Physical anomalous dimensions at small $x^\star$

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Abstract. I present a theoretical discussion of the uncertainties related to the QCD analysis of the proton structure function $F_2(x, Q^2)$ at small x. The role played by the 'unphysical' gluon density is pointed out. It is shown how the study of more observables can reduce the theoretical uncertainty and, in particular, an alternative method of analysis, based on the introduction of physical anomalous dimensions, is suggested.

1 Introduction

One of the main outcome of the physics programme carried out at HERA is the observed striking rise of the proton structure function $F_2(x, Q^2)$ [1] at small values of the Bjorken variable x ($2 \cdot 10^{-5} < x < 10^{-2}$) and high values of the momentum transfer Q^2 ($Q^2 \gtrsim 2$ GeV^2).

The HERA data on F_2 represent the first experimental observation of a cross section increasing faster than logarithmically with the energy (see, for instance, [2]). This high-energy behaviour in the hard-scattering regime is expected if the underlying dynamics is driven by self-interacting massless vector bosons, the gluons. Thus, the steep rise of F_2 certainly confirms one of the basic prediction of perturbative QCD [3].

However, the main reason why the HERA data have attracted much theoretical attention goes beyond this point. The issue, indeed, is whether the striking rise of F_2 at small x calls forth a theoretical interpretation in terms of non-conventional QCD dynamics. In this context, non-conventional QCD stands for any approach (based either on the original BFKL equation [4] or on $k_\perp$-factorization [5–10]) in which the small-x behaviour of $F_2(x, Q^2)$ is studied by resumming logarithmic corrections of the type $(\alpha_S \ln x)^n$ to *all orders* in the strong coupling α_S. By contrast, no small-x resummation is performed within the conventional QCD (or DGLAP [3, 11]) approach: the parton densities of the

proton at a fixed input scale Q_0^2 are evolved in Q^2 according to the Altarelli-Parisi equation evaluated in *fixed-order* perturbation theory.

The theoretical motivation for the non-conventional approach based on resummation is clear. Since multiple gluon radiation in the final state produces perturbative contributions of the type $(\alpha_S \ln x)^n$, as soon as x is sufficiently small (i.e. $\alpha_S \ln 1/x \sim 1$), the fixed-order expansion in α_S must become inadequate to describe the QCD dynamics. Thus, in principle, the non-conventional approach is certainly more accurate at asymptotically-small values of x. The question is whether, in practice, in the HERA kinematic region we are already approaching this asymptotic regime.

In my opinion it is quite difficult to answer this question in the context of the QCD analysis of the sole F_2. Indeed, the small-x rise of F_2 can be obtained as the result of two combined effects: the increase of perturbative scaling violation in the small-x region and the intrinsic non-perturbative steepness of the gluon density. These perturbative and non-perturbative components are mixed up not only on the phenomenological side but, more importantly, on theoretical basis. Since the QCD description of a single observable, namely F_2, requires the introduction of two non-perturbative inputs, quark and gluon densities, the distinction between perturbative and non-perturbative components is strongly dependent on their own definition rather than on the underlying dynamics.

A better understanding of QCD physics at small x can be achieved by considering more observables and thus (over-)constraining the definition of the parton densities. In particular, by simply using two hadronic observables one can formulate the dynamics of scaling violation enterely in terms of perturbative quantities that play the role of physical anomalous dimensions. These anomalous dimensions are unambiguously computable in QCD perturbation theory and thus they theoretically appear as golden quantities for comparing the conventional and non-conventional approaches.

The outline of the paper is as follows. In Sect. 2, I briefly review the theoretical and phenomenological status of the QCD analysis of $F_2(x, Q^2)$. In particular, I qualitatively discuss the theoretical uncertainties relative to different perturbative approaches and to the small-x behaviour of the

gluon density. In Sect. 3, I introduce the physical anomalous dimensions that control the Q^2 evolution of $F_2(x, Q^2)$ and of the longitudinal structure function $F_L(x, Q^2)$. The main features of these anomalous dimensions are discussed in Sect. 3.1, while in Sect. 3.2, I present their explicit expressions in resummed perturbation theory at small x. Additional observations on the relationship between physical anomalous dimensions and parton model are considered in Sect. 4. Section 5 deals with physical anomalous dimensions for heavy-flavour structure functions. In particular, it points out the kinematical features of the physical anomalous dimensions for observables that depend on several large-momentum scales. Some general comments are left to Sect. 6.

2 The proton structure function F_2 and the gluon density

The master equations for the perturbative-QCD study of the proton structure function at small x are as follows

$$F_2(x, Q^2) = \langle e_f^2 \rangle \tilde{f}_S(x, Q^2) + \ldots + \mathcal{O}(1/Q^2) \ , \tag{1}$$

$$\frac{dF_2(x, Q^2)}{d \ln Q^2} = \langle e_f^2 \rangle \int_x^1 dz \, \big[\, P_{SS}(\alpha_S(Q^2), z) \, \tilde{f}_S \left(x/z, Q^2 \right)$$
$$+ P_{Sg}(\alpha_S(Q^2), z) \, \tilde{f}_g \left(x/z, Q^2 \right) \big]$$
$$+ \ldots + \mathcal{O}(1/Q^2) \ , \tag{2}$$

$$\frac{d \tilde{f}_g(x, Q^2)}{d \ln Q^2} = \int_x^1 dz \, \big[P_{gq}(\alpha_S(Q^2), z) \, \tilde{f}_S \left(x/z, Q^2 \right)$$
$$+ P_{gg}(\alpha_S(Q^2), z) \, \tilde{f}_g \left(x/z, Q^2 \right) \big] \ , \tag{3}$$

where e_f is the electric charge of each quark with flavour f, $\langle e_f^2 \rangle = (\sum_{f=1}^{N_f} e_f^2)/N_f$ and N_f is the number of active flavours. In (1–3) I am using the same notation as in [10]. Thus, the singlet density $\tilde{f}_S$ and the gluon density $\tilde{f}_g$ are related to the usual quark (antiquark) and gluon densities f_{q_f} ($f_{\bar{q}_f}$) and f_g by the following relations

$$\tilde{f}_S(x, Q^2) = x \sum_f \big[f_{q_f}(x, Q^2) + f_{\bar{q}_f}(x, Q^2) \big] \ ,$$

$$\tilde{f}_g(x, Q^2) = x f_g(x, Q^2) \ , \tag{4}$$

and the quark splitting function P_{SS} and P_{Sg} are given in terms of the customary Altarelli-Parisi splitting functions P_{ab} as follows

$$P_{Sg}(\alpha_S, x) = 2 N_f P_{q_i g}(\alpha_S, x) \ ,$$

$$P_{SS}(\alpha_S, x) = \sum_j \big[P_{q_i q_j}(\alpha_S, x) + P_{q_i \bar{q}_j}(\alpha_S, x) \big] \ . \tag{5}$$

The dots and the term $\mathcal{O}(1/Q^2)$ on the right-hand side of (1, 2) respectively denote the flavour non-singlet component and higher-twist contributions. The contribution of the non-singlet component to (1, 2) is quantitatively negligible at small-x and it will be neglected throughout the paper. However, its inclusion is formally straightforward (see, for instance, (5.3-5.5) in [10]).

The basis for (1–3) is provided by the factorization theorem of mass singularities [12]. According to this theorem the (*perturbatively calculable*) splitting functions $P_{ab}(\alpha_S, x)$ and the (*phenomenological*) parton densities $\tilde{f}_a(x, Q^2)$ are not separately physical observables. Only proper combinations (convolutions) of them (for instance, the right-hand sides of (1, 2)) are related to measurable quantities. Therefore one has some freedom (ambiguity) in defining splitting functions and parton densities. This freedom is called factorization-scheme dependence and follows from the fact that hadron scattering cross-sections cannot be computed within a purely perturbative framework. The factorization theorem states that at high momentum transfer Q, the perturbatively non-calculable component of *all* the cross sections is factorizable in *few* universal (process independent) parton distributions. These parton distributions can be defined using experimental information on an *equal* number of hadronic observables at a certain scale. Having that done, the high-Q^2 behaviour of all the hadronic cross sections can be unambiguously (modulo power suppressed corrections) computed by using perturbation theory.

Equations (1) and (2) refer to the so-called DIS factorization scheme[1][13]. In this scheme, (1) actually represents the definition of the singlet-quark density $\tilde{f}_S$. The true dynamical information is instead contained in the scaling violations of F_2 as described by (2) and by the analogous evolution equation (3) for the gluon density.

The Altarelli-Parisi splitting functions entering into (2, 3) are computable in QCD perturbation theory as a power series expansion in α_S:

$$P_{ab}(\alpha_S, x) = \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{2\pi} \right)^n \, P_{ab}^{(n-1)}(x) \ , \tag{6}$$

and the coefficients $P_{ab}^{(n-1)}(x)$ in this series can be calculated (at least, in principle) to any order n in α_S.

Conceptually, the content of the QCD analysis of F_2 according to the master equations written above is the following. One first computes the splitting functions in (6) to a given perturbative accuracy. Then, by using the experimental information on F_2 and $dF_2/d \ln Q^2$, from (2, 3) one can determine the quark and gluon densities as functions of x and Q^2. Finally, (3) enters as self-consistency check of the QCD evolution equations.

In practice, the QCD analysis proceeds as follows. One assigns a certain parametrization for the parton densities at a given input scale Q_0^2. Then, inserting this parametrization into (1–3), one can fit the input parameters to the experimental data.

It is worth emphasizing a point that is independent of the actual procedure used in the QCD analysis. Whilst the F_2 data uniquely determine the (DIS scheme) quark density, the measurement of $dF_2/d \ln Q^2$ does not give access directly to the determination of the gluon density $\tilde{f}_g$, but rather to that of the product (convolution) $P_{Sg} \otimes \tilde{f}_g$. Since P_{Sg} is evaluated in a certain theoretical framework (that is, at a given perturbative order or in resummed perturbation theory, in a certain factorization scheme and so forth), the ensuing $\tilde{f}_g$ turns out to be 'theory-dependent'.

