OR Spectrum (2003) 25: 497–519
DOI: 10.1007/s00291-003-0135-5

OR Spectrum

Optimal lower bounds on the contribution margin in the case of stochastic order arrival

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Abstract. In customer order driven production, decisions on the acceptance of customer orders usually have to be based on variable costs and contribution margins (abbreviated CM), since in the short term only these quantities can be influenced. If we assume that customer orders arrive according to a stochastic process and that the decisions on order acceptance have to be made on each order separately, a customer order usually should be accepted only if its contribution margin exceeds a positive lower bound. This paper shows by means of a stochastic model that, under certain assumptions, this lower bound on the contribution margin can be determined using full costing, provided that the available capacity (constant over time) and the arrival process are balanced. This insight justifies, to a certain extent, the use of full costs to support decisions on the short-term production volume, which is a behaviour that can be observed in practice rather frequently. We also demonstrate the extension of the modelling approach to state-dependent lower bounds on the contribution margin.

Key words: Cost accounting – Dynamic pricing – Order acceptance – Stochastic models – Cost allocation

1 Description of the problem

Master production scheduling is an important planning problem in industry, since the master production schedule (MPS) determines both the cash flow of the enterprise and the material flow through the entire logistic chain. For customer order driven production, the MPS consists of the accepted customer orders and their quoted due dates. This makes order acceptance and due date setting important decision problems in manufacturing planning and control for this type of production (for a discussion of the decision structure, see Kingsman, 2000).

In order to maximize its profit, the firm has to decide on acceptance or rejection of an order based on its contribution margin (in the following abbreviated CM), its capacity requirement and the state of the manufacturing system. Thus we have to develop decision rules supporting the acceptance/rejection decision based on these pieces of information. In this paper we deal with decision rules for order acceptance/rejection where the decision is based mainly on contribution margin and capacity requirement of the orders, and we model the manufacturing system as a single-stage queueing system. Thus we rely on research that can be found in cost accounting and in operations research literature.

For deterministic situations, rules for deciding on order acceptance in order to maximise the contribution margin are described extensively in cost accounting literature (e.g., Coenenberg, 1997). An order is to be accepted if its contribution margin exceeds the opportunity costs of accepting the order. Exact calculation of the opportunity costs requires information on all available orders and leads to a model that considers all orders and all relevant restrictions simultaneously.

We restrict our attention to the stochastic case: If the orders arrive according to a stochastic process and the decision on acceptance/rejection must be made immediately, the acceptance decision can only be based on decision rules that maximize, e.g., the expected profit per period. These decision rules usually impose lower bounds on the contribution margin of the orders. Optimal values for the lower bound of the contribution margin can be derived from stochastic models, as described, e.g., in Wijngaard and Miltenburg (1997), Carrizosa et al. (1998) [see also the *Revenue Management* approach (Harris/Pinder, 1995), which is based on similar logic]. Two observations are crucial in this respect:

- The optimal lower bound on the CM highly depends on the current situation (state of the manufacturing system, expectations on future orders and their profitability). It can only be calculated by stochastic models requiring information that is difficult to obtain and to express in quantitative terms.
- Empirical studies indicate that order acceptance decisions frequently are supported by full costing systems.[1] This leads to the conclusion that managers consider the allocated fixed costs as an estimate of the opportunity costs that are difficult to measure (Zimmerman, 1979, p. 510f). The empirical work of Wouters (1994) indicates that managers are aware of this interpretation of the allocated fixed costs.

Thus an important aspect of order acceptance is the question in which cases and to what extent order acceptance decisions based on full costs (including allocated fixed costs) can be optimal.[2]

The purpose of this paper is twofold: First, we derive a model that shows that under certain assumptions order acceptance based on full costs can indeed be optimal if the lower bound on the contribution margin is independent of the current state of the manufacturing system over a certain range (detailed explanation

[1] See, e.g., Lange and Schauer (1996), Mills (1988), Govindarajan and Anthony (1983). For a literature survey on this topic, see Wouters (1994).

[2] We assume that information on costs is utilized optimally for the decision process (for this problem, see Wiese, 1994). Furthermore we assume that decisions are made according to expected values.

below). Second, we show the extension of the applied modelling technique to state-dependent lower bounds on the CM.

The paper is organized as follows: In Section 2 we describe the relevant literature and cast light on the motivation for our research. Section 3 presents the model that proves the optimality of order acceptance decisions based on full costing under certain assumptions. In Section 3.2 the acceptance decision is based only on the CM of an order, Section 3.3 extends the analysis to simultaneous consideration of CM and capacity requirement of the orders. In Section 4 we extend the modelling approached to state-dependent order acceptance rules. Section 5 presents conclusions and topics for future research.

2 Relevant literature

The question of whether full costing can be used to support order acceptance decisions has been treated in the literature in relation to two interrelated decision problems which we term (1) deciding on the *lower bound on the CM* and (2) *pricing*.

We define deciding on the *lower bound of the contribution margin* as follows: For each customer order $j$the contribution margin $y_j$is known. We have to decide on the CM lower bound that is required for accepting the order. If the CM is lower than this lower bound then the order is rejected.

Miller and Buckman (1984) consider a service department that offers to satisfy requests from the operating departments. The operating departments decide to have the request satisfied by the service department if the benefit of the request is higher than the variable costs associated with meeting the request plus the *transfer price* that is charged by the service department. The transfer price is the decision variable and is assumed to be independent of the state of the system. The service department is modelled as an M/M/s/s queueing system (Erlang's loss system with s parallel servers). The service rate for one server μ is given, the number of parallel servers s is a decision variable. The authors conclude that – provided the capacity s is optimal – " ... if the cost of capacity function $C(s)$ is of the form $a\ s^{\alpha}$, then the expected value of opportunity costs will be greater than or equal to α times the allocated capacity costs based on straight-line depreciation and 100 percent utilization." (Miller and Buckman, 1987, p. 637). If α is close to 1, then the allocated fixed costs are a good approximation of the average opportunity costs (see Miller and Buckman, 1987, p. 633).

Dewan and Mendelson (1990) analyze a similar model that includes delay costs and considers capacity as a continuous variable. For simultaneous optimization of transfer price and capacity they conclude that if the service department is modelled as an M/M/1 queueing system the optimal transfer price is equal to the marginal capacity cost. Again, only for 100% utilization can the total capacity costs be recovered (see Dewan and Mendelson, 1990, p. 1510), that is, the service department is a "deficit center". Stidham (1992) analyzes the properties of the optimal solution for more general assumptions and develops an algorithm for solving the model.

Balachandran and Srinidhi (1990) show that under specific assumptions it is optimal to charge each order the quotient of fixed costs per period and number of orders produced. The model assumes a specific nonlinear cost function, sufficient capacity for accepting all arriving orders, and a single-server system.

