that passengers are likely to change to fast transportation modes (e.g., high-speed trains) as early as possible and leave them as late as possible. The dependencies between the passengers' paths is considered in equilibrium models. We do not review the literature on traffic assignment here but refer to Patriksson (1994), Carraresi et al. (1995), Peeta and Ziliaskopoulos (2001), Desaulniers and Hickman (2007), and Babonneau and Vial (2008) and references therein.

However, the *real* passengers' weights along every edge strongly depend on the line concept which is to be designed, leading to a chicken-egg-problem. Hence, new approaches in which the passengers' routes are part of the optimization and are *not* fixed beforehand, become more and more popular. Such approaches integrate line planning and traffic assignment. The underlying mathematical model is the *change & go graph* developed in Schöbel and Scholl (2006a). It allows to include the route choice as (individual) shortest path problem for every OD-pair within the optimization.

Notation 5 *Given a public transportation network PTN* $= (V, E)$ *and a set of potential lines* $\mathcal{L}^0$, *its corresponding* **change & go graph** *consists of a set of nodes*

$$\left\{(v, l) : v \in \mathbf{V} \textit{is a station of line } l \in \mathcal{L}^0\right\}$$

for every line-station combination and a set of edges

$$\{((v, l_1), (u, l_2)) : \textit{ Either } (v = u) \textit{ or } (l_1 = l_2 \textit{ and } (u, v) \in E)\},$$

i.e., two nodes of the change & go graph are linked if they are consecutive stations within the same line or if they belong to the same station. The latter case allows to model transfers between the corresponding lines.

By calculating shortest paths in the change & go graph the individual traveling time of a passenger traveling through the public transportation network can be evaluated keeping also track of the number of transfers necessary. Note that different weights for different transfers are possible, e.g., one can model long transfer times at large stations.

Technical constraints

Various technical restrictions ensuring the operability of a line concept are also treated in line planning models. Symmetric to the lower edge frequency requirements (see (4)) often *upper edge frequency requirements* are used. For a line concept $(\mathcal{L}, f)$ these are given as

$$\text{(UEF)} \quad \sum_{l \in \mathcal{L}: e \in l} f_l \leq f_e^{\max} \quad \text{for all } e \in E \tag{5}$$

where $f_e^{\max}$ models restrictions such as security headways or noise avoidance. It sometimes is also used to bound the costs of the line concept in passenger-oriented models. Note that (LEF) and (UEF) have first been introduced by Wegel (1974).

Similar to the edge frequency requirements, also *lower/upper node frequency requirements* may be considered. They restrict the number of vehicles stopping at particular stations using lower bounds $f_v^{\min}$, or upper bounds $f_v^{\max}$, respectively. For a line concept $(\mathcal{L}, f)$ they are defined as

$$\text{(LNF)} \quad \sum_{l \in \mathcal{L}: v \in l} f_l \geq f_v^{\min} \quad \text{for all } v \in V, \tag{6}$$

$$\text{(UNF)} \quad \sum_{l \in \mathcal{L}: v \in l} f_l \leq f_v^{\max} \quad \text{for all } v \in V. \tag{7}$$

Feasibility of lines

Line planning models also differ with respect to the restrictions on the lines which may be chosen for the line concept. In the literature, two major possibilities to model *feasibility of lines* are considered.

The first one assumes that a *line pool* $\mathcal{L}_0$ of potential lines is given. It may have been collected by the transportation company and consists of what is considered as "reasonable" lines. All of the lines in the line pool may be established. The goal is to choose a subset of lines from the line pool, i.e., it is required that $\mathcal{L} \subseteq \mathcal{L}_0$. This approach can be seen as the second phase of a two-step approach: the lines are constructed in a first phase, and a line plan is chosen from this set in a second phase. A distinction into these two steps and heuristics constructing lines for the first phase have for example been suggested in Ceder and Wilson (1986). Note that most line planning models use this approach and hence deal with the selection of a set of lines out of a given line pool $\mathcal{L}_0$. Various models and algorithms exist for constructing a set of reasonable routes, many of them are based on shortest path procedures, some of them driven by the passengers' demand. We refer to Kepaptsoglou and Karlaftis (2009, p. 498–499) for an overview about such methods.

