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Optimal intervention for epidemic models with general infection and removal rate functions

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Abstract. In this paper, known results on optimal intervention policies for the general stochastic epidemic model are extended to epidemic models with more general infection and removal rate functions. We consider first policies allowing for the isolation of any number of infectives from the susceptible population at any time, secondly policies allowing for the immunisation of the entire susceptible population at any time, and finally policies allowing for either of these interventions. In each case the costs of infection, isolation and immunisation are assumed to have a particular, rather simple, form. Sufficient conditions are given on the infection and removal rate functions of the model for the optimal policies to take the same simple form as in the case of the general stochastic epidemic model. More general costs are briefly discussed, and some numerical examples given. Finally, we discuss possible directions for further work.

Keywords: Epidemics – Epidemic costs – Optimal intervention – Isolation policies – Immunisation policies – Vaccination policies

1. Introduction

One of the main reasons for studying mathematical models of disease spread is the hope that improved understanding of the transmission mechanism may lead to more effective control strategies. However, in the stochastic epidemic modelling literature surprisingly little attention seems to have been paid to the explicit evaluation of alternative intervention policies. Wickwire (1977) provides a review of applications of mathematical control theory to disease models up to 1976,

and more recent contributions include Lefèvre (1981), Greenhalgh (1986a, b, 1987, 1988) and Cai and Luo (1994).

The control theoretic approach assigns costs to both intervention and infection, the optimal policy being that which minimises their total combined cost. Most simply, the cost of infection may be taken to be proportional to the number of infections which occur. The cost of intervention will depend upon the specific type of intervention under consideration, which may involve controlling epidemic parameters such as the rate at which individuals make contact with each other, or instantaneously changing the states of some individuals (impulse control). In this paper we look at impulse controls, the possible interventions being either to isolate infective individuals from the susceptible population or to immunise susceptibles. The various costs involved are assumed to have quite specific simple forms, more general cost functions being considered briefly in Sect. 5.

The most widely studied continuous time stochastic epidemic model is that known as the general stochastic epidemic (see, for example, Bailey, 1975, chapter 6). We suppose that we have a closed, homogeneously mixing population divided into susceptible, infective and removed individuals, and denote by $X(t)$, $Y(t)$, $Z(t)$ the numbers of susceptible, infective and removed individuals in the population at time t. Since the population is closed, $X(t) + Y(t) + Z(t)$ remains constant for all $t \geqq 0$, so that provided we know the total number of individuals present initially, $X(0) + Y(0) + Z(0)$, the epidemic is completely described by the process $\{(X(t), Y(t)), t \geqq 0\}$, which is assumed to be a continuous-time Markov process with transition probabilities

$$\left.\begin{aligned} &\Pr((X(t+\delta t), Y(t+\delta t)) = (x-1, y+1) | (X(t), Y(t)) = (x, y)) \\ &\quad = \beta x y\, \delta t + o(\delta t), \\ &\Pr((X(t+\delta t), Y(t+\delta t)) = (x, y-1) | (X(t), Y(t)) = (x, y)) \\ &\quad = \gamma y\, \delta t + o(\delta t), \end{aligned}\right\} \tag{1.1}$$

all other transitions having probability $o(\delta t)$, and the parameters $\beta > 0$, $\gamma > 0$ being known as the infection rate and removal rate, respectively. The process terminates when the number of infectives becomes zero, which will almost surely happen within finite time.

One way in which this model can be extended is to allow different infection and removal rate functions than those given by (1.1), replacing the constant parameters β and γ by appropriate functions β_{xy} and γ_{xy}. For instance, Severo (1969) took $\beta_{xy} = \beta x^{-b} y^{a-1}$, $\gamma_{xy} = \gamma y^c$, with β, γ, a, b, c constant. Another possibility would be to take

$\beta_{xy} = \beta/(x + y)^{\alpha}$, $\gamma_{xy} = \gamma$ for constants α, β, γ. Taking $\alpha = 1$ gives the model considered by Gleissner (1988) and Ball and O'Neill (1993), while when $\alpha = 0$ the model reduces to the general stochastic epidemic model. The cases $\alpha = 0, 0.5, 1$ were studied in a discrete-time setting by Saunders (1980a, b). More generally, Saunders (1980a, b) allowed β_{xy} to be any function of $(x + y)$, and the continuous-time version of this model was analysed by O'Neill (1997).

For the general stochastic epidemic model, Abakuks (1972, 1973, 1974) studied intervention policies involving either isolation of infective individuals from the susceptible population, immunisation of susceptible individuals, or both. The cost of an individual becoming infected was regarded as fixed and equal to the unit of cost, relative to which the costs of isolation and immunisation were defined. Dynamic programming was then used to investigate the form of the optimal policy. The aim of this paper is to see how far results such as those of Abakuks (1972, 1973, 1974) can be extended to models with more general infection and removal rate functions, and what conditions on the rate functions are necessary for such results to remain valid. An optimal control problem for a model with general infection rate function, but in which infectives which have recovered do not become immune but instead return to the susceptible state, was investigated by Lefèvre (1981). The form of intervention considered involved control of the infection rate and recovery rate parameters of the process, in contrast to the impulse control with which we concern ourselves.

2. Isolation policies

2.1. Optimality equations and the cost of an uncontrolled epidemic.

Consider the model given by (1.1) with β and γ each allowed to depend upon x and y. We assume that $\beta_{xy} > 0$ for all (x, y) with $x \geqq 1$, $y \geqq 1$, and that $\gamma_{xy} > 0$ for all (x, y) with $x \geqq 0$, $y \geqq 1$. Defining

$$p_{xy} = \frac{\beta_{xy} xy}{\beta_{xy} xy + \gamma_{xy} y}, \qquad q_{xy} = \frac{\gamma_{xy} y}{\beta_{xy} xy + \gamma_{xy} y},$$

then the embedded jump chain of the epidemic process has transitions

$$(X, Y) \to (X - 1, Y + 1) \text{ with probability } p_{XY},$$

$$(X, Y) \to (X, Y - 1) \text{ with probability } q_{XY}.$$

Suppose that the cost of an individual being infected is fixed, and take this as the unit of cost. We consider policies which at any time

allow us to isolate any number of infectives, each at a cost $L > 0$. In fact, if the epidemic is in state (x, y), we need only decide whether to isolate a single infective, or do nothing until the next transition occurs, since isolation of several infectives can be achieved by repeatedly choosing to isolate single infectives. Defining $V_L(x, y)$ to be the expected future cost of adopting an optimal policy when the epidemic is in state (x, y), and $W_L(x, y)$ to be the expected future cost of waiting for one transition to occur naturally and adopting an optimal policy from then onwards, then

$$V_L(x, 0) = 0 \quad \text{for } x \geqq 0, V_L(0, y) = 0 \quad \text{for } y \geqq 0,$$

$$W_L(x, y) = p_{xy}(1 + V_L(x - 1, y + 1)) + q_{xy} V_L(x, y - 1) \quad \text{for } x, y \geqq 1, \tag{2.1}$$

$$V_L(x, y) = \min\{W_L(x, y), L + V_L(x, y - 1)\} \quad \text{for } x, y \geqq 1. \tag{2.2}$$

In any state (x, y) with $x, y \geqq 1$, if $W_L(x, y) < L + V_L(x, y - 1)$ the optimal policy is to do nothing, if $W_L(x, y) > L + V_L(x, y - 1)$ the optimal policy is to isolate an infective, and if $W_L(x, y) = L + V_L(x, y - 1)$ then we can equally well do either. We shall adopt the convention that if $W_L(x, y) = L + V_L(x, y - 1)$, we isolate an infective at (x, y).