Wijngaard and Miltenburg (1997) describe a model of packaging lines where there is some overcapacity that can be used for accepting extra sales opportunities. The model shows that "the minimum reward per unit of capacity ... that is required before an additional order is accepted is higher than the cost of increasing the capacity ..." (p. 17f).

In contrast to deciding on the lower bound on the CM, we define the *pricing* problem as the decision on the price of products and services, depending on product type, capacity requirement, etc.

Banker and Hughes (1994) analyze a model which assumes that expected demand for each product is linear in its price. Actual demand is random. One property of the optimal solution is that "given the knowledge of the demand parameters ....., the optimal price of each product is a function only of the activity based cost" (Banker and Hughes, 1994, p. 488). Jahnke and Chwolka (1999) conclude from their model that in the scenario under consideration full costs are a sensible base for pricing decisions (p. 3). They also consider the case where there is no precise knowledge of the functional relationship between price and demand. Göx (2002) analyzes pricing and capacity planning of a monopolist firm for uncertain demand. The model optimizes the decisions depending on the types of capacity restrictions (short-term capacity expansion possible or not possible) and on the time of the availability of demand information. Pavia (1995) uses a multi-product model to show that it can be optimal to allocate no fixed costs to certain product types.

Karmarkar and Pitbladdo (1993) consider a model where firms compete on a market with linear relationship between price and demand. They do not find support for the suggestion that allocation of overhead costs is a means of rationing scarce capacity (p. 1042).

We can conclude that whether order acceptance and pricing should be based on full costing depends strongly on the assumptions made. For the decision problem that primarily interests us, that of setting the minimum contribution margin (or the combination of the minimum contribution margin and capacity) for a stochastic arrival of orders, most research work concludes that, inasmuch as each order is to be charged its proportionate share of capacity costs, these capacity costs relate to 100% capacity utilization. A job j that requires c_j hours of capacity is charged c_j hours of capacity (compare the numerical example in Dewan and Mendelson, 1990, p. 1510).[3] Since stochastic systems under the usual premises can *never* be utilized to 100%, the capacity costs can never be fully recovered. Full costing that charges all arising costs to jobs that were actually processed or to actual hours of work thus proves unsuited for the computation of minimum contribution margin. With the exception of the work of Balachandran and Srinidhi (1990), which is based on very specific premises, to our knowledge no model has been formulated by which such a (usual) full costing system actually yields the optimal minimum contribution margin.

[3] Note the different result in Wijngaard and Miltenburg (1997).

This is the first goal of the present paper. In the next section we employ a stochastic model to demonstrate that – given certain assumptions, in particular proportional capacity costs, stochastic arrival of orders, and optimal tuning of available capacity to the characteristics of the arrival process – order acceptance on the basis of full costing (which charges to the jobs the fixed costs of capacity in relation to the *actual* mean capacity utilization) represents the optimal behavior.

The model is developed in the next section.

3 The model

3.1 Structure of the model

We consider a manufacturing system which we model as a single-stage, multiple-server queueing system. Capacity is constant over time. Customer orders arrive according to a Poisson process with arrival rate λ [orders per time unit]. Acceptance or rejection of an order does not influence the arrival of other orders. The decision on acceptance or rejection must be made immediately.

Each customer order j is characterized by its contribution margin (CM) y_j and by its capacity requirement c_j. The expected total required capacity ($\lambda\overline{c}$ with $\overline{c}$ = average capacity requirement of an order) can exceed the available capacity. At the time of arrival each customer knows the waiting time and hence the delivery time (= flow time) of the order; that is, due date setting has been performed by the manufacturer. If the waiting time or flow time exceeds a certain limit, the order is withdrawn. Thus in the single-server case the customer can indirectly observe the remaining work at the server which is – for a single-server system and FCFS service discipline – the promised waiting time. This model of customer behaviour and due-date setting requires queueing models with a limit on the waiting time (or flow time) where an order enters the queue if the waiting time (or flow time) is below a certain limit and otherwise is lost. To our knowledge no general results are available for this type of queueing system (for an analysis of the heavy traffic case, see Kushner, 2001; Plambeck et al., 2000; overview of controlled queueing networks in Stidham, 2002, p. 207 f). So we approximate the system using a *finite-capacity queueing system*: If at the time of arrival there are $(r-1)$ or less orders in the system, the customer accepts the resulting waiting time, otherwise the order is lost.

If management maximizes the expected CM per period, it encounters the decision problem of order acceptance: It can be optimal to reject an order even in the case of idle capacity if the arrival of more profitable orders can be expected.[4] We have to derive decision rules that maximize the expected CM per period.

The system under consideration is depicted in Figure 1.

CM and capacity requirement of the orders (random variables Y and C) follow a two-dimensional distribution with probability density function $f_{Y,C}(y, c)$. Since the acceptance of the orders is based only on CM and capacity requirement, the

[4] For simplicity we ignore the costs of waiting time (for this aspect, see Banker et al., 1988).

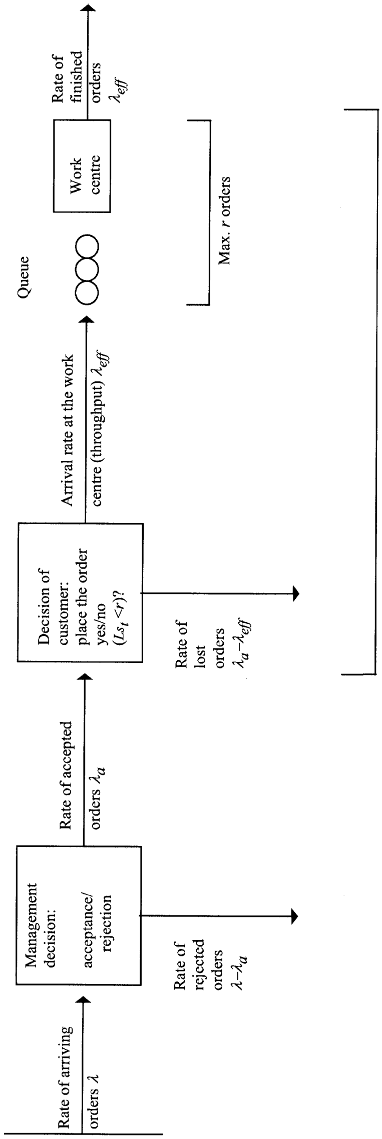

Fig. 1. The system under consideration. Ls_t .... Number of orders in the system at time t

acceptance/rejection decisions split the Poisson arrival process and the accepted orders arrive according to a Poisson process with rate λ_a. For the accepted orders the customer decides on placing or withdrawing the order according to the rule described above. The decision rule employed by the customer and the way the work centre is operated lead to a finite-capacity queueing system (M/G/s/r system) with arrival rate λ_a and throughput λ_{eff}. The difference $\lambda_a - \lambda_{eff}$ is the rate of lost orders that arrive when the system is full $(Ls_t = r)$. If we assume that the available capacity (modelled as a continuous variable) is subject to long-term changes by investment decisions (the available capacity is a decision variable, but it is constant over the planning horizon considered in our model), we have to analyze three topics:

- The rule for order acceptance/rejection that maximizes the expected CM per period for given capacity,
- the combination of acceptance/rejection rule and capacity that maximizes the expected profit per period (expected CM less capacity costs),
- the extent to which the optimal decisions can be approximated using simple decision rules. As derived from our literature review, especially the quality of decisions based on full costs has to be evaluated.

