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On the option interpretation of rational harvesting planning

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Abstract. We consider the determination of the harvesting strategy maximizing the present expected value of the cumulative yield from the present up to extinction. By relying on a combination of stochastic calculus, ordinary nonlinear programming, and the classical theory of diffusions, we show that if the underlying population evolves according to a logistic diffusion subject to a general diffusion coefficient, then there is a single threshold density at which harvesting should be initiated in a singular fashion. We derive the condition which uniquely determines the threshold and show that harvesting should be initiated only when the option value of further preserving another individual falls below its opportunity cost. In this way, we present a real option interpretation of rational harvesting planning. We also consider the comparative static properties of the value of the harvesting opportunity and state a set of usually satisfied conditions under which increased stochastic fluctuations (demographic or environmental) decrease the expected cumulative yield from harvesting and increase the optimal harvesting threshold, thus postponing the rational exercise of the irreversible harvesting decision.

1. Introduction

Determining socially acceptable harvesting policies is undoubtly one of the most challenging and most controversial problems in the management of renewable resources. The main source of disagreement is whether a rationally planned harvesting policy should be based principally on ecologically important factors or on purely economic principles. More precisely, in the presence of capital markets and a continually present harvesting effort, economically rational harvesting often leads to the biological overexploitation of the harvested population (cf. [2], [10], Section 2.3, [23], and [24]). This has led to the emergence of two separate approaches for studying the effects of harvesting on population growth and its long run behavior. In ecology, the most preferred approach to study this problem is to consider the expected consequences of a harvesting policy on the dynamic behavior of the pop-

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ulation density independently on whether the implemented policy is optimal or not (cf. [7], [9], [28], [32], and [33]). In turn in economics, the harvesting problem is usually approached by relying on capital theoretic models which take into account the capital markets and their role as an alternative source of potential income for the harvester. Thus, in those studies the harvester is viewed as an investor having the opportunity to invest either on a harvesting venture yielding a (possibly known) rate of return or buying bonds yielding a safe interest income. Economical studies on this topic can be roughly divided into two separate classes. Namely, those modelling the harvested resource as an input for production (cf. [10], Sections 2.4, and 2.5, [11], chapter 5, [30], Part III) or those modelling the harvested resource as the supplied output ([2], [6], [23], [24], and [25]). In both cases the objective of the harvester is generally assumed to be the maximization of the present expected value of the cumulative yield from harvesting over an arbitrary time horizon (it can be either finite or infinite; cf. [10], Section 2.5). As is well-known, the optimal harvesting problem results in a linear variational problem which can be solved by relying on various approaches from which the most popular ones are the classical calculus of variations (cf. [10], chapter 2) and dynamic programming (cf. [2], [6], and [25]). The resulting optimal policy is singular in the sense that there is an optimal population density towards which the population should be driven by harvesting at a maximal rate as long as the population density is above the optimal threshold density. At this density, the population is harvested just enough to keep the population at the optimal threshold (the harvester lives off the marginal increases in the population density). The traditional approaches to this problem have, however, a limitation which plays a major role in realistic decision making problems. Namely, the models are generally assumed to be deterministic (see [2], [6], [23], [24], and [25] for exceptions). Thus, in such approach the harvester does not face uncertainty about the consequences of his actions (for example, increased extinction risk), nor does he face unanticipated stochastic shocks affecting the dynamic fluctuation of the density of the harvested population (environmental or demographic shocks, unanticipated catastrophes).

In light of these arguments, it is our purpose in this study to consider the determination of the optimal harvesting strategy of a harvester facing a stochastically fluctuating population. In accordance with the traditional economical studies of rational harvesting planning, we assume that the objective of the harvester is to find a harvesting plan maximizing the present expected value of the cumulative yield. However, in order to also take into account the existing environmental and demographic noise, we assume that the population evolves according to a stochastic approximation of the classical logistic model of population growth and that the time horizon of the harvester is from the present up to extinction (cf. [2], [6], [23], [24], and [25]). In order to gain both mathematical generality and biological tractability (cf. [2], [6], [23], [24], [25], [34], and [35]), we do not specify exactly the form of the diffusion coefficient (i.e. the volatility coefficient). In this way, we are able to maintain the analysis more generally valid than in previous studies of this problem and to provide the optimal solutions for a great variety of problems belonging into this class independently of their local stochastic behavior. By relying on a combination of modern stochastic calculus, the classical theory of diffusions, and ordinary nonlinear programming techniques (cf. [5]), we demonstrate that it is a general property of models belonging into this class that, whenever the per capita growth rate of the population at low densities is greater than the discount rate, there is a single optimal harvesting threshold at which harvesting should be initiated at a maximal rate. Below this critical density the population is left unharvested and evolving according to the law of the population in the absence of harvesting. In other words, the resulting optimal harvest policy is singular (cf. [15], chapter VIII). It is worth pointing out that these type of harvesting policies are difficult, if possible at all, to be implemented in reality (see [6] for an explanation). In accordance with the literature on real options and optimal harvesting, we find that the immediate depletion of a population can be optimal only if the net convenience yield accrued from preserving the population is non-positive for all densities. In other words, if the expected net growth rate of the population is non-positive for all densities, then a rational harvester should exercise the harvesting opportunity immediately at full capacity and deplete the entire population instantaneously. In accordance with the modern theory of real options (cf. [12]), we demonstrate that harvesting is discontinued whenever its option value falls short its opportunity cost, which we interpret accordingly as the option value of preservation (cf. [2] and [6]). Put differently, we show that harvesting should be initiated whenever the marginal option value of preservation vanishes. In contrast to models considering the rational exercise of investment opportunities, we demonstrate that the present expected value of the cumulative yield is an increasing and concave function of the current population density and that increased stochastic fluctuations decrease its value. Thus, increased volatility decreases the cumulative yield independently of the source of the fluctuations (whether they are demographic or environmental). We also consider the long-run effect of rational harvesting planning. As intuitively is clear, harvesting has a significant impact on the long-run dynamic behavior of the population density. Especially, if the unharvested population is never expected to go extinct in finite time, then neither is the harvested population expected to do so. However, it is worth pointing out that since the considered population process constitutes a continuous approximation of a real discrete population density, the extinction for a real population may actually occur in finite expected time (cf. [23], and [24]). Moreover, it turns out that harvesting may lead to the introduction of a long-run stationary distribution towards which the population evolves in the long-run even while such a distribution would not exist in the absence of harvesting. Thus, the long-run stationary behavior of the population density is significantly altered as both the shape and the support of the approached long run stationary distribution are changed by the presence of harvesting.

The contents of this study are as follows. In section two we present the general model and solve it explicitly in terms of the fundamental solutions of an ordinary second order differential equation. In section three we illustrate our theoretical results explicitly by relying on a model with environmental stochasticity first introduced in [34] and later considered in an optimal harvesting problem in [25]. We solve the arising problem explicitly and study its consequences numerically. Finally, our section four concludes.