For any given region in the (x, y) plane with $x + y \leqq M$, equations (2.1) and (2.2) may be solved recursively by ordering the states $(1, 1)$, $(1, 2), \ldots, (1, M - 1), (2, 1), (2, 2), \ldots, (2, M - 2), \ldots, (M - 1, 1)$.

As well as the cost of an optimal policy, we will also be interested in the cost of an uncontrolled epidemic. If we define $C(x, y)$ to be the expected future cost of an epidemic starting from (x, y) under a policy of never intervening, then $C(x, y)$ is given by

$$C(x, 0) = 0 \quad \text{for } x \geqq 0, C(0, y) = 0 \quad \text{for } y \geqq 0,$$

$$C(x, y) = p_{xy}(1 + C(x - 1, y + 1)) + q_{xy} C(x, y - 1) \quad \text{for } x, y \geqq 1.$$

Since $C(x, y)$ is the expected number of susceptibles ever to become infected, starting with x susceptibles, and there is a positive probability that not all of the susceptibles become infected (in fact, with probability $\prod_{j=1}^{y} q_{xj} > 0$ no susceptible ever becomes infected), then

$$V_L(x, y) \leqq C(x, y) < x.$$

2.2. Structure of the cost function and form of the optimal policy

In order to investigate the form of the optimal policy, we need to look at the structure of the optimal expected cost function $V_L(x, y)$. We first

show that increasing the number of infectives present can never result in a decrease in expected cost under the optimal isolation policy.

Lemma 2.1. *$V_L(x, y+1) \geqq V_L(x, y)$ for $x, y \geqq 0$.*

Proof. For $x = 0$, $y \geqq 0$, we have $V_L(0, y+1) = V_L(0, y) = 0$, so the result holds for $x = 0$. Suppose inductively that for some fixed $x \geqq 1$, $V_L(x-1, y)$ is a non-decreasing function of y. Then $V_L(x, 1) \geqq 0 = V_L(x, 0)$, so now suppose inductively that for some fixed $y \geqq 1$ we have $V_L(x, y) \geqq V_L(x, y-1)$. From (2.1),

$$\begin{aligned} &W_L(x, y+1) - W_L(x, y) \\ &\quad = p_{xy}(V_L(x-1, y+2) - V_L(x-1, y+1)) + q_{xy}(V_L(x, y) - V_L(x, y-1)) \\ &\qquad + (p_{x,y+1} - p_{xy})(1 + V_L(x-1, y+2) - V_L(x, y)) \\ &\quad \geqq (p_{x,y+1} - p_{xy})(1 + V_L(x-1, y+2) - V_L(x, y)), \end{aligned} \tag{2.3}$$

by the inductive hypotheses. But from (2.1) we also have

$$W_L(x, y+1) - V_L(x, y) = p_{x,y+1}(1 + V_L(x-1, y+2) - V_L(x, y)). \tag{2.4}$$

Multiplying (2.3) by $p_{x,y+1}$ and substituting from (2.4) gives

$$\begin{aligned} &p_{x,y+1}(W_L(x, y+1) - W_L(x, y)) \geqq (p_{x,y+1} - p_{xy})(W_L(x, y+1) - V_L(x, y)) \\ &\quad p_{xy}(W_L(x, y+1) - V_L(x, y)) \geqq p_{x,y+1}(W_L(x, y) - V_L(x, y)) \\ &\quad \Rightarrow W_L(x, y+1) - V_L(x, y) \geqq 0, \end{aligned}$$

since from (2.2) we know that $V_L(x, y) \leqq W_L(x, y)$ and we assumed that $p_{xy} > 0$ for $x, y \geqq 1$. Hence

$$V_L(x, y+1) = \min\{W_L(x, y+1), L + V_L(x, y)\} \geqq V_L(x, y), \tag{2.5}$$

and by induction on y and then on x the result follows. □

Using Lemma 2.1 we can now establish the following result.

Theorem 2.2. *Provided $p_{x,y+1} \leqq p_{xy}$ for $x, y \geqq 1$, then for each $x \geqq 0$ there exists an integer $s(x) \geqq 0$ such that the optimal isolation policy is to isolate all infectives if $1 \leqq y \leqq s(x)$, but to isolate none otherwise. Furthermore, for $x \geqq 1$, $s(x) < x/L$.*

Proof. For $x = 0$, take $s(0) = 0$. For $x, y \geqq 1$ we have from (2.1) that

$$\begin{aligned} &W_L(x, y+1) - W_L(x, y) \\ &\quad = p_{x,y+1}(V_L(x-1, y+2) - V_L(x-1, y+1)) \\ &\qquad + q_{x,y+1}(V_L(x, y) - V_L(x, y-1)) \\ &\qquad + (p_{x,y+1} - p_{xy})(1 + V_L(x-1, y+1) - V_L(x, y-1)). \end{aligned} \tag{2.6}$$

From (2.2) and Lemma 2.1, $W_L(x, y) \geqq V_L(x, y) \geqq V_L(x, y-1)$, and substituting into (2.1) gives

$$p_{xy}(1 + V_L(x-1, y+1)) + q_{xy} V_L(x, y-1) \geqq V_L(x, y-1)$$

$$\Rightarrow 1 + V_L(x-1, y+1) - V_L(x, y-1) \geqq 0, \tag{2.7}$$

since $p_{xy} > 0$.

Also, (2.2) implies that $V_L(x-1, y+2) \leqq L + V_L(x-1, y+1)$. Substituting this inequality together with (2.7) into (2.6) (and recalling our assumption that $p_{x,y+1} \leqq p_{xy}$),

$$W_L(x, y+1) - W_L(x, y) \leqq p_{x,y+1} L + q_{x,y+1}(V_L(x, y) - V_L(x, y-1)).$$

Now suppose that at (x, y), the optimal policy does not isolate any infectives. Then $V_L(x, y) = W_L(x, y) < L + V_L(x, y-1)$, so that

$$W_L(x, y+1) - V_L(x, y) < p_{x,y+1} L + q_{x,y+1} L = L,$$

which implies that at $(x, y+1)$ the optimal policy again does not isolate any infectives.

For $x \geqq 1$, define $s(x) = \max\{y\text{: isolate at } (x, y)\}$, and define $s(x) = 0$ if there is no y for which it is optimal to isolate at (x, y). Then we have shown that the optimal isolation policy is to isolate all infectives if $1 \leqq y \leqq s(x)$, at a cost of Ly, but to isolate none otherwise. But $V_L(x, y) < x$, so $V_L(x, s(x)) = Ls(x) < x$, and $s(x) < x/L$, as claimed. □

The condition $p_{x,y+1} \leqq p_{xy}$ means that adding one infective to the population increases the chance that the next transition will be a removal rather than an infection. One would expect that increasing the number of infectives present would increase both the rate at which new infections occur and the rate at which removals occur. However, whereas the infection process depends upon both susceptibles and infectives, the removal process could plausibly be expected to depend only upon the number of infectives present, so it seems reasonable that the addition of an infective has more effect upon the removal rate than upon the infection rate. In the case of the general stochastic epidemic model, $p_{x,y+1} = \beta x/(\beta x + \gamma). = p_{xy}$ for $x, y \geqq 1$, and Theorem 2.2 applies. A fuller discussion of the condition for validity of Theorem 2.2 appears in Section 2.3.

Having demonstrated the existence of the isolation boundary $s(x)$ (under certain conditions on $\{p_{xy}\}$), we would like to know the shape of this boundary. The following property of the expected cost function $V_L(x, y)$ is useful in this regard.

Lemma 2.3. *Provided* $p_{x+1,y} \geqq p_{xy}$ *for* $x, y \geqq 1$ *then* $V_L(x+1, y) \geqq V_L(x, y)$ *for* $x, y \geqq 0$.