We start our analysis using the most straightforward decision rule.

3.2 Decision rule: lower bound on the contribution margin

3.2.1 Formulation of the model

The CM of the customer orders is a random variable Y with probability distribution function $F_Y(y)$ and density function $f_Y(y)$. The capacity required for an order is independent of the CM with mean $\bar{c}$. We have to decide on the lower bound on the CM y_{min}, that must be attained for an order to be accepted. The model can be formulated as follows:

The rate of accepted orders λ_a is the fraction of the arriving orders λ for which the CM exceeds y_{min}:

$$\lambda_a = \lambda(1 - F_Y(y_{min})) \tag{1}$$

The distribution of the CM of the accepted orders is the conditional distribution of the CM given that the order meets the requirements for acceptance. We term the probability density function $f_{Y|Accept}(y)$:

$$\begin{aligned} f_{Y|Accept}(y) &= 0 && \text{for} \quad y < y_{\min} \\ f_{Y|Accept}(y) &= \frac{f_Y(y)}{1 - F_Y(y_{\min})} && \text{for} \quad y \geq y_{\min} \end{aligned} \tag{2}$$

Hence the expected CM of an accepted order $E[Y|Accept]$:

$$E[Y|Accept] = \int_{y_{\min}}^{v} y\, f_{Y|Accept}(y)dy \tag{3}$$

with v the maximum CM (maximum value of the distribution).

Since the decision of the customer to place or to withdraw an order is independent of the CM, the expected total CM per period $E[Y_P]$ is then:

$$E[Y_P] = E[Y|Accept]\lambda_{eff} \tag{4}$$

λ_{eff} can be calculated from λ_a using finite-capacity queueing models. In the following we shall not need the precise functional relationship between λ_{eff} and λ_a. We assume that the ratio of λ_a and λ_{eff} is a function of the traffic intensity ρ and is independent of the processing time distribution:

$$\lambda_{eff} = \lambda_a h(\rho) \tag{5}$$

with

$$\rho = \frac{\lambda_a}{\mu} \tag{6}$$

$\mu =$ service rate of one server.

$h(\rho)$ is the fraction of the accepted orders that are actually placed by the customer if the traffic intensity is ρ.

The following conditions must hold:

$$h(0) = 1$$
$$\lim_{\rho\to\infty} h(\rho) = 0$$

The assumptions that must be satisfied in order to obtain a factor $h(\rho)$ that is independent of the processing time distribution can be described as follows:

For the Poisson arrival process which we assume throughout the paper the probability that the system is full (all places occupied) p_r is also the probability that a customer finds all places occupied,[5] which means:

$$h(\rho) = 1 - p_r.$$

$h(\rho)$ (or p_r, respectively) is independent of the processing time distribution in the following cases:

- for exponentially distributed service times and one server (M/M/1/r system; Neumann, 1977, p. 396),
- for Erlang's loss system M/G/s/s (Tijms, 1994, p. 326),
- for the M/M/s/r model if $\rho < s$ (Papadopoulos et al., 1993, p. 365).

This restricts the model to exponentially distributed service times if a queue is allowed, which is a strong assumption. Furthermore, in Section 3.3 we shall assume that the capacity requirement of an order is an element of the acceptance rule, which means that the service time distribution of the accepted orders depends on the acceptance rule and cannot be subject to assumptions. So we have to analyze the general model M/G/s/s+N (s places at the servers and N places in the queue).

It can be shown that for this model the probability that a customer finds all places occupied (p_{s+N}) depends on the service time distribution (Tijms, 1994,

[5] "PASTA" (Poisson arrivals see time averages) property; see Buzacott and Shantikumar (1993, p. 54), Tijms (1994, p. 73 ff.).

p. 327; Seelen et al., 1985). The simulation results described in Tijms (1994, p. 327) for a M/G/s/s+N system with $s = 5$ and $N = 6$ and 10 places in the queue indicate only a weak dependence of $h(\rho)$ on the coefficient of variation of the service time distribution (the throughput changes by less than 5% if the coefficient of variation increases from 0 to 2). Our numerical experiments indicate that for an $M/G/1/1+N$ system the impact of the coefficient of variation can be slightly higher, but the results derived in this paper seem to be rather insensitive to this dependence. So we use Eq. (5) as an approximation for the queueing system $M/G/s/s+N$.

The service rate μ for one server depends on the capacity requirement of the orders (required standard hours of work) and on the capacity of the work centre. It follows from our assumptions that the distribution of the capacity requirement per order is independent of $y_{\min}$. Thus the average capacity requirement per order is constant, and we can express the capacity of the work centre as the service rate μ. Like Dewan and Mendelson (1990) we assume that μ is a continuous variable. Furthermore we assume that the fixed costs per period are proportional to the available capacity, which is also the basis for activity-based costing (see Schneeweiss, 1998, p. 278). Empirical studies only partly support this assumption [see Zugarramurdi et al. (2002) and the studies cited there, which indicate that in many cases there are economies of scale, and Kölbel and Schulze (1960)][6]. We denote the fixed costs per period for one unit of capacity as K. The total fixed costs per period K_{fix} are then:

$$K_{fix} = K\mu \,. \tag{7}$$

Numerical example 1. K is the fixed capacity costs per period for $\mu = 1$ (that is, capacity for serving one order per period is available). If the costs for this capacity are $K = 5$ per period and the available capacity is $\mu = 120$ orders per period, then the fixed capacity costs per period are $K_{fix} = 5$ times $120 = 600$. For a multiple-server system the parameter K must be modified accordingly.

The expected profit per period $E[G]$ is the difference between expected CM and fixed costs:

$$E[G] = E[Y_P] - K_{fix} \tag{8}$$

3.2.1 Analysis of the model

First we analyze the optimal lower bound on the CM y_{min} for given capacity (service rate) μ. The expected CM per period can be calculated by substituting (1) to (3) and (5) into (4). This yields:

$$E[Y_P] = \lambda h(\rho) \int_{y_{\min}}^{v} y\, f_Y(y)\, dy \tag{9}$$

Analytical calculation of the optimal value for y_{min} hardly seems possible, but the optimal solution for $\lambda \to 0$ and $\lambda \to \infty$ is given by the following

Theorem 1. For $\lambda \to 0$ the optimal y_{min} approaches the minimum value of the CM distribution.

[6] For an overview of these cost relation functions, see Wildemann, 1982, p. 131 ff

For $\lambda \to \infty$ the optimal y_{min} approaches the maximum value of the CM distribution.

Proof. The proof is given in Missbauer (2000).

