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Multiparametric bifurcations of an epidemiological model with strong Allee effect

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Abstract In this paper we completely study bifurcations of an epidemic model with five parameters introduced by Hilker et al. (Am Nat 173:72–88, 2009), which describes the joint interplay of a strong Allee effect and infectious diseases in a single population. Existence of multiple positive equilibria and all kinds of bifurcation are examined as well as related dynamical behavior. It is shown that the model undergoes a series of bifurcations such as saddle-node bifurcation, pitchfork bifurcation, Bogdanov–Takens bifurcation, degenerate Hopf bifurcation of codimension two and degenerate elliptic type Bogdanov–Takens bifurcation of codimension three. Respective bifurcation surfaces in five-dimensional parameter spaces and related dynamical behavior are obtained. These theoretical conclusions confirm their numerical simulations and conjectures by Hilker et al., and reveal some new bifurcation phenomena which are not observed in Hilker et al. (Am Nat 173:72–88, 2009). The rich and complicated dynamics exhibit that the model is very sensitive to parameter perturbations, which has important implications for disease control of endangered species.

Keywords Epidemic model · Allee effect · Degenerate elliptic type · Bogdanov–Takens bifurcation · Codimension three · Degenerate Hopf bifurcation

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1 Introduction

In 1931, Allee (1931) observed that many animal and plant species suffer a decrease of population growth rate as their populations reach small sizes or low density, in particular, the population exhibits a "critical size or density", below which the population declines on average, and above which it increases on average. This phenomenon in biology is called strong Allee effect, which is particularly relevant for endangered species, small or invasive populations. It can be caused by a number of mechanisms, for instance, higher juvenile mortality by genetic inbreeding and difficulties in finding mating partners at low population densities (see Courchamp et al. 2008). On the other hand, infectious diseases are becoming increasing recognized as a major threaten factor in population viability (see Anderson and May 1979; Daszak et al. 1999; de Castro and Bolker 2005; Haydon et al. 2002; Hilker et al. 2007, 2005; Hudson et al. 2001 and references therein). Hence, some species suffer from both disease and strong Allee effect, as observed, for example, the island fox species in Clifford et al. (2006) and Angulo et al. (2007). And the joint interplay of disease and Allee effects has been investigated only recently by mathematical model (see Deredec and Courchamp 2006; Friedman and Yakubu; Hilker et al. 2005, 2009; Thieme et al. 2009 and references therein). Let us recall the mathematical model proposed by Hilker et al. for the synergetic interplay of Allee effects and disease infection in a host population. The reader is referred to Hilker et al. (2009) for details about the description of the following model,

$$\begin{cases} \frac{dx}{dt} = r(1-x)(x-u)x - \alpha y, \\ \frac{dy}{dt} = [-\alpha - d - ru + (\sigma - 1)x - \sigma y]y, \end{cases} \tag{1.1}$$

where $x(t)$ is the dimensionless total population density at time $t \geq 0$ which is composed of infected $y(t)$ and susceptible $x(t) - y(t)$, the carrying capacity is normalized to unity, the parameter $r > 0$ adjusts per capita growth rate without disease, $0 < u < 1$ being the Allee threshold, $\alpha > 0$ being disease-induced mortality, $\sigma > 0$ being infectious rate and $d > 0$ determines the effect of density dependence and independence in the demographic functions.

Hilker et al. (2009) studied the existence of endemic equilibria of system (1.1) by graphical nullcline analysis, found complicated dynamical behaviors of system (1.1) by numerical continuation and simulations, and conjecture that these dynamical behaviors are associated with Bogdanov–Takens bifurcations. Motivated by their work, we systematically study bifurcations and dynamics of system (1.1) by theoretic analysis. We provide a detail qualitative and bifurcation analysis of model (1.1) depending on all five parameters, give the necessary and sufficient condition for the existence of endemic equilibria, clarify several threshold quantities for disease persistence, extinction, and the possibility of multiple stable steady states, prove that system (1.1) can undergo a degenerate elliptic type Bogdanov–Takens bifurcation of codimension three and a series of other bifurcations such as saddle-node bifurcation, pitchfork bifurcation, Bogdanov–Takens bifurcation, Hopf bifurcation and degenerate Hopf bifurcations, and obtain the respective dynamics of system (1.1) for each bifurcation surfaces in five-dimensional parameter spaces. These theoretical conclusions confirm the conjecture of Hilker et al. (2009) and reveal some new bifurcation phenomena, which are not observed in Hilker et al. (2009), such as pitchfork bifurcation, degenerate Hopf bifurcation and degenerate elliptic type Bogdanov–Takens bifurcation of codimension three. The methods developed in this paper provide an approach to study bifurcations and dynamics of general mathematical models with multi-parameters.