2. Optimal harvesting

Consider a stochastically fluctuating population with a density $\{X(t); t \in [0, \tau(0))\}$, where $\tau(0) = \inf\{t \geq 0 : X(t) = 0\}$ denotes the possibly infinite extinction date, defined on a complete filtered probability space $(\Omega, P, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{F})$ and described on $\mathbf{R}_+$ by the (Itô-) stochastic differential equation

$$dX(t) = \mu X(t)(1 - \gamma X(t))dt + \sigma(X(t))dW(t) - dZ(t), \quad X(0) = x \ , \qquad (1)$$

where $\mu > 0$ denotes the per capita growth rate of the population at low densities, $\gamma^{-1} > 0$ denotes the carrying capacity of the environment, $Z(t)$ denotes the cumulative harvesting effort, and the mapping $\sigma : \mathbf{R}_+ \mapsto \mathbf{R}$ denoting the infinitesimal diffusion coefficient of X is a given Lipschitz-continuous mapping on $\mathbf{R}_+$ satisfying the condition $\{\sigma^{-1}(0)\} \cap (0, \gamma^{-1}) = \emptyset$ (i.e. $\sigma(x) \neq 0$ on $(0, \gamma^{-1})$). We call a harvesting strategy Z *admissible* if it is non-negative, non-decreasing, right-continuous, and $\{\mathscr{F}_t\}$-adapted, and denote the set of admissible controls as Λ. Moreover, since the considered population process is defined up to the extinction date, we observe that $X(t) > 0$ for all $t \in [0, \tau(0))$ and, therefore, that $Z(t) \leq X(t)$. It is now clear that under these assumptions the population density X evolves in the absence of harvesting according to a regular, time-homogeneous, and linear diffusion process for which the infinitesimal diffusion coefficient σ does not vanish on $(0, \gamma^{-1})$. If a singularity $\bar{x} \geq \gamma^{-1}$, where $\sigma(\bar{x}) = 0$, exists then it is assumed to be a natural boundary for the diffusion process describing the density of the population (cf. [8], pp. 14–17, and [22], pp. 226–242, for a complete boundary classification for linear diffusions). Thus, we assume that while the process may evolve towards the upper boundary $\bar{x}$, it is never expected to hit it in finite time. Moreover, in accordance with reality (cf. [2], [6], [23], [24], and [25]), we assume that the upper boundary ∞ of the state-space of the population density process X is natural. Thus, even while the population density may be expected to increase, it is never expected to become infinitely high in finite time. Moreover, we know that the basic characteristics of the diffusion X are (cf. [8], chapter II, [17], chapter 4)

$$S'(x) = \exp\left(-\int^x \frac{2\mu s(1-\gamma s)ds}{\sigma^2(s)}\right) \qquad (2)$$

denoting its scale density and

$$m'(x) = \frac{2}{\sigma^2(x)S'(x)}$$

denoting the density of its speed measure. These characteristics determine the behavior of the underlying population process X (cf. [8], chapter II, [22], chapter 15). Especially, we know that if the population X evolves towards a long run stationary distribution with support on $(0, \bar{x})$, where $\bar{x} \geq \gamma^{-1}$ denotes the upper boundary of the state-space of X (i.e. $\bar{x}$ has to be a singular point for X), then the density, denoted as $p(x)$, of the stationary distribution is given as the solution of the ordinary differential (adjoint) equation (cf. [22], pp. 220–222)

$$\frac{1}{2}\frac{d^2}{dx^2}[\sigma^2(x)p(x)] - \frac{d}{dx}[\mu x(1-\gamma x)p(x)] = 0 \ , \qquad (3)$$

subject to a set of appropriate boundary conditions (cf. [8], p. 18). As intuitively is clear and will be proved in the subsequent analysis, harvesting has a significant impact on the form of this distribution whenever it exists. Moreover, it is worth emphasizing that it is not difficult to construct examples where the population process does not posess a stationary distribution in the absence of harvesting but does have one in the presence of harvesting. Thus, as will become apparent from our subsequent analysis, harvesting has a strong impact on the asymptotic behavior of the harvested species.

By following the recent literature on optimal harvesting of stochastically fluctuating populations (cf. [2], [6], [23], [24], and [25]) consider now a harvester whose sole objective is to maximize the *present expected value of the cumulative yield* from the present up to extinction. That is, consider the (singular) stochastic control problem

$$V(x) = \sup_{Z\in\Lambda} E_x \int_0^{\tau(0)} e^{-rs} dZ(s) \ , \tag{4}$$

where $r \geq 0$ denotes the discount rate which in this study is assumed to measure both economic discounting, denoted as r_1, and an exogeneously determined density-independent catastrophes rate, denoted as r_2 (i.e. the catastrophe can be interpreted as a discrete Poisson event). Put formally, we assume that $r = r_1 + r_2$. In a purely capital theoretic approach neglecting the potentially catastrophical phenomena, the economic discounting term r_1 constitutes the factor measuring the opportunity cost of investing in an asset rendering a profit flow increasing at the expected growth rate of the population instead of investing a similar amount of capital in bonds yielding the sure rate of return r_1 (cf. [12], pp. 114–117). As is clear from the definition of the discount factor in our model, potential catastrophes increase this opportunity cost; a result which is in accordance with the findings on the rational pricing of defaultable bonds (cf. [13]). Before proceeding, we state an auxiliary verification lemma

Lemma 1. *Let $U : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be a twice continuously differentiable function satisfying the conditions*

$$\text{(i) } U'(x) \geq 1 \quad \text{for all } x \in \mathbf{R}_+;$$
$$\text{(ii) } ((\mathscr{A} - r)U)(x) \leq 0 \quad \text{for all } x \in \mathbf{R}_+ \ ,$$

where

$$\mathscr{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu x(1-\gamma x)\frac{d}{dx} \tag{5}$$

is the differential operator representing the infinitesimal generator of X. Then, $V(x) \leq U(x)$ for all $x \in \mathbf{R}_+$.

Proof. Follows directly from Lemma 1 in [5]. □

Lemma 1 shows that if we find a r-superharmonic mapping U growing faster than the identity mapping $x \mapsto x$, then such a mapping dominates the value of the harvesting strategy. Unfortunately, this does not give us the value explicitly even while it hints how the value function should be constructed. In order to accomplish this task, define the net convenience yield from holding a reservoir (cf. [12], p. 115, for an economic interpretation of this factor) as the mapping $\theta : \mathbf{R}_+ \mapsto \mathbf{R}$ described by the equation

$$\theta(x) = \mu x(1 - \gamma x) - rx \quad . \tag{6}$$

It is now clear that θ is twice continuously differentiable and strictly concave on $\mathbf{R}_+$. Moreover, we find that (cf. [2], [6], and [25])

Lemma 2. *If the per capita growth rate of the population at low densities is smaller than or equal to the discount rate, then it is optimal to deplete the entire population instantaneously. That is, if $\mu \leq r$, then $Z(0) = x$, $\tau(0) = 0$, and $V(x) = x$.*