If the arrival rate is very low, nearly all orders (we assume positive CM) can be accepted. For an arrival rate approaching infinity only the most profitable orders should be accepted.

Now we analyze the applicability of full costing, which requires assumptions about the available capacity. We assume that the available capacity is optimal for arrival rate λ. For our model this means that μ and y_{min} have to be determined simultaneously. The objective is maximization of the expected profit per period, which can be calculated from (8) by substituting (7) for K_{fix} and (9) for $E[Y_P]$:

$$E[G] = \lambda h(\rho) \int_{y_{\min}}^{v} y\, f_Y(y)\, dy - K\mu \tag{10}$$

From this objective function we can derive the following

Theorem 2. Simultaneous optimization of y_{min} and μ for maximizing $E[G]$ according to (10) yields the following optimal value for y_{min} (denoted $y^*_{\min}$):

$$y^*_{\min} = \frac{K\mu}{\lambda\left(1 - F_Y(y_{\min})\right) h\left(\rho\right)} \tag{11}$$

Proof. Appendix 1.

The denominator of (11) is the throughput λ_{eff} (Eqs. (5), (1)), the numerator is the fixed costs K_{fix} (Eq. (7)). Hence the optimal lower bound on the CM $y^*_{\min}$ is:

$$y^*_{\min} = \frac{K_{fix}}{\lambda_{eff}} \tag{12}$$

This means that the optimal lower bound on the CM can be calculated by full costing; all fixed costs per period can be allocated to the produced orders. If the variable costs of all orders are identical, the costs per order are the total costs divided by the number of orders that are produced. In the case of non-identical variable costs of the orders, equal amounts of fixed costs are allocated to each order.

Numerical example 2. We continue numerical Example 1 (Sect. 3.2.1) and assume an M/M/1 system with $r = 2$ places in the system and arrival rate $\lambda = 100$. The CM per order is normally distributed with mean $= 10$ and standard deviation $= 3$. The formulas for the M/M/1/r system (Neumann, 1977, p. 396) yield

$$h(\rho) = \begin{cases} \frac{1-\rho^r}{1-\rho^{r+1}}, & \rho \neq 1 \\ \frac{r}{r+1}, & \rho = 1 \end{cases} \tag{13}$$

Using numerical methods to maximize the expected profit per period (10), we obtain the following optimal values: $y^*_{\min} = 6.6146, \mu^* = 65.4747, E[G] = 203.555, K_{fix} = 327.373$ and $\lambda_{eff} = 49.4928$. $y^*_{\min}$ can be calculated with Eq. (12). Since $\lambda_{eff} y^*_{\min} = K_{fix}$, all fixed costs can be allocated to the orders. The objective function is depicted in Figure 2.

By assumption, the decision rule derived here allocates the fixed costs to the orders without considering their capacity requirement. In the next section we relax this assumption.

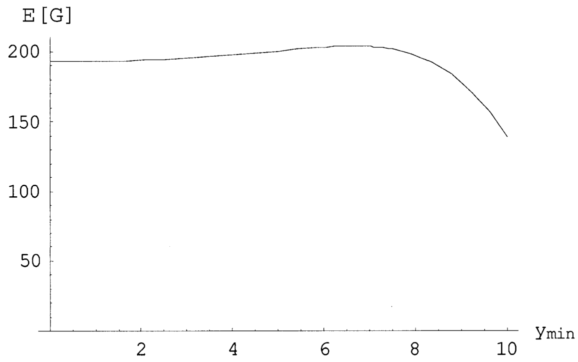

Fig. 2. Solution to numerical Example 2: $E[G]$ as a function of y_{min} for the optimal value of μ

3.3 Decision rules encompassing contribution margin and capacity requirement

3.3.1 Formulation of the model

Again we assume a Poisson arrival process of orders. CM y_j and required capacity c_j per order follow a two-dimensional distribution with probability density function $f_{Y,C}(y,c)$. We consider only orders with non-negative CM, that is:

$$f_{Y,C}(y,c) = 0 \qquad\qquad \text{if} \quad y < 0 \quad \text{or} \quad c < 0 \,.$$

This presents the decision problem of determining the subset of orders that has to be accepted in order to maximize the expected profit per period. Thus we have to decide on the conditions on CM and capacity requirement that must be satisfied by the accepted orders. It is certain that if an order j with a CM y_j and capacity requirement c_j is to be accepted, then an order i with $y_i \geq y_j$ and $c_i \leq c_j$ is to be accepted as well. But the type of the restriction(s) defining the acceptance set and the rejection set is not known.

In this paper we do not analyze the type of these restriction(s). We assume that the subset of orders that are to be accepted is defined by *one linear restriction*, that is, a linear combination of CM and capacity requirement of the orders. An order j is to be accepted if it satisfies the following condition:

$$c_j \leq a + b\, y_j \tag{14}$$

a and b are the parameters of the decision rule and the variables that we have to optimize. This is depicted in Figure 3.

We assume a *linear* decision rule partly for sake of simplicity. Furthermore, decisions based on cost accounting systems usually follow linear rules, e.g., lower bound on the CM or on the CM per unit of a bottleneck resource.

In the following we calculate the expected CM per period as a function of a and b, and calculate the expected profit per period as a function of a, b and available capacity. This will provide the basis for an analysis of the optimal rule for order acceptance.

First we calculate the probability distribution of CM and capacity requirement of the *accepted* orders as a function of a and b. Since $b \geq 0$ (see above), the

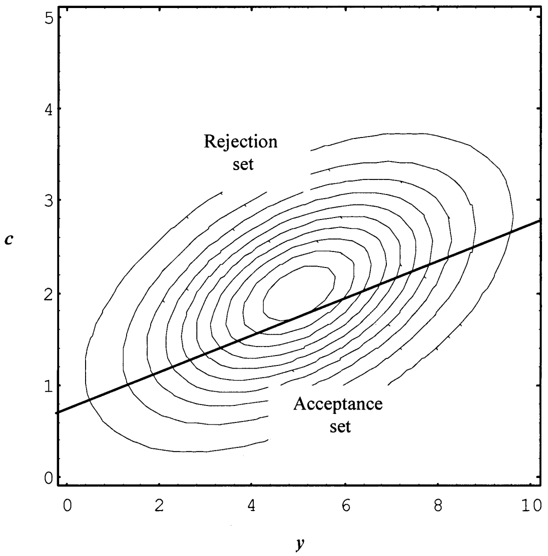

Fig. 3. Linear decision rule for order acceptance. The contour plot depicts the probability density function $f_{Y,C}(y,c)$

proportion of accepted orders (Q) is (1 minus integral of the probability density of the rejection set):

$$Q = 1 - \int_0^{\infty} \left(\int_0^{(c-a)/b} f_{Y,C}(y,c)\, dy \right) dc \tag{15}$$

Note that $(c-a)/b$ is the inverse function of the acceptance restriction. The integration variable c ranges from MAX$[0;a]$ to ∞. Since $f_{Y,C}(y,c)$ is 0 for negative values of y or c, we can write 0 instead of MAX$[0;a]$.