Proof. For $x = 0$, $y \geqq 0$, we have $V_L(1, y) \geqq 0 = V_L(0, y)$. Fix $x \geqq 1$ and suppose inductively that $V_L(x, y) \geqq V_L(x-1, y)$ for $y \geqq 0$. For $y = 0$, $V_L(x+1, 0) = V_L(x, 0) = 0$. Fix $y \geqq 1$ and suppose inductively that $V_L(x+1, y-1) \geqq V_L(x, y-1)$. Then from (2.1),

$$\begin{aligned} W_L(x+1, y) - W_L(x, y) = {} & p_{xy}(V_L(x, y+1) - V_L(x-1, y+1)) \\ & + q_{xy}(V_L(x+1, y-1) - V_L(x, y-1)) \\ & + (p_{x+1,y} - p_{xy})(1 + V_L(x, y+1) \\ & - V_L(x+1, y-1)) \\ \geqq {} & (p_{x+1,y} - p_{xy})(1 + V_L(x, y+1) \\ & - V_L(x+1, y-1)) \end{aligned} \tag{2.8}$$

by our inductive hypotheses. Replacing x by $x+1$ in (2.7) and substituting into (2.8), together with the assumption that $p_{x+1,y} \geqq p_{xy}$, gives

$$W_L(x+1, y) - W_L(x, y) \geqq 0.$$

Substituting this into (2.2), together with the inductive hypothesis that $V_L(x+1, y-1) \geqq V_L(x, y-1)$, gives $V_L(x+1, y) \geqq V_L(x, y)$. By induction on y and x, the result follows. □

We can apply Lemma 2.3 to show that under appropriate conditions on $\{p_{xy}\}$ the isolation boundary $s(x)$ is non-decreasing in x.

Theorem 2.4. *Provided that* $p_{x,y+1} \leqq p_{xy} \leqq p_{x+1,y}$ *for* $x, y \geqq 1$, *then* $s(x+1) \geqq s(x)$ *for* $x \geqq 0$.

Proof. At $(x, s(x))$ the optimal policy is to isolate all infectives, so $V_L(x, s(x)) = Ls(x)$. From Lemma 2.3, $V_L(x+1, s(x)) \geqq V_L(x, s(x)) = Ls(x)$. But a policy of isolating all infectives at $(x+1, s(x))$ has cost $Ls(x)$, so that the optimal policy cannot cost more than this, so $V_L(x+1, s(x)) = Ls(x)$ and the optimal policy at $(x+1, s(x))$ is to isolate all infectives. Hence $s(x) \leqq s(x+1)$. □

For $s(x)$ to be non-decreasing in x means that the amount of effort worth putting in to protect any given number of susceptibles x from infection (the cost $Ls(x)$) is at least as great as the effort worth putting in to protect any smaller number of susceptibles.

The condition $p_{xy} \leqq p_{x+1,y}$ means that adding one susceptible to the population increases the chance that the next transition will be an infection rather than a removal. It seems likely that for most realistic models, increasing the number of susceptibles present will increase the rate at which new infections occur while having no effect upon the rate at which infectives are removed, so that the condition will indeed be satisfied. In the case of the general stochastic epidemic model, $p_{xy} < p_{x+1,y}$ for $x, y \geqq 1$ and Theorem 2.4 applies. Further discussion of the condition appears in Sect. 2.3.

For an epidemic starting from a state (x, y) with $y \leqq s(x)$, the optimal isolation policy is to immediately terminate the epidemic by isolation of all infectives. If an epidemic starts with $y > s(x)$, then we allow the epidemic to proceed uninterrupted until $Y_t \leqq s(X_t)$ for the first time and then terminate the process by isolating all infectives. If $s(\cdot)$ is non-decreasing then it follows that $Y_t = s(X_t)$ when the process first crosses the isolation boundary.

One final property of the expected cost function $V_L(x, y)$ which we will need later is the following.

Lemma 2.5. *Provided that* $\sum_{y=1}^{\infty} p_{xy} = \infty$ for all $x \geqq 1$, then for each $x \geqq 0$ we have $V_L(x, y) \to x$ as $y \to \infty$.

Proof. First note that as $V_L(x, y) < x$ and $V_L(x, y+1) \geqq V_L(x, y)$ by Lemma 2.1, $\lim_{y\to\infty} V_L(x, y)$ certainly exists.

Since $V_L(0, y) = 0$ for $y \geqq 0$, the result holds for $x = 0$. Fix $x \geqq 1$, and suppose inductively that $V_L(x-1, y) \to x-1$ as $y \to \infty$.

If $V_L(x, y) = L + V_L(x, y-1)$ for infinitely many y values, then $V_L(x, y) \to \infty$ as $y \to \infty$, but since $V_L(x, y) < x$ for all y, this cannot happen. Recalling (2.2), it follows that there exists y_0 with $V_L(x, y) = W_L(x, y)$ for all $y \geqq y_0$. Thus from (2.1), for $y \geqq y_0$,

$$V_L(x, y) = p_{xy}(1 + V_L(x-1, y+1)) + q_{xy} V_L(x, y-1). \tag{2.9}$$

We have assumed inductively that $V_L(x-1, y) \to x-1$ as $y \to \infty$, so for any $\varepsilon > 0$ there exists $y_1 \geqq y_0$ such that for $y \geqq y_1$, $V_L(x-1, y) \geqq x - 1 - \varepsilon$. Thus for $y \geqq y_1 - 1$, (2.9) implies

$$V_L(x, y) \geqq p_{xy}(x - \varepsilon) + q_{xy} V_L(x, y-1).$$

Iterating this inequality, we have that for $y \geqq y_1$,

$$\begin{aligned} V_L(x, y) &\geqq p_{xy}(x-\varepsilon) + q_{xy}(p_{x,y-1}(x-\varepsilon) + q_{x,y-1} V_L(x, y-2)) \\ &= (p_{xy} + q_{xy} p_{x,y-1})(x-\varepsilon) + q_{xy} q_{x,y-1} V_L(x, y-2). \end{aligned}$$

Repeated iteration now gives

$$V_L(x, y) \geqq (p_{xy} + q_{xy}p_{x,y-1} + \cdots + q_{xy}q_{x,y-1}\cdots q_{x,y_1+1}p_{xy_1})(x-\varepsilon) + \left(\prod_{j=y_1}^{y} q_{xy}\right)V_L(x, y_1 - 1)$$

$$= \left(1 - \prod_{j=y_1}^{y} q_{xy}\right)(x-\varepsilon) + \left(\prod_{j=y_1}^{y} q_{xy}\right)V_L(x, y_1 - 1). \tag{2.10}$$

Letting y tend to infinity in (2.10), and noting that the condition $\sum_{y=1}^{\infty} p_{xy} = \infty$ is equivalent to $\prod_{y=1}^{\infty} q_{xy} = 0$ (see Williams, 1991, p. 40), we have $\lim_{y\to\infty} V_L(x, y) \geqq x - \varepsilon$. But $V_L(x, y) < x$, so $\lim_{y\to\infty} V_L(x, y) = x$, and by induction on x the result follows. □

A rather simpler proof of Lemma 2.5, along the lines of Lemma 2.1 of Abakuks (1973), is possible if we assume that for all $x \geqq 1$, $\lim_{y\to\infty} p_{xy}$ exists and is non-zero. However, for some of the models which we consider, for instance when $\beta_{xy} = \beta/(x + y)$. and $\gamma_{xy} = \gamma$, although $\lim_{y\to\infty} p_{xy} = 0$ our condition on $\{p_{xy}\}$ is satisfied.