In the second approach, no predefined lines are given, but the lines are constructed from scratch within the optimization process. This has often been done heuristically (see Sect. 3.5) and more recently also in an exact way (Borndörfer et al. 2007). Note that constructing the lines from scratch does usually not mean that all possible paths in the PTN may be chosen as lines. Often, there are rules that have to be respected during the construction, or lines have to satisfy certain criteria regarding their shapes.

The characteristics mentioned in these sections are summarized in two tables. Table 1 collects the objective functions which are common in line planning models while Table 2 lists some of the basic constraints.

2.2 Complexity of line planning models

In order to discuss the complexity of line planning models we define the following *basic model*:

Table 1 Objective functions in line planning models

Cost-V	Variable costs of lines are considered	See Eq. (1)
Cost-F	Fixed costs of lines are considered	
Cost-FV	Fixed and variable costs of lines are considered	
Pass-DT	Number of direct travelers	Notation 4
Pass-RT	Riding time	Notation 4
Pass-TT	Traveling time	Notation 4

Table 2 Basic constraints in line planning models

BUD	Budget constraint	(2)
CAP	Capacity constraint	(3)
CAP'	Capacity constraint for single lines	Page 495
LEF	Lower edge frequency requirements	(4)
UEF	Upper edge frequency requirements	(5)
LNF	Lower node frequency requirements	(6)
UNF	Upper node frequency requirements	(7)
CON	Connected path for every OD-pair	Page 495

(LP-basic) *Given a PTN, a set* $\mathcal{L}^0$ *of potential lines, and lower and upper frequencies* $f_e^{\min} \leq f_e^{\max}$ *for all* $e \in E$, *find a line concept* $(\mathcal{L}, f)$ *with* $\mathcal{L} \subseteq \mathcal{L}^0$, $f_l \in \mathbb{N}_0 \ \forall\, l \in \mathcal{L}$, *and* $f_e^{\min} \leq \sum_{l \in \mathcal{L}: e \in l} f_l \leq f_e^{\max}$ *for all* $e \in E$.

Although (LP-basic) looks rather simple, it is already NP-hard as shown in Bussieck (1998) by *exact cover by 3-sets (X3C)*, even for the special case that $f_e^{\min} = f_e^{\max} = 1$ for all $e \in E$. Since most other models contain (LEF) and (UEF) as constraints, this result shows NP-hardness for most line planning models.

There are two special cases of (LP-basic) which are easy to solve:

- If upper frequencies are neglected (i.e., $f_e^{\max} = \infty$ for all $e \in E$) a feasible solution exists if every edge with positive $f_e^{\min}$ is contained in at least one line, and it can be found easily by adding enough frequency to the lines until all lower edge frequency requirements are satisfied.
- If all paths of the PTN are allowed as potential lines (LP-basic) is always feasible and can also be solved in polynomial time by taking each edge as a line, e.g., with frequency $f_e^{\min}$.

These simple cases are sometimes used as starting points for heuristic approaches.

We now add a cost objective function to (LP-basic) and obtain a cost-oriented model.

Basic cost model (LP-cost) *Given a PTN, a set* $\mathcal{L}^0$ *of potential lines, lower and upper frequencies* $f_e^{\min} \leq f_e^{\max}$ *for all* $e \in E$, *and costs* cost_l for each line $l \in \mathcal{L}^0$ find a line concept $(\mathcal{L}, f)$ which satisfies the constraints of (LP-basic) and minimizes the costs $c(\mathcal{L}, f) = \sum_{l \in \mathcal{L}} \text{cost}_l\, f_l$.

In contrast to (LP-basic), (LP-cost) is NP-hard even in the two special cases

- in which we have no upper frequencies, and
- in the case where all paths of the PTN are allowed as lines.

The first is even true if $\text{cost}_l = 1$ for all lines and $f_e^{\min} = 1$ for all edges as can be seen by reduction to *set covering*. Note that (LP-cost) without upper frequencies is a multi-covering problem and can hence be solved by integer programming approaches for multi-covering. The latter special case is NP-hard by reduction to *Hamiltonian path*.

We remark that by reduction to *vertex cover* Claessens et al. (1998) showed NP-hardness of (LP-cost) without upper frequencies, but with an additional capacity constraint (CAP).

We now turn our attention to passenger-oriented models. Many of the early papers about line planning deal with the *direct travelers approach* (see Dienst 1978). In this model, it is assumed that customers use *preferable paths* (e.g., shortest paths) which are known beforehand and are fixed for each OD-pair (u, v), $u, v \in V$. The costs in the direct travelers approach are bounded due to upper edge frequency requirements (UEF). The problem is NP-hard since (LP-basic) is included as a special case.