The probability density function of the distribution of CM and capacity requirement of the accepted orders, denoted $f_{Y,C|Accept}(y,c)$, is the conditional distribution given the acceptance restriction (14) is satisfied:

$$\begin{aligned} f_{Y,C|Accept}(y,c) &= 1/Q f_{Y,C}(y,c) \qquad && \text{if} \quad c \le a + b\,y \\ f_{Y,C|Accept}(y,c) &= 0 && \text{otherwise} . \end{aligned} \tag{16}$$

Now we can calculate the probability density functions of the CM of the accepted orders $f_{Y|Accept}(y)$ and of the capacity requirement of the accepted orders $f_{C|Accept}(c)$, which are the margin distributions of $f_{Y,C|Accept}(y,c)$. We also calculate the mean CM of the accepted orders $E[Y|Accept]$ and the mean capacity requirement of the accepted orders $E[C|Accept]$:

$$f_{Y|Accept}(y) = \int_0^{a+by} f_{Y,C|Accept}(y,c)dc \tag{17}$$

$$f_{C|Accept}(c) = \int_{(c-a)/b}^{\infty} f_{Y,C|Accept}(y,c)dy \tag{18}$$

$$E[Y|Accept] = \int_0^{\infty} y\, f_{Y|Accept}(y)\, dy \tag{19}$$

$$E[C|Accept] = \int_0^\infty c f_{C|Accept}(c)\, dc \tag{20}$$

The parameters of the queueing model can be calculated as follows:
For given arrival rate λ the rate of accepted orders λ_a is (analogous to (1)):

$$\lambda_a = \lambda Q \tag{21}$$

The accepted orders arrive according to a Poisson process (see Sect. 3.2). The throughput λ_{eff} follows (5). Note that now the distribution of the service times of the accepted orders usually depends on the acceptance rule and (5) holds exactly only if no queue is allowed. Otherwise (5) is an approximation (see Sect. 3.2).

Since the mean capacity requirement per order depends on the acceptance rule, we define the capacity of one server L as a continuous variable and consider L as the rate at which one unit of work content of the orders (e.g., one standard hour) is processed (unit of measurement of L is, e.g., standard hours per hour). If $L = 1$, a capacity requirement of c units of time (e.g., standard hours) means a service time of c units of time, for $L = 2$ the service time is $c/2$, etc. The service rate is then:

$$\mu = \frac{L}{E[C|Accept]} \tag{22}$$

Numerical Example 3. We use the example of Section 3.2.1, where the service rate is $\mu = 120$ orders per period. If we assume the mean capacity requirement per order $E[C|Accept] = 7$ hours of work, then we obtain from (22): $L = 7^{*}120 = 840$ [hours of work]. If capacity is doubled (to 1680), then μ doubles to 240. If the capacity is 840 and $E[C|Accept]$ increases from 7 to 9 then μ reduces to $840/9 = 93.33$.

Calculation of the expected CM per period according to (4) and of the expected profit per period according to (8) is the same as in Section 3.2. The fixed costs are (analogous (7)):

$$K_{fix} = K\ L \tag{23}$$

3.3.2 Analysis of the model

The expected profit per period $E[G]$ can be calculated from (8) substituting (4) for the expected CM per period and (23) for the fixed costs. In (4) we have to substitute (19), (17), (16) and (15) for $E[Y|Accept]$ and (5), (21), (15), (6), (22), (20), (18) for λ_{eff}. After performing these substitutions, $E[G]$ is a function of a, b and L. Analytical calculation of the optimal values for the variables seems extremely difficult, so we formulate a hypothesis about the optimal solution and then try to prove it.

We formulate the following hypothesis: The optimal decision rule follows from full costing. Hence we have to calculate the optimal values for the variables if the hypothesis were correct.

A full costing system that allocates fixed costs according to the capacity requirement of the orders usually will use *hourly cost rates* of the facilities: The total fixed costs per period are divided by the total hours of work performed during this period, which results in the cost rate per hour of work that is required from the facility.

In our model, a decision rule for order acceptance that accepts an order if the CM is at least the allocated capacity costs obtained by hourly cost rates is represented as follows:

Acceptance restriction for order j:

$$y_j \geq \textit{allocated fixed costs per hour of work} \cdot c_j$$
$$\Rightarrow \frac{y_j}{c_j} \geq \textit{allocated fixed costs per hour of work}$$

Acceptance restriction in the model:

$$c_j \leq a + by_j$$
$$\Rightarrow \frac{y_j}{c_j} \geq \frac{1}{b} - \frac{a}{b\,c_j} \tag{24}$$

The allocated fixed costs per hour of work is the same for all orders and hence independent of c_j (this is our hypothesis!). From this follows: $a = 0$. The allocated fixed costs per hour according to (24) is then $1/b$.

If this hypothesis is correct, the following optimal value for the allocated fixed costs per hour $1/b$ results:

– The fixed costs per period are given by (23).
– The total work performed per period is the throughput (number of orders) λ_{eff} according to (5), (21), (15) times the mean capacity requirement of an accepted order $E[C|Accept]$.

Thus the allocated fixed costs per hour of work in such a cost accounting system are

$$\textit{allocated fixed costs per hour} = \frac{1}{b} = \frac{K_{fix}}{\lambda_{eff} E[C|Accept]} \tag{25}$$

Since $E[C|Accept] = L/\mu$ (from (22)), we can write:

$$\textit{allocated fixed costs per hour} = \frac{1}{b} = \frac{K_{fix}}{\lambda_{eff} \frac{L}{\mu}} \tag{26}$$

So we can formulate our hypothesis as follows: If the decision variables a, b and L are determined so as to maximize the expected profit per period $E[G]$, the optimal values a^*, b^* should be as follows:

$$a^* = 0$$
$$b^* = \frac{\lambda_{eff} \frac{L}{\mu}}{K_{fix}} \tag{27}$$

From an analysis of the objective function ($E[G]$ as a function of a, b and L) we derive Theorem 3.

Theorem 3. Optimizing a and b simultaneously yields the optimal value $a^* = 0$ independently of the available capacity L.

Proof. Appendix 2.

An order should be accepted if its CM per unit of work required exceeds a certain lower bound.

Theorem 4. If for $a = 0$ b and L are optimized simultaneously, the optimal value b^* is given by (27).

Proof. Appendix 3.

Stated more precisely: In Appendices 2 and 3 we prove that for the values a^* and b^* mentioned above the partial derivatives of the objective function are zero. We cannot prove that the objective function is convex (this seems to be extremely difficult) and thus cannot rule out local optima with certainty.

Theorems 3 and 4 mean that, provided the available capacity is adjusted optimally, the lower bound on the CM per unit of work can be calculated by means of full costing. Fixed costs are allocated to the orders according to the capacity required from the resource.