Bifurcation theory of dynamical systems has been developed for more than half century, which is to study the changes in the qualitative or topological structure of a given dynamical system with parameters as the parameters vary. If a small smooth change of parameters made to a special value of parameters (called the bifurcation value) of a dynamical system causes a sudden 'qualitative' or topological change in its dynamic behavior, then we call the system undergoes a bifurcation in the small neighborhood of this bifurcation value. Otherwise, we say the system is structure stable. We refer the reader to the books Andronov et al. (1971), Chow and Hale (1982), Chow et al. (1994), Kuznetsov (1998) for details. As an application of bifurcation theory, bifurcation phenomenon in epidemiological models or ecological models have important consequences for disease control or species management, respectively, (see for example Baer et al. 2006; Etoua and Rousseau 2010; Hadeler and van den Driessche 1997; Xiao 2005; Xiao and Ruan 1999; Xiao and Zhang 2007 and references therein). Since Bogdanov (1981) and Takens (1974) studied the 2-parameters generic family of nilpotent cusp of codimension two and obtained a complete classification of topological phase portraits for the family (it is usually called the Bogdanov–Takens bifurcation, denoted by BT bifurcation for short), many researchers have devoted their effort to generalize Bogdanov and Takens' pioneer results to n-parameters generic family of nilpotent equilibrium of codimension n with $n \geq 3$ and there have been a series of achievements and unprecedented challenges on the theme, for instance, degenerate BT bifurcation of codimension 3: saddle, elliptic and focus cases (see for instance Dumortier et al. 2001, 1987, 1991; Li and Rousseau 1989; Mardešić 1992; Xiao 1993, 2008).

In the bifurcation analysis of model (1.1), we are interested in bifurcations near the endemic equilibria. We shall prove that the system has a nilpotent endemic equilibrium of codimension at most 3 for all allowable parameter values, and the nilpotent endemic equilibrium is elliptic if it exists. It is also shown that there exist three independent parameters of $(r, u, d, \alpha, \sigma)$ such that the system undergoes the degenerate elliptic type BT bifurcation of codimension 3 even if it is a 5-parameters family. The bifurcation diagram for this degenerate BT bifurcation of codimension 3 in Dumortier et al. (1991) and the stability of disease-free equilibria have shown that system (1.1) has rich bifurcation phenomenon and complicated dynamics, for instance, system (1.1) can has homoclinic loop or two limit cycles for some values of parameters, respectively. The existence of limit cycles (isolated periodic orbits) for epidemic models is interesting and significant both in mathematics and applications since the existence of stable limit cycle provides a satisfactory explanation for those species communities in which epidemic is observed to break out in a rather reproducible periodic manner. This may have profound implications for disease control and biological conservation.

To the best of our knowledge, system (1.1) is the first example in models which exhibits only elliptic type degenerate BT bifurcation of codimension 3 and no focus type degenerate BT bifurcation of codimension 3. This is different from the observation in Baer et al. (2006). We shall provide explicit smooth transformations to reduce the system with an nilpotent equilibrium into a normal form of codimension 3, which is very useful for the bifurcation analysis. On the other hand, degenerate Hopf bifurcation of codimension 2 of system (1.1) is studied and the explicit conditions for the existence of two limit cycles are given.

This paper is organized as follows. General properties of system (1.1), such as boundedness of solutions, existence and stability of disease-free or endemic equilibria, and saddle-node bifurcation are discussed in Sect. 2. In Sect. 3 we make theoretic studies on the bifurcation of an elliptic type degenerate BT equilibrium of codimension 3, which is an endemic equilibria of system (1.1). In Sect. 4 we prove that the system can undergo some bifurcations of codimension 2 or 1 such as BT bifurcation, degenerate Hopf bifurcation and Hopf bifurcation, and provide an approach to study Hopf bifurcation and degenerate Hopf bifurcation. A brief biological interpretation and conclusion are given in the last section.

2 General analysis of system (1.1)

In this section, we make general analysis on dynamics of system (1.1). From the point of view of biology, we only restrict our attention to system (1.1) in the closed first quadrant in the (x, y) plane, denoted by $\mathbb{R}^2_+$. However, the positive y-axis (i.e. $x = 0$ and $y > 0$) is not invariant for system (1.1) and the vector fields of system (1.1) on the positive y-axis are from right to left. Hence, the x coordinates of solution curves crossing y-axis of system (1.1) will be negative as t increases. Based on the ecological meaning, we adopt the following convention.

(H1): $x(t) \equiv 0$ for all $t \geq t_0$ if there exists a positive time t_0 such that the solution $(x(t), y(t))$ of system (1.1) satisfies $x(t_0) = 0$.

We first state a lemma which shows that system (1.1) is as "well behaved" as one intuits from biology if (H1) holds. We omit the standard proof.