(Direct-travelers) *Given a PTN, a set $\mathcal{L}^0$ of potential lines, lower and upper frequencies $f_e^{\min} \le f_e^{\max}$ for all $e \in E$, and an OD-matrix W_{uv} together with fixed paths for each OD-pair (u, v) for all $u, v \in V$. The goal is to find a line concept $(\mathcal{L}, f)$ which satisfies the constraints of (LP-basic) and maximizes the number of direct travelers.*

More recently, models have been suggested that look not only at direct travelers but consider the traveling times of all passengers in the system. Such traveling time models usually allow that passengers are routed freely within the optimization. Costs are controlled by a budget constraint.

(Traveling-time) *Given a PTN, a set $\mathcal{L}^0$ of potential lines, costs c_l for each line $l \in \mathcal{L}^0$, a budget B, and an OD-matrix W_{uv} for all $u, v \in V$. The goal is to find a line concept $(\mathcal{L}, f)$ which satisfies (BUD) and minimizes the sum of traveling times of the passengers.*

Also this problem is NP-hard, even if the PTN is a linear graph and all costs c_l are equal to one; see Schöbel and Scholl (2006a).

3 Literature review

Having discussed the most common models, we will now review literature on line planning. We will sketch generalizations of the basic models and review solution approaches. For literature up to the beginning of the 1990s, we refer to the surveys by Israeli and Ceder (1995) and Chua (1984) for more application-oriented approaches.

Our survey is structured as follows. We start with literature on the classical models discussing cost-oriented models in Sect. 3.1 and passenger-oriented models in Sect. 3.2. We then proceed with more recent models, in particular game-theoretic models in Sect. 3.3 and location-based models for line planning in Sect. 3.4. Section 3.5 is devoted to iterative approaches.

3.1 Cost-oriented models

Extensive research on cost-oriented models is presented by Claessens et al. (1998); see also Zwaneveld (1997), Claessens (1994), and Zwaneveld et al. (1996). The goal of this work was to determine lines that minimize the operational costs subject to service constraints and capacity requirements. The (railway) model presented in Claessens et al. (1998) determines not only the lines and their frequencies, but also the type of train operating a line and the number of cars for each train. The costs investigated include fixed costs per car per hour (e.g., cost of capital, fixed maintenance), variable costs per car per kilometer (as energy costs, cleaning costs, or costs for the ticket collector), and variable costs per train per kilometer (e.g., for the driver and for energy). These costs may vary for different types of cars and trains. Using decision variables of type X_l^{tc} which are defined to be one, if line l is served by vehicles of type t with c cars, a nonlinear formulation is presented. In order to linearize the model, its dimension is further increased, and variables of type X_l^{tfc} are used which further include the frequency f of the lines. The algorithm suggested is based upon constraint satisfaction and a branch & bound procedure. Computations on real-world data of the Dutch railway system are reported.

The model has been the basis for many other publications that will be described in the following: Two different linearizations and a cut & branch algorithm for its solution have been developed in Bussieck (1998). Numerical results using instances from the German and the Dutch railway system are reported.

A branch & cut approach (neglecting the upper frequency requirements) based on the models of Claessens et al. (1998) and Bussieck (1998) is proposed in Goossens et al. (2004) and Goossens (2004). Practical experiences with the Dutch Railway system show that the approach runs in reasonable time for real-world instances. Goossens (2004) further investigates an extension called *multi-line planning problem* in which not all trains need to stop at all stations; see also Goossens et al. (2006). The problem is modeled as a multi-commodity flow problem with a flow for each type of train. The numerical results show that its solution is better than the solution obtained from solving a sequence of single-type line planning problems.

A fast procedure for solving the cost model is proposed in Bussieck et al. (2004). The authors use a nonlinear formulation of the model of Claessens et al. (1998). They derive lower bounds for three different linearizations. In their algorithm, they combine linear and nonlinear programming techniques to develop a (heuristic) variable fixing procedure based on finding a feasible solution in the nonlinear model followed by a standard branch & bound approach. Valid inequalities further strengthen their formulation. In their numerical study, they show that their algorithm is able to generate good solutions within a rather small computation time.