2.3. Discussion of the conditions

In order that Theorems 2.2 and 2.4 apply to a particular epidemic model, we require $p_{x,y+1} \leqq p_{xy} \leqq p_{x+1,y}$ for $x, y \geqq 1$. In terms of the transition rate functions β_{xy}, γ_{xy} these conditions become

$$\frac{\beta_{x,y+1}}{\gamma_{x,y+1}} \leqq \frac{\beta_{xy}}{\gamma_{xy}} \leqq \frac{(x+1)\beta_{x+1,y}}{x\gamma_{xy}}.$$

In the case of Lemma 2.5, the condition $\sum_{y=1}^{\infty} p_{xy} = \infty$ is equivalent to $\sum_{y=1}^{\infty}(\beta_{xy}/\gamma_{xy}.) = \infty$.

For the model of Severo (1969), with $\beta_{xy} = \beta x^{-b} y^{a-1}$, $\gamma_{xy} = \gamma y^c$, then it is straightforward to show that Theorem 2.2 applies if $a \leqq c + 1$. When $c = 0$, so that the rate at which removals occur is simply proportional to the number of infectives present, we thus require $a \leqq 1$, meaning that the rate at which infections occur increases no faster than linearly with the number of infectives present. If $a > 1$, it may not be true that each extra infective added to the population causes less of an increase in risk to the susceptibles than the previous infective. Even if at (x, y) the isolation of an infective is not worth the cost L, it may nevertheless be optimal at $(x, y + 1)$ to isolate an infective, and so Theorem 2.2 does not apply. For Theorem 2.4 we

additionally require $b \leqq 1$. This means that the rate at which infections occur does not actually decrease as the number of susceptibles present increases, which certainly seems a natural condition. Provided this is the case, Theorem 2.4 tells us that the number of infectives which it is worthwhile to remove in order to protect the susceptibles does not decrease as the number of susceptibles present increases. The condition for Lemma 2.5 to hold for this model is that $a \geqq c$.

For the model with $\beta_{xy} = \beta/(x + y)^{\alpha}$, $\gamma_{xy} = \gamma$, Theorem 2.2 holds provided that $\alpha \geqq 0$. That is, the rate at which infections occur increases no faster than linearly with the numbers of infectives and susceptibles present. Theorem 2.4 further requires $\alpha \leqq 1$, so that the rate at which infections occur does not decrease as the numbers of susceptibles or infectives present increase. For Lemma 2.5 we require $\alpha \leqq 1$. Provided $0 \leqq \alpha \leqq 1$, Theorems 2.2 and 2.4 and Lemma 2.5 all apply. The extreme cases $\alpha = 0$ and $\alpha = 1$ give, respectively, the general stochastic epidemic model and the model of Gleissner (1988) and Ball and O'Neill (1993), with $\alpha = 0.5$ giving the model of Saunders (1980a, b).

More generally, consider the model of O'Neill (1997), in which individuals change their behaviour in response to perceived epidemic spread (measured by the number of removals to have occurred), so that $\beta_{xy} = \beta_{x+y}$, $\gamma_{xy} = \gamma$. The condition for Theorem 2.2 now becomes simply $\beta_{x+y+1} \leqq \beta_{x+y}$ for $x, y \geqq 1$. For a model with basic infection rate βxy, as the epidemic spreads and $(x + y)$ decreases we would expect β to decrease also, reflecting increased awareness of risk, so that β_{x+y} would be a non-decreasing function of $(x + y)$. In this case, Theorem 2.2 is not applicable (unless β_{x+y} is constant). However, if the basic infection rate is some other function such as $\beta xy/(x + y)$, with the parameter β decreasing in response to disease spread, then Theorem 2.2 may apply. For Theorem 2.4, we additionally require $\beta_{x+y} \leqq (x + y)\beta_{x+y+1}/(x + y - 1)$ for $x, y \geqq 1$.

Although the conditions for Theorems 2.2 and 2.4 may not be the least restrictive possible, it can be seen that they are not particularly restrictive. That conditions of some sort are necessary is shown by numerical examples in Sect. 6. The condition for Lemma 2.5 seems least restrictive of our conditions, and in fact is a necessary as well as a sufficient condition. To see this, recall first that $V_L(x, y) \leqq C(x, y)$. Now $C(x, y)$ is the expected number of initial susceptibles ever to become infected, and with probability $\prod_{j=1}^{y} q_{xj}$ no susceptible ever becomes infected, so that $C(x, y) \leqq x(1 - \prod_{j=1}^{y} q_{xj})$. Hence if $\prod_{j=1}^{\infty} q_{xj} > 0$ then $\lim_{y \to \infty} C(x, y) < x$.

2.4. An alternative method for computing the isolation boundary

Having shown that if $p_{x,y+1} \leqq p_{xy} \leqq p_{x+1,y}$ for $x, y \geqq 1$ then the optimal isolation policy is in fact an optimal stopping rule, and that the isolation boundary $s(\cdot)$ is non-decreasing, then provided our epidemic model satisfies a certain technical condition (2.11) below we can apply a result of Clancy (1999) to give an alternative way of computing $s(x)$. From Theorem 2.2 of Clancy (1999), we have the following.

Theorem 2.6. *Suppose there exist functions* $P(\cdot), Q(\cdot) > 0$ *such that*

$$p_{xy}P(x) = q_{xy}Q(x+y) \quad \text{for } x, y \geqq 1, \tag{2.11}$$

with $P(1), P(2), \ldots$ *all distinct.*

For $r(\cdot) \geqq 0$ *some non-decreasing function, define the stopping time* T *by*

$$T = \inf\{t \geqq 0 : Y_t \leqq r(X_t)\}. \tag{2.12}$$

Then for an epidemic starting from state (x, y) *with* $y \geqq r(x)$, *for any function* $l(\cdot)$,

$$\mathbf{E}[l(X_T)|(x,y)] = \sum_{n=0}^{x} \left\{ \tilde{G}_n \prod_{i=r(n)+1}^{x+y-n} q_{ni} \Big/ \prod_{i=1}^{x-n} \left(1 - \frac{P(n+i)}{P(n)}\right) \right\}$$

where $\tilde{G}_0, \tilde{G}_1, \ldots$ *are defined recursively by the triangular system of linear equations*

$$\sum_{n=0}^{m} \left\{ \tilde{G}_n \prod_{i=r(n)+1}^{m+r(m)-n} q_{ni} \Big/ \prod_{i=1}^{m-n} \left(1 - \frac{P(n+i)}{P(n)}\right) \right\} = l(m) \quad (m = 0, 1, \ldots).$$

For the general stochastic epidemic, (2.11) may be satisfied by taking $P(x) = 1/\beta x$, $Q(x+y) = 1/\gamma$. When $\beta_{xy} = \beta x^{-b} y^{a-1}$, $\gamma_{xy} = \gamma y^c$, then provided $a = c + 1$ and $b \neq 1$ we can take $P(x) = x^{b-1}$, $Q(x+y) = \beta/\gamma$. For $\beta_{xy} = \beta/(x+y)^\alpha$, $\gamma_{xy} = \gamma$, take $P(x) = \gamma/\beta x$, $Q(x+y) = 1/(x+y)^\alpha$.

If $p_{x,y+1} \leqq p_{xy} \leqq p_{x+1,y}$ for $x, y \geqq 1$ then our optimal isolation policy is to stop the epidemic by isolating all infectives at the time T defined by (2.12) with $s(\cdot)$ in place of $r(\cdot)$. Since $s(\cdot)$ is non-decreasing, then provided $y \geqq s(x)$ initially we will have $Y_T = s(X_T)$, and the total cost of the epidemic will be $x - X_T + LY_T = x - X_T + Ls(X_T)$. Thus the expected total cost is given by $V_L(x, y) = x - \mathbf{E}[l(X_T)|(x, y)]$ with $l(m) = m - Ls(m)$. Once we know the values of $s(1), s(2), \ldots, s(x)$ we can use Theorem 2.6 to compute the value of $V_L(x, y)$, or indeed the expectation of any function of the cost. Furthermore, since the distribution of X_T is discrete, the total cost $x - X_T + Ls(X_T)$ is also a discrete random variable. So long as (2.11)

holds, the probability mass function of X_T, and hence of total cost, may be computed using equations (2.5) of Clancy (1999), a triangular system of linear equations in the final state probabilities $\Pr(X_T = \omega)$, $\omega = 0, 1, 2, \ldots, x$.