Lemma 2.1 *If convention* $(H1)$ *holds, then the solutions of system* (1.1) *are nonnegative and bounded, that is, for each solution* $(x(t, x_0, y_0), y(t, x_0, y_0))$ *of* (1.1) *with initial condition* $(x(0, x_0, y_0), y(0, x_0, y_0)) = (x_0, y_0) \in \mathbb{R}^2_+$, *there exists a* $T(x_0, y_0) \geq 0$ *such that* $0 \leq x(t, x_0, y_0) \leq 1,\ 0 \leq y(t, x_0, y_0) \leq \max\{0, \frac{1}{\sigma}(\sigma - 1 - \alpha - d - ru)\}$ *for* $t \geq T(x_0, y_0)$.

System (1.1) always has three disease-free (or boundary) equilibria : $O(0, 0)$, $A(u, 0)$ and $E_0(1, 0)$ for all allowable parameters. In epidemiology, the basic reproduction number of an infection is a useful metric, which can be defined as the number of secondary infections produced by a single infected during its entire infectious period in a completely susceptible population. Hilker et al. (2009) obtain the basic reproduction number R_0 and the critical host density P_T for system (1.1) as follows

$$R_0 = \frac{\sigma}{\alpha + d + ru + 1}, \quad P_T = \frac{\alpha + d + ru}{\sigma - 1}.$$

Obviously, $R_0 \le 1$ if and only if $\sigma - 1 - \alpha - d - ru \le 0$. And the condition $\sigma - 1 - \alpha - d - ru \le 0$ leads to the extinction of infections $y(t)$ of system (1.1). Hence if $R_0 \le 1$, system (1.1) does not have endemic equilibria and the infectious disease can not establish in the host population. Let us see the stability of disease-free equilibria of system (1.1) as $R_0 \le 1$. We compute the Jacobian matrix of system (1.1) at a disease-free equilibrium $(x, 0)$ and denote it by $J_d(x, 0)$,

$$J_d(x,0) = \begin{pmatrix} \alpha f'(x) & -\alpha \\ 0 & (\sigma - 1)(x - P_T) \end{pmatrix},$$

where $f(x) = \frac{r}{\alpha}(1 - x)(x - u)x$. Then the characteristic equation of $J_d(x, 0)$ is

$$(\lambda - \alpha f'(x))(\lambda - (\sigma - 1)(x - P_T)) = 0, \tag{2.1}$$

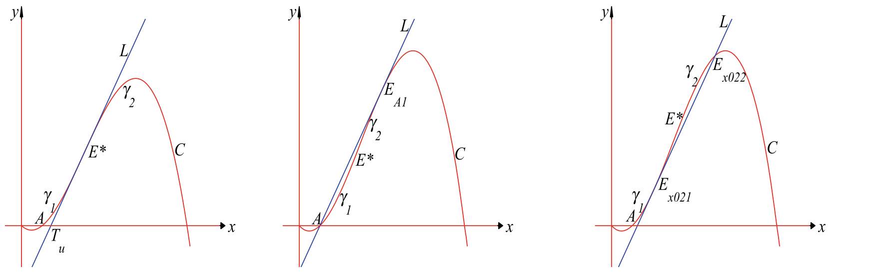
where x is the x coordinate of the disease-free equilibrium. From (2.1), we have

Theorem 2.1 *If $R_0 \le 1$, then system* (1.1) *has only three disease-free equilibria $O(0,0)$, $A(u,0)$ and $E_0(1,0)$. Both $O(0,0)$ and $E_0(1,0)$ are stable nodes, and $A(u,0)$ is a saddle. Bistable appears for system* (1.1).

From Theorem 2.1 we can see whether or not the disease-free host population goes to extinction depending on the initial population number. If the initial population number is in the left of the stable manifold of $A(u, 0)$ then it will go to extinction, otherwise, it will be survival. Thus, if $R_0 \le 1$ the infectious disease could not affect the dynamics of system (1.1) which is determined only by Allee effect. This leads to bistable phenomenon.

Note that $R_0 \le 1$ is equivalent to $P_T \ge 1$ if $\sigma > 1$. By the formula of R_0 (or P_T), we know that R_0 (P_T, respectively) is a decreasing (an increasing, respectively) function of the Allee threshold u. Thus, the basic reproduction number R_0 could lessen to one or less than one if let u increase starting at zero and other parameters keep unchanged. Thus, R_0 for the host with Allee effect is less than that for the host without Allee effect. Theorem 2.1 shows that the disease can not establish in the host population if $R_0 \le 1$. Biologically, this conclusion shows that the presence of an Allee effect in host populations might play a stabilizing and protective role in the invasion of disease. And the critical host density $P_T \ge 1$ which prevents the population from reaching the necessary density for disease establishment even though the infectious rate σ is not small.