Proof. Apply the Doléans-Dade-Meyer change of variables formula to the identity mapping $x \mapsto x$. In that case, we find that for all admissible controls $Z \in \Lambda$ and all $x \in \mathbf{R}_+$ we have (cf. [31], p. 74)

$$E_x[e^{-rT_R} X(T_R)] = x + E_x \int_0^{T_R} e^{-rs} X(s)[\mu - r - \mu\gamma X(s)]ds$$
$$-E_x \int_0^{T_R} e^{-rs} dZ(s) \quad ,$$

where $T_R = \tau(0) \wedge R \wedge \tau(R)$, and $\tau(R) = \inf\{t \geq 0 : X(t) \geq R\}$. By reordering terms, we find that for all admissible controls $Z \in \Lambda$ and all $x \in \mathbf{R}_+$ we have

$$E_x \int_0^{T_R} e^{-rs} dZ(s) = x - E_x[e^{-rT_R} X(T_R)] + E_x \int_0^{T_R} e^{-rs} \theta(X(s))ds \ \leq x \quad ,$$

because of our assumption $\mu \leq r$. By letting now R tend to infinity we obtain by monotone convergence that for all admissible controls $Z \in \Lambda$ and all $x \in \mathbf{R}_+$

$$E_x \int_0^{\tau(0)} e^{-rs} dZ(s) \leq x \quad ,$$

completing our proof. □

Lemma 2 states a familar result form the modern theory of real options. Namely, that if the net marginal convenience yield from holding reservoirs is non-positive for all population densities, then waiting is never optimal and the population should be instantaneously depleted. In other words, in case the expected net yield accrued from holding part of the harvested population alive is always negative, then it is always suboptimal to wait for future potential unexpected increases of the population density X. This result is somewhat alarming since it shows that potential catastrophes increase the required rate of return accrued from the population and, therefore, decrease the incentives to wait and postpone the harvesting decision. It is also obvious from the proof of Lemma 2 that due to the nonnegativity of the process X and monotone convergence, we find that for all $x \in \mathbf{R}_+$

$$V(x) \leq x + \sup_{Z(t)\in\Lambda} E_x \int_0^{\tau(0)} e^{-rs}\theta(X(s))ds \ . \tag{7}$$

That is, the value of the optimal harvesting policy can never exceed the sum of the current population density and the maximized present expected value of the cumulative convenience yields from the present up to the extinction date. Interestingly, it is also clear from our results that if $\lim_{R\uparrow\infty} E_x[e^{-rT_R}X(T_R)] = 0$ for all $Z \in \Lambda$ and all $x \in \mathbf{R}_+$ (a *transversality condition*), then we have that

$$V(x) = x + \sup_{Z(t)\in\Lambda} E_x \int_0^{\tau(0)} e^{-rs}\theta(X(s))ds,$$

that is, that the value of the optimal harvesting strategy is equal to the sum of the current density and the maximized present expected value of the future convenience yields accrued from preserving part of the population. Thus, if the present expected net population density vanishes in the long run independently of the implemented harvesting strategy, then maximizing the present expected value of the cumulative yield from the present up to extinction is equivalent with maximizing the sum of the current population density (measuring the current harvesting potential) and the present expected value of the cumulative future net convenience yields from holding inventories. This result illustrates the close connection between rational harvesting planning and optimal cash flow control (cf. [4]). Moreover, (7) implies that

Lemma 3. *If* $\mu > r > 0$ *then for all* $x \in \mathbf{R}_+$ *and* $Z(t) \in \Lambda$ *we have that*

$$x \leq V(x) \leq x + \frac{(\mu-r)^2}{4\mu\gamma r} \ .$$

Therefore,

$$\lim_{x\uparrow\infty} \frac{V(x)}{x} = 1 \ .$$

Proof. Choosing $Z(0) = x$ implies that $V(x) \geq x$ for all $x \in \mathbf{R}_+$. On the other hand, by noticing that $\theta(x) \leq \frac{(\mu-r)^2}{4\mu\gamma}$ for all $x \in \mathbf{R}_+$ and invoking (7) yields the required result. The limit condition follows directly by dividing the inequality by x and letting x tend to infinity. □

Remark. It is worth noticing that in the absence of discounting, that is, if $r = 0$ then the inequality of Lemma 3 reads as

$$x \leq V(x) \leq x + \frac{\mu}{4\gamma} E_x[\tau^*(0)],$$

where $\tau^*(0)$ denotes the expected extinction date in the absence of harvesting. Thus, the value of the optimal harvesting strategy is bounded whenever extinction is attainable in finite expected time.

Lemma 3 demonstrates that under logistic growth, the value of the optimal strategy is bounded for any admissible strategy. Thus, no matter how complicated the optimal strategy turns out to be, its value will always remain between two known boundaries. As intuitively is clear, the upper boundary measures the sum of the current harvesting potential x and the maximum present expected value of the cumulative net convenience yields accrued from postponing the harvesting decision and keeping the population alive. Moreover, it is also clear from the lemma, that the value function is going to eventually grow linearly thus indicating that harvesting should be optimal at high densities (which is indeed the case, as we will later prove). Before proceeding in our analysis, we state the following definition:

Definition 1. ([8], chapter II, and [17], Section 4.6, and [26], Section II.3) The *Green-kernel* $G_r : \mathbf{R}_+^2 \mapsto \mathbf{R}_+$ of the diffusion X is defined as

$$G_r(x, y) = \int_0^\infty e^{-rt} p(t; x, y) dt \ ,$$

where $p(t; x, y)$ is the transition density of X defined with respect to its speed measure m. There are two linearly independent functions (the *fundamental solutions*), $\psi(x)$ and $\varphi(x)$, with $\psi(x)$ increasing and $\varphi(x)$ decreasing, spanning the set of solutions of the ordinary differential equation $((\mathscr{A} - r)u)(x) = 0$. The Green-kernel $G_r(x, y)$ can be rewritten in terms of these solutions in the alternative form

$$G_r(x, y) = \begin{cases} B^{-1}\psi(x)\varphi(y), & x < y \\ B^{-1}\psi(y)\varphi(x), & x \geq y \end{cases}$$

where

$$B = \frac{\psi'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}\psi(x) > 0$$

is the *constant Wronskian* determinant of the fundamental solutions.

By following now the approach presented in [5] (see also [2], [3], and [4]), we now consider the associated nonlinear programming problem

$$R(x) = \sup_{b \geq 0} E_x \int_0^{\tau(0)} e^{-rs} \theta(\hat{X}(s)) ds \ , \tag{8}$$

where $\hat{X}$ denotes the (uncontrolled) diffusion X constrained to be killed at the origin and reflected at the upper boundary b. That is, we will consider the maximization problem of the present expected value of the cumulative convenience yields from the present up to extinction under the constraint that there is an upper boundary at which the process is reflected. In accordance with the definition first introduced in [6] and with the terminology of the modern theory of real options (cf. [12], p. chapters 5 and 6), we call the mapping $R : \mathbf{R}_+ \mapsto \mathbf{R}_+$ *the option value of preservation* since it measures the expected present value of the cumulative yield accrued from postponing the harvesting decision and retaining part of the population alive. Therefore, it can also be interpreted as *the option value of waiting to* *harvest* since it essentially measures the expected intertemporal gains accrued from leaving the harvesting decision unexercised. By relying now on Definition 1, we find that the Green representation of the Markovian functional in (8) for $x \in (0, b]$ is

$$R(x) = \sup_{b \geq 0} \int_0^b G_r^{(0,b]}(x, y)\theta(y)m'(y)dy \ , \tag{9}$$

where $G_r^{(0,b]}(x, y)$ denotes the Green-kernel of the constrained process $\tilde{X}$ (cf. [8], pp. 27–31). It is worth noticing that $G_r^{(0,b]}(x, y)$ can be written in terms of the fundamental solutions of the original unconstrained population process X as