Theorem 2.6 can also be used to determine the values of $s(1)$, $s(2), \ldots$ recursively. The same argument used by Abakuks (1973) for the general stochastic epidemic model shows that if we take $\tilde{G}_0 = s(0) = 0$ then we can find $s(1), \tilde{G}_1, s(2), \tilde{G}_2, \ldots$ by alternately applying the two formulae

$$s(x) = \min\left\{ y \geqq s(x-1) : p_{x,y+1}(x - Ly) - \sum_{n=1}^{x-1} \left\{ (q_{n,x+y+1-n} - q_{x,y+1}) \tilde{G}_n \prod_{i=s(n)+1}^{x+y-n} q_{ni} \Big/ \prod_{i=1}^{x-n} \left(1 - \frac{P(n+i)}{P(n)} \right) \right\} < L \right\},$$

$$\tilde{G}_x = x - Ls(x) - \sum_{n=1}^{x-1} \left\{ \tilde{G}_n \prod_{i=s(n)+1}^{x+s(x)-n} q_{ni} \Big/ \prod_{i=1}^{x-n} \left(1 - \frac{P(n+i)}{P(n)} \right) \right\}.$$

Whether for computation of the isolation boundary or analysis of the distribution of cost, the formulae above are chiefly of theoretical interest, since for numerical computation they are no better than direct application of equations such as (2.1) and (2.2).

3. Total immunisation policies

A complementary problem to that of Sect. 2 is to consider policies which at any time allow us to immunise any number of susceptibles. For the general stochastic epidemic model, this problem was investigated by Abakuks (1972). Equations corresponding to (2.1) and (2.2) can easily be written down, and solved numerically for any specific parameter values. However, the optimal policies found in this way have a rather more complicated form than the optimal isolation policy, and thus are harder to treat analytically. Furthermore, the optimal policy may in some cases involve periods of non-intervention separated by immunisations of single susceptibles, which is unlikely to be a practical policy to implement. So for reasons of both mathematical tractibility and practical implementation, we follow Abakuks (1974) in restricting attention to total immunisation policies, which at any time allow us to either immunise all of the susceptibles present, thereby terminating the epidemic, or do nothing.

Suppose that the cost of immunising x susceptibles is $A + Kx$, where $A, K \geqq 0$ and $A + K > 0$. We may think of A as the initial cost

of setting up an immunisation programme, and K the additional cost per immunisation. Defining $V_{A,K}(x, y)$ to be the expected future cost of adopting an optimal policy when the epidemic is in state (x, y), and $W_{A,K}(x, y)$ to be the expected future cost of waiting for one transition to occur and adopting an optimal policy from then onwards, then corresponding to equations (2.1) and (2.2) we have

$$V_{A,K}(x, 0) = 0 \quad \text{for } x \geqq 0, \qquad V_{A,K}(0, y) = 0 \quad \text{for } y \geqq 0,$$

$$W_{A,K}(x, y) = p_{xy}(1 + V_{A,K}(x - 1, y + 1)) + q_{xy} V_{A,K}(x, y - 1) \quad \text{for } x, y \geqq 1, \tag{3.1}$$

$$V_{A,K}(x, y) = \min\{W_{A,K}(x, y), A + Kx\} \quad \text{for } x, y \geqq 1. \tag{3.2}$$

In any state (x, y) with $x, y \geqq 1$, if $W_{A,K}(x, y) < A + Kx$ the optimal policy is to do nothing, if $W_{A,K}(x, y) > A + Kx$ the optimal policy is to immunise all of the x susceptibles present, and if $W_{A,K}(x, y) = A + Kx$ then we can equally well do either. We shall adopt the convention that if $W_{A,K}(x, y) = A + Kx$, we immunise the susceptibles at (x, y).

Exactly as for isolation policies, we have $V_{A,K}(x, y) \leqq C(x, y) < x$ for $x \geqq 1$, so if $K \geqq 1$ then $V_{A,K}(x, y) < A + Kx$ for $x \geqq 1$ and the optimal policy never immunises. We shall assume from now on that $K < 1$.

Corresponding to Lemma 2.1, we have the following.

Lemma 3.1. $V_{A,K}(x, y + 1) \geqq V_{A,K}(x, y)$ for $x, y \geqq 0$.

Proof. Identical to the proof of Lemma 2.1, except that now (2.5) becomes

$$V_{A,K}(x, y + 1) = \min\{W_{A,K}(x, y + 1), A + Kx\} \geqq V_{A,K}(x, y). \quad \square$$

Corresponding to Theorem 2.2, we can now demonstrate the existence of an immunisation boundary as follows.

Theorem 3.2. *For each $x \geqq 0$ there exists an integer $t(x)$, $0 \leqq t(x) \leqq \infty$, such that the optimal policy is to immunise the susceptibles if $y > t(x)$, but not otherwise.*

Proof. For $x = 0$, take $t(0) = \infty$. Suppose that at (x, y) it is optimal to immunise $(x \geqq 1)$, so $V_{A,K}(x, y) = A + Kx$. Then by Lemma 3.1, $V_{A,K}(x, y + 1) \geqq V_{A,K}(x, y) = A + Kx$. But from (3.2), $V_{A,K}(x, y + 1) \leqq A + Kx$, so $V_{A,K}(x, y + 1) = A + Kx$ and at $(x, y + 1)$ it is again optimal to immunise. The result follows. $\square$

As for isolation policies, we would like to know the shape of the immunisation boundary, and to this end we have the following.

Theorem 3.3. *Suppose* $\sum_{y=1}^{\infty} p_{xy} = \infty$ *for* $x \geqq 1$. *Defining*

$$R = \min\{x \in \mathbb{Z} : x > A/(1-K)\}, \tag{3.3}$$

then $t(x) = \infty$ *for* $0 \leqq x < R$, *but* $t(x) < \infty$ *for* $x \geqq R$.

Proof. For $x = 0$, we know that $t(0) = \infty$.

For $1 \leqq x < R$, $y \geqq 0$, we have $V_{A,K}(x, y) < x \leqq A + Kx$, so that at (x, y) the optimal policy is not to immunise. That is, $t(x) = \infty$ for $1 \leqq x < R$.

For $x = R$, $y \leqq t(R)$, the optimal policy is not to immunise, and since it is also optimal not to immunise for $x < R$, it is not possible for an epidemic starting from one of these states to reach a state where it is optimal to immunise. Hence $V_{A,K}(R, y) = C(R, y)$ for $y \leqq t(R)$. But under the assumption that $\sum_{y=1}^{\infty} p_{xy} = \infty$ for $x \geqq 1$, a slightly simpler form of the proof of Lemma 2.5 shows that $\lim_{y\to\infty} C(x, y) = x$ for $x \geqq 0$, so that in particular $\lim_{y\to\infty} C(R, y) = R > A + KR$. On the other hand, $V_{A,K}(R, y) \leqq A + KR$, so it cannot be true that $V_{A,K}(R, y) = C(R, y)$ for all $y \geqq 1$, and we must have $t(R) < \infty$.