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[1] Equation (3) does not involve (at least, directly) any physical observable. Thus, it takes the same form in any scheme. Of course, the gluon splitting functions P_{gq} and P_{gg} have to be consistently computed in the corresponding factorization scheme

2.1 Fixed-order perturbation theory

Only the first two terms $P_{ab}^{(0)}(x)$, $P_{ab}^{(1)}(x)$ of the perturbative expansion (6) of the splitting functions are exactly known (i.e. known for any values of x) [14]. In the conventional approach these leading order (LO) and next-to-leading order (NLO) terms are used as theoretical inputs for the QCD analysis of F_2. It turns out that the HERA data can be succesfully described [15–19] by parton densities having the following small-x behaviour

$$\tilde{f}_S(x, Q_0^2) \simeq x^{-\lambda_S} \ , \quad \tilde{f}_g(x, Q_0^2) \simeq x^{-\lambda_g} \tag{7}$$

with $\lambda_S \sim \lambda_g = 0.2 \div 0.3$ at the input scale $Q_0^2 \sim 4$ GeV^2.

Up to the second order in α_S, the quark splitting functions P_{SS}, P_{Sg} in (2) are essentially flat at small x, whilst the gluon splitting functions are steeper and behave as follows[2]

$$P_{gg}(\alpha_S, x) \simeq \frac{C_A}{C_F} P_{gq}(\alpha_S, x) \simeq \frac{\bar{\alpha}_S}{x} \ , \tag{8}$$

where $C_A = N_c$, $C_F = (N_c^2 - 1)/(2N_c)$, $N_c = 3$ is the number of colours and I have defined $\bar{\alpha}_S \equiv C_A \alpha_S / \pi$. Thus, the phenomenological success of the NLO QCD approach tells us that the rise of F_2 at small x is due to the DGLAP evolution in the gluon channel (i.e. it is due to (3)) combined with a steep behaviour ($\sim x^{-0.2}$) of the input densities at $Q_0^2 \sim 4$ GeV^2.

2.2 Resummed perturbation theory

The basis for the non-conventional QCD approach is provided by the BFKL equation [4]. Starting from it, a formalism that is able to combine *consistently* small-x resummation with the QCD factorization theorem has been set up in the last few years. This formalism, known as $k_\perp$-factorization or high-energy factorization, was first discussed to leading-order accuracy in [5–8] and then was extended to higher-orders in [9, 10]. In the high-energy factorization approach, one ends up with the usual QCD evolution equations (namely, (1–3) in the case of the proton structure function F_2) but the splitting functions $P_{ab}(\alpha_S, x)$ in (6) (and, in general, the process-dependent coefficient functions: see Sects. 3–5) are no longer evaluated in fixed-order perturbation theory. They are indeed supplemented with the all-order resummation of the *leading* ($\frac{1}{x}\alpha_S^n \ln^{n-1} x$), *next-to-leading* ($\frac{1}{x}\alpha_S^n \ln^{n-2} x$) and, possibly, subdominant ($\frac{1}{x}\alpha_S^n \ln^m x$, $m < n-2$) contributions at small x. Note, also, that this resummation can be performed by having full control of the factorization-scheme dependence of splitting (and coefficient) functions and parton densities [8, 10, 20].

The present theoretical status of small-x resummation is the following[3]. The leading-logarithmic (LL) contributions to the gluon splitting functions $P_{gg}(\alpha_S, x)$, $P_{gq}(\alpha_S, x)$ are known [4, 8, 21]. Their resummation leads to a very steep (power-like) asymptotic behaviour:

$$P_{gg}(\alpha_S, x)|_{\text{asym.}} \simeq \frac{C_A}{C_F} P_{gq}(\alpha_S, x)|_{\text{asym.}} \sim \bar{\alpha}_S \, x^{-(1+\lambda_L)} \ , \tag{9}$$

where the power $1 + \lambda_L = 1 + 4\bar{\alpha}_S \ln 2 \simeq 1 + 2.65\alpha_S$ is the so-called intercept of the perturbative QCD pomeron. The complete next-to-leading logarithmic (NLL) contributions to the gluon splitting functions are not yet known and calculations are in progress [22–24]. In particular, the contributions proportional to N_f in P_{gg} have been evaluated recently [24]. Owing to the gluon dominance at high energy, the quark splitting functions $P_{Sg}(\alpha_S, x)$, $P_{SS}(\alpha_S, x)$ do not contain LL contributions. However, the NLL terms are completely known [9, 10] to all orders in perturbation theory.

Having developed a resummed perturbative expansion to the same (modulo the still unknown NLL terms in the gluon sector) degree of theoretical accuracy as the fixed-order perturbative expansion, one can set up a fully consistent non-conventional QCD approach [10]. This is accomplished [25, 26] by adding leading and next-to-leading logs to one- and two-loop contributions (after subtracting the resummed logarithmic terms, in order to avoid double counting) in the splitting functions $P^{ab}(\alpha_S, x)$, thus obtaining a perturbative framework that is everywhere at least as good as the fixed-order expansion, and much better as x becomes small.

After the first numerical analyses [27] within the $k_\perp$-factorization framework, phenomenological studies based on resummed perturbation theory have been performed during the last year [25, 26, 28]. They have shown that, likewise the conventional QCD analysis, the non-conventional approach can accomodate the parton densities to provide a description of the HERA data on F_2.

The naïve explanation for that could be that the inclusion of the resummed logarithmic corrections produces a small effect in the Altarelli-Parisi splitting functions. Actually, this is not the case.

Indeed, it is true that the resummation of the leading terms $\frac{1}{x}\alpha_S^n \ln^{n-1} x$ in the gluon splitting functions has a moderate impact on the scaling violations of F_2 in the kinematical range presently investigated at HERA. The situation is however different in the quark channel, that is, in the evolution equation (2). The measured large value of $dF_2(x, Q^2)/d \ln Q^2$ at small x calls for a quite steep product (convolution) $P_{Sg} \otimes \tilde{f}_g$. In the conventional (fixed-order) perturbative analysis this condition can be fulfilled only by choosing a quite steep input distribution $\tilde{f}_g$. After resummation of the next-to-leading terms $\frac{1}{x}\alpha_S^n \ln^{n-2} x$, the quark splitting functions $P_{Sg}(\alpha_S, x)$ and $P_{SS}(\alpha_S, x)$ are much steeper than the corresponding splitting functions evaluated in two-loop order[4]. Thus, the non-conventional approach can succeed in describing the small-x rise of F_2 by using parton densities that at the input scale Q_0^2 are less steep than those needed in the fixed-order approach.

From this result one may conclude that the conventional and non-conventional approaches are phenomenologically equivalent: the HERA data on F_2 at sufficiently high Q^2 cannot distinguish steep input densities from steep dynamical evolution.

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[2] The scheme dependence of the splitting functions appears only starting from two-loop order. In particular, in two-loop order this dependence is pretty mild at small x

[3] I refer to Sects. 3 and 5 for the resummation in the process-dependent coefficient functions

[4] I refer to [29] for a more detailed discussion on the small-x behaviour of the resummed splitting functions

The conclusion is instead different from a theoretical viewpoint. Since the resummation of the NLL contributions leads to a large effect on F_2, there is no justification for truncating the QCD perturbative expansion at NLO: the fixed-order expansion approach is thus theoretically disfavoured. The only caveat against such a firm conclusion is that the NLL terms in the gluon sector are still unknown: they may lead to a large and opposite effect with respect to those in the quark channel.

At the same time, since the known NLL contributions produce large corrections on F_2, one can expect that subleading terms may still have a sizeable effect. Thus, at present, perturbative QCD predictions for the small-x behaviour of F_2 suffer from substantial theoretical uncertainties [25]. A better understanding of subleading contributions is necessary to reduce these uncertainties [25, 30].

The QCD analysis of the sole proton structure function F_2, moreover, is affected by an even larger indeterminacy related to the difficulty in disentangling perturbative and non-perturbative effects. In order to clarify this point, let me briefly consider the issue of the factorization-scheme dependence [20, 29, 31].

2.3 Factorization-scheme dependence

As discussed in the first part of this Section, the parton densities are not physical observables and, in particular, they are not calculable in perturbation theory. In the perturbative framework they are defined apart from an overall perturbative function. Thus, starting from the DIS factorization scheme considered so far, one can introduce a new factorization scheme of DIS type[5] (i.e. a scheme in which the physical identification of the quark density with the proton structure function as in (1) remains valid) by defining a new gluon density $\tilde{f}_g^{(new)}$ as follows [29]

$$
\tilde{f}_g^{(new)}(x, Q^2) = \tilde{f}_g(x, Q^2) + \int_x^1 \frac{dz}{z}
$$
$$
\times \left[u(\alpha_S(Q^2), z)\, \tilde{f}_g(x/z, Q^2) \right.
$$
$$
\left. + \frac{C_F}{C_A}\, v(\alpha_S(Q^2), z)\, \tilde{f}_S(x/z, Q^2) \right]. \qquad (10)
$$

Here $u(\alpha_S, z)$ and $v(\alpha_S, z)$ are functions that can be expanded as power series in α_S and vanish for $\alpha_S = 0$. As for their functional dependence on z, it is quite arbitrary. The only constraints are that $u(\alpha_S, z)$ and $v(\alpha_S, z)$ contain at most NLL terms of the type $\alpha_S(\alpha_S \ln z)^n$ for $z \to 0$ and that these NLL terms are equal in u and v.

The dynamical evolution equations (2) and (3) can be written in terms of the new gluon density (10) and of new Altarelli-Parisi splitting functions. The above constraints guarantee that the new splitting functions have the same LL behaviour as the DIS-scheme splitting functions, so that the dominant perturbative dynamics is left unchanged.

The freedom of arbitrarily choosing the factorization scheme is not a particular feature of small-x dynamics. The

transformation in (10) can be applied in the small-x as well as in the large-x regions. In general, its effect amounts to a redefinition of the input parton densities that is perturbatively under control. The effect, instead, can be quite large in the small-x region because each power of α_S can be accompanied by an enhancing logarithmic factor of $\ln 1/x$.

In order to quantify the theoretical uncertainty related to the scheme dependence, let us consider the simplest case in which the splitting functions in (2, 3) are evaluated to LL accuracy. Thus, we can perform the scheme transformation in (10) by choosing any NLL functions u and v and, in particular, we can set $u(\alpha_S, z) = v(\alpha_S, z) = A\alpha_S z^{-K\alpha_S}$, where A and K are constants of order unity. Assuming the extreme case of flat input densities, this leads to the following factorization-scheme uncertainty

$$
\delta\tilde{f}_g(x) = \tilde{f}_g^{(new)}(x) - \tilde{f}_g(x)
$$
$$
= \frac{A}{K}\, (x^{-K\alpha_S} - 1) \sim \frac{A}{K}\, x^{-K\alpha_S} \quad . \qquad (11)
$$

This implies that, from the QCD analysis of F_2 to LL accuracy, one cannot argue whether steep input densities have a non-perturbative origin or rather mimic higher-order perturbative effects. Of course, using the NLL expressions for the splitting functions one reduces the factorization-scheme uncertainty by a factor of α_S. For the case considered above one obtains $\delta\tilde{f}_g(x) \sim \alpha_S x^{-K\alpha_S}$ that, however, still represents a substantial indeterminacy.