Numerical Example 4. We assume that y_j and c_j follow a two-dimensional normal distribution with $E[Y] = 10,\ E[C] = 8,\ \sigma[Y] = 3,\ \sigma[C] = 2$, coefficient of correlation $= 0.7$. Using (13) (for the M/M/1/r-System) as an approximation and using the data of Example 1 and 2 with $r = 2$ places in the system, we obtain by numerical methods: $a = 0, b = 1.23, L = 545.89$. The fixed costs per unit of time $1/b = 0.8124$. Again, the total fixed costs can be allocated to the orders, since the following equation holds: $\lambda_{eff}\ \frac{1}{b}\ E[C|Accept] = K_{fix}$. The M/M/1/r-approximation underestimates the throughput by 6% maximum, but the coefficient of variation of $f_{C|Accept}(c)$ only varies between 0.4 and 0.25 over the relevant range of b.

4 State-dependent lower bounds on the contribution margin

In this section we describe the extension of the modelling technique described above to an order acceptance policy with state-dependent lower bounds on the CM. Dynamic pricing problems frequently occur in practice, and extensive literature is available on this topic, especially for cases when a given stock of items must be sold by a deadline (for a survey, see Gallego and van Ryzin, 1994; Chatwin, 2000). Since the aim of this section is the extension of our modelling approach, we do not discuss this literature here.

4.1 Formulation of the model

We consider the problem of deciding on the lower bound of the CM. The assumptions are the same as in Section 3.2 with one modification: On arrival of an order the number of orders in the system (denoted n) and the CM of the arriving order are known. The order is accepted if its CM exceeds the state-dependent lower bound $\hat{y}_n$, otherwise it is rejected (for a model with discrete customer classes, see Miller, 1969). The $\hat{y}_n$ are independent of the capacity requirement c_j. If r customers are in the system, the customer withdraws the order. We model this behaviour as an *M/M/1 queueing system with impatient customers*: Each arriving order enters the system with probability π_n with n being the number of customers in the system at the time of arrival. With probability $1 - \pi_n$ the customer is lost. Management has to decide on the $\hat{y}_n,\ n = 0, ..., r-1$ (which determines the π_n). The assumed behaviour of the customers means that $\pi_r = 0$.

The model can be derived as follows:
The fraction of accepted customers if n customers are in the system (denoted Q_n) is the fraction of customers with a CM exceeding the lower bound $\hat{y}_n$:

$$Q_n = 1 - F_Y(\hat{y}_n) \qquad n = 0, ..., r-1 \tag{28}$$
$$Q_r = 0$$

The fraction of orders that enter the system Q (orders placed by the customer and accepted) is then (note the PASTA – property of the Poisson arrival process):

$$Q = \sum_{n=0}^{r-1} p_n Q_n \ . \tag{29}$$

Hence the throughput λ_{eff}:

$$\lambda_{eff} = \lambda Q \tag{30}$$

The state probabilities p_n can be calculated from queueing models where the arriving customers enter the system with a state-dependent probability Q_n. For a M/M/1 model the p_n are (Neumann and Morlock, 1993, p. 676):

$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \rho^n \prod_{\tau=0}^{n-1} Q_\tau} \tag{31}$$

$$p_n = \rho^n\, p_0 \prod_{\tau=0}^{n-1} Q_\tau \qquad n = 1, ..., r \tag{32}$$

Since $Q_n = 0$ for $n \geq r$, in Eq. (31) only the summands for $n = 1$ to r are relevant, all other summands are zero. The traffic intensity is

$$\rho = \lambda / \mu \ . \tag{33}$$

For given values of λ, μ, and $\hat{y}_n$, which determines the traffic intensity ρ and the Q_n (Eq. 28), the Eqs. (29), (31) and (32) are a system of $r + 2$ equations in the variables Q and $p_n (n = 0, ..., r)$. It can be solved by repeated substitution and also yields the throughput λ_{eff} (Eq. (30)).

Next, we have to calculate the probability distribution of the CM of the orders entering the system.

The probability density function of the CM of the orders entering the system if n customers are in the system at the time of arrival $(f_{Y|(Enter,\, n)})$ is:

$$\begin{aligned} f_{Y|(Enter,n)}(y) &= \tfrac{f_Y(y)}{1-F_Y(\hat{y}_n)} \qquad \text{for} \quad y \geq \hat{y}_n \\ &= 0 \qquad \text{for} \quad y < \hat{y}_n \qquad n = 1, ..., r-1 \end{aligned} \tag{34}$$

Expected CM of these orders $(E[Y|(Enter, n)])$:

$$E[Y|(Enter, n)] = \int_{\hat{y}_n}^{v} y\, f_{Y|(Enter,n)}(y)\, dy = \frac{1}{1 - F_Y(\hat{y}_n)} \int_{\hat{y}_n}^{v} y f_Y(y)\, dy \qquad n = 1, ..., r-1 \tag{35}$$

The expected CM of an order entering the system $E[Y|Enter]$ is the weighted average of the $E[Y|(Enter, n)]$ with p_n times Q_n as the weights:

$$E[Y|Enter] = \frac{1}{\sum_{n=0}^{r-1} p_n\, Q_n} \sum_{n=0}^{r-1} p_n\, Q_n\, E[Y|(Enter, n)] \tag{36}$$

The expected CM per period, fixed costs and expected profit per period are calculated from (4), (7) and (8) with $E[Y|Accept] = E[Y|Enter]$ according to (36).

4.2 Solution of the model

Analytical solution of the optimization problems

$$\max_{\hat{y}_0,\dots,\hat{y}_{r-1}} \quad E[Y_p] \tag{37}$$

and

$$\max_{\mu,\hat{y}_0,\dots,\hat{y}_{r-1}} \quad E[G] \tag{38}$$

seems to be difficult and is a topic for future research. Therefore we use numerical methods to extend our example introduced above.

Numerical example. We use the same data as in Example 2 and calculate $\mu, \hat{y}_1, \hat{y}_2$ simultaneously according to (38). The optimal values are $\mu = 64.4991, \hat{y}_0 = 4.7269, \hat{y}_1 = 8.2457$. The expected profit $E[G]$ increases from 203.555 in Example 2 to 209.345. The expected number of orders that are accepted when n orders are in the system is $\lambda Q_n p_n$. Thus all fixed costs can be allocated to the accepted orders if the following equation holds:

$$\lambda \sum_{i=0}^{r-1} (\hat{y}_i Q_i p_i) = K_{fix}$$

For our numerical example this is the case. A general proof of this property is not available at the moment.

5 Conclusions and directions for future research

In this paper we have shown that under certain assumptions the optimal lower bound on the contribution margin of an order that is required to accept the order can be determined by a full costing system provided that it is independent of the state of the manufacturing system (as long as not all places in the queueing system are occupied). The results lend support to the hypothesis that the acceptance of full costing in practice is due to the fact that allocated fixed costs can be a good estimate of the expected opportunity costs of accepting an order. But this result only holds if a number of assumptions are satisfied, especially with regard to an optimal balance of capacity and demand. Relaxing some of the assumptions and performing sensitivity analyses if assumptions are violated remains to be done.