Now fix $x > R$ and suppose that $t(x-1) < \infty$, but that $t(x) = \infty$. For $y \geqq t(x-1)$,

$$\begin{aligned} V_{A,K}(x, y) &= W_{A,K}(x, y) \\ &= p_{xy}(1 + V_{A,K}(x-1, y+1)) + q_{xy} V_{A,K}(x, y-1) \\ &= p_{xy}(1 + A + K(x-1)) + q_{xy} V_{A,K}(x, y-1). \end{aligned}$$

Repeated iteration of this relationship, as in the proof of Lemma 2.5, gives

$$\begin{aligned} V_{A,K}(x, y) &= \left(1 - \prod_{j=t(x-1)+1}^{y} q_{xj}\right)(A + Kx + 1 - K) \\ &\quad + \left(\prod_{j=t(x-1)+1}^{y} q_{xy}\right) V_{A,K}(x, t(x-1)) \\ &\to A + Kx + 1 - K \quad \text{as } y \to \infty, \end{aligned}$$

since we have assumed $\sum_{y=1}^{\infty} p_{xy} = \infty$, so that $\prod_{y=1}^{\infty} q_{xy} = 0$.

But $A + Kx + 1 - K > A + Kx$, and $V_{A,K}(x, y) \leqq A + Kx$, so we have a contradiction. Thus if $t(x-1) < \infty$ we must have $t(x) < \infty$, and the result follows. □

In the case of total immunisation policies for the general stochastic epidemic model Abakuks (1974) conjectured that the immunisation boundary $t(x)$ is non-increasing in x, but was only able to prove this for certain parameter values. For our more general model it need not be

the case that $t(x)$ be non-increasing (see Sect. 6 for a numerical example).

4. Isolation or total immunisation policies

Having considered isolation policies and total immunisation policies separately, suppose now that both options are available. That is, at any time we may isolate any number of infectives, each at cost L, immunise the entire susceptible population, at cost $A + Kx$, or do nothing. We assume as before that $L > 0$, $A \geqq 0$, $0 \leqq K < 1$, and $A + K > 0$. Writing $V_{L,A,K}(x, y)$ for the expected future cost of adopting an optimal policy starting from state (x, y) and $W_{L,A,K}$ for the expected future cost of waiting for one transition to occur and adopting an optimal policy from then onwards, then

$$V_{L,A,K}(x, 0) = 0 \quad \text{for } x \geqq 0,\ V_{L,A,K}(0, y) = 0 \quad \text{for } y \geqq 0,$$

$$W_{L,A,K}(x, y) = p_{xy}(1 + V_{L,A,K}(x - 1, y + 1)) + q_{xy} V_{L,A,K}(x, y - 1) \quad \text{for } x, y \geqq 1,$$

$$V_{L,A,K}(x, y) = \min\{W_{L,A,K}(x, y), L + V_{L,A,K}(x, y - 1), A + Kx\} \quad \text{for } x, y \geqq 1.$$

In any state (x, y), if $W_{L,A,K}(x, y) < \min\{L + V_{L,A,K}(x, y-1), A + Kx\}$ we do nothing, if $L + V_{L,A,K}(x, y) < A + Kx$ and $L + V_{L,A,K}(x, y) \leqq W_{L,A,K}(x, y)$ we isolate an infective, and if $A + Kx \leqq \min\{W_{L,A,K}(x, y), L + V_{L,A,K}(x, y - 1)\}$ we immunise all of the susceptibles.

Properties of the optimal policy can be established using similar arguments to those of Sects. 2 and 3, and are collected together as follows.

Theorem 4.1.

(i) $V_{L,A,K}(x, y + 1) \geqq V_{L,A,K}(x, y)$ *for* $x, y \geqq 0$.

(ii) *For each* $x \geqq 0$ *there exists an integer* $T(x)$, $t(x) \leqq T(x) \leqq \infty$, *such that the optimal policy immunises the susceptibles if* $y > T(x)$, *but not otherwise.*

(iii) *With* R *defined by* (3.3), *then provided* $\sum_{y=1}^{\infty} p_{xy} = \infty$ *for* $x \geqq 1$ *we have* $T(x) = \infty$ for $0 \leqq x < R$, *but* $T(x) < \infty$ *for* $x \geqq R$.

(iv) *Provided* $p_{x,y+1} \leqq p_{xy}$ *for* $x, y \geqq 1$, *then for each* $x \geqq 0$ *there exists an integer* $S(x)$, $0 \leqq S(x) \leqq \min\{s(x), T(x)\}$, *such that the optimal policy isolates all infectives if* $1 \leqq y \leqq S(x)$, *but isolates none otherwise.*

(v) *For* $x \geqq 1$, *defining* $\phi(x) = \max\{y \in \mathbb{Z} : y < (A + Kx)/L\}$, *then* $S(x) \leqq \phi(x) \leqq T(x)$.

(vi) *Provided* $p_{x+1,y} \geqq p_{xy}$ *for* $x, y \geqq 1$ *then* $V_{L,A,K}(x+1, y) \geqq V_{L,A,K}(x, y)$ *for* $x, y \geqq 0$.

(vii) *Provided* $p_{x,y+1} \leqq p_{xy} \leqq p_{x+1,y}$ *for* $x, y \geqq 1$, *then* $S(x+1) \geqq S(x)$ *for* $x \geqq 0$.

Proof.

(i) As Lemmas 2.1 and 3.1.

(ii) The proof that $T(x)$ exists is as for Theorem 3.2. To see that $T(x) \geqq t(x)$, note that the class of policies now under consideration includes the total immunisation policies of Sect. 3, so that $V_{L,A,K}(x, y) \leqq V_{A,K}(x, y)$ for $x, y \geqq 0$. If $T(x) = \infty$ then $T(x) \geqq t(x)$, so suppose now $T(x) < \infty$ for some $x \geqq 1$. Since $V_{A,K}(x, T(x)+1) \geqq V_{L,A,K}(x, T(x)+1) = A + Kx$, then $V_{A,K}(x, T(x)+1) = A + Kx$, the optimal total immunisation policy at $(x, T(x)+1)$ is to immunise, and $t(x) \leqq T(x)$.

(iii) The proof that $T(x) = \infty$ for $0 \leqq x < R$ and $T(R) < \infty$ is as for Theorem 3.3, but with $V_L(x, y)$ in place of $C(x, y)$. For $x > R$, observe that since $V_{L,A,K}(x, y) < x$ we can have $V_{L,A,K}(x, y) = L + V_{L,A,K}(x, y-1)$ for at most finitely many y values, so that if $T(x) = \infty$ then $V_{L,A,K}(x, y) = W_{L,A,K}(x, y)$ for all sufficiently large y. Exactly as in the proof of Theorem 3.3, a contradiction follows, so that $T(x) < \infty$ for $x > R$.

(iv) From part (iii) we know that the optimal policy immunises the susceptibles for $y > T(x)$, but not for $y \leqq T(x)$. For $y < T(x)$ we proceed as in the proof of Theorem 2.2 to show that if the optimal policy does not isolate any infectives at (x, y), then it does not isolate at $(x, y+1)$, and the existence of $S(x)$ follows. Since the optimal policy for $y > T(x)$ is to immunise and not to isolate then $S(x) \leqq T(x)$. To show that $S(x) \leqq s(x)$, note that the class of policies now under consideration includes the isolation policies of Sect. 2, so that $V_{L,A,K}(x, y) \leqq V_L(x, y)$ for $x, y \geqq 0$. For $x \geqq 0$, since $V_L(x, S(x)) \geqq V_{L,A,K}(x, S(x)) = LS(x)$, then $V_{A,K}(x, S(x)) = LS(x)$, the optimal isolation policy at $(x, S(x))$ is to isolate all infectives, and $s(x) \geqq S(x)$.

(v) Fix $x \geqq 1$. For $y \leqq \phi(x)$ we have $Ly < A + Kx$, so that it is cheaper to isolate all infectives than to immunise the susceptibles, the optimal policy certainly does not immunise, and so $T(x) \geqq \phi(x)$. For $y > \phi(x)$, on the other hand, to immunise is no more expensive than to isolate all infectives, and the optimal policy cannot be to isolate, so $S(x) \leqq \phi(x)$.