Recently, Torres et al. (2008a) analyzed a cost model on simple network topologies motivated by the structure of the PTN in Quito. It is a cost-oriented model in which a sum of fixed costs and variable costs is to be minimized and only lower edge frequency requirements are present. As generalization, the authors allow that fast lines need not stop at all stations. The model has been investigated for simple network structures as corridors and trees. It is shown that the model is NP-hard even in these cases (unless the fixed costs are zero, the fleet is homogeneous and only lines containing each arc forward and backward are considered). Nevertheless, the numerical results indicate that problems whose underlying PTN has a simple structure can be solved by an IP solver in reasonable time. Also Torres et al. (2008b) deal with the feeder lines for the same system in Quito. The model only considers variable costs for the lines and is again investigated for simple line pools and simple PTNs. A polynomial case is identified on trees if all lines of the line pool start in the same terminal station. In Borndörfer et al. (2009), the so-called *line connectivity problem* is defined which is a line planning problem in which the only requirement (CON) is that all passengers can (somehow) travel between their origins and their destinations. The authors point out that the Steiner Tree Problem is a special case of the line connectivity problem and hence of many line planning problems.

3.2 Passenger-oriented models

Direct travelers approach

The maximization of the number of direct travelers was the focus of many of the early research papers. The first formulation of (Direct-traveler) as an integer program is given in Dienst (1978). The solution method proposed in this work is a branch & bound approach which builds a line partition by adding lines one after another. As next node in the branch & bound tree the line with the maximal current direct travelers is chosen in a greedy manner.

The model in Dienst (1978) assumes infinite capacity of the trains. This assumption is not needed in the direct travelers approach presented by Bussieck et al. (1996) and Bussieck (1998). They add a capacity constraint which ensures that all direct travelers can be transported. The resulting models they present are formulated as integer programs and solved by integer programming techniques:

In Bussieck et al. (1996), the edge frequencies are fixed by setting $f_e^{\min} = f_e^{\max}$ for all $e \in E$. An integer programming formulation is presented which is based on variables d_{ijl} representing the number of *direct* travelers between i and j that use line $l \in \mathcal{L}^0$. Aggregating these d_{ijk} variables over all lines together with relaxation techniques leads to a model that can be solved by a cutting plane approach. A slight modification of the model is investigated in Zimmermann et al. (1997). Here, $f_e^{\min} \leq f_e^{\max}$ is allowed such that the edge frequencies are not fixed in advance. Moreover, a budget constraint is introduced. The resulting model is formulated as an integer program and solved by column generation and valid inequalities. Numerical results using data of German, Dutch, and Swiss railways are reported. Many more details can be found in Bussieck (1998), where the problem of maximizing the number of direct travelers subject to lower and upper edge frequency requirements and capacity constraints (CAP') is called *revised direct travelers approach*. Preprocessing and constraint generation techniques are suggested, and an extensive polyhedral analysis is provided. Moreover, a software package for solving the problem is developed.

Minimizing the traveling and the riding time

The first integer programming approach which includes routing of passengers while minimizing the traveling time was presented in Schöbel and Scholl (2003); see also Schöbel and Scholl (2006a) and Scholl (2005). The goal is to design the lines in such a way that the sum of all traveling times of all passengers is minimal, respecting a budget constraint (BUD). Note that the budget constraint is crucial; otherwise, one would establish direct lines (or the shortest possibility that can be achieved using the lines from the line pool) for any passenger. In order to formulate the problem as integer program, the *change & go graph* (see Notation 5) is used leading to variables $x_{O,D,a}$ indicating whether the passengers traveling from origin O to destination D use the arc a of the change & go graph in their shortest path. The problem is tackled by a Dantzig–Wolfe decomposition approach. Different relaxations are compared theoretically and numerically. For practical instances, the authors report that even the linearization is time-consuming to solve. In Schmidt and Schöbel (2010), the complexity of integrating the routing decisions of the passengers into the line planning problem is investigated, and it is shown that the resulting problem is NP-hard even in very special cases. Pseudo-polynomial algorithms are provided for the case of linear networks with some further assumptions.

Independent of Scholl (2005), Borndörfer and Pfetsch (2006) and Borndörfer et al. (2007, 2008) present a model in which passenger paths can also be freely routed. Their objective is to minimize the riding time (neglecting the time for transfers) to which they add fixed costs and variable costs for the line system. They present two multi-commodity flow formulations: The first formulation concerns the case in which all paths are allowed as lines and is the first integer programming model in which both the passengers' paths and the paths of the lines are not fixed in advance but determined within the optimization. Their integer programming formulation is related to a service network model originally presented in Kim and Barnhart (1997). Flow variables for the passengers and flow variables for the lines ensure that the lines are constructed from scratch and the passengers are routed freely through the network.