$$G_r^{(0,b]}(x, y) = \begin{cases} \tilde{B}^{-1}\varphi(y, b)\psi(x, 0), & x < y \\ \tilde{B}^{-1}\varphi(x, b)\psi(y, 0), & x \geq y \end{cases}$$

where

$$\varphi(x, b) = \varphi(x) - \frac{\varphi'(b)}{\psi'(b)}\psi(x)$$

denotes the decreasing and

$$\psi(x, 0) = \psi(x) - \frac{\psi(0)}{\varphi(0)}\varphi(x)$$

denotes the increasing fundamental solution of the ordinary differential equation $((\mathscr{A} - r)u)(x) = 0$ under the assumption of killing at 0 and reflection at b, and

$$\tilde{B}^{-1} = \frac{B^{-1}}{\varphi(0, b)}\varphi(0) = \frac{B^{-1}}{\psi'(b, 0)}\psi'(b) \ ,$$

denotes the Wronskian of $\varphi(x, b)$ and $\psi(x, 0)$. We can now demonstrate that

Lemma 4. *Assume that $\mu > r$. Then there is an optimal threshold, denoted $b^* \in ((\mu - r)/(2\gamma\mu), (\mu - r)/(\gamma\mu))$, satisfying the first order condition*

$$r \int_0^{b^*} \psi(y, 0)[\mu y(1 - \gamma y) - ry]m'(y)dy = [\mu b^*(1 - \gamma b^*) - rb^*]\frac{\psi'(b^*, 0)}{S'(b^*)} \ . \tag{10}$$

Alternatively, the first order condition (10) can be written as

$$r \int_0^{b^*} \frac{\psi'(y, 0)}{S'(y)}[\mu - r - 2\gamma\mu y]dy = 0 \ . \tag{11}$$

Moreover, the option value of preservation $R(x)$ satisfies on $(0, b^*]$ the conditions

(i) $R'(x) \geq 0$ for all $x \in (0, b^*)$, and
(ii) $R(0) = \lim_{x\uparrow b^*} R'(x) = \lim_{x\uparrow b^*} R''(x) = 0$.

Proof. Assume first that the density process has no singularities on $\mathbf{R}_+$. By following then the approach of [5] and invoking the strong Markov property of diffusions, rewrite (9) in the form

$$R(x) = \tilde{R}(x) - \tilde{R}'(b)\frac{\psi(x,0)}{\psi'(b,0)} \ , \tag{12}$$

where

$$\tilde{R}(x) = \int_0^\infty G_r^{(0,\infty)}(x,y)\theta(y)m'(y)dy$$

denotes the option value of postponing the harvesting opportunity indefinitely. By differentiating now (12) with respect to the boundary point b we find that a candidate for an optimal reflection boundary, denoted now b^*, has to satisfy the first order necessary condition

$$\begin{aligned}\frac{\partial R(x)}{\partial b}\Big|_{b=b^*} &= -\frac{\psi(x,0)}{\psi'^2(b^*,0)}[\tilde{R}''(b^*)\psi'(b^*,0) - \tilde{R}'(b^*)\psi''(b^*,0)] \\ &= -\frac{2S'(b^*)\psi(x,0)}{\sigma^2(b^*)\psi'^2(b^*,0)}\left[r\int_0^{b^*}\psi(y,0)\theta(y)m'(y)dy - \theta(b^*)\frac{\psi'(b^*,0)}{S'(b^*)}\right] \\ &= 0 \ ,\end{aligned}$$

proving (10). (11) is then obtained by noticing that

$$\theta(b^*)\frac{\psi'(b^*,0)}{S'(b^*)} = r\int_0^{b^*}\theta(b^*)\psi(y,0)m'(y)dy + \theta(b^*)\frac{\psi'(0,0)}{S'(0)} \ ,$$

collecting terms, and invoking Fubini's theorem. To prove the existence and uniqueness of b^*, define the parameters $x_0 = \frac{\mu-r}{2\mu\gamma}$ and $x_1 = \frac{\mu-r}{\mu\gamma}$, and notice that the mapping

$$h(x) = r\int_0^x \frac{\psi'(y,0)}{S'(y)}\theta'(y)dy$$

satisfies the condition $h(0) = 0$, and the inequality

$$h(x_0) = r\int_0^{x_0} \frac{\psi'(y,0)}{S'(y)}\theta'(y)dy > 0 \ ,$$

since θ is increasing on $(0, x_0)$. Moreover, we find by the mean value theorem for integrals that

$$\begin{aligned}h(x_1) &= r\int_0^{x_0} \frac{\psi'(y,0)}{S'(y)}\theta'(y)dy + r\int_{x_0}^{x_1} \frac{\psi'(y,0)}{S'(y)}\theta'(y)dy \\ &= r\left(\frac{\psi'(\eta_1,0)}{S'(\eta_1)} - \frac{\psi'(\eta_2,0)}{S'(\eta_2)}\right)\theta(x_0) \ ,\end{aligned}$$

where $\eta_1 \in (0, x_0)$ and $\eta_2 \in (x_0, x_1)$. By noticing now that

$$\frac{d}{dx}\frac{\psi'(x,0)}{S'(x)} = r\psi(x,0)m'(x) > 0 \ ,$$

we find that $h(x_1) < 0$ implying that (11) has at least one root in (x_0, x_1). However, since

$$h'(x) = \theta'(x)\frac{\psi'(x,0)}{S'(x)} < 0$$

on (x_0, ∞) we find that the monotonicity of h on (x_0, ∞) then prove the uniqueness of the root b^*. Moreover, this demonstrates that also the second-order (sufficiency) local concavity condition

$$\frac{\partial^2 R(x)}{\partial b^2}\Big|_{b^*\text{opt.}} = \frac{2\psi(x,0)\theta'(b^*)}{\sigma^2(b^*)\psi'(b^*,0)} < 0$$

is met, thus proving the optimality of b^*. By differentiating (12) with respect to x, we find that

$$R'(x) = \psi'(x,0)\left[\frac{\tilde{R}'(x)}{\psi'(x,0)} - \frac{\tilde{R}'(b^*)}{\psi'(b^*,0)}\right] . \tag{13}$$

On the other hand, we have that

$$\frac{d}{dx}\left[\frac{\tilde{R}'(x)}{\psi'(x,0)}\right] = \frac{2S'(x)h(x)}{\sigma^2(x)\psi'^2(x,0)} ,$$

proving that $R'(x) \geq 0$ for all $x \leq b^*$. It remains to show that $R(x)$ satisfies part (ii). The conditions $R(0) = R'(b^*) = 0$ are clear by definition. By differentiating (9) twice with respect to x and letting $x \uparrow b^*$ we find that

$$R''(b^*) = B^{-1}\frac{\psi'(b^*,0)\varphi''(b^*,b^*)}{\psi'(b^*,0)}\int_0^{b^*}\psi(y,0)\theta(y)m'(y)dy - \frac{2\theta(b^*)}{\sigma^2(b^*)} .$$

By noticing now that $\psi'(b^*)\varphi''(b^*,b^*) = \frac{2rBS'(b^*)}{\sigma^2(b^*)}$ and then invoking condition (10) completes our proof in the non-singular case.