In order to gain more theoretical accuracy one should compute higher perturbative orders in the Altarelli-Parisi splitting functions. The same goal can be achieved in a simpler manner by eliminating the factorization-scheme uncertainty, that is, by relating the gluon density to other physical observables.

Actually, there is one more reason for studying the small-x behaviour of physical observables other than F_2. As a matter of fact, the large effect on F_2 of the NLL contributions in the quark channel might be, in a sense, spurious or, more precisely, related to the use of certain factorization schemes [20, 29].

For instance (and quite strikingly), one can choose the functions $u(\alpha_S, z)$ and $v(\alpha_S, z)$ in such a way that all the NLL terms in the quark channel are removed from the (new) quark splitting functions P_{Sg}, P_{SS} and absorbed into the redefinition (10) of the gluon density [29]. The only price one has to pay consists in the introduction of additional NLL terms in the gluon splitting functions. It turns out that the effect of these additional terms is quantitatively small [26]. Therefore, within this factorization scheme (called SDIS scheme in [29]) the non-conventional approach to NLL accuracy and the conventional approach to NLO are, in practice, indistinguishable (apart from the caveat on the unknown NLL terms in the gluon channel) as for the QCD analysis of F_2. The introduction of the SDIS scheme thus provides a more formal argument to explain the phenomenological equivalence of the conventional and non-conventional approaches that has been pointed out in Sect. 2.2.

This equivalence may appear as due to a particular algebraic trick or to fine-tuning of the factorization scheme with no physical content. Actually, this is not necessarily the case. The factorization-scheme dependence, rather than an ambiguity in higher-order perturbative coefficients, has

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[5] More general factorization schemes, like for instance the $\overline{\text{MS}}$ scheme, are considered in [10, 31]

to be regarded more physically as a parametrization of our ignorance in factorizing perturbative from non-perturbative physics. At present, the proton structure function F_2 (i.e. (1–3)) provides experimental/theoretical information that is not sufficiently accurate to disentangle perturbative from non-perturbative dynamics at small-x.

In order to have better control on the perturbative dynamics, one should consider the small-x behaviour of other physical observables. Indeed, by means of the transformation from the DIS to the SDIS schemes (almost) no trace of small-x perturbative contributions is left in F_2 and all the resummation effects are moved to other physical quantities. These effects can be sizeable. It may also happen that the resummed contributions are almost universal, in the sense that, in the kinematic regions that are experimentally accessible, they produce very similar quantitative effects in all physical observables. In this case, these contributions can be consistently absorbed into the non-perturbative parton densities and fixed-order perturbation theory can be safely used throughout.

3 Factorization-theorem invariants at small x

The theoretical motivations for studying the small-x behaviour of several different physical observables have been pointed out in the previous Section. As discussed in [5, 32] and furtherly elaborated on in [24], this study can be performed by considering properly defined K-factors (ratios of hadronic cross sections), which are factorization-scheme independent. Analogously, one can introduce factorization-scheme invariants that relate the scaling violations of different structure functions. These invariants are discussed in the rest of this paper.

Among the observables that one can consider, the longitudinal structure function F_L of the proton is becoming increasingly topical. On the experimental side, data on $F_L(x, Q^2)$ at small x will be available soon from HERA. On the theoretical side, this quantity is known to a sufficient accuracy.

In order to make more explicit this statement about the theoretical accuracy of F_L, let me recall that, using the factorization theorem of mass singularities, F_L is given as follows

$$F_L(x, Q^2) = \langle e_f^2 \rangle \int_x^1 \frac{dz}{z} \left[C_L^S(\alpha_S(Q^2), z) \, \tilde{f}_S\left(x/z, Q^2\right) \right.$$
$$\left. + C_L^g(\alpha_S(Q^2), z) \, \tilde{f}_g\left(x/z, Q^2\right) \right]$$
$$+ \dots + \mathcal{O}(1/Q^2) \ , \tag{12}$$

where $\tilde{f}_S$ and $\tilde{f}_g$ are the same parton densities that enter into (1–3) and, as in (2, 3), the dots and the term $\mathcal{O}(1/Q^2)$ respectively denote flavour non-singlet and higher-twist contributions. In any given factorization scheme the coefficient functions C_L^S and C_L^g in (12) are computable in QCD perturbation theory according to the following power series expansion

$$C_L^a(\alpha_S, x) = \sum_{n=1}^{+\infty} \left(\frac{\alpha_S}{2\pi}\right)^n \, C_L^{a\,(n-1)}(x) \ . \tag{13}$$

Both the LO and NLO coefficients $C_L^{a\,(0)}(x)$, $C_L^{a\,(1)}(x)$ have been computed for any value of x [33]. Correspondingly, in resummed perturbation theory all the NLL terms $\ln^{n-1} x$ in $C_L^{a\,(n)}(x)$ are known [10]. Owing to this theoretical information, (1–3) can be supplemented with (12) thus eliminating the factorization scheme uncertainty. To this purpose one should introduce the parton densities $\tilde{f}_S$ and $\tilde{f}_g$ extracted from (1–3) into (12). Theoretical consistency simply requires that splitting functions and coefficient functions are evaluated to the corresponding accuracy, that is, to NLO in the conventional approach and including NLL terms in resummed perturbation theory.

3.1 Physical anomalous dimensions

The unphysical role played by the parton densities within this context is clear. Indeed, one can write down evolution equations that involve only physical observables and perturbative quantities. Starting from (1, 12) and performing straightforward algebraic manipulations, one first express the parton densities $\tilde{f}_S$ and $\tilde{f}_g$ as functions of F_2 and F_L. Then, inserting the expressions derived in this manner into (2, 3), one obtains the following dynamical equations

$$\frac{dF_2(x, Q^2)}{d\ln Q^2} = \int_x^1 \frac{dz}{z} \left[\Gamma_{22}(\alpha_S(Q^2), z) \, F_2\left(x/z, Q^2\right) \right.$$
$$\left. + \Gamma_{2L}(\alpha_S(Q^2), z) \, F_L\left(x/z, Q^2\right) \right]$$
$$+ \dots + \mathcal{O}(1/Q^2) \ , \tag{14}$$
$$\frac{dF_L(x, Q^2)}{d\ln Q^2} = \int_x^1 \frac{dz}{z} \left[\Gamma_{L2}(\alpha_S(Q^2), z) \, F_2\left(x/z, Q^2\right) \right.$$
$$\left. + \Gamma_{LL}(\alpha_S(Q^2), z) \, F_L\left(x/z, Q^2\right) \right]$$
$$+ \dots + \mathcal{O}(1/Q^2) \ , \tag{15}$$

From a formal viewpoint (14, 15) may appear equivalent to (2, 3). However, (14, 15) relate the scaling violations of two physical observables, namely F_2 and F_L, to the actual value of the same observables. It follows that the kernels $\Gamma_{ij}(\alpha_S(Q^2), x)$ (with $i, j = 2, L$) are *physical observables* as well. Owing to the formal resemblance to the Altarelli-Parisi splitting functions, the kernels $\Gamma_{ij}(\alpha_S(Q^2), x)$ can be considered as *physical splitting functions*.

The main physical properties of the kernels $\Gamma_{ij}(\alpha_S(Q^2), x)$ are that $i)$ each of them is consistently computable in QCD perturbation theory (modulo higher-twist corrections that are suppressed by some power of $1/Q$ in the hard-scattering regime) and $ii)$ each of them is a factorization-theorem invariant, i.e. it does not depend on both the factorization scheme and the factorization scale. In other words, from the viewpoint of perturbative QCD, each $\Gamma_{ij}(\alpha_S(Q^2), x)$ is completely analogous to the celebrated ratio

$$R_{e^+e^-} = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \ , \tag{16}$$

in e^+e^- annihilation.

The perturbative expansions of the physical splitting functions are the following

$$\Gamma_{LL}(\alpha_S, x) = \sum_{n=1}^{+\infty} \left(\frac{\alpha_S}{2\pi}\right)^n \Gamma_{LL}^{(n-1)}(x)$$

$$= \frac{\alpha_S}{2\pi} \left[\Gamma_{LL}^{(0)}(x) + \frac{\alpha_S}{2\pi} \Gamma_{LL}^{(1)}(x) + \ldots \right] , \qquad (17)$$

$$\Gamma_{L2}(\alpha_S, x) = \sum_{n=1}^{+\infty} \left(\frac{\alpha_S}{2\pi}\right)^{n+1} \Gamma_{L2}^{(n-1)}(x)$$

$$= \left(\frac{\alpha_S}{2\pi}\right)^2 \left[\Gamma_{L2}^{(0)}(x) + \frac{\alpha_S}{2\pi} \Gamma_{L2}^{(1)}(x) + \ldots \right] , \quad (18)$$

$$\Gamma_{2L}(\alpha_S, x) = \sum_{n=1}^{+\infty} \left(\frac{\alpha_S}{2\pi}\right)^{n-1} \Gamma_{2L}^{(n-1)}(x)$$

$$= \left[\Gamma_{2L}^{(0)}(x) + \frac{\alpha_S}{2\pi} \Gamma_{2L}^{(1)}(x) + \ldots \right] , \qquad (19)$$

$$\Gamma_{22}(\alpha_S, x) = \sum_{n=1}^{+\infty} \left(\frac{\alpha_S}{2\pi}\right)^n \Gamma_{22}^{(n-1)}(x)$$

$$= \frac{\alpha_S}{2\pi} \left[\Gamma_{22}^{(0)}(x) + \frac{\alpha_S}{2\pi} \Gamma_{22}^{(1)}(x) + \ldots \right] . \qquad (20)$$

Note that the expansions for the diagonal kernels Γ_{LL}, Γ_{22} are completely analogous to that in (6) for the Altarelli-Parisi splitting functions. The mismatch in the overall power of α_S between the expansions for the diagonal and non-diagonal (Γ_{L2}, Γ_{2L}) kernels is due to the fact that, from a perturbative viewpoint (or, equivalently, because of the validity of the Callan-Gross relation, $F_L = 0$, in the naïve parton model), the longitudinal structure function has to be considered as a physical quantity of relative order α_S with respect to F_2, i.e. $F_L \sim \alpha_S F_2$. Taking this into account, a conventional QCD calculation should consistently consider the contributions $\Gamma_{ij}^{(0)}(x)$ in (17–20) as lowest-order terms, $\Gamma_{ij}^{(1)}(x)$ as next-order terms and so forth.