Their second formulation using a variable for each potential line and for each potential passenger path is the basis for their branch & price approach. The authors show that the resulting pricing problem is NP-hard (Borndörfer et al. 2008). Generating the lines dynamically the problem is solved by a column generation approach which has been tested using the bus system of the city of Potsdam, Germany. In Borndörfer and Pfetsch (2006), some special cases of the models in Borndörfer et al. (2007, 2008) are investigated, namely a model with unsplittable routing (all passengers of the same OD-pair take the same route) and a model where passengers travel along shortest paths. The models are solved by heuristics and compared theoretically and numerically.

Another formulation also allowing the passengers to be freely routed was developed in Nachtigall and Jerosch (2008). Similar to Schöbel and Scholl (2006a), they consider the traveling times of all passengers including a penalty for transfers. Instead of routing passengers in a change & go graph, they look for a combination of partial routes which significantly reduces the number of variables needed making the model numerically tractable. The resulting variables are of type $x_{O,D,u,v,l}$ indicating if the passengers traveling from O to D use the line l between the stations u and v. In their model, they minimize the traveling time using upper edge frequency and upper node frequency requirements (UEF, UNF) and a budget constraint (BUD) in order to bound the costs. Again, column generation is used as solution approach together with a sophisticated pricing operation. The approach was tested using the bus network of Berlin, Germany.

Other passenger-oriented approaches

Given upper frequency requirements, Puhl and Stiller (2007) look for a line concept which maximizes the number of passengers that can be transported, i.e., they investigate how many passengers can maximally be routed along all possible lines. They show that their problem results in a path constrained network flow problem and that it is as hard to approximate as *maximum clique*. Polynomially solvable special cases are identified.

Klier and Haase (2008) suggest to use frequency-dependent transfer times as input data. Using a small example, they show how these may be calculated in a pre-processing step in which different line combinations for every OD-pair are evaluated, and up to five possible paths per OD-pair are generated. These paths are weighted by the expected number of passengers that are likely to use them. The objective maximizes the total number of expected passengers.

3.3 Game-theoretic models

In order to obtain delay-resistant line plans, Schöbel and Schwarze (2006b) and Schwarze (2008) present a game-theoretic approach to line planning. The lines correspond to the players that decide about their frequencies. Their individual benefit functions include the delay-resistance of their lines which does depend on all lines using the same infrastructure and hence on the decision of the other players. An equilibrium ensures that the frequencies are equally distributed over the network, and hence, delays due to capacity conflicts are less likely.

Another game-theoretic approach has been suggested in Kontogiannis and Zaroliagis (2008a,b). Similar to Schöbel and Schwarze (2006b), a potentially large number of line operators is given which have their lines fixed and try to maximize their personal benefits. In contrast to the former model, Kontogiannis and Zaroliagis (2008a) introduce a *network operator* whose duty is the management of the infrastructure. The goal of the network operator is to achieve a social optimum by maximizing the sum of benefits of the operators (which he does not know). The authors show how this can be accomplished in certain situations by suggesting a pricing scheme for the usage of the shared infrastructure which is robust with respect to the different (uncertain) benefit functions of the single operators.

3.4 Location-based models

In location theory [see Hamacher and Drezner (2001) for an introduction and overview about the field], researchers consider the following problem: Given a network, where to locate a path in the network such that the distances from the path to a given set of demand points is minimized? This problem can be interpreted as locating a rapid-transit line in a given transportation network. The classical objective in location theory is to minimize the *access times* to the new line (location of a *median path*). However, recently, also OD-based objective functions were investigated, and applications in transportation planning are described. In these studies, the new path is interpreted as a high speed line, and the objective is to locate it in such a way that the distances for the OD-pairs (which may use the new high-speed line if it is beneficial for them) are minimal. The papers mentioned in this section follow this approach and consider the planning of *one high speed line* in an existing transportation network.

A passenger-oriented approach called *trip coverage model* has been proposed by Laporte et al. (2005, 2007). The goal is to design lines which are able to compete with the private mode (e.g., using private cars or bicycles). The objective is to maximize the number of passengers using public transportation. The model hence incorporates not only an assignment but also a modal split procedure within the line planning model. The authors present integer programming formulations and a case study. Locating a metro line in Seville is considered in Laporte et al. (2009); metaheuristics for solving the problem are suggested in Martínez et al. (2005). A solution approach using Bender's decomposition is proposed in Marin and Jaramillo (2009).