The representation of $R(x)$ above assumes that the population density process does not have a singularity on $\mathbf{R}_+$. If such a singularity, denoted now as $\bar{x} > \gamma^{-1}$, exists then the assumed naturality of this boundary for the density process $X(t)$ guarantees that $R(x)$ can in that case be as written as in (12) with

$$\tilde{R}(x) = \int_0^{\bar{x}} G_r^{(0,\bar{x})}(x,y)\theta(y)m'(y)dy .$$

The rest of the proof is then completely analogous with the proof in the non-singular case. □

Lemma 4 proves that for a *diffusion approximation of the classical Verhulst-Pearl model of logistic population growth*, there typically exists a critical density below carrying capacity at which the option value of preservation falls below the value of the opportunity to harvest. As in traditional models of future-oriented rational harvesting planning, our results show that the optimal threshold will lie above the thresold x_0 at which the convenience yield is maximized but below the threshold x_1 at which the convenience yield from retaining part of the population unharvested vanishes. Specially, our results demonstrate that if $\mu < 2r$, then the optimal threshold b^* will always be smaller than $(2\gamma)^{-1}$ which is the density at which the growth rate of the population is maximized. Unfortunately, it is impossible to state a set of simple parametric conditions guaranteeing the contrary result. It is, however, clear that it is essentially the relative sizes between the population growth rate at low densities and the size of the stochastic fluctuations which determine whether the optimal threshold is above $(2\gamma)^{-1}$ or not. Thus, *stochastic fluctuations may lead to the implementation of a biologically sustainable harvesting policy*. It is also worth noticing that our results demonstrate that the applied representation is indeed a maximal one since (8) is *structurally stable* (cf. [5]) in the sense that

$$R'(x) = \sup_{b \geq 0} \left[\tilde{R}'(x) - \tilde{R}'(b) \frac{\psi'(x,0)}{\psi'(b,0)} \right] ,$$

demonstrating that also the marginal option value of preservation $R'(x)$ can be interpreted as the solution of a standard nonlinear programming problem (i.e. differentiation and maximization commute; cf. [24] for a similar result). On the basis of these results it is now clear that the critical harvesting threshold b^* is chosen so as to maximize both the option value of preservation and its marginal value for all densities below the critical threshold. Put formally,

Theorem 1. *Assume that $\mu > r$. Then the optimal harvesting strategy is*

$$Z(t) = \max_{0 \leq s \leq t} (X(s) - b^*)^+ \quad , \tag{14}$$

where $b^* \in (x_0, x_1)$ is the unique interior root of the necessary condition (10). Moreover, the present expected value of the cumulative yield resulting from the optimal policy reads as

$$V(x) = \begin{cases} x + \frac{\theta(b^*)}{r}, & x \geq b^* \\ x + R(x), & x < b^* \end{cases} , \tag{15}$$

where $R(x)$, denoting the option value of preservation, is defined as in (9).

Proof. By relying on Lemma 4, it is now straightforward to verify that V satisfies the conditions of the verification Lemma 1. Thus, it is a majorant for the value of the optimal harvesting policy. On the other hand, it is clear that in the case of this section, the stochastic differential equation (1) with reflection at b^* admits a unique solution (cf. [16], Section 1.6). Thus we notice by the Doléans- Dade-Meyer change of variables formula that

$$\begin{aligned} E_x[e^{-rT_R} V(X(T_R))] = V(x) + E_x \int_0^{T_R} e^{-rs}((\mathscr{A} - r)V)(X(s))ds \\ - E_x \int_0^{T_R} e^{-rs} V'(X(s))dZ(s) \\ + E_x \sum_{0 < s \leq T_R} e^{-rs}[V(X(s)) - V(X(s-)) \\ - V'(X(s-))(X(s) - X(s-))] \ , \end{aligned}$$

where $T_R = \tau(0) \wedge R \wedge \tau(R)$. Since the local time $Z(t) = \max_{0\le s\le t}(X(s) - b^*)^+$ increases only at the boundary b^*, $V'(b^*) = 1$, V is bounded on $(0, b^*]$, and $((\mathscr{A} - r)V)(X(t)) = 0$ outside a t-set of Lebesgue measure zero, we find by letting $R \uparrow \infty$ that

$$V(x) = E_x \int_0^{\tau(0)} e^{-rs} dZ(s) \ ,$$

completing the proof of our Theorem (see [15], p. 328, for a description of the considered *Skorokhod-problem*). □

Theorem 1 proves that for any diffusion approximation of the type (1) of the classical Verhulst–Pearl model of logistic population growth, there is a unique optimal harvesting threshold at which the harvesting opportunity should be exercised at a maximal rate in order to keep the population below the threshold b^*. This result demonstrates why standard models studying the maximization of the present expected value of the cumulative yield from the present up to extinction (cf. [2], [6], [23], [24], and [25]) end up having a single upper threshold at which harvesting is initiated in a singular fashion. Moreover, it is now clear from our Theorem 1 that

Corollary 1. *Assume that the conditions of Theorem 1 are satisfied. Then the optimal harvesting strategy is of the form (14) where the optimal harvesting threshold b^* is the unique root of the algebraic equation*

$$\psi''(b^*, 0) = 0 \ . \tag{16}$$

Moreover, the present expected value of the cumulative yield (15) can be rewritten as

$$V(x) = \begin{cases} x + \frac{\theta(b^*)}{r}, & x \ge b^* \\ \frac{\psi(x,0)}{\psi'(b^*,0)}, & x < b^* \end{cases} \tag{17}$$

Proof. It is now clear from Theorem 1 that on $(0, b^*]$ the value $V(x)$ is the solution of the ordinary differential equation $((\mathscr{A} - r)V)(x) = 0$ subject to the boundary constraints $V(0) = 0$ and $V'(b^*) = 1$. Moreover, since $V'(x) = 1$ on (b^*, ∞) and the value is continuous, we have (17). Finally, the smooth-fit principle $V''(b^*) = 0$ then proves (16). □

Corollary 1 presents the results of our Theorem 1 in a more familiar form (cf. [6], [23], [24], and [25]). Since the present expected value of the cumulative yield has to be r-harmonic on the non-action region where harvesting is suboptimal, it can be written in terms of the fundamental solutions of the ordinary differential equation $((\mathscr{A} - r)V)(x) = 0$. By using this representation, we find that the cumulative yield can be written on $(0, b^*]$ alternatively in the form

$$V(x) = x + \frac{\psi(x, 0)}{\psi'(b^*, 0)} - x = x + \int_0^x \left(\frac{\psi'(y, 0)}{\psi'(b^*, 0)} - 1 \right) dy \ , \tag{18}$$

demonstrating that the option value of preservation can be alternatively written as

$$R(x) = \int_0^x \left(\frac{\psi'(y, 0)}{\psi'(b^*, 0)} - 1 \right) dy \ .$$

It is worth noticing that extinction risk plays a significant role in the determination of the harvesting threshold only if the lower boundary 0 is either regular or entrance. If this is not the case, then according to Corollary 1 the optimal threshold simply satisfies the condition $\psi''(b^*) = 0$ (cf. [6], and [25]). Moreover, by relying on Theorem 1 we also find that (cf. [6], [23], and [24])