Obviously, since the kernels Γ_{ij} are physical observables, they are renormalization-group invariant quantities. It follows that, if computed in fixed-order perturbation theory they should exibits the customary dependence on the renormalization scale μ. Thus, to be more precise, in the evolution equations (14, 15) one has to perform the replacement $\Gamma_{ij}(\alpha_S(Q^2), x) \rightarrow \Gamma_{ij}(\alpha_S(\mu^2), Q^2/\mu^2, x)$. Equations (17–20) refer to the perturbative expansion of Γ_{ij} for $\mu = Q^2$. In general one obtains:

$$\Gamma_{ij}\left(\alpha_S(\mu^2), \frac{Q^2}{\mu^2}, x\right)$$
$$= \left(\frac{\alpha_S(\mu^2)}{2\pi}\right)^p \left[\Gamma_{ij}^{(0)}(x) + \frac{\alpha_S(\mu^2)}{2\pi} \right.$$
$$\left. \times \left(\Gamma_{ij}^{(1)}(x) - p \, \Gamma_{ij}^{(0)}(x) \, 2\pi\beta_0 \ln \frac{Q^2}{\mu^2} \right) + \ldots \right], \qquad (21)$$

where $12\pi\beta_0 = 11C_A - 2N_f$ is the first coefficient of the QCD β-function and $p = 1$ for Γ_{LL} and Γ_{22}, $p = 0$ for Γ_{2L}, $p = 2$ for Γ_{L2}.

This discussion on the perturbative features of the kernels $\Gamma_{ij}(\alpha_S, x)$ can be summarised by saying that they are infrared and collinear safe quantities. Thus, as in the case of the ratio $R_{e^+e^-}$, the x-dependent perturbative coefficients $\Gamma_{ij}^{(n)}$ are computable by first principles starting from parton-level Feynman diagrams and without carrying out any factorization procedure of mass singularities. Nonetheless, since higher-order perturbative calculations for Altarelli-Parisi splitting functions and process-dependent coefficient functions are already available, it is more convenient to relate directly the $\Gamma_{ij}^{(n)}$'s to these quantities.

To the purpose of simplifying the notation it is also useful to introduce the N-moments. For any function $g(x)$, I define its N-moments g_N in the usual way:

$$g_N \equiv \int_0^1 dx \, x^{N-1} \, g(x) . \qquad (22)$$

Thus, for instance, the evolution equations (2, 3) become:

$$\frac{dF_{2,N}(Q^2)}{d\ln Q^2} = \langle e_f^2 \rangle \left[\gamma_{SS,\,N}(\alpha_S(Q^2)) \, \tilde{f}_{S,\,N}(Q^2) \right.$$
$$\left. + \gamma_{Sg,\,N}(\alpha_S(Q^2)) \, \tilde{f}_{g,\,N}(Q^2) \right] , \qquad (23)$$

$$\frac{d\tilde{f}_{g,\,N}(Q^2)}{d\ln Q^2} = \gamma_{gq,\,N}(\alpha_S(Q^2)) \, \tilde{f}_{S,\,N}(Q^2)$$
$$+ \gamma_{gg,\,N}(\alpha_S(Q^2)) \, \tilde{f}_{g,\,N}(Q^2) , \qquad (24)$$

where the *anomalous dimensions* $\gamma_{ab,\,N}(\alpha_S)$ are related to the $N+1$-moments of the Altarelli-Parisi splitting functions, that is,

$$\gamma_{ab,N}(\alpha_S) \equiv \int_0^1 dx \, x^N P_{ab}(\alpha_S, x) = P_{ab,\,N+1}(\alpha_S) . \qquad (25)$$

Analogously, the dynamical equations (14, 15) can be rewritten as follows

$$\frac{dF_{2,N}(Q^2)}{d\ln Q^2} = \Gamma_{22,\,N}(\alpha_S(Q^2)) \, F_{2,\,N}(Q^2)$$
$$+ \Gamma_{2L,\,N}(\alpha_S(Q^2)) \, F_{L,\,N}(Q^2) , \qquad (26)$$

$$\frac{dF_{L,\,N}(Q^2)}{d\ln Q^2} = \Gamma_{L2,\,N}(\alpha_S(Q^2)) \, F_{2,\,N}(Q^2)$$
$$+ \Gamma_{LL,\,N}(\alpha_S(Q^2)) \, F_{L,\,N}(Q^2) , \qquad (27)$$

where $\Gamma_{ij,\,N}(\alpha_S)$ are the *physical anomalous dimensions*, i.e. the N-moments of the physical splitting functions $\Gamma_{ij}(\alpha_S, x)$.

The physical anomalous dimensions are related to $\gamma_{ab,\,N}$ and to the longitudinal coefficient functions in (12) by the following equations

$$\Gamma_{LL,\,N} = \left[\gamma_{gg,\,N} + \frac{C_{L,\,N}^S}{C_{L,\,N}^g} \gamma_{Sg,\,N} + \frac{d\ln C_{L,\,N}^g}{d\ln Q^2} \right]_{\text{DIS}} , \qquad (28)$$

$$\Gamma_{L2,\,N} = \left[C_{L,\,N}^g \gamma_{gq,\,N} - C_{L,\,N}^S \gamma_{gg,\,N} \right.$$
$$+ C_{L,\,N}^S \left(\gamma_{SS,\,N} - \frac{C_{L,\,N}^S}{C_{L,\,N}^g} \gamma_{Sg,\,N} \right)$$
$$\left. + C_{L,\,N}^S \left(\frac{d\ln C_{L,\,N}^S}{d\ln Q^2} - \frac{d\ln C_{L,\,N}^g}{d\ln Q^2} \right) \right]_{\text{DIS}} , \qquad (29)$$

$$\Gamma_{2L,\,N} = \left[\frac{\gamma_{Sg,\,N}}{C_{L,\,N}^g} \right]_{\text{DIS}} , \qquad (30)$$

$$\Gamma_{22,\,N} = \left[\gamma_{SS,\,N} - \frac{C^S_{L,\,N}}{C^g_{L,\,N}}\gamma_{Sg,\,N}\right]_{\mathrm{DIS}}, \qquad (31)$$

where I have used the shorthand notation $\Gamma_{ij,\,N} = \Gamma_{ij,\,N}(\alpha_S(Q^2))$, $\gamma_{ab,\,N} = \gamma_{ab,\,N}(\alpha_S(Q^2))$, $C^a_{i,\,N} = C^a_{i,\,N}(\alpha_S(Q^2))$. In (28–31) the subscript DIS on the right-hand side means that the quantities inside the square brackets have to be evaluated in the DIS factorization scheme. Obviously, this does not mean that $\Gamma_{ij,\,N}$ are scheme dependent. The only point is that their expressions in terms of $\gamma_{ab,\,N}$ and $C^a_{L,\,N}$ are more cumbersome if $\gamma_{ab,\,N}$ and $C^a_{L,\,N}$ are given in a different factorization scheme.

Both the Altarelli-Parisi splitting functions and the longitudinal coefficient functions [33] are known up to two-loop order. Therefore, using (28–31), one can obtain the two lowest-order terms $\Gamma^{(0)}_{ij}$, $\Gamma^{(1)}_{ij}$ of the physical anomalous dimensions.

3.2 Behaviour at small x

Let me now consider the small-x behaviour of the physical anomalous dimensions. From power-counting arguments, it follows that the most singular terms in the perturbative coefficients $\Gamma^{(n)}_{ij}(x)$ behave as $\Gamma^{(n)}_{ij}(x) \sim x\,P^{(n)}(x) \sim (\ln x)^n$ or, equivalently, $\Gamma^{(n)}_{ij,\,N} \sim (1/N)^{n+1}$ in N-moment space[6]. As in the case of the Altarelli-Parisi splitting functions, one expects two entries in the matrix of the physical splitting functions that contain leading logarithms. These two entries are those more directly related to the gluon channel and, hence, they appear in the evolution equation (15) for the longitudinal structure function. Thus, we have $\Gamma^{(n)}_{LL}(x) \sim \Gamma^{(n)}_{L2}(x) \sim (\ln x)^n$. The evolution equation (14) is instead more related to the quark dynamics and thus the corresponding anomalous dimensions contain only NLL terms, i.e. $\Gamma^{(n)}_{2L}(x) \sim \Gamma^{(n)}_{22}(x) \sim (\ln x)^{n-1}$.

As in the case of the fixed-order perturbative expansions, (28–31) can be used to obtain resummed logarithmic expressions at small x for the physical anomalous dimensions. The resummation programme carried out in [5, 10] leads to analytic formulae given in terms of the LL contributions to the gluon anomalous dimensions, that is,

$$\gamma_{gg,\,N}(\alpha_S) = \gamma_N(\alpha_S) + \mathcal{O}\left(\alpha_S(\alpha_S/N)^n\right). \qquad (32)$$

Here, $\gamma_N(\alpha_S)$ is the BFKL anomalous dimension [4, 21] and is obtained by solving the implicit equation

$$1 = \frac{\bar{\alpha}_S}{N}\chi(\gamma_N(\alpha_S)), \qquad (33)$$

where the characteristic function $\chi(\gamma)$ is expressed in terms of the Euler ψ-function as follows

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma). \qquad (34)$$

Having recalled these results, I am now in a position of presenting all-order resummed formulae for the physical anomalous dimensions, starting from the leading components Γ_{LL} and Γ_{L2}.

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[6] Note that logarithmic contributions of the type $\ln^{n-1}x$ in x-space correspond to multiple poles $(1/N)^n$ in N-space

Since quark splitting functions and longitudinal coefficient functions are subleading at small x, (28) and (32) immediately gives

$$\Gamma_{LL,\,N}(\alpha_S) = \gamma_N(\alpha_S) + \mathcal{O}\left(\alpha_S(\alpha_S/N)^k\right). \qquad (35)$$

Note also that, using the known next-to-leading results for γ_{Sg} and C^a_L, as soon as the next-to-leading contributions to γ_{gg} will be evaluated, one can provide the full NLL corrections to Γ_{LL} [24].