In the rest of our study, we shall focus on the ecologically more interesting case that $R_0 > 1$ (i.e. $\sigma > \alpha + d + ru + 1$) and the Allee threshold is far from the carrying capacity, that is, $0 < u < \frac{1}{2}$. However, $R_0 > 1$ is equivalent to $0 < P_T < 1$. Henceforth, we always assume that



(a) a triple point **(b)** a double point **(c)** a double point and a cross point

Fig. 1 The tangent points of the nontrivial nullclines C and L

$$\text{(H2):} \quad 0 < u < \frac{1}{2}, \quad 0 < P_T < 1.$$

Under hypothetical condition (H2), we consider the existence of endemic equilibria (i.e. positive equilibria) for system (1.1). Even though it is algebraically impossible to find explicit sufficient and necessary conditions for the number of endemic equilibria of system (1.1) depending on all parameters, Hilker et al. (2009) obtained the existence and number of endemic equilibria by graphical nullcline. Here we further make theoretic analysis of system (1.1) and give a complete classification on the number of endemic equilibria and the corresponding explicit sufficient and necessary conditions. This is essential for study bifurcations of system (1.1).

Note that the existence of endemic equilibria of system (1.1) is equivalent to that of intersection points of the nontrivial nullclines C and L of system (1.1) in the interior of the first quadrant $\mathbb{R}^2_+$, where

$$C = \left\{(x, y) : y = f(x),\ f(x) = \frac{r}{\alpha}(1 - x)(x - u)x,\ 0 \le x < +\infty\right\},$$
$$L = \left\{(x, y) : y = g(x),\ g(x) = \frac{\sigma - 1}{\sigma}(x - P_T),\ 0 \le x < +\infty\right\}.$$

We first make analysis on the geometric character of the cubic curve C. The cubic curve C crosses the x-axis at three points $O(0, 0)$, $A(u, 0)$ and $E_0(1, 0)$, which are disease-free equilibria of system (1.1). And the curve C has a maximum point, a minimum point and a unique inflection at $E^*(x^*, y^*)$, where $x^* = \frac{u+1}{3}$, $y^* = \frac{r}{27\alpha}(u + 1)(1 - 2u)(2 - u)$.

Hence, $E(x, y)$ is an endemic equilibrium of system (1.1) if and only if $E(x, y)$ is an intersection point of the cubic curve C and the straight line L in the interval $u < x < 1$. And $E^*(x^*, y^*)$ is a unique endemic equilibrium of system (1.1) if the straight line L is a tangent line of C passing through the inflection point E^*, which intersects x-axis at $(T_u, 0)$ and E^* is a triple tangent point of L and C, see Fig. 1a, where $T_u = \frac{(u+1)^3}{9(u^2-u+1)}$. Note that $0 < u < \frac{1}{2}$. Thus $u < T_u < 1$.

Since E^* divides the curve C in the interval $u \le x < 1$ into two parts, we let γ_1 be the portion of the curve C with $u \le x < x^*$, and let γ_2 be the portion of the curve C with $x^* < x < 1$.

We further consider if the curve C and the line L has a double point with $u < x < 1$. It is clear that there are no double points of L and C if the line L crosses x-axis in the interval $T_u < x < 1$. Thus, L and C has a unique cross point in the interval $u < x < 1$ if $T_u < P_T < 1$. And the line L may be tangent to the curve C in the interval $u < x < 1$ only if $0 < P_T < T_u$. Next we seek the conditions of being tangent between the line L and the curve C in the interval $u < x < 1$.

Suppose that the line L is tangent to the curve C in the interval $u < x < 1$. Then the following equations

$$\begin{cases} f(x) = g(x) > 0, \\ f'(x) = \frac{\sigma-1}{\sigma} \end{cases} \tag{2.2}$$

must have solutions in the interval $u < x < 1$.

It is clear that the existence of solutions to (2.2) in the interval $u < x < 1$ is equivalent to that of the following equation

$$f(x) - f'(x)(x - P_T) = 0, \quad 0 < P_T < T_u, \tag{2.3}$$

where $f'(x) = -\frac{r}{\alpha}\left(3x^2 - 2(1+u)x + u\right)$.

By Cardan formula, we compute the discriminant Δ of Eq. (2.3) and obtain that

$$\Delta = \frac{P_T}{3888(u^2 - u + 1)}(1 - P_T)(u - P_T)(T_u - P_T).$$

Hence, there exists a unique real root $\widetilde{x_0}$ with $u < \widetilde{x_0} < 1$ of Eq. (2.3) if $0 < P_T < u$. Since $\Delta = 0$ as $P_T = u$, Eq. (2.3) has two different real solutions u and $\frac{1}{2}$. And if $u < P_T < T_u$, then $\Delta < 0$ and Eq. (2.3) has only two different roots x_{01} and x_{02} with $u < x_{02} < \frac{u+1}{3} < x_{01} < 1$ by computation.