Furthermore, we have demonstrated the extension of the applied modelling technique to a state-dependent order acceptance policy. Further development of this modelling approach should encompass simultaneous consideration of contribution margin, capacity requirement and state of the manufacturing system, in order to obtain optimal decisions or reasonable long-term decision rules. Modelling the impact of order acceptance on the time-dependent state of the manufacturing system is also required for solving other decision problems (e.g., due date assignment, order release); thus future research must aim at integrating this type of models into the research on optimal order acceptance policies.

We are aware of the limitations of the research methodology applied here, since rejecting (or not acquiring) a customer order can have grave consequences on the future behavior of customers; this can be modelled only to a limited extent. Thus the decision problem considered here requires integration of OR models and research on human behavior as well.

Appendix 1: Proof of Theorem 2

The partial derivatives of (10) with respect to y_{min} and μ yield the following optimality conditions:

$$\begin{aligned}\frac{\partial E[G]}{\partial y_{\min}} &= -y_{\min}\lambda h(\rho) f_Y(y_{\min}) - \frac{\lambda^2\left(\int_{y\,\min}^{v} y f_Y\,(y)\,dy\right) h'(\rho)}{\mu} f_Y\,(y_{\min}) \\ &= 0 \end{aligned} \tag{39}$$

$$\frac{\partial E\,[G]}{\partial \mu} = -K \frac{\lambda^2\,(1-F_Y\,(y_{\min}))\left(\int_{y\,\min}^{v} y\, f_Y\,(y)\;dy\right) h'(\rho)}{\mu^2} = 0 \tag{40}$$

with $\rho = \frac{\lambda}{\mu}(1 - F_Y(y_{\min}))$; $h'(\rho)$ is the first derivative of $h(\rho)$ with respect to ρ.

We define the variable u:

$$u = \frac{1}{\mu}\lambda^2 \left(\int_{y\,\min}^{v} y f_Y\,(y)\,dy\right) h'(\rho)\,, \tag{41}$$

because this factor can be found in (40) and in the second term of (39). Substituting u into (40) yields

$$-K - \frac{1 - F_Y(y_{\min})}{\mu} u = 0 \tag{42}$$

and u is

$$u = \frac{K\mu}{-1 + F_Y(y_{\min})}\,. \tag{43}$$

This can be used to simplify the second summand in (39). So we obtain from (39):

$$-y_{\min}\lambda h\,(\rho)\, f_Y\,(y_{\min}) + f_Y\,(y_{\min}) \frac{K\mu}{1 - F_Y(y_{\min})} = 0 \tag{44}$$

Pulling out common factors and simplifying yields:

$$\frac{f_Y\,(y_{\min})}{1 - F_Y(y_{\min})}\left[K\mu + y_{\min}\lambda\,(-1 + F_Y(y_{\min}))\,h\,(\rho)\right] = 0 \tag{45}$$

Ignoring the trivial solution $y_{\min} = f_Y^{-1}(0)$, we obtain the solution

$$\hat{y}_{\min} = \frac{K\mu}{\lambda\left(1 - F_Y\left(y_{\min}\right)\right)\, h\left(\rho\right)} \tag{46}$$

$\square$

Appendix 2: Proof of Theorem 3

The expected profit per period $E[G]$ can be calculated as described in the text. Simplification yields:

$$E[G] = -KL + \lambda h\left(\rho\right) \int_0^{\infty} y \int_0^{a+by} f_{Y,C}\left(y,c\right) dc\, dy\ , \tag{47}$$

with

$$\rho = \frac{\lambda \int_0^{\infty} c \int_{\frac{-a+c}{b}}^{\infty} f_{Y,C}\left(y,c\right)\, dy\, dc}{L} \tag{48}$$

The partial derivative of $E[G]$ with respect to a, divided by the constant λ:

$$\frac{\partial E[G]}{\partial a}\frac{1}{\lambda} = h(\rho) \int_0^{\infty} y\, f_{Y,C}(y, a + b\, y) dy + \tag{49}$$
$$+ \frac{\lambda \left(\int_0^{\infty} y \int_0^{a+by} f_{Y,C}(y,c)\, dc\, dy \right) \left(\int_0^{\infty} c\, f_{Y,C}\left(\frac{-a+c}{b}, c\right) dc\right) h'\left(\rho\right)}{b\, L}$$

The partial derivative of $E[G]$ with respect to b, divided by the constant λ:

$$\frac{\partial E[G]}{\partial b}\frac{1}{\lambda} = h(\rho) \int_0^{\infty} y^2 f_{Y,C}\left(y, a + by\right) dy + \tag{50}$$
$$+ \frac{\lambda \left(\int_0^{\infty} y \int_0^{a+by} f_{Y,C}\left(y,c\right) dc\, dy \right) \left(\int_0^{\infty} c\left(-a+c\right) f_{Y,C}\left(\frac{-a+c}{b}, c\right) dc\right) h'\left(\rho\right)}{b^2\, L}$$

We define

$$u = \int_0^{\infty} y \int_0^{a+by} f_{Y,C}\left(y,c\right) dc\, dy \tag{51}$$

This term can be found in both partial derivatives. Substituting u in (49) yields:

$$\frac{\partial E[G]}{\partial a}\ \frac{1}{\lambda} = \tag{52}$$
$$= h\left(\rho\right) \int_0^{\infty} y\, f_{Y,C}\left(y, a + b\, y\right) dy + \frac{\lambda u \left(\int_0^{\infty} c\, f_{Y,C}\left(\frac{-a+c}{b}, c\right) dc\right) h'\left(\rho\right)}{b\, L}$$

Substituting u in (50) yields:

$$\frac{\partial E[G]}{\partial b}\ \frac{1}{\lambda} = h\left(\rho\right) \int_0^{\infty} y^2\, f_{Y,C}\left(y, a + b\, y\right) dy + \tag{53}$$
$$+\ \frac{\lambda u \left(\int_0^{\infty} c\left(-a+c\right) f_{Y,C}\left(\frac{-a+c}{b}, c\right) dc\right) h'\left(\rho\right)}{b^2\, L}$$

Letting (53) be zero yields the value for u provided b is optimal:

$$u = -\frac{b^2 Lh(\rho) \int_0^\infty y^2 f_{Y,C}(y, a + by)\, dy}{\lambda \left(\int_0^\infty c(-a + c) f_{Y,C}\left(\frac{-a+c}{b}, c\right) dc \right) h'(\rho)}$$

Substituting this into (52) yields simplified:

$$\frac{\partial E[G]}{\partial a}\frac{1}{\lambda} = h(\rho)\left(\int_0^\infty y\, f_{Y,C}(y, a + by)dy - \right. \left. - \frac{b(\int_0^\infty c\, f_{Y,C}(\frac{-a+c}{b}, c)dc) \int_0^\infty y^2\, f_{Y,C}(y, a + by)dy}{\int_0^\infty c(-a + c) f_{Y,C}(\frac{-a+c}{b}, c)dc} \right) \tag{54}$$

We have to set this term to zero and calculate a from this equation. Since $h(\rho) > 0$ for finite ρ, the term

$$\int_0^\infty y f_{Y,C}(y, a + by)\, dy - \frac{b\left(\int_0^\infty c f_{Y,C}\left(\frac{-a+c}{b}, c\right) dc\right) \int_0^\infty y^2 f_{Y,C}(y, a + by)\, dy}{\int_0^\infty c(-a + c) f_{Y,C}\left(\frac{-a+c}{b}, c\right) dc} \tag{55}$$

must be zero. Analytical solution of this equation hardly seems possible, so we try to prove the theorem $a = 0$.