Incidentally, (35) clearly shows that the BFKL anomalous dimensions $\gamma_N(\alpha_S)$, being related to the small-x behaviour of Γ_{LL}, is a physical quantity. On the contrary, the gluon anomalous dimensions $\gamma_{gg,\,N}$ are factorization-scheme dependent and, in general, one might expect that this scheme-dependence affects also their LL behaviour.

As can be seen from (29–31), to the purpose of evaluating the LL terms in Γ_{L2}, as well as the next-to-leading terms in Γ_{2L} and Γ_{22}, it is *not* sufficient to know the leading contributions in the gluon channel, i.e. in γ_{gq} and γ_{gg}. One has to use the full information provided by the next-to-leading order resummation performed in [10]. This feature emphasizes once more that the standard anomalous dimensions and coefficient functions are not physical observables. The power counting of small-x logarithms is different for physical observables and leading and next-to-leading logarithms in anomalous dimensions and coefficient functions get (slightly) mixed up. Without the NLL calculations in [10], no LL analysis of physical quantities at small x can be carried out.

Using the results for γ_{gq}, γ_{Sg}, C^a_L obtained in [10], (29) gives the following expression for the LL terms in Γ_{L2}

$$\Gamma_{L2,\,N}(\alpha_S) = \frac{\alpha_S}{2\pi}\left\{\left(\frac{C_F}{C_A}C^{g\,(0)}_{L,\,N} - C^{S\,(0)}_{L,\,N}\right)\gamma_N(\alpha_S) \right.$$
$$\left. + \mathcal{O}\left(\alpha_S(\alpha_S/N)^k\right)\right\}, \qquad (36)$$

where $C^{a\,(0)}_{L,\,N}$ are the N-moments of the lowest-order coefficient functions $C^{a\,(0)}_L(x)$ in (13). One can see that, eventually, also the small-x resummation in Γ_{L2} turns out to be proportional to the BFKL anomalous dimension. As for the NLL terms in Γ_{L2}, part of them can be obtained from those known [10] for quark anomalous dimensions and coefficient functions. The remaining terms require the evaluation of the gluon anomalous dimensions γ_{gg} and γ_{gq} to NLL order *and* the computation of the DIS-scheme coefficient functions C^a_L to *next-to-next-to-leading* logarithmic (NNLL) accuracy! This feature is consistent with the factorization-scheme dependence of (1–3). Indeed, it is straightforward to check that, after having fixed the factorization scheme to NLL accuracy in (1, 2), only the NLL contributions of P_{gg} in (3) are unambiguously defined: by properly choosing the scheme transformation in (10), one still has the freedom of arbitrarily defining the NLL terms in the non-diagonal gluon splitting function P_{gq}. In other words, the sole calculation of the still unknown gluon anomalous dimensions to NLL order will not be sufficient to provide a consistent theoretical framework for the analysis of physical observables to NLL accuracy in resummed perturbation theory.

The evaluation of the next-to-leading contributions to the physical anomalous dimensions Γ_{2L}, Γ_{22} enterely relies on

the calculations of the quark anomalous dimensions in [9] and of the longitudinal coefficient functions in [10]. Using these results and (30, 31), one obtains[7]

$$\frac{\alpha_S}{2\pi}\,\Gamma_{2L,\,N}(\alpha_S) = \frac{\alpha_S}{2\pi}\left[\frac{1}{1-\gamma_N(\alpha_S)} + \frac{3}{2}\,\gamma_N(\alpha_S)\right]$$
$$+ \mathcal{O}\left(\alpha_S^2(\alpha_S/N)^k\right)\,, \tag{37}$$

$$\Gamma_{22,\,N}(\alpha_S) = \frac{\alpha_S}{2\pi}\left\{\left(\frac{C_F}{C_A}\,C_{L,\,N}^{g\,(0)} - C_{L,\,N}^{S\,(0)}\right)\right.$$
$$\times\left[\frac{1}{1-\gamma_N(\alpha_S)} + \frac{3}{2}\,\gamma_N(\alpha_S)\right]$$
$$\left.+\left(\gamma_{SS,\,N}^{(0)} - \frac{C_F}{C_A}\,\gamma_{Sg,\,N}^{(0)}\right)\right\}$$
$$+ \mathcal{O}\left(\alpha_S^2(\alpha_S/N)^k\right)\,. \tag{38}$$

Equations (37, 38) provide resummed analytical formulae for Γ_{2L} and Γ_{22} in terms of the BFKL anomalous dimension in (32). Note that, if one compares the right-hand sides of these equations with the corresponding expressions for γ_{SS}, γ_{Sg}, C_L^a in [10], one can see that (37, 38) are remarkably simpler. These equations have to be considered as the main scheme-invariant output of the next-to-leading order resummation in the quark channel.

Having presented the main features of the physical anomalous dimensions Γ_{ij} both in fixed-order and in resummed perturbation theory, let me add some comments on the dynamical equations (14, 15).

The first comment regards the theoretical accuracy at small x. Suppose, for instance, that the physical anomalous dimensions $\Gamma_{ij,\,N}(\alpha_S)$ are evaluated only to LL order in resummed perturbation theory. This implies the following theoretical indeterminacy $\delta\Gamma_N/\Gamma_N = \mathcal{O}(\alpha_S(\alpha_S/N)^k)$ or, equivalently, $\mathcal{O}(\alpha_S^2(\alpha_S\ln x)^k)$ in x space. In order to make a direct comparison with the discussion in Sect. 2.3, we can parametrize this uncertainty in terms of a singular function of the type $A\alpha_S^2\,x^{-K\alpha_S}$. Owing to the convolution structure in (14, 15) and considering the extreme case of flat structure functions, this leads to the following theoretical uncertainty

$$\frac{\delta F_{i=2,L}(x)}{F_{i=2,L}(x)} \sim \frac{A}{K}\,\alpha_S\,x^{-K\alpha_S}\,. \tag{39}$$

Comparing (11) and (39), we can see that the replacement of unphysical parton densities with physical observables (and the ensuing elimination of the factorization-scheme dependence) allows one to gain a factor of α_S in the nominal theoretical accuracy. Of course, this is due to the fact that the dynamical evolution equations (14, 15) to LL accuracy contain more theoretical information than (1–3) to the same accuracy. As a matter of fact, the evaluation of the physical anomalous dimensions to LL order is equivalent to the knowledge of leading-order splitting functions and next-to-leading order coefficient functions (see the discussion above (36)).

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[7] Note that, consistently with the logarithmic accuracy of the right-hand sides of (36, 38), the N-moments of the lowest-order anomalous dimensions and coefficient functions can be replaced with their values at $N = 0$, namely $\gamma_{SS,\,N=0}^{(0)} = 0$, $\gamma_{Sg,\,N=0}^{(0)} = C_{L,\,N=0}^{g\,(0)} = \frac{4}{3}T_R N_f$, $C_{L,\,N=0}^{S\,(0)} = C_F$

Other comments regard phenomenological aspects. The evolution equation (15) for F_L is physically analogous to the evolution equation for the gluon density. This analogy is particularly clear at small x, because the physical anomalous dimensions Γ_{LL} and Γ_{L2} turn out to be proportional to the BFKL anomalous dimension. Thus the effects of small-x resummation in (15) can directly be inferred from those studied in [25] for the gluon density.

The evolution equation (14) for F_2 is physically analogous to the evolution equation for the quark density. From the expressions in (37, 38) we see that the small-x resummation effects increase the amount of scaling violation. Equation (37), for instance, can be rewritten as follows

$$\Gamma_{2L,\,N}(\alpha_S) = 1 + 2.5\,\gamma_N(\alpha_S) + \sum_{n=2}^{+\infty}(\gamma_N(\alpha_S))^n\,. \tag{40}$$

Thus, besides the resummation accomplished by the BFKL anomalous dimension, there are further enhancing effects due to the positive definite (although, not large) coefficients in the series (40). Phenomenological studies of these purely perturbative (i.e. independent of the parton densities) effects appear interesting.

In general (14, 15) relate measurable values of observables, F_2, F_L and their derivatives with respect to Q^2, to perturbative quantities, the physical anomalous dimensions. Thus, in the hard scattering regime, these equations provide absolute predictions of perturbative QCD. In practice, the measurement of $dF_L/d\ln Q^2$ can be quite difficult. In this respect, once F_L is measured at a certain value of Q^2, from (15) one can obtain its value at any Q^2 and then one can use (14) as a test of perturbative QCD that is free from non-perturbative parameters. Phenomenological analyses along these lines are in progress [34].

4 Parton picture and the unphysical gluon density

The mathematical steps that are necessary to go from (1–3, 12) to (14, 15) are pretty straightforward and, naively, one would be led to conclude that the former equations are in one-to-one correspondence with the latter. This is not the case. In order to clarify this point let me first consider a case in which a one-to-one correspondence between physical and partonic observables can really be established.

Suppose we want to evaluate the high-Q^2 behaviour of a hadronic observable F_C other than, say, F_2 and F_L. Suppose also that it is a flavour-singlet observable measured in lepton-hadron scattering processes. Thus, within the partonic framework, we should consider a factorization formula analogous to (12). Writing this formula directly in N-space, we have

$$F_{C,\,N}(Q^2) = C_{C,\,N}^S(\alpha_S(Q^2))\,\tilde{f}_{S,\,N}(Q^2)$$
$$+ C_{C,\,N}^g(\alpha_S(Q^2))\,\tilde{f}_{g,\,N}(Q^2)\,. \tag{41}$$

Using the parton densities as determined from the scaling violations of (for instance) F_2 and F_L, the perturbative QCD prediction for F_C in (41) amounts to the computation of *two* factorization-scheme dependent quantities: the coefficients functions $C_{C,\,N}^S(\alpha_S)$ and $C_{C,\,N}^g(\alpha_S)$.