When $0 < P_T < u$, by the geometric character of C and L, we obtain that in the interval $u < x < 1$ the line L and the curve C has only a point of tangency at $E_{x_0} = (\widetilde{x_0}, f(\widetilde{x_0})) \in \gamma_2$ if $\frac{\sigma-1}{\sigma} = f'(\widetilde{x_0})$.

When $P_T = u$, there are two cases for the points of tangency for L and C in $u < x < 1$. One case is that the line L and the curve C has only a point of tangency at $E_{A1} = (\frac{1}{2}, \frac{r}{8\alpha}(1 - 2u)) \in \gamma_2$ if $\frac{\sigma-1}{\sigma} = \frac{r}{4\alpha}$, see Fig. 1b. And the other case is that the line L and the curve C has a point of tangency at $A(u, 0)$ and a cross point at $E_{A2} = (1 - u, \frac{r}{\alpha}u(1 - 2u)(1 - u)) \in \gamma_2$ if $\frac{\sigma-1}{\sigma} = \frac{ru(1-u)}{\alpha}$.

When $u < P_T < T_u$, by calculation we find that there are also two cases for the points of tangency for L and C in $u < x < 1$. If $\frac{\sigma-1}{\sigma} = f'(x_{01})$, then the line L and the curve C has a point of tangency at $E_{x011} = (x_{01}, f(x_{01})) \in \gamma_2$ and a cross point at $E_{x012} = (x_{012}, f(x_{012})) \in \gamma_1$, where x_{012} is a simple root of the equation $f(x) - f'(x_{01})(x - x_0) = 0$.

If $\frac{\sigma-1}{\sigma} = f'(x_{02})$, then the line L and the curve C has a point of tangency at $E_{x021} = (x_{02}, f(x_{02})) \in \gamma_1$ and a cross point at $E_{x022} = (x_{022}, f(x_{022})) \in \gamma_2$, see Fig. 1c, where x_{022} is a simple root of the equation $f(x) - f'(x_{02})(x - x_0) = 0$.

Note that

$$\frac{ru(1-u)}{\alpha} < f'(x_{02}) < \frac{r}{3\alpha}(u^2-u+1), \quad \frac{r}{4\alpha} < f'(x_{01}) < \frac{r}{3\alpha}(u^2-u+1),$$

and $f'(x_{02}) < f'(x_{01})$. Therefore, the line L and the curve C have three different cross points in the interval $u < x < 1$ if $f'(x_{02}) < \frac{\sigma-1}{\sigma} < f'(x_{01})$.

Summing up the above analysis, we have the following lemma on the existence and non-existence of endemic equilibria of system (1.1).

Lemma 2.2 *Assume that* $(H2)$ *holds. Then system* (1.1) *has at most three endemic equilibria and at least no endemic equilibria. More precisely,*

(i) *system* (1.1) *has three endemic equilibria if and only if*

$$f'(x_{02}) < \frac{\sigma-1}{\sigma} < f'(x_{01}), \ u < P_T < T_u. \tag{2.4}$$

The three endemic equilibria are cross points of the line L *and the curve* C.

(ii) *System* (1.1) *has two endemic equilibria if and only if one of the following conditions holds.*

$$0 < \frac{\sigma-1}{\sigma} < f'(\widetilde{x_0}), \quad 0 < P_T < u; \tag{2.5}$$

$$\frac{ru(1-u)}{\alpha} < \frac{\sigma-1}{\sigma} < \frac{r}{4\alpha}, \quad P_T = u; \tag{2.6}$$

$$\frac{\sigma-1}{\sigma} = f'(x_{01}), \quad u < P_T < T_u; \tag{2.7}$$

$$\frac{\sigma-1}{\sigma} = f'(x_{02}), \quad u < P_T < T_u. \tag{2.8}$$

The two endemic equilibria are cross points of L *and* C *if one of* (2.5) *and* (2.6) *holds. And if one of* (2.7) *and* (2.8) *holds, then one endemic equilibrium is a point of tangency of* L *and* C *and the other is a cross point of* L *and* C.

(iii) *System* (1.1) *has one endemic equilibrium if and only if one of the following conditions holds.*

$$T_u < P_T < 1; \tag{2.9}$$

$$\text{either } \frac{\sigma-1}{\sigma} < f'(x_{02}) \ \text{ or } \ f'(x_{01}) < \frac{\sigma-1}{\sigma}, \text{ and } u < P_T < T_u; \tag{2.10}$$

$$\frac{\sigma-1}{\sigma} = \frac{ru(1-u)}{\alpha}, \quad P_T = u; \tag{2.11}$$

$$\frac{\sigma-1}{\sigma} = \frac{r}{3\alpha}(u^2-u+1), \quad P_T = T_u; \tag{2.12}$$

$$\frac{\sigma-1}{\sigma}=f'(\widetilde{x_0}),\quad 0<P_T<u; \tag{2.13}$$

$$\frac{\sigma-1}{\sigma}=\frac{r}{4\alpha},\quad P_T=u; \tag{2.14}$$

If one of (2.9)–(2.11) *holds, then the unique endemic equilibrium is a cross point of L and C. If one of* (2.12)–(2.14) *holds, then the unique endemic equilibrium is a tangency point of L and C, which is at* E^*, E_{x_0} *and* E_{A1}, *respectively.*