A systematic analysis of minimizing the traveling times for a given set of OD-pairs by fastening some of the edges in an existing network has been done by Schmidt and Schöbel (2009). In their model, they allow all possible paths as lines and use a budget constraint which fixes the total length of the new high-speed lines but not their number. It turns out that planning a single high-speed line in a tree network or in a simple circle is polynomially solvable; all other variants are NP-hard. The paper also investigates the location of other structures, e.g., of a high-speed tree.

3.5 Heuristics for various line planning problems

So far we discussed publications that all started with defining a model for line planning and then presented approaches for its solution. However, also the other way round is followed in the literature: driven by applications, and often without specifying a clear model, many heuristics are proposed. They aim at constructing a line concept taking into account various objective functions greedily in each of their steps. We mention some of the ideas used.

Concerning constructive procedures, the basic idea can already be found in Lampkin and Saalmans (1967). Starting from a simple set of lines that does not cover all stops in the network, uncovered stops are inserted sequentially until a connected network is obtained. In every step, the costs of the insertion should be small. Another idea is to start with a line plan containing every edge as a single line and to combine these lines. In Wegel (1974), this is done maximizing the number of direct travelers in every step.

Silman et al. (1974) and Dubois et al. (1979) propose the *skeleton method* in which they choose the endpoints of the lines and a few intermediate stops which are then joined by shortest paths. Depending on the actual goal, the shortest paths can be chosen with respect to minimal length or minimal traveling time.

The algorithm of Sonntag (1977, 1979) [similar to the one of Patz (1925)] starts with a line plan which contains a line for each OD pair. Lines are then iteratively eliminated, while new lines may be constructed by joining parts of lines. Passengers are routed to other paths in every iteration keeping track of their traveling times.

Mandl (1980) directly starts with a feasible set of lines which is heuristically improved by a neighborhood search allowing to exchange parts of the lines or to add or remove stations from lines. Starting from an empty set of lines Simonis (1981) iteratively constructs a line concept in which lines are contained in the set of shortest paths for a given set of OD-pairs and maximizes the number of direct travelers in each step. Pape et al. (1995) propose the *dual set method*. They first identify *core lines* containing a large number of travelers which are then combined to a partial line plan. Other lines are then added to make sure that all edges are covered; in each step, the number of direct travelers is maximized. Quak (2003) further refines this procedure by adding a timetabling step behind. He presents a good overview about constructive heuristics and a case study.

A related area of research concerns literature on the *transit route network design problem* (TRNDP) which is defined as a "process which implies clear and consistent techniques for designing a public transportation network" (see Kepaptsoglou and Karlaftis 2009), i.e., the focus is on defining an approach rather than a model. TRNDP literature deals with the definition of the *routes* of the vehicles (which can be later seen as lines) and their frequencies, but may also comprise other planning stages such as network design aspects, timetabling, vehicle scheduling, or even the definition of fares in the tariff system. Nevertheless, the basic restrictions considered in literature on the TRNDP are similar to restrictions considered in line planning [i.e., (LEF) and (UEF) are often mentioned], but in TRNDP the constraints considered are usually much more detailed. They take into account, e.g., load factors, restrictions on the shape, directness, maximum length and number of routes and even legal requirements such as bus transit industry guidelines; see Zhao and Zeng (2006) for an overview on different types of constraints. The goals of TRNDP procedures are cost-oriented, passenger-oriented, or a combination of both (often called *welfare*). Also other objectives have been considered such as capacity maximization, minimization of the fleet size, or energy conservation. It is worth mentioning that there are also approaches following multi-objective goals (see e.g., Baaj and Mahmassani 1991; Israeli and Ceder 1995; Bielli et al. 2002; Chakroborty 2003; Fan and Machemehl 2006; Mauttone and Urquharta 2009). The procedures proposed for finding a solution to the TRNDP often stem from real-world (bus) applications and contain similar elements as the line planning heuristics, e.g., route generation algorithms where in every step new vertices are inserted (e.g., Baaj and Mahmassani 1991; Mauttone and Urquharta 2009), local improvements, genetic algorithms, and other metaheuristics. We refer to Kepaptsoglou and Karlaftis (2009) for a recent and excellent survey on the topic including a list of more than 60 papers on the TRNDP.