Alternatively, we can use (1, 12), or (1, 2), to rewrite (41) as follows

$$F_{C,N}(Q^2) = K_{C2,N}(\alpha_S(Q^2)) F_{2,N}(Q^2)$$
$$+ K_{CL,N}(\alpha_S(Q^2)) F_{L,N}(Q^2) \; , \qquad (42)$$

$$F_{C,N}(Q^2) = \overline{K}_{C2,N}(\alpha_S(Q^2)) F_{2,N}(Q^2)$$
$$+ \frac{1}{\Gamma_{2C,N}(\alpha_S(Q^2))} \frac{d \ln F_{2,N}(Q^2)}{d \ln Q^2} \; , \qquad (43)$$

where, using the same notation as in (28–30), we have:

$$K_{C2,N} = \frac{1}{\langle e_f^2 \rangle} \left[C_{C,N}^S - \frac{C_{L,N}^S}{C_{L,N}^g} C_{C,N}^g \right]_{\text{DIS}} ,$$

$$K_{CL,N} = \frac{1}{\langle e_f^2 \rangle} \left[\frac{C_{C,N}^g}{C_{L,N}^g} \right]_{\text{DIS}} . \qquad (44)$$

or:

$$\overline{K}_{C2,N} = \frac{1}{\langle e_f^2 \rangle} \left[C_{C,N}^S - \frac{\gamma_{SS,N}}{\gamma_{Sg,N}} C_{C,N}^g \right]_{\text{DIS}} ,$$

$$\Gamma_{2C,N} = \langle e_f^2 \rangle \left[\frac{\gamma_{Sg,N}}{C_{C,N}^g} \right]_{\text{DIS}} , \qquad (45)$$

Equations (42) and (43) are equivalent to (41). The only difference is that the factorization-scheme dependence embodied in the coefficient functions C_C^S and C_C^g (and in the parton densities) has been explicitly eliminated by replacing the parton densities with physical quantities (F_2 and F_L in (42) or F_2 and its Q^2-derivative in (43)) and introducing the K-factors K_{C2}, K_{CL} or $\overline{K}_{C2}$, Γ_{2C}. These K-factors are factorization-scheme independent and have the same perturbative properties of the physical anomalous dimensions (one of them, Γ_{2C}, actually coincides with a physical anomalous dimension). As for the study of small-x physics and the comparison between fixed-order and resummed perturbation theory, the K-factors are certainly preferred [5, 32] with respect to the coefficient functions in (41). However, the perturbative QCD prediction for F_C in (42) or (43) still involves the computation of *two* K-factors that replace the two coefficient functions.

The counting of 'degrees of freedom' is instead different in the case of scaling violations. In order to describe the scaling violations of F_2 and F_L according to the partonic formulae in (1, 2, 3, 12), one has to assign two input parton densities $\tilde{f}_S(x, Q_0^2)$, $\tilde{f}_g(x, Q_0^2)$ (related to the low-Q^2 behaviour of F_2 and F_L or of F_2 and its Q^2-slope) and to compute *six*[1] quantities in QCD perturbation theory: four flavour-singlet splitting functions $P_{ab}(\alpha_S, x)$ and two coefficient functions $C_L^S(\alpha_S, x)$, $C_L^g(\alpha_S, x)$. On the contrary, the solution of the dynamical evolution equations (14, 15) requires two non-perturbative initial conditions at the input scale Q_0^2 and the calculation of only *four* quantities, the physical anomalous dimensions $\Gamma_{ij}(\alpha_S, x)$, in QCD perturbation theory. It is evident that the parton picture in (1–3, 12) introduces spurious perturbative QCD effects.

Obviously, there is nothing wrong with the parton model or with the (light-cone) Wilson expansion for deep-inelastic

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[1] This number becomes eight in factorization schemes (like the $\overline{\text{MS}}$ scheme) that are different from the DIS scheme

lepton-hadron scattering. The spurious effects noticed above simply follow from the ambiguity in the definition of singlet-quark and gluon densities. From a field theory viewpoint, the ambiguity is related to the mixing under renormalization of singlet-quark and gluon operators. Owing to the mixing matrix, the renormalization prescription has to be specified by four (two, in DIS-type schemes) arbitrary perturbative functions. In the partonic framework this ambiguity is unavoidable and ultimately related to the fact that *no physical current* with point-like coupling to gluons does exist.

The unphysical perturbative contributions that are introduced through the definition of the gluon density are responsible for the theoretical uncertainty pointed out in Sect. 2.3. In order to furtherly clarify this aspect, let me discuss another possible effect in resummed perturbation theory.

Suppose that in a certain factorization scheme the resummation of non-leading logarithmic contributions in the quark or gluon anomalous dimensions produces a singularity in the N-plane at a positive value $N = \overline{N}(\alpha_S) = c_k \alpha_S^k + \dots$. Solving the evolution equation (24) from the input scale Q_0^2 to the hard scale Q^2, these higher-order terms factorize into Q_0^2-dependent and Q^2-dependent contributions. In particular, the Q_0^2-dependent factor will contain the singularity at $N = \overline{N}(\alpha_S(Q_0^2))$ and this singularity will dominate the small-x behaviour of $F_2(x, Q^2)$ at *any* Q^2 unless it is cancelled by a zero in the input parton densities $\tilde{f}_{a,N}(Q_0^2)$. In this case, however, since the parton densities are assumed to be positive definite, they must have a singularity at a value of N larger than $\overline{N}(\alpha_S(Q_0^2))$. As a result the small-$x$ behaviour of $F_2(x, Q^2)$ turns out to be controlled by an unphysical singularity that is simply due to the choice of the factorization scheme.

This is certainly an extreme effect. Nonetheless, it shows that the spurious perturbative functions that are introduced in the partonic picture may lead to *obstructions* that cannot any longer be removed within the same framework (i.e. without releasing the positivity constraint on the parton densities).

5 Heavy-flavour structure functions

Most of the discussion in Sect. 3 on physical anomalous dimensions can be repeated for other structure functions in deep-inelastic lepton-proton scattering, for instance, the heavy-flavour structure functions $F_2^{Q\overline{Q}}$ or $F_L^{Q\overline{Q}}$. These structure functions are completely analogous to the customary structure functions F_2, F_L with the only additional constraint that heavy quarks of mass M are produced in the final state.

To be precise, all the theoretical formulae in the previous Sections refer to the kinematical region $Q^2 \gg M^2$ and thus neglect corrections of relative order M^2/Q^2. In order to take into account the mass effects, one should perform the replacement $F_i \to F_i + F_i^{Q\overline{Q}}$. However, in the following F_i still denotes the massless contribution to the structure function.

There are advantages and disadvantages in substituting the heavy-quark structure functions for F_L in the study of scaling violations. On the experimetal side [35], the charm contribution F_2^{cc} to the small-x behaviour of the proton structure function is certainly more easily measurable than

F_L. This feature has to be contrasted with a larger theoretical uncertainty [36] due to the unknown precise value of the charm mass and with related complications considered below.

In the heavy-flavour case, the analogue of the collinear-factorization formula (12) is ($i = 2, L$):

$$F_i^{Q\bar{Q}}(\xi, Q^2; M^2)$$
$$= \int_\xi^1 \frac{dz}{z} \left[C_i^{Q\bar{Q}, g}(\alpha_S(Q^2), \xi/z; Q^2/M^2) \, \tilde{f}_g(z, Q^2) \right.$$
$$\left. + C_i^{Q\bar{Q}, S}(\alpha_S(Q^2), \xi/z; Q^2/M^2) \, \tilde{f}_S(z, Q^2) \right] \quad . \quad (46)$$

Note that $F_i^{Q\bar{Q}}$ and the coefficient functions $C_i^{Q\bar{Q}, a}$ depend on the mass M. Also note that in (46) I have defined $F_i^{Q\bar{Q}}$ as function of Q^2, M^2 and the *inelasticity variable* ξ, which is related to the customary Bjorken variable x by $\xi = x(1 + 4M^2/Q^2)$. From a theoretical viewpoint the scaling variable ξ is preferred to x because it fulfils the kinematical constraint $0 \le \xi \le 1$. Thus, considering N-moments with respect to ξ, (46) is diagonalized as follows

$$F_{i,N}^{Q\bar{Q}}(Q^2; M^2) = C_{i,N}^{Q\bar{Q}, g}(\alpha_S(Q^2); Q^2/M^2) \, \tilde{f}_{g,N}(Q^2)$$
$$+ C_{i,N}^{Q\bar{Q}, S}(\alpha_S(Q^2); Q^2/M^2) \, \tilde{f}_{S,N}(Q^2) \, . \quad (47)$$

The functions $C_i^{Q\bar{Q}, a}$ have a perturbative expansion similar to (13). There are two main differences with respect to the case of C_L^a. For fixed α_S the heavy-quark coefficient functions are not scale invariant because of their explicit dependence on Q^2/M^2. The expansion for $C_{i,N}^{QQ, S}(\alpha_S; Q^2/M^2)$ starts in $\mathcal{O}(\alpha_S^2)$ (i.e. $n \ge 2$) and thus only gluons contribute to (46) at LO (the sensitivity to the gluon density is somehow enhanced).

The coefficient functions $C_i^{Q\bar{Q}, a\,(n-1)}$ have been fully computed up to NLO ($n = 0, 1$) in [37]. The corresponding resummed formulae to NLL accuracy[2] were obtained in [5].

The evolution equations that involve the physical anomalous dimensions $\Gamma^{Q\bar{Q}}$ for the pair of observables $\{F_2, F_i^{Q\bar{Q}}\}$ are the following

$$\frac{dF_2(x, Q^2)}{d \ln Q^2}$$
$$= \int_x^1 \frac{dz}{z} \left[\Gamma_{22}^{Q\bar{Q}}(\alpha_S(Q^2), x/z; Q^2/M^2) \, F_2(z, Q^2) \right.$$
$$\left. + \Gamma_{2i}^{Q\bar{Q}}(\alpha_S(Q^2), x/z; Q^2/M^2) \, F_i^{Q\bar{Q}}(z, Q^2; M^2) \right] \quad , \quad (48)$$

$$\frac{dF_i^{Q\bar{Q}}(\xi, Q^2; M^2)}{d \ln Q^2}$$
$$= \int_\xi^1 \frac{dz}{z} \left[\Gamma_{i2}^{Q\bar{Q}}(\alpha_S(Q^2), \xi/z; Q^2/M^2) \, F_2(z, Q^2) \right.$$
$$\left. + \Gamma_{ii}^{Q\bar{Q}}(\alpha_S(Q^2), \xi/z; Q^2/M^2) \, F_i^{Q\bar{Q}}(z, Q^2; M^2) \right] \quad , \quad (49)$$

or, equivalently, in N-space:

---

[2] The explicit formula for $C_{L,N}^{Q\bar{Q}, a}$ was not reported in [5] and can be found in [38]

$$\frac{dF_{2, N}(Q^2)}{d \ln Q^2}$$
$$= \Gamma_{22, N}^{Q\bar{Q}}(\alpha_S(Q^2); Q^2/M^2) \, F_{2, N}(Q^2)$$
$$+ \Gamma_{2i, N}^{Q\bar{Q}}(\alpha_S(Q^2); Q^2/M^2) \, F_{i, N}^{Q\bar{Q}}(Q^2; M^2) \, , \quad (50)$$

$$\frac{dF_{i, N}^{Q\bar{Q}}(Q^2; M^2)}{d \ln Q^2}$$
$$= \Gamma_{i2, N}^{Q\bar{Q}}(\alpha_S(Q^2); Q^2/M^2) \, F_{2, N}(Q^2)$$
$$+ \Gamma_{ii, N}^{Q\bar{Q}}(\alpha_S(Q^2); Q^2/M^2) \, F_{i, N}^{Q\bar{Q}}(Q^2; M^2) \, . \quad (51)$$

The relation between the physical anomalous dimensions $\Gamma^{Q\bar{Q}}$ and the customary splitting and coefficient functions is similar to that in (28–31) for the case of $\{F_2, F_L\}$, apart from the replacement $C_{L, N}^a \to C_{i, N}^{Q\bar{Q}, a}/\langle e_f^2 \rangle$. Using the same notation as in (28–31), we have:

$$\Gamma_{ii, N}^{Q\bar{Q}} = \left[\gamma_{gg, N} + \frac{C_{i, N}^{Q\bar{Q}, S}}{C_{i, N}^{Q\bar{Q}, g}} \gamma_{Sg, N} + \frac{d \ln C_{i, N}^{Q\bar{Q}, g}}{d \ln Q^2} \right]_{\text{DIS}} \, , \quad (52)$$

$$\Gamma_{i2, N}^{Q\bar{Q}} = \frac{1}{\langle e_f^2 \rangle} \left[C_{i, N}^{Q\bar{Q}, g} \gamma_{gq, N} - C_{i, N}^{Q\bar{Q}, S} \gamma_{gg, N} \right.$$
$$+ C_{i, N}^{Q\bar{Q}, S} \left(\gamma_{SS, N} - \frac{C_{i, N}^{Q\bar{Q}, S}}{C_{i, N}^{Q\bar{Q}, g}} \gamma_{Sg, N} \right)$$
$$\left. + C_{i, N}^{Q\bar{Q}, S} \left(\frac{d \ln C_{i, N}^{Q\bar{Q}, S}}{d \ln Q^2} - \frac{d \ln C_{i, N}^{Q\bar{Q}, g}}{d \ln Q^2} \right) \right]_{\text{DIS}} \, , \quad (53)$$

$$\Gamma_{2i, N}^{Q\bar{Q}} = \langle e_f^2 \rangle \left[\frac{\gamma_{Sg, N}}{C_{i, N}^{Q\bar{Q}, g}} \right]_{\text{DIS}} \, , \quad (54)$$

$$\Gamma_{22, N}^{Q\bar{Q}} = \left[\gamma_{SS, N} - \frac{C_{i, N}^{Q\bar{Q}, S}}{C_{i, N}^{Q\bar{Q}, g}} \gamma_{Sg, N} \right]_{\text{DIS}} \, . \quad (55)$$

5.1 Perturbative features

The dynamical evolution equations (48, 49) (or (50),(51)) are analogous to (14, 15) (or (26),(27)). Note, however, a main and important difference: now the physical kernels $\Gamma^{Q\bar{Q}}$ depend not only on α_S but also on Q^2/M^2. This dependence is due to the mass-dependence of the heavy-flavour coefficients functions $C_i^{Q\bar{Q}, a}$. In particular, using the identity:

$$\frac{d \ln C_{i, N}^{Q\bar{Q}, a}(\alpha_S(Q^2); Q^2/M^2)}{d \ln Q^2}$$
$$= \frac{d \ln \alpha_S(Q^2)}{d \ln Q^2} \frac{\partial \ln C_{i, N}^{Q\bar{Q}, a}(\alpha_S(Q^2); Q^2/M^2)}{\partial \ln \alpha_S(Q^2)}$$
$$+ \frac{\partial \ln C_{i, N}^{Q\bar{Q}, a}(\alpha_S(Q^2); Q^2/M^2)}{\partial \ln Q^2} \, , \quad (56)$$

we can see that the perturbative expansions of the physical anomalous dimensions $\Gamma_{ii}^{Q\bar{Q}}$ and $\Gamma_{i2}^{Q\bar{Q}}$ are the following

$$\Gamma_{ii,\,N}^{Q\bar{Q}}(\alpha_S;Q^2/M^2) = \frac{d\ln C_{i,\,N}^{Q\bar{Q},\,g(0)}(Q^2/M^2)}{d\ln Q^2}$$

$$+\sum_{n=1}^{+\infty}\left(\frac{\alpha_S}{2\pi}\right)^n \Gamma_{ii,\,N}^{Q\bar{Q}(n-1)}(Q^2/M^2)$$

$$= \frac{d\ln C_{i,\,N}^{Q\bar{Q},\,g(0)}(Q^2/M^2)}{d\ln Q^2}$$

$$+\frac{\alpha_S}{2\pi}\left[\Gamma_{ii,\,N}^{Q\bar{Q}(0)}(Q^2/M^2)\right.$$

$$\left. +\frac{\alpha_S}{2\pi}\,\Gamma_{ii,\,N}^{Q\bar{Q}(1)}(Q^2/M^2)+\dots\right] \;, \quad (57)$$

$$\Gamma_{i2,\,N}^{Q\bar{Q}}(\alpha_S;Q^2/M^2) = \sum_{n=1}^{+\infty}\left(\frac{\alpha_S}{2\pi}\right)^{n+1}\Gamma_{i2,\,N}^{Q\bar{Q}(n-1)}(Q^2/M^2)$$

$$= \left(\frac{\alpha_S}{2\pi}\right)^2\left[\Gamma_{i2,\,N}^{Q\bar{Q}(0)}(Q^2/M^2)\right.$$

$$\left. +\frac{\alpha_S}{2\pi}\,\Gamma_{i2,\,N}^{Q\bar{Q}(1)}(Q^2/M^2)+\dots\right] \;. \quad (58)$$

Comparing (57) and (17), we see that $\Gamma_{ii}^{Q\bar{Q}}$ contains a lowest-order contribution (the first term on the right-hand side of (57)) that is absent in Γ_{LL}. This contribution leads to *kinematical* scaling violations, that is, to scaling violations that are independent of the running of $\alpha_S(Q^2)$ and are simply due to the production kinematics of the heavy-quark pair. Thus, as for the 'true' dynamical scaling violations, the perturbative QCD calculation should consistently consider the coefficients $\Gamma^{Q\bar{Q}(0)}$ in (57, 58) as LO terms, $\Gamma^{Q\bar{Q}(1)}$ as NLO terms and so forth.

Apart from the explicit (Q^2/M^2)-dependence of the coefficients $\Gamma^{Q\bar{Q}(n-1)}(Q^2/M^2)$, the perturbative expansions of $\Gamma_{2i}^{Q\bar{Q}}$ and $\Gamma_{22}^{Q\bar{Q}}$ are completely analogous to those in (19) and (20), respectively.

Owing to its kinematical origin, the first term on the right-hand side of (57) can be eliminated from the physical anomalous dimensions. To this purpose it is sufficient to rescale $F_i^{Q\bar{Q}}$ in (50, 51) by the (factorization-scheme independent) coefficient $C_i^{Q\bar{Q},\,g(0)}$ and, thus, to consider $F_{2,\,N}$ and $F_{i,\,N}^{Q\bar{Q}}/C_{i,\,N}^{Q\bar{Q},\,g(0)}$ as dynamical variables. Nonetheless, this rescaling is not sufficient to exactly put the physical anomalous dimensions $\Gamma^{Q\bar{Q}}$ on equal terms with those in (17–20). The interplay between kinematical and dynamical scaling violations in the heavy-flavour case cannot be avoided beyond the LO. The NLO coefficients $\Gamma_{ii,\,N}^{Q\bar{Q}(1)}$ and $\Gamma_{i2,\,N}^{Q\bar{Q}(1)}$ will always depend on the *next-to-next-to-leading* order (NNLO) (!) coefficient functions $C_i^{Q\bar{Q},\,a(2)}$. Since these have not yet been computed, a fully consistent NLO study of the dynamical evolution equation (49) is, strictly speaking, not feasible at present.

This discussion of the perturbative features of the physical anomalous dimensions $\Gamma^{Q\bar{Q}}$ is not peculiar to the heavy-quark case. It applies to the physical anomalous dimensions of any structure function that depends on some other large-momentum scale besides Q^2.

5.2 Small-x resummation

The power counting of the logarithmic behaviour of $\Gamma^{Q\bar{Q}}$ at small x is similar to that of the physical anomalous dimensions relating F_2 and F_L.

The two entries $\Gamma_{ii}^{Q\bar{Q}}$ and $\Gamma_{i2}^{Q\bar{Q}}$ have LL contributions. These can be obtained by using (52, 53) and the known resummed formulae for the heavy-quark coefficient functions [5] and the quark anomalous dimensions [9]. I find:

$$\Gamma_{ii,\,N}^{Q\bar{Q}}(\alpha_S;Q^2/M^2) = \gamma_N(\alpha_S) + \frac{\partial \ln H^{(i)}(\gamma_N(\alpha_S);Q^2/M^2)}{\partial \ln Q^2}$$

$$+\mathcal{O}\left(\alpha_S(\alpha_S/N)^k\right) \;, \quad (59)$$

$$\Gamma_{i2,\,N}^{Q\bar{Q}}(\alpha_S;Q^2/M^2) = \frac{1}{\langle e_f^2\rangle}\,\frac{\alpha_S}{2\pi}\left\{\frac{C_F}{C_A}\,C_{i,\,N}^{Q\bar{Q},\,g(0)}(Q^2/M^2)\right.$$

$$\times\left[\Gamma_{ii,\,N}^{Q\bar{Q}}(\alpha_S;Q^2/M^2)\right.$$

$$\left. -\frac{d\ln C_{i,\,N}^{Q\bar{Q},\,g(0)}(Q^2/M^2)}{d\ln Q^2}\right]$$

$$\left. +\mathcal{O}\left(\alpha_S(\alpha_S/N)^k\right)\right\} \;, \quad (60)$$

where the functions $H^{(i)}(\gamma;Q^2/M^2)$ in (59) are simply proportional to the K-factors $K_N^{(i)}(Q^2/M^2)$ introduced in the second paper of [5]. Their eplicit expressions are:

$$H^{(2)}(\gamma;Q^2/M^2)$$

$$= \left(\frac{Q^2}{4M^2}\right)^{1-\gamma}\left\{2(1+\gamma)\frac{M^2}{Q^2}\right.$$

$$+ \left[2+3\gamma-3\gamma^2-2(1+\gamma)\frac{M^2}{Q^2}\right]$$

$$\left. \cdot\left(1+\frac{Q^2}{4M^2}\right)^{\gamma-1}F(1-\gamma,1/2;3/2;\frac{Q^2}{Q^2+4M^2})\right\} \;, \quad (61)$$

$$H^{(L)}(\gamma;Q^2/M^2)$$

$$= \left(\frac{Q^2}{4M^2}\right)^{1-\gamma}\frac{4M^2}{Q^2+4M^2}\left\{\left(1-\gamma+\frac{6M^2}{Q^2}\right)\right.$$

$$+ \left[\gamma(1-\gamma)\frac{Q^2}{2M^2}-2(1-\gamma)-\frac{6M^2}{Q^2}\right]$$

$$\left. \times\left(1+\frac{Q^2}{4M^2}\right)^{\gamma-1}F(1-\gamma,1/2;3/2;\frac{Q^2}{Q^2+4M^2})\right\}, \quad (62)$$

where $F(a,b;c;z)$ is the hypergeometric function.