For $a = 0$, (55) is:

$$\int_0^\infty y f_{Y,C}(y, by)\, dy - \frac{b\left(\int_0^\infty c f_{Y,C}\left(\frac{c}{b}, c\right) dc\right) \int_0^\infty y^2 f_{Y,C}(y, by)\, dy}{\int_0^\infty c^2 f_{Y,C}\left(\frac{c}{b}, c\right) dc} \tag{56}$$

(56) can be analyzed as follows: $f_{Y,C}(\frac{c}{b}, c)$ are the values of $f_{Y,C}(y, c)$ for all combinations of contribution margin and capacity requirement which meet the restriction $c = by$. By substituting $c = by$, we transform both integrals with c as integration variable in (56) into integrals with y as integration variable:

$$\int_0^\infty c f_{Y,C}\left(\frac{c}{b}, c\right) dc = \int_0^\infty b^2 y f_{Y,C}(y, by)\, dy \tag{57}$$

$$\int_0^\infty c^2 f_{Y,C}\left(\frac{c}{b}, c\right) dc = \int_0^\infty b^3 y^2 f_{Y,C}(y, by)\, dy \tag{58}$$

Substituting (57) and (58) into numerator and denominator, respectively, of the fraction in (56) yields:

$$\int_0^\infty y f_{Y,C}(y, by)\, dy - \frac{b\left(\int_0^\infty b^2 y f_{Y,C}(y, by)\, dy\right) \int_0^\infty y^2 f_{Y,C}(y, by)\, dy}{\int_0^\infty b^3 y^2 f_{Y,C}(y, by)\, dy} \tag{59}$$

Pulling out b^2 and b^3 from the integrals demonstrates that (59) is zero.

Appendix 3: Proof of Theorem 4

According to Theorem 3 we set $a = 0$.

The partial derivative of $E[G]$ with respect to b, divided by the constant λ:

$$\frac{\partial E[G]}{\partial b}\frac{1}{\lambda} = h(\rho)\int_0^\infty y^2 f_{Y,C}(y, by)\,dy + \\ + \frac{\lambda\left(\int_0^\infty y\int_0^{by} f_{Y,C}(y,c)\,dc\,dy\right)\left(\int_0^\infty c^2 f_{Y,C}\left(\frac{c}{b},c\right)dc\right)h'(\rho)}{b^2 L} \tag{60}$$

The partial derivative of $E[G]$ with respect to L:

$$\frac{\partial E[G]}{\partial L} = \\ = -K - \frac{\lambda^2\left(\int_0^\infty y\int_0^{by} f_{Y,C}(y,c)\,dc\,dy\right)\left(\int_0^\infty c\int_{\frac{c}{b}}^\infty f_{Y,C}(y,c)\,dy\,dc\right)h'(\rho)}{L^2} \tag{61}$$

Again, u (term (51) from Appendix 2) is an element of both partial derivatives. So we substitute

$$u = \int_0^\infty y\int_0^{by} f_{Y,C}(y,c)\,dc\,dy \tag{62}$$

and calculate u for the optimal value of L (that is, $\frac{\partial E[G]}{\partial L}$ (from (61)) $= 0$):

$$u = -\frac{KL^2}{\lambda^2\left(\int_0^\infty c\int_{\frac{c}{b}}^\infty f_{Y,C}(y,c)\,dy\,dc\right)h'(\rho)} \tag{63}$$

In (60) we substitute u from (62) for the first expression in parenthesis of the numerator and obtain:

$$\frac{\partial E[G]}{\partial b}\frac{1}{\lambda} = h(\rho)\int_0^\infty y^2 f_{Y,C}(y,by)\,dy + \frac{\lambda u\left(\int_0^\infty c^2 f_{Y,C}\left(\frac{c}{b},c\right)dc\right)h'(\rho)}{b^2 L} \tag{64}$$

In (64) we substitute (63) for u and obtain:

$$\frac{\partial E[G]}{\partial b}\frac{1}{\lambda} = h(\rho)\int_0^\infty y^2 f_{Y,C}(y,b\,y)\,dy - \\ - \frac{KL\int_0^\infty c^2 f_{Y,C}\left(\frac{c}{b},c\right)dc}{b^2\lambda\int_0^\infty c\int_{\frac{c}{b}}^\infty f_{Y,C}(y,c)\,dy\,dc} \tag{65}$$

$\int_0^\infty c^2 f_{Y,C}\left(\frac{c}{b},c\right)dc$ can be transformed into an integral with y as integration variable (as in Appendix 2).

Pulling out b from the integral and simplifying yields:

$$\frac{\partial E[G]}{\partial b}\frac{1}{\lambda} = \left(h(\rho) - \frac{bKL}{\lambda\int_0^\infty c\int_{\frac{c}{b}}^\infty f_{Y,C}(y,c)\,dy\,dc}\right)\int_0^\infty y^2 f_{Y,C}(y,by)\,dy \tag{66}$$

Since $\int_0^\infty y^2 f_{Y,C}(y,by)\,dy$ cannot be equal to zero, the optimality condition for b is:

$$h(\rho) - \frac{bKL}{\lambda\int_0^\infty c\int_{\frac{c}{b}}^\infty f_{Y,C}(y,c)\,dy\,dc} = 0 \tag{67}$$

We have to prove that the optimal value for b resulting from (67) is equal to the supposed value from Eq. (27):

$$b^*(\text{supposed}) = \frac{\lambda_{eff}\, L}{\mu K_{fix}}. \tag{68}$$

We denote the optimal value for b resulting from (67) as b^*. We obtain from (67):

$$b^* = \frac{\lambda h\,(\rho) \int_0^\infty c \int_{\frac{c}{b}}^\infty f_{Y,C}\,(y,c)\,dy\,dc}{KL}. \tag{69}$$

Performing the respective substitutions for μ, K_{fix} and λ_{eff} in (68) demonstrates that b^* (69) is equal to b^* *(supposed)* (68), which is the reciprocal of the fixed costs per unit of capacity. This completes the proof. □

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