4 Ongoing research in line planning

Robustness in line planning

Nowadays, robustness is more and more an issue in optimization (see Bertsimas and Sim 2004; Ben-Tal et al. 2009). Given an optimization problem, the classical concepts of strict robustness ask for a solution which is feasible for all possible changes in the input data (sometimes also called *scenarios*). It turns out that hedging against all uncertainties is often much too conservative such that more applicable robustness concepts are needed for many real-world applications. In particular, for optimization of public transportation, new concepts to analyze robustness of a plan and to develop robust plans have been proposed; see the concepts of *recovery robustness* (Liebchen et al. 2009; Erera et al. 2009; Cicerone et al. 2009) and of *light robustness* (Fischetti and Monaci 2009; Schöbel 2010).

A line concept may be called *robust* if it is still good in case of changes of the input data. There are two different types of uncertainties that are to be considered:

- Disturbances during operation. Sometimes a line concept has to be updated during operation. This may happen if a special event takes place or if a track is not available due to maintenance or accidents. Scenarios to be taken into account hence only differ in a few parameters from the expected values of the input parameters, but these may change drastically.
- Approximation of the input parameters. As in many optimization problems, the input parameters may be inaccurate. This concerns costs or traveling times, as well as the information about the passengers. Hence, most of the input parameters are uncertain, but it can be assumed that each single uncertainty is rather small.

In the game-theoretic approaches described in Sect. 3.3, robust line plans are generated with respect to delays or with respect to the benefit functions of the operators. More about robustness in public transportation can be found in the publications of the European project ARRIVAL and in Ahuja et al. (2009).

Integration with other planning phases

Another issue is *integrated model* in which the line planning phase is solved simultaneously with some other phases of the planning process (depicted in Fig. 1). Due to the complexity of the integrated problem, exact solution procedures cannot be expected. We mention some papers containing aspects of integration.

The earliest planning phase is the network design process which includes planning the stations. Integration of line planning and the location of the stations has been treated in Chapter 6 of Goossens (2004) and also in some of the location-oriented approaches. Line planning, while respecting network design issues and platform assignment, is treated in Carey (1994). The integration of line planning with timetabling and network design is analyzed in Barber et al. (2008). Integration of line planning and vehicle scheduling has been investigated in Claessens et al. (1998). The three planning steps line planning, timetabling, and vehicle scheduling have been considered in Israeli and Ceder (1995) and Quak (2003), while Liebchen (2008) even investigates the integration of the four planning steps line planning, timetabling, vehicle- and crew-scheduling. An integrated model and a heuristic changing the order of the classical planning steps has been proposed in Michaelis and Schöbel (2009). On the other hand, line planning has been also considered within timetabling; see Liebchen and Möhring (2007) and Lindner (2000).

5 Conclusion and directions of future research

In this paper, we reviewed existing literature on line planning problems. We discussed different modeling aspects of line planning and identified the basic characteristics of the various existing models making it easier to compare different algorithms and approaches.

As already mentioned, robustness of line planning models and their integration with other planning phases are challenging fields of current and future research. In order to evaluate aspects of integration, we are currently developing a simulation tool called LinTim (Goerigk et al. 2011). Based on simulated but close-to-real-world data, it can be used to analyze line planning, timetabling, vehicle scheduling, and delay management in an integrated way and to answer questions concerning the delay-resistance of lines or the effect of line planning on timetabling.

We already pointed out that the two main objectives of line planning contradict each other: A line concept which is convenient for the passengers is costly, and a line concept which uses a small budget usually does not offer the service that customers would like to have. It could hence be a promising task to combine these two objectives within a bicriteria model and to identify Pareto solutions of such line planning models.

Finally, the line planning models have to go further to practice to make them applicable to the real-world planning process. This can only be achieved by discussing and adapting the current line planning models keeping in mind (at least) the following points: For real-world applications, it would be beneficial to be able to determine the capacities of the vehicles instead of assuming them to be fixed. This can be done analogously to the extended cost model in Claessens et al. (1998) and should also be included in other models. Also it needs to be further considered that a good public transportation system may generate new demand [a first investigation about this issue can be found in Laporte et al. (2005)]. A more philosophical questions is if peak and off-peak demand should be treated by different line concepts or if it is appropriate to use the same line concept all over the day.

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