(vi) As Lemma 2.3.

(vii) As in the proof of Theorem 2.4 we have $V_{L,A,K}(x+1, S(x)) = LS(x)$ so that the optimal policy at $(x+1, S(x))$ is either to isolate all infectives or to immunise all susceptibles. Since $S(x) \leqq \phi(x) \leqq \phi(x+1) \leqq T(x+1)$, the optimal policy does not immunise at $(x+1, S(x))$, so it must isolate, hence $S(x+1) \geqq S(x)$. □

Provided $p_{x,y+1} \leqq p_{xy}$ for $x, y \geqq 1$, then the (x, y) plane is divided into three regions by the boundaries $S(x)$, $T(x)$, where $0 \leqq S(x) \leqq T(x) \leqq \infty$. For $y \leqq S(x)$, the optimal policy is to isolate all infectives. For $S(x) < y \leqq T(x)$, we allow the epidemic to proceed uninterrupted. For $y > T(x)$, we immunise the entire susceptible population.

In the case of the general stochastic epidemic Abakuks (1974) was able to show that $S(x) = T(x) = \phi(x)$ for all sufficiently large x. For our more general model it seems that this need no longer be the case. An example in which $T(x) - S(x)$ increases with x is given in Sect. 6.

5. More general costs

The particular forms of costs which we have considered so far, while reasonably plausible, are motivated as much by mathematical convenience as realism. Both the cost of disease and the cost of intervention could well take some more complicated form. For more complicated cost functions, determining the structure of the optimal policy will in general be far harder, but a few results along the lines of Sects. 2, 3 and 4 can be obtained, as outlined below.

First of all, consider the cost of infection. In general, the problem of finding an optimal intervention policy for more complicated infection costs will be far less tractable than the simple case of unit cost per infection. However, in the particular case when an individual with infectious period of length I generates cost $u + vI$ for constants $u, v \geqq 0$ with $u + v > 0$, these individual costs being summed to give the overall infection cost, then the problem can be reduced to that of unit cost per infection provided $\gamma_{xy} = \gamma$, constant, for $x \geqq 0$, $y \geqq 1$. The condition that γ be constant means that whenever an individual becomes infected it remains so for a random time which is Exponentially distributed with mean $1/\gamma$, independent of the behaviour of the rest of the population. The argument is the same as that given for the general stochastic epidemic in Abakuks (1973).

Turning to intervention costs, if we consider only policies which at any time allow us to either terminate the epidemic or do nothing, then in general the cost of terminating the epidemic when in state (x, y) will

be some function $k(x, y) \geqq 0$. Denoting by $V_{k(\cdot,\cdot)}(x, y)$ the expected future cost of an optimal policy starting from state (x, y) then we can immediately write down optimality equations corresponding to (3.1) and (3.2). Provided that $k(x, y+1) \geqq k(x, y)$ for $x, y \geqq 1$ then exactly as for Lemma 3.1 we can show that $V_{k(\cdot,\cdot)}(x, y)$ is non-decreasing in y. If $p_{x+1,y} \geqq p_{xy}$ for $x, y \geqq 1$ and $k(x+1, y) \geqq k(x, y)$ for $x, y \geqq 1$ then $V_{k(\cdot,\cdot)}(x, y)$ is also non-decreasing in x, the proof being as for Lemma 2.3.

In the case of a total immunisation policy with cost $k(x)$ then Theorem 3.2 still applies, the proof being as before. That is, an immunisation boundary $t(x)$ exists such that the optimal policy is to terminate the epidemic by immunisation if and only if $y > t(x)$, where $0 \leqq t(x) \leqq \infty$ for $x \geqq 0$. Furthermore, defining $R = \min\{x \in \mathbb{Z}: x > k(x)\}$ (with $R = \infty$ if $x \leqq k(x)$ for all x) then provided $k(x+1) < k(x) + 1$ for $x \geqq 0$ (so that $x > k(x)$ for all $x \geqq R$) and $\sum_{y=1}^{\infty} p_{xy} = \infty$ for $x \geqq 1$ we have $t(x) = \infty$ for $1 \leqq x < R$ but $t(x) < \infty$ for $x \geqq R$ (proof as Theorem 3.3).

6. Numerical examples

Figure 1 shows some typical optimal policies when both isolation of infectives and immunisation of the entire susceptible population are allowed (costs being as in Sect. 4). In each case the conditions for Theorems 2.2 and 2.4 (or parts (iv) and (vii) of Theorem 4.1) are satisfied, so the isolation boundary $S(x)$ exists and is non-decreasing in x. The upper, dark grey, area shows those states (x, y) where it is optimal to immunise $(y > T(x))$, the lower, paler grey area shows where it is optimal to isolate $(1 \leqq y \leqq S(x))$, and the white area shows where the optimal policy is to do nothing $(S(x) < y \leqq T(x))$. In the cases of Fig. 1(i) (the general stochastic epidemic) and 1(ii) ($\beta_{xy} = \beta/\sqrt{x+y}$, $\gamma_{xy} = \gamma$), then if the only intervention allowed is total immunisation, the immunisation boundary $t(x)$ is non-increasing in the region of the (x, y) plane shown, as Abakuks (1974) conjectured would always be the case for the general stochastic epidemic. For the general stochastic epidemic Abakuks (1974) has shown that $T(x) = S(x)$ for sufficiently large x. In Fig. 1(ii), the boundaries $T(x)$ and $S(x)$ do seem to come together as x increases, but even computing the optimal policy as far as $x, y \leqq 250$ we still find $T(x) - S(x) = 1$, though for such large x, y values numerical errors may affect our results. In Fig. 1(iii) ($\beta_{xy} = \beta/(x+y)$, $\gamma_{xy} = \gamma$) and in Fig. 1(iv) ($\beta_{xy} = \beta/\sqrt{xy}$, $\gamma_{xy} = \gamma$) the immunisation boundary $T(x)$ is increasing, and in fact if the only

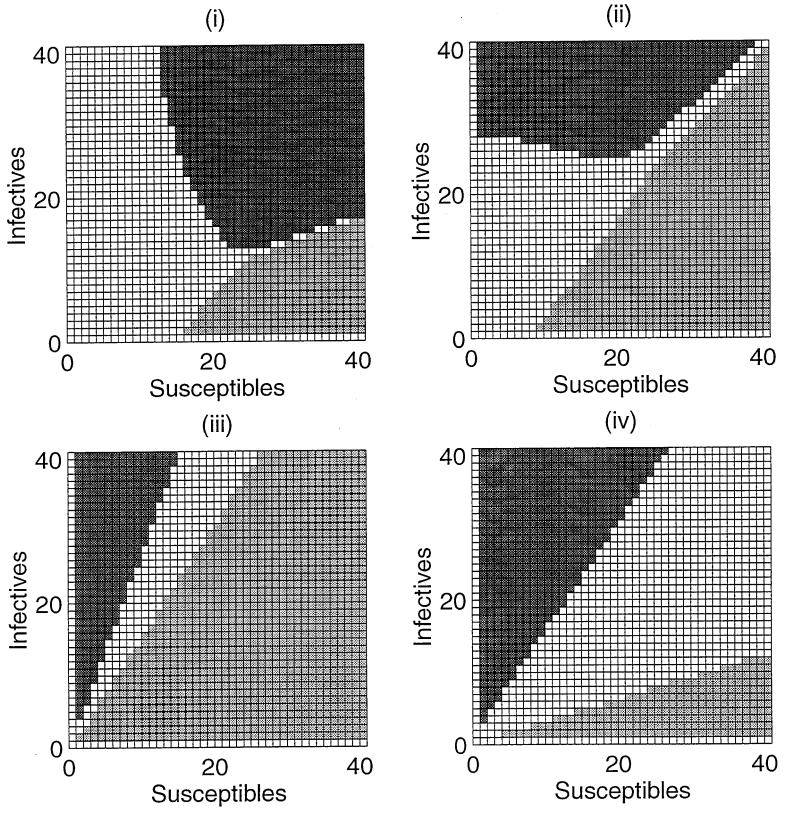

Fig. 1. Optimal isolation-or-total-immunisation policies. Dark grey area indicates states (x, y) where it is optimal to immunise the susceptible population, pale grey area where it is optimal to isolate an infective, white area where it is optimal not to intervene. Costs as in Section 4.