(iv) *System* (1.1) *has no endemic equilibria if and only if*

$$\frac{\sigma-1}{\sigma}>f'(\widetilde{x_0}),\quad 0<P_T\le u, \tag{2.15}$$

where $f'(\widetilde{x_0})=\frac{r}{4\alpha}$ *as* $P_T=u$.

In what follows we study the stability of equilibria and phase portraits of system (1.1). First we consider the stability of disease-free equilibria for system (1.1) if $0<P_T<1$. From the eigenvalues of (2.1), we have the following.

Lemma 2.3 *Disease-free equilibria of system* (1.1) *has the following stabilities, respectively.*

(i) $O(0,0)$ *is a stable hyperbolic node and* $E_0(1,0)$ *is a hyperbolic saddle if* $0<P_T<1$*;*

(ii) $A(u,0)$ *is a unstable hyperbolic node if* $0<P_T<u$*;* $A(u,0)$ *is a hyperbolic saddle if* $u<P_T<1$ *and* $A(u,0)$ *is a saddle-node if* $P_T=u$.

In Table 2 of Hilker et al. (2009), Hilker et al. claimed that $E_0(1,0)$ is a unstable node if $0<P_T<1$. This is not true. From Lemma 2.3, we know that $E_0(1,0)$ is a hyperbolic saddle if $0<P_T<1$. Further we obtain the following global conclusions on dynamics of system (1.1).

Theorem 2.2 *If* $(H2)$ *and* (2.15) *hold, then system* (1.1) *has only disease-free equilibria and no endemic equilibria. Disease-free equilibria* $O(0,0)$ *is a stable node,* $A(u,0)$ *is a unstable node or saddle-node, and* $E_0(1,0)$ *is a hyperbolic saddle. Almost all orbits of system* (1.1) *in the interior of* $\mathbb{R}^2_+$ *tend to* $O(0,0)$ (*see Fig.* 2).

Biologically, this conclusion shows that the joint interplay between infectious diseases and Allee effects is deathblow for the host population if the critical host density P_T is low and the infectious rate σ is large. This leads that the whole population goes to extinction.

In the following we investigate the stability of endemic equilibria of system (1.1). We denote the Jacobian matrix of system (1.1) at an endemic equilibrium $E(x,y)$ by $J_e(x,y)$, where $J_e(x,y)=\begin{pmatrix}\alpha f'(x) & -\alpha\\ (\sigma-1)y & -\sigma y\end{pmatrix}$. Then the characteristic equation of $J_e(x,y)$ is

$$\lambda^2+(\sigma f(x)-\alpha f'(x))\lambda-\alpha\sigma f(x)\left(f'(x)-\frac{\sigma-1}{\sigma}\right)=0, \tag{2.16}$$

where x is the x coordinate of the endemic equilibrium.

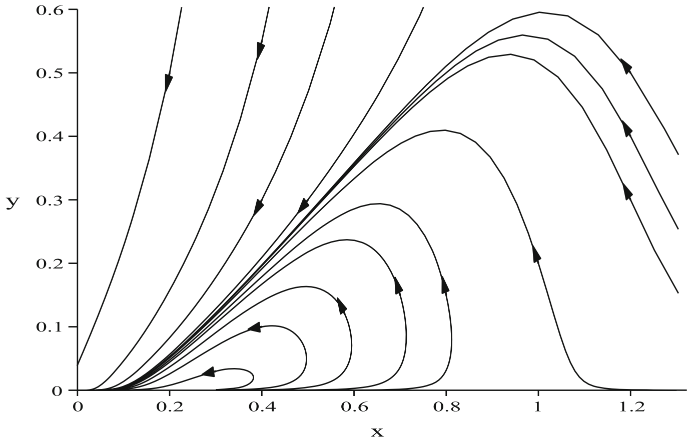


Fig. 2 Extinction of the population in system (1.1) as $R_0 > 1$, $P_T = u$ and $\frac{\sigma-1}{\sigma} > \frac{r}{4\alpha}$, where the values of parameters for the figure are $r = 1, u = 0.3, d = 0.2, \alpha = 1, \sigma = 6$

From the distribution of eigenvalues of characteristic equation (2.16), we now derive the topological nature of endemic equilibria of system (1.1) as follows.