Corollary 2. *Assume that the mapping* $\frac{2\mu x(1-\gamma x)}{\sigma^2(x)}$ *is locally integrable at the carrying capacity* γ^{-1} *and that* 0 *is an attracting boundary for the population in the absence of harvesting. Then, in the absence of discounting, the optimal harvesting policy is*

$$Z(t) = \max_{0\leq s\leq t} (X(s) - \gamma^{-1})^+ \quad .$$

Moreover, the value of the optimal strategy $V : \mathbf{R}_+ \mapsto \mathbf{R}_+$ *is monotonically increasing, concave, and reads as*

$$V(x) = \begin{cases} x - \gamma^{-1} + \frac{S(\gamma^{-1})}{S'(\gamma^{-1})}, & x \geq \gamma^{-1} \\ \frac{S(x)}{S'(\gamma^{-1})}, & x < \gamma^{-1} \end{cases} ,$$

where $S'(x)$ *denoting the scale density of* X *is defined as in (2) and*

$$S(x) = \int_0^x S'(y)dy$$

denotes the scale function of X.

Proof. Analogous with the proof of Theorem 1. □

Corollary 2 proves the intuitively clear result that in the absence of capital markets and potential catastrophes the optimal harvesting threshold is equal to the carrying capacity of the population whenever the lower boundary 0 is attracting. Thus, discounting speeds up harvesting and accelerates extinction by increasing the opportunity cost of leaving the harvesting opportunity unexercised. The reason for this result is obvious from a capital theoretic point of view. Whenever the harvester is intertemporally indifferent between different generations, the only source of "interest income" is the population density and, therefore, by exercising the opportunity at carrying capacity the harvester maximizes the potentially productive capacity.

We have not yet characterized the general comparative static properties of the expected cumulative yield under the rational harvesting policy. As usually, the relationship between discounting and the value of the harvesting opportunity is negative, as is demonstrated in

Theorem 2. *Denote now as* $V_r(x)$ *the value of the harvesting opportunity under the discount rate* r *and assume that* $\tilde{r} \geq r$. *Then* $V_r(x) \geq V_{\tilde{r}}(x)$.

Proof. We know that $V_r(x)$ satisfies for all $x \in \mathbf{R}_+$ the variational inequalities $V_r(x) \geq 1$ and $((\mathscr{A} - r)V_r)(x) \leq 0$. We observe then by the non-negativity of the value that under our assumptions $((\mathscr{A} - \tilde{r})V_r)(x) = ((\mathscr{A} - r + r - \tilde{r})V_r)(x) \leq (r - \tilde{r})V_r(x) \leq 0$ for all $x \in \mathbf{R}_+$. Thus, the required result follows from Lemma 1. □

Theorem 2 states a familiar result from investment theory. Namely, that increased discounting decreases the value of the harvesting opportunity by increasing the opportunity cost of investment. Thus, the higher the rate of return of alternative capital assets is, the lower is the value of the harvesting opportunity and, therefore, the lower are the incentives to hold the option to harvest alive. Another key factor affecting the rational harvesting decision is the size of the stochastic fluctuations of the underlying diffusion X. In order to describe the relationship between volatility and the rational harvesting policy, we first prove the following auxiliary result characterizing the curvature of the expected cumulative yield:

Theorem 3. *Assume that*

(i) the mapping $\sigma : \mathbf{R}_+ \mapsto \mathbf{R}$ is continuously differentiable with Lipschitz-continuous derivative and the mapping $\sigma'(x)$ satisfies the usual Novikov-condition (cf. [36], Section 8.6), and
(ii) the extinction boundary 0 is natural for the density process $X(t)$.

Then, the value of the optimal harvesting strategy is increasing and concave on $\mathbf{R}_+$. Moreover, the cumulative expected present value of the future convenience yields from the present up to extinction, i.e. $R(x)$, is also concave on $(0, b^*)$.

Proof. The result is a direct implication of Theorem 5 in [5]. □

Theorem 3 demonstrates that the expected marginal cumulative yield is positive but diminishing. A central implication of Theorem 3 is now summarized in

Theorem 4. *Assume that the conditions of Theorem 3 are met. Then, increased stochastic fluctuations decreases or leaves unchanged the expected cumulative yield from harvesting and increases or leaves unchanged the optimal harvesting threshold. That is, if $\tilde{\sigma} : \mathbf{R}_+ \mapsto \mathbf{R}$ satisfies the condition $\tilde{\sigma}(x) \geq \sigma(x)$ on $\mathbf{R}_+$, $\tilde{V}(x)$ denotes the value of the harvesting opportunity, and $\tilde{b}$ denotes the optimal harvesting threshold in the presence of greater stochastic fluctuations, then $\tilde{b} \geq b^*$ and $\tilde{V}(x) \leq V(x)$ on $\mathbf{R}_+$.*

Proof. As was shown in Theorem 3, the value $V(x)$ of the harvesting opportunity is concave in x. Moreover, as was shown in Theorem 1 it also satisfies for all $x \in \mathbf{R}_+$ the conditions $V'(x) \geq 1$ and

$$\frac{1}{2}\sigma^2(x)V''(x) + \mu x(1-\gamma x)V'(x) - rV(x) \leq 0 \ .$$

Consider now the population subject to greater stochastic fluctuations measured by the infinitesimal diffusion coefficient $\tilde{\sigma}(x)$. In that case the differential operator describing the infinitesimal generator of the underlying population process reads as

$$\tilde{\mathscr{A}} = \frac{1}{2}\tilde{\sigma}^2(x)\frac{d^2}{dx^2} + \mu x(1-\gamma x)\frac{d}{dx} \ .$$

The concavity of the value $V(x)$ then implies that $((\tilde{\mathscr{A}}-r)V)(x) = ((\tilde{\mathscr{A}}-\mathscr{A}+\mathscr{A}-r)V)(x) \leq \frac{1}{2}(\tilde{\sigma}^2(x)-\sigma^2(x))V''(x) \leq 0$. Therefore, $V(x)$ satisfies the conditions of Lemma 1 and we find that for all $x \in \mathbf{R}_+$ we have

$$V(x) \geq \tilde{V}(x) = \sup_{Z\in\Lambda} E_x \int_0^{\tau(0)} e^{-rs}dZ(s) \ ,$$

where

$$dX(t) = \mu X(t)(1-\gamma X(t))dt + \tilde{\sigma}(X(t))dW(t) - dZ(t), \quad X(0) = x \ .$$

To prove that $\tilde{b} \geq b^*$, notice that on $(\max(\tilde{b}, b^*), \infty)$ we have that

$$V(x) - \tilde{V}(x) = \frac{\theta(b^*)-\theta(\tilde{b})}{r} \ .$$

Since $V(x) \geq \tilde{V}(x)$ for all $x \in \mathbf{R}_+$ we find that $\theta(b^*) \geq \theta(\tilde{b})$. However, since the optimal threshold is attained in the decreasing part of the mapping θ we find that $b^* \leq \tilde{b}$ completing the proof of our theorem. □

Theorem 4 states a set of usually satisfied conditions under which increased uncertainty has a negative impact on the expected cumulative yield from harvesting. This result is of interest since it contradicts a usual result from the literature on real options stating that increased uncertainty increases the value of investment opportunities. Moreover, we also find that increased uncertainty increases the optimal harvesting threshold and, therefore, postpones the exercise of the harvesting opportunity. Thus, as in ordinary models of real investment opportunities, we find that increased uncertainty increases the option value of waiting and, therefore, postpones the rational exercise of the opportunity. These results are of interest since they prove that *the sign of the relationship between uncertainty and rational harvesting planning is negative in any logistic diffusion model* satisfying the appropriate smoothness requirements. At least to the best knowledge of the author, this is the first time such results are rigurously proven in models considering rational (singular) harvesting planning.