Owing to the dependence on Q^2/M^2, the LL behaviour of $\Gamma_{ii}^{Q\bar{Q}}$ (unlike that of Γ_{LL} in (35)) is not simply given by the BFKL anomalous dimension $\gamma_N(\alpha_S)$. The resummation of the LL terms in $\Gamma_{ii}^{Q\bar{Q}}$ is achieved through the (α_S/N)-dependence of $\gamma_N(\alpha_S)$ and the γ-dependence of the function $H^{(i)}(\gamma;Q^2/M^2)$ on the right-hand side of (59). The LL contributions to $\Gamma_{i2}^{Q\bar{Q}}$ in (60) are proportional to $\Gamma_{ii}^{Q\bar{Q}}$ after subtraction of its lowest-order kinematic contribution (cf. (57)).

The evaluation of $\Gamma_{ii}^{Q\bar{Q}}$ and $\Gamma_{i2}^{Q\bar{Q}}$ to NLL accuracy would require the calculation of the gluon anomalous dimensions γ_{ga} to NLL order and that of the heavy-flavour coefficient functions to NNLL order. The NNLL accuracy in $C_i^{Q\bar{Q}\,a}$ is demanded by the interplay between kinematical and dynamical scaling violations, as discussed in Sect. 5.1.

The anomalous dimensions $\Gamma_{2i}^{Q\bar{Q}}$ and $\Gamma_{22}^{Q\bar{Q}}$ contain only NLL terms at small x. These are explicitly given by the following expressions

$$
\frac{\alpha_S}{2\pi}\,\Gamma_{2i,\,N}^{Q\bar{Q}}(\alpha_S; Q^2/M^2)
$$
$$
= \frac{\alpha_S}{2\pi}\,\frac{N_f T_R \langle e_f^2 \rangle}{e_Q^2}\,\left[2 + 3\gamma_N(\alpha_S) - 3\gamma_N^2(\alpha_S)\right]
$$
$$
\cdot\frac{\sqrt{\pi}\;\Gamma(1 + \gamma_N(\alpha_S))}{\Gamma(1/2 + \gamma_N(\alpha_S))\,H^{(i)}(\gamma_N(\alpha_S); Q^2/M^2)}
$$
$$
+ \mathcal{O}\left(\alpha_S^2(\alpha_S/N)^k\right)\ , \tag{63}
$$

$$
\Gamma_{22,\,N}^{Q\bar{Q}}(\alpha_S; Q^2/M^2)
$$
$$
= \frac{\alpha_S}{2\pi}\left\{\frac{C_F}{C_A}\,\frac{C_{i,\,N}^{Q\bar{Q},\,g\,(0)}(Q^2/M^2)}{\langle e_f^2 \rangle}\,\Gamma_{2i,\,N}^{Q\bar{Q}}(\alpha_S; Q^2/M^2)\right.
$$
$$
\left. + \left(\gamma_{SS,\,N}^{(0)} - \frac{C_F}{C_A}\,\gamma_{Sg,\,N}^{(0)}\right)\right\} + \mathcal{O}\left(\alpha_S^2(\alpha_S/N)^k\right)\ , \tag{64}
$$

where e_Q is the heavy-quark electric charge and $\Gamma(z)$ is the Euler Γ-function. The resummation of the logarithmic contributions in (63) is embodied in the (α_S/N)-dependence of BFKL anomalous dimension $\gamma_N(\alpha_S)$ and the γ_N-dependence of the functions $H^{(i)}$, according to (61, 62). In (64) the physical anomalous dimension $\Gamma_{22}^{Q\bar{Q}}$ to NLL accuracy is expressed in terms of $\Gamma_{2i}^{Q\bar{Q}}$ through a relation that is analogous to that between Γ_{22} and Γ_{2L} in (37, 38).

6 Summary and discussion

In this contribution I have discussed how the study of different observables can contribute to our understanding of the dynamics of high-energy hadronic interactions in the hard-scattering regime. The main motivation for considering different observables is that from the analysis of a single quantity is difficult to disentangle perturbative from non-perturbative QCD physics. Of course, we aim to describe both perturbative and non-perturbative physics but keeping separate the two aspects can simplify theoretical and phenomenological investigations.

In Sect. 2 the interplay between perturbative and non-perturbative dynamics has been pointed out in the context of QCD analyses of the small-x behaviour of the proton structure function $F_2(x, Q^2)$. The factorization theorem of mass singularities provides a representation of F_2 in terms of phenomenological parton densities and perturbatively computable splitting and coefficient functions. As long as the latter have well-behaved perturbative expansions, this representation is highly predictive. In the small-x regime, however, higher perturbative orders are strongly enhanced by logarithmic contributions so that, in principle, resummation procedures are mandatory. Thus a physical issue arises: where

is the boundary between perturbative and non-perturbative phenomena in the hard scattering regime? It is quite difficult to tackle this issue by studying the small-x behaviour of the sole F_2. Indeed, as discussed in Sects. 2.1 and 2.2, the small-x rise of F_2 produced by resumming LL and NLL contributions in the perturbative expansion of the splitting functions is, in many respects (and with the present theoretical and experimental accuracy), indistinguishable from a similar rise due to steep parton densities whose Q^2 evolution is performed according to NLO perturbation theory. This uncertainty is formally taken into account by the factorization-scheme dependence, as discussed in Sect. 2.3. Owing to this dependence, the gluon density may play the role of a hidden variable that, in the case of F_2, relates different perturbative QCD approaches, namely resummed and fixed-order perturbation theory.

A better theoretical control on perturbative physics can be achieved by exploiting the very physical content of the factorization theorem, that is, the universality (process independence) of the parton densities. This means that the same parton densities and the same perturbative approach have to be used to study the small-x behaviour of different physical observables. Universality is particularly evident in the framework of the physical anomalous dimensions introduced in Sect. 3. Here I have discussed in detail the case of F_2 and F_L but the method is completely general.

For any given set $\tilde{f}_a$ of parton densities one should consider a set of an equal number of hadronic observables F_a. Thus, one can work out the factorization procedure in matrix form as follows

$$
F = C\,\tilde{f}\ , \tag{65}
$$

where $C = C_{ab}$ is the coefficient function matrix and the simple product structure on the right-hand side is usually valid in N-moment space. Then, it is straightforward to derive the following evolution equations

$$
\frac{dF}{d\ln Q^2} = \Gamma\,F\ , \tag{66}
$$

where the matrix Γ of physical anomalous dimensions for the given set of hadronic observables is related to C and to the customary matrix γ of anomalous dimensions as follows

$$
\Gamma = \frac{dC}{d\ln Q^2}\,C^{-1} + C\,\gamma\,C^{-1}\ . \tag{67}
$$

While C and γ are separately factorization-scheme dependent, the physical anomalous dimensions (67) are factorization-scheme invariant. As any other infrared and collinear safe observable, they are perturbatively computable apart from corrections that are suppressed by some inverse power of Q in the hard-scattering regime.

In Sect. 3.1 I have considered the physical anomalous dimensions relating the singlet components of F_2 and F_L. These are the most important contributions at small x but the physical anomalous dimensions matrix can be introduced in any kinematic region of x. It is just sufficient to start from (65) by including flavour non-singlet parton densities and hadronic observables.

In the small-x region the theoretical and phenomenological importance of the evolution equations (66) follows from the fact that the small-x perturbative dynamics is completely

controlled by the physical anomalous dimensions. No spurious perturbative effect (see the discussion in Sect. 4) and no subtle interplay between perturbative logarithms and steepness of parton densities takes place in (66). The physical anomalous dimensions can be evaluated both in fixed-order perturbation theory and in resummed perturbation theory. For any given set of observables and kinematic region of x, one can thus compare the two approaches and study the theoretical accuracy of the perturbative expansion. In particular, this is important to establish where the resummed perturbative corrections start to dominate with respect to perturbative terms that are not logarithmically-enhanced at small x. Having the perturbative dynamics under control, one can go back to the partonic picture of (65) and investigate more safely the small-x behaviour of the non-perturbative parton densities.

In Sect. 3.2, I have presented resummed expressions for the physical anomalous dimensions Γ_{LL}, Γ_{L2}, Γ_{2L}, Γ_{22}. Perturbative calculations to NLL accuracy are available for other hadronic observables and, in particular, for the heavy-flavour structure functions $F_2^{Q\bar{Q}}$ and $F_L^{Q\bar{Q}}$ [5]. In Sect. 5, I have considered the corresponding physical anomalous dimensions $\Gamma^{Q\bar{Q}}$. Using the theoretical approach of [5–7] one can investigate the small-x behaviour of other physical anomalous dimensions. In my opinion, some phenomenological studies within the framework of physical anomalous dimensions are warranted.

The complete feasibility of this program demands precise experimental determinations of (at least) two inclusive observables at small x. Measurements of other observables with a sufficient accuracy to be compared with F_2 are certainly an experimental challenge. Nonetheless, this effort is likely to be essential to understand the transition between perturbative and non-perturbative dynamics at small x.

Note added. The use of physical anomalous dimensions has been previously advocated in the literature (see, for instance, [39]) although in different context. I wish to thank Georges Grunberg for having pointed out those references to me.

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