Theorem 2.3 *The endemic equilibrium $E(x, y)$ is degenerate if it is a tangent point of L and C. Otherwise, the endemic equilibrium $E(x, y)$ is an elementary equilibrium. More precisely, the endemic equilibrium $E(x, y)$ is a center or weak focus if $f'(x) - \frac{\sigma-1}{\sigma} < 0$ and $\sigma f(x) - \alpha f'(x) = 0$; the endemic equilibrium is a hyperbolic node or focus if $f'(x) - \frac{\sigma-1}{\sigma} < 0$ and $\sigma f(x) - \alpha f'(x) \neq 0$, and the endemic equilibrium is a hyperbolic saddle if $f'(x) - \frac{\sigma-1}{\sigma} > 0$.*

From Lemma 2.2 and Theorem 2.3, we easily check that the parameters satisfying condition (2.7), (2.8), (2.13) or (2.14), respectively come to being a hypersurface in five-dimensional parameter space $(\alpha, d, r, u, \sigma)$. These hypersurfaces are saddle-node bifurcation surfaces of system (1.1). As parameters $(\alpha, d, r, u, \sigma)$ vary in the small neighborhood of the these hypersurfaces and cross the respective hypersurfaces, system (1.1) undergoes saddle-node bifurcation. Moreover, the parameters satisfying condition (2.12) consist of a hypersurface, which is pitchfork bifurcation surface. As parameters $(\alpha, d, r, u, \sigma)$ vary in the small neighborhood of this hypersurface and transversal to it, system (1.1) undergoes pitchfork bifurcation. Let us see the Fig. 3. We choose P_T and $\frac{\sigma-1}{\sigma}$ as two bifurcation parameters. When $(P_T, \frac{\sigma-1}{\sigma})$ vary in the range

$$\Omega_1 = \left\{ \left(P_T, \frac{\sigma-1}{\sigma} \right) : \ T_u \leq P_T < 1, \ \sigma > \right\},$$

system (1.1) always has an endemic equilibrium. And as bifurcation parameters $(P_T, \frac{\sigma-1}{\sigma})$ pass through the point $Q = (T_u, \frac{r}{3\alpha}(u^2 - u + 1))$ and go into the domain Ω_2 bounded by L_a, L_b and $P_T = u$, system (1.1) has three endemic equilibria.

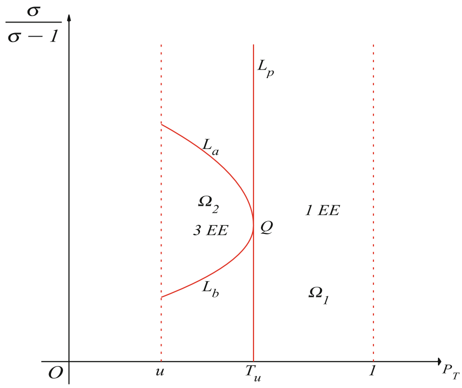


Fig. 3 The bifurcation diagram of equilibria of system (1.1), where P_T and $\frac{\sigma-1}{\sigma}$ are bifurcation parameters, $Q(T_u, \frac{r}{3\alpha}(u^2-u+1))$ is a bifurcation point, 1 EE represents "one endemic equilibrium" and 3 EE means "three endemic equilibria"

This implies that endemic equilibria of system (1.1) always exist as $(\alpha, d, r, u, \sigma)$ vary in the direction from Ω_1 through Q into Ω_2, or vice versa. Hence, in the case the disease can establish in the host population.

3 Elliptic type degenerate BT bifurcation of codimension three

In this section we first give a lemma, which provides a series of explicit smooth transformations to derive a normal form with terms up to and including fourth order for the codimension three BT equilibrium. Then we apply the lemma to system (1.1) and obtain that the system has a nilpotent equilibrium of codimension at most 3 for all allowable parameters. Further we verify that the degenerate equilibrium is an elliptic type equilibrium of codimension 3 if it exists. Last we choose three independent parameters of $(r, u, d, \alpha, \sigma)$ such that the system undergoes the degenerate BT bifurcation of codimension 3 in the elliptic case.

Note that Xiao and Ruan (1999) have provided a series of explicit smooth transformations to derive a normal form with terms up to and including third order for a nilpotent (or double-zero eigenvalue) equilibrium of planar systems and obtain the normal form of nilpotent equilibrium of codimension 2. Here we follow their method to provide a series of explicit smooth transformations to derive a normal form for the nilpotent equilibrium of codimension 3.