(i) $\beta_{xy} = \beta, \gamma_{xy} = \gamma$, with $\gamma/\beta = 20, A = 5, K = 0.5, L = 1.5$.

(ii) $\beta_{xy} = \beta/\sqrt{x + y}, \gamma_{xy} = \gamma$, with $\gamma/\beta = 5, A = 0, K = 0.8, L = 0.8$.

(iii) $\beta_{xy} = \beta/(x + y), \gamma_{xy} = \gamma$, with $\gamma/\beta = 20, A = 0, K = 0.05, L = 0.02$.

(iv) $\beta_{xy} = \beta/\sqrt{xy}, \gamma_{xy} = \gamma$, with $\gamma/\beta = 20, A = 0, K = 0.1, L = 0.1$.

intervention allowed is total immunisation the boundary $t(x)$ is also increasing. It appears that now the two boundaries $S(x)$ and $T(x)$ move apart as x increases rather than coming together, that is, $T(x) - S(x)$ increases with x.

Figure 2 shows that, while the conditions for Theorems 2.2 and 2.4 may not be the least restrictive possible, some conditions are necessary. The graphs show computed isolation policies in the case when isolation of infectives is the only intervention permitted, with costs as in Sect. 2. The grey area indicates those states (x, y) where it is optimal to

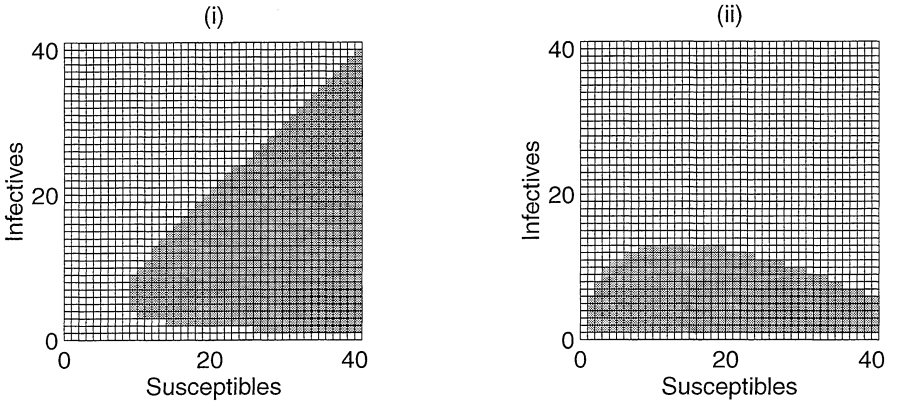

Fig. 2. Optimal isolation policies. Grey area indicates states (x, y) where it is optimal to isolate an infective, white area where it is optimal not to intervene. Costs as in Section 2.
(i) $\beta_{xy} = \beta y$, $\gamma_{xy} = \gamma$, with $\gamma/\beta = 50$, $L = 1$.
(ii) $\beta_{xy} = \beta/(x + y)^2$, $\gamma_{xy} = \gamma$, with $\gamma/\beta = 1$, $L = 0.02$.

isolate an infective, the white area those states where it is optimal not to intervene. In the case of Fig. 2(i) ($\beta_{xy} = \beta y$, $\gamma_{xy} = \gamma$), Theorem 2.2 does not apply, and there exist states (x, y) where it is optimal to isolate some but not all of the infectives present. Fig. 2(ii) ($\beta_{xy} = \beta/(x + y)^2$, $\gamma_{xy} = \gamma$) shows an example where Theorem 2.2 applies but Theorem 2.4 does not, so that the isolation boundary $s(x)$ can decrease as x increases.

7. Discussion

In this paper, we have found conditions under which the particularly simple forms of optimal intervention policy found by Abakuks (1973, 1974) for the general stochastic epidemic model apply to models with less standard infection and removal rate functions. There are many directions in which it would be desirable to extend this work.

Firstly, the model itself is somewhat restrictive. The assumption of a closed population amounts to an assumption that the time scale over which the epidemic occurs is sufficiently short that births, deaths due to other causes than the disease under consideration, immigration and emigration may be neglected. The homogeneous mixing assumption will become less tenable as population size increases, so that we are effectively assuming the population is reasonably small. Thus while our model may provide a reasonable description of the spread of disease through a small population over a short period of time, larger

populations or longer time scales require more complicated models, which are likely to be less mathematically tractable.

Another limitation of our model is the assumption that once an individual has had the disease, it necessarily becomes immune to further infection. In practice some (or all) individuals may well return to the susceptible state. Furthermore, we have not distinguished between those who recover from the disease, becoming immune to further infection, and those who die from their illness. This is particularly relevant since our intervention consists of transferring individuals from either the susceptible or the infective state to the 'removed' state. If naturally 'removed' individuals are in fact dead, then we may wish to distinguish between these and the individuals removed by our intervention. Although for some animal diseases artificially 'removed' individuals may indeed be killed, this will not usually be the case. A model which distinguishes between individuals who have recovered and are immune to further infection on the one hand, and individuals who have died from the disease on the other, is the subject of Picard and Lefèvre (1993) and Lefèvre and Picard (1993).

For an epidemic spreading in a small population over a short period of time, when our model may provide a reasonable description of the process, we still have the problem that in order to implement our optimal intervention strategy we require perfect information on the current state of the population. In practice, perfect information is unlikely to be instantaneously (or even eventually) available to us. Any information which we do have is likely to carry some financial cost, so that ideally we should extend out optimization problem to incorporate the cost of gathering information, to be balanced against the cost of implementing a sub-optimal policy due to ignorance of the true state of the population. Similarly, we are unlikely to know exactly either the functional form of β_{xy} and γ_{xy}, or the true values of any numerical parameters such as, in the case of the general stochastic epidemic, β and γ. If we want to compute the boundaries $s(x)$, $t(x)$, $S(x)$ or $T(x)$, then we need perfect information about these transition rate functions. However, the main objective throughout this paper has been to describe the optimal policies qualitatively, and for this we only need to be satisfied that conditions such as $p_{x,y+1} \leqq p_{xy} \leqq p_{x+1,y}$ hold true.

As far as allowable intervention is concerned, we have made the simplifying assumption that both immunisation of susceptibles and isolation of infectives can be carried out instantaneously with 100% effectiveness. Suppose now that each isolation is successful with probability $\rho < 1$, independently of all other events. Then the number of isolation attempts need to successfully isolate one individual will be a Geometrically distributed random variable with mean $1/\rho$. If each

isolation attempt has cost L, the expected cost of one successful isolation is therefore L/ρ, and the results of Sect. 2 on isolation policies apply as before with L replaced by L/ρ. This assumes that having tried and failed to isolate an individual, the probability that the next attempted isolation will be successful is unaltered. More generally, we can simply take L to be the expected cost of successfully isolating one individual, however many attempts this may take, and apply the results of Sect. 2 as before. A similar argument can be applied to immunisation costs if immunisation is not always 100% effective. Of course, we still have to assume that we know the expected cost of a successful intervention, and that we can tell whether or not an intervention has been successful.

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