It is now of interest to consider the long-run behavior of the harvested population. It is a standard result in the theory of linear diffusions that since for $x \in (l, b^*]$

$$E_x[e^{-r\tau(l)}; \tau(l) < \infty] = \frac{\varphi(x, b^*)}{\varphi(l, b^*)} \ ,$$

we find that

$$P_x[\tau(l) < \infty] = \lim_{r\downarrow 0} \frac{\varphi(x, b^*)}{\varphi(l, b^*)} \ .$$

Therefore, by letting l decrease to zero, we find that *if* 0 *is either natural or entrance, then the population is never going to become extinct in finite time under the optimal harvesting policy*. In the remaining cases (i.e. when 0 is either regular or exit) we have that

$$P_x[\tau(0) < \infty] = \lim_{r\downarrow 0} \frac{\varphi(x, b^*)}{\varphi(0, b^*)} \ . \tag{19}$$

It is clear from this analysis that the expected extinction date can be (but does not have to be) finite only if the limit in (19) equals 1. Otherwise, the (unconditionally) expected extinction date is always infinite. If the lower boundary is attainable in finite expected time, then the expected extinction date of the rationally harvested population reads as (cf. [22], pp. 192–202)

$$E_x[\tau(0)] = \int_0^x S(y)m'(y)dy + S(x)\int_x^{b^*} m'(y)dy \ ,$$

where $S(x) = \int_0^x S'(y)dy$. It is also worth noticing that for all $x \in (0, b^*]$ we find that

$$E_x[e^{-r\tau(0)}; \tau(0) < \infty] = \frac{\varphi(x, b^*)}{\varphi(l, b^*)} > \frac{\varphi(x)}{\varphi(0)} = E_x[e^{-r\tilde{\tau}(0)}; \quad \tilde{\tau}(0) < \infty] \ ,$$

where $\tilde{\tau}(0) = \inf\{t \geq 0 : X(t) = 0\}$ denotes the extinction date of the population in the absence of harvesting. Thus, by letting $r \downarrow 0$ we find that

$$P_x[\tau(0) < \infty] > P_x[\tilde{\tau}(0) < \infty] \ .$$

In other words, the *probability of extinction in finite time is higher in the presence of harvesting than in the absence of it* whenever the lower boundary 0 is either regular or exit. Our main results on the long-run distributional behavior of the population density is summarized in

Theorem 5. *Assume that the lower boundary* 0 *of the state space of the population density* X *is either entrance or natural, non-attracting, and satisfies the condition* $\lim_{l\downarrow 0}\int_l^x m'(y)dy < \infty$ *for any* $x \in (l, b^*]$. *Then the population density of the rationally harvested population evolves towards a long-run stationary distribution with density*

$$p(x) = \frac{m'(x)}{\int_0^{b^*} m'(s)ds} \ .$$

Proof. This is a direct consequence of (3) and the assumptions on the boundary behavior of the population density X (cf. [22], pp. 220–222). □

Remark. Theorem 5 shows how harvesting affects the long run behavior of the harvested population (cf. [6], [7], [9], [27], [32], and [35]). The nonnegativity of the harvesting rate guarantees that the harvested population is dominated by the density of the population density in the absence of harvesting. Therefore, as intuitively is clear, harvesting will always speed up extinction. However, it is an interesting property of the implemented harvesting strategy, that it may lead to the introduction of a long-run stationary distribution even in cases where such a distribution does not exist for the unregulated population. This is a property which is usually neglected in theoretical studies of singular harvesting strategies (cf. [23], [24], and [25]) even while it plays a significant role in determining the future expected population densities for populations evolving “close” to the stationary distribution.

3. Optimal harvesting in the presence of environmental noise

To illustrate our results, consider as in [25] a population evolving according the standard (Verhulst–Pearl) logistic population growth model subject to environmental stochasticity. In that case, the dynamics of the harvested population are described by the stochastic differential equation (cf. [25], [34], and [35])

$$dX(t) = \mu X(t)(1-\gamma X(t))dt + \sigma X(t)(1-\gamma X(t))dW(t) - dZ(t),$$
$$X(0) := x \ . \quad (20)$$

As was proven in [25], if $\mu > r$, then the optimal harvesting problem (4) has a unique well-defined solution. In the spirit of our Corollary 1, we can now demonstrate that this solution can be stated explicitly in terms of standard hypergeometric functions as summarized in

Lemma 5. *(A) Assume that $\mu > r$. Then the optimal harvesting strategy is of the form (14) where the optimal harvesting threshold b^* is the unique root of the algebraic equation $\psi''(b^*) = 0$, where*

$$\psi(x) = \left(\frac{\gamma x}{1-\gamma x}\right)^{\alpha_1} F(a, b, c; -\frac{\gamma x}{1-\gamma x})$$

denotes the increasing fundamental solution of the ordinary second-order differential equation

$$\frac{1}{2}\sigma^2 x^2(1-\gamma x)^2 U''(x) + \mu x(1-\gamma x)U'(x) - rU(x) = 0 \ ,$$

F is the standard hypergeometric function,

$$a = 1 - \frac{\alpha_2}{2} + \frac{\alpha_1}{2} - \frac{1}{2}\sqrt{(\alpha_2^2 - 2\alpha_2(2+\alpha_1)) + (2-\alpha_1)^2},$$
$$b = 1 - \frac{\alpha_2}{2} + \frac{\alpha_1}{2} + \frac{1}{2}\sqrt{(\alpha_2^2 - 2\alpha_2(2+\alpha_1)) + (2-\alpha_1)^2},$$
$$c = 1 - \alpha_2 + \alpha_1,$$
$$\alpha_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0,$$

and

$$\alpha_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0 \ .$$

Moreover, the present expected value of the cumulative yield (15) resulting from the optimal policy reads as

$$V(x) = \begin{cases} x + \frac{\theta(b^*)}{r}, & x \geq b^* \\ \frac{\psi(x,0)}{\psi'(b^*,0)}, & x < b^* \end{cases}$$

(B) Increased uncertainty, that is, increased σ decreases the value $V(x)$ and increases the optimal harvesting threshold b^*.

Proof. In order to derive the value function explicitly, we have to determine the increasing solution of the ordinary second order differential equation

$$\frac{1}{2}\sigma^2 x^2(1-\gamma x)^2 U''(x) + \mu x(1-\gamma x)U'(x) - rU(x) = 0 \ .$$

By making the transformation

$$U(x) = H(\gamma x/(1-\gamma x)) \tag{21}$$

and noticing that $1+\gamma x/(1-\gamma x) = 1/(1-\gamma x)$, we find that (21) can be rewritten as

$$(1+y)y^2 H''(y) + (2y + \frac{2\mu}{\sigma^2}(1+y))yH'(y) - \frac{2r}{\sigma^2}(1+y)H(y) = 0 \ , \tag{22}$$

where $y = \gamma x/(1-\gamma x)$. Denote now the positive and the negative root of the quadratic equation $z^2 - (1-\frac{2\mu}{\sigma^2})z - \frac{2r}{\sigma^2} = 0$ as

$$\alpha_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

and

$$\alpha_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \ ,$$

respectively. Then (22) reads as

$$(1+y)y^2 H''(y) + (2y + (1-\alpha_1-\alpha_2)(1+y))yH'(y) + \alpha_1\alpha_2(1+y)H(y) = 0 \ .$$

By making now a second transformation $H(y) = y^\zeta G(y)$, where ζ is an unknown constant to be determined, we find that (22) can be rewritten in the form

$$(1+y)yG''(y) + (2\zeta + (1-\alpha_1-\alpha_2) + (3+2\zeta-\alpha_1-\alpha_2)y)G'(y) + 2\zeta G(y) = 0$$

whenever $\zeta = \alpha_1$ or $\zeta = \alpha_2$. By making then the transformation $G(y) = J(\hat{y})$, where $\hat{y} = -y$ then finally yields

$$(1-\hat{y})\hat{y}J''(\hat{y}) - (2\zeta + (1-\alpha_1-\alpha_2) - (3+2\zeta-\alpha_1-\alpha_2)\hat{y})J'(\hat{y}) - 2\zeta J(\hat{y}) = 0$$

which is a form of the standard hypergeometric equation (cf. [1], p. 562, and [18], pp. 465–470). Choosing $\zeta = \alpha_1$ and solving the equation

$$a + b = 2 + \alpha_1 - \alpha_2$$
$$ab = 2\alpha_1$$

then completes the proof of part (A) of our theorem. Part (B) then follows directly from Theorem 4. □

Lemma 5 states explicitly in terms of the standard hypergeometric function the present expected value of the cumulative yield of harvesting. By following the analysis in [6] we now illustrate numerically in Table 1 below the results of Lemma 4 by relying on the example of the Antarctic fin whale (*Balaenoptera physalus*) presented in [10] (pp. 49–50). To this end, we assume that $\mu = 8\%$, $\gamma^{-1} = 400000$, and $x = 70000$ (where γ^{-1} and x are interpreted as the number of individuals).

As can be directly seen from Table 1, our numerical results support our Theorem 4 stating that the sign of the relationship between environmental stochasticity and the harvesting incentives is negative. That is, increased infinitesimal fluctuations σ decrease the present expected value of the cumulative yield. Moreover, Table 1 also indicates that increased uncertainty increases the optimal harvesting threshold b^* and, therefore, postpones the rational exercise of the harvesting opportunity. Interestingly, it is clear that in the absence of harvesting the population does not have a long run stationary distribution. However, if $\mu > \sigma^2/2$, that is, if the carrying capacity is an attracting boundary, then in the presence of harvesting the population density tends towards the invariant truncated β-distribution with support $(0, b^*)$ and density

$$p(x) = \frac{x^{2\mu/\sigma^2-2}(1-\gamma x)^{-2\mu/\sigma^2-2}}{\gamma^{1-2\mu/\sigma^2}\int_0^{\gamma b^*} s^{2\mu/\sigma^2-2}(1-s)^{-2\mu/\sigma^2-2}ds} \ . \tag{23}$$

Therefore, as was argued in the previous section, even while the implemented harvesting strategy is instantaneous and singular, it alters dramatically the long run behavior of the population density. Put formally, in the absence of harvesting the population density evolves towards the carrying capacity γ^{-1} whenever $\mu > \sigma^2/2$. However, in the presence of harvesting, the process converges towards a random variable $X(\infty)$ distributed according to the density (23) and posessing an expected value

$$E[X(\infty)] = \frac{\int_0^{b^*} y^{2\mu/\sigma^2-1}(1-\gamma y)^{-2\mu/\sigma^2-2}dy}{\gamma^{1-2\mu/\sigma^2}\int_0^{\gamma b^*} s^{2\mu/\sigma^2-2}(1-s)^{-2\mu/\sigma^2-2}ds} \ .$$

Table 1. The impact of increased stochastic fluctuations and discounting on the optimal harvesting threshold and cumulative yield

r	σ	b^*	$V(70000)$
3%	0.1	130252	171549
3%	0.4	186342	140142
3%	0.7	223814	100430
3%	1.0	236677	85415
5%	0.1	78667	92438
5%	0.4	111774	86705
5%	0.7	132547	78721
5%	1.0	140665	74856

4. Summary and conclusions

We considered the determination of the harvesting strategy maximizing the present expected value of the cumulative yield (i.e. the present expected value of the cumulative catch) from the present up to the potentially finite extinction date of the harvested population. We demonstrated that if the underlying population follows a stochastic approximation of the classical logistic (Verhulst-Pearl) model of population growth then typically there is a unique harvesting threshold at which harvesting should be initiated at full capacity in order to keep the population density below the optimal thresold density. This optimal threshold was shown to be attained at the point where the option value of further preserving another individual falls below its opportunity cost. This result, which is familiar from the recent theory on real options and the valuation of irreversible investment opportunities, shows the way in which the irreversibility of the harvesting opportunity and the stochasticity of the population growth create an explicitly defined option value for both harvesting and preservation. In accordance with intuitive thinking and the theory of real investment opportunities, the harvesting threshold at which harvesting should be initiated is higher under stochastic than under deterministic population growth. Moreover, while the optimal harvesting threshold is below carrying capacity, it is above the critical density at which the maximum sustainable yield is attained. Thus, deterministic models neglecting the stochastic fluctuations affecting real populations may recommend the implementation of harvesting policies which both overestimate the “true” growth capacity of a population and neglect the potential extinction risk affecting all real populations. In accordance with the modern literature on real investment opportunities, we also found that increased volatility decreases the value of the harvesting opportunity and increases the optimal harvesting threshold by increasing the value of waiting and, therefore, by increasing the incentives to wait and postpone the rational exercise of the harvesting opportunity. Interestingly, we also found that the optimal policy may lead to the introduction of a long run stationary distribution for the harvested population even in such cases where such a distribution would not exist in the absence of harvesting. Thus, as intuitively is clear, the implemented harvesting strategy has a significant impact on the stochastic dynamics of the population density as it both shrinks the state space and alters the (upper) boundary behavior of the population density.

While the model considered in this study generalizes the recent studies on the present expected value of the cumulative yield-maximizing harvesting strategies, it does not take into account two important factors affecting real populations. Namely, it neglects both the structure of a population and the effect of competition on the population dynamics. It is a well-reported phenomenon that regulatory constraints on implemented harvesting strategies may have a strong impact on both the age-, sex-, and size-distribution of a population. Similarly, harvesting pressure affects critically the population dynamics of competing populations. This is especially evident in predator-prey systems, where harvesting may have a pronouncedly distortionary effect on the population dynamics by altering the response of either the predator or the prey to changes in their relative population densities. Unfortunately, such generalizations are out of the scope of the present analysis and, therefore, left for future research.

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