Consider the following system

$$\begin{cases} \dot{x} = y + a_{30}x^3 + a_{03}y^3 + a_{21}x^2y + a_{12}xy^2 + a_{40}x^4 \\ \qquad +a_{04}y^4 + a_{31}x^3y + a_{13}xy^3 + a_{22}x^2y^2 + R_1(x, y), \\ \dot{y} = b_{11}xy + b_{30}x^3 + b_{03}y^3 + b_{21}x^2y + b_{12}xy^2 + b_{40}x^4 \\ \qquad +b_{04}y^4 + b_{31}x^3y + b_{13}xy^3 + b_{22}x^2y^2 + R_2(x, y), \end{cases} \tag{3.1}$$

where $b_{11} \neq 0, a_{ij},\ b_{ij},\ i, j = 0, \ldots, 4,\ 3 \leq i + j \leq 4$ are real parameters, and $R_1(x, y)$ and $R_2(x, y)$ are smooth functions of their arguments with at least fifth-order terms of (x, y). Then we have

Lemma 3.1 *Assume that $b_{11}b_{30} \neq 0$. Then there exists a small neighborhood U_0 of $(0,0)$ such that in this neighborhood U_0 system (3.1) is locally topologically equivalent to*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = b_{11}xy + b_{30}x^3 + (b_{21} + 3a_{30})x^2y + (b_{40} - b_{11}a_{30})x^4 \\ \qquad +(4a_{40} + b_{31} + \frac{1}{3}b_{11}a_{21} + \frac{1}{6}b_{11}b_{12})x^3y + R(x, y), \end{cases} \tag{3.2}$$

where $R(x, y)$ is a smooth function of their arguments with at least fifth-order terms of (x, y).

Moreover, assume that $5b_{30}(b_{21}+3a_{30}) - 3b_{11}(b_{40} - b_{11}a_{30}) \neq 0$. Then the equilibrium $(0,0)$ of system (3.1) is a degenerate saddle of codimension 3 if $b_{30} > 0$; it is a degenerate focus or center of codimension 3 if $b_{30} < 0$ and $b_{11}^2 + 8b_{30} < 0$; and it is a degenerate elliptic of codimension 3 if $b_{30} < 0$ and $b_{11}^2 + 8b_{30} \geq 0$.

Proof It is clear that $(0,0)$ is an equilibrium of system (3.1) with two zero eigenvalues. We perform a near-identity smooth change of coordinates

$$\begin{cases} X = x - (\frac{1}{3}a_{21} + \frac{1}{6}b_{12})x^3 - (\frac{1}{2}a_{12} + \frac{1}{2}b_{03})x^2y, \\ Y = y + a_{30}x^3 + a_{03}y^3 - \frac{1}{2}b_{12}x^2y - b_{03}xy^2. \end{cases}$$

This transformation is invertible in a small neighborhood of the origin, and its inverse can be found by the method of unknown coefficients. Thus, system (3.1) becomes

$$\begin{cases} \dot{X} = Y + a_{40}X^4 + a_{04}Y^4 + [a_{31} - \frac{1}{2}b_{11}(a_{12} + b_{03})]X^3Y \\ \qquad +a_{13}XY^3 + a_{22}X^2Y^2 + R_3(X, Y), \\ \dot{Y} = b_{11}XY + b_{30}X^3 + (b_{21} + 3a_{30})X^2Y + (b_{40} - b_{11}a_{30})X^4 \\ \qquad +[b_{31} - b_{11}(-\frac{1}{3}a_{21} - \frac{1}{6}b_{12})]X^3Y + (b_{13} + 2b_{11}a_{03})XY^3 \\ \qquad +b_{04}Y^4 + [b_{22} - \frac{1}{2}b_{11}b_{03} + \frac{1}{2}b_{11}a_{12}]X^2Y^2 + R_4(X, Y), \end{cases} \tag{3.3}$$

where $R_3(X, Y)$ and $R_4(X, Y)$ are smooth functions of their arguments with at least fifth-order terms of (X, Y).

In order to kill non-resonant quartic terms of system (3.3), we let

$$\begin{cases} z_1 = X + \left(-\frac{1}{4}a_{31} - \frac{1}{12}b_{11}a_{12} + \frac{1}{6}b_{11}b_{03} - \frac{1}{12}b_{22}\right)X^4 + a_{04}Y^4 \\ \qquad - \left(\frac{1}{3}a_{22} + \frac{1}{6}b_{13}\right)X^3Y - (\frac{1}{2}a_{13} + \frac{1}{2}b_{04})X^2Y^2, \\ z_2 = Y + a_{40}X^4 + (-\frac{1}{3}b_{22} + \frac{1}{6}b_{11}b_{03} - \frac{1}{6}b_{11}a_{12})X^3Y \\ \qquad -b_{04}XY^3 - (\frac{1}{2}b_{13} + b_{11}a_{03})X^2Y^2 \end{cases}$$

in the small neighborhood of the origin. System (3.3) is transformed into

$$\begin{cases} \dot{z_1} = z_2 + R_5(z_1, z_2), \\ \dot{z_2} = b_{11}z_1z_2 + b_{30}z_1^3 + (b_{21} + 3a_{30})z_1^2z_2 + (b_{40} - b_{11}a_{30})z_1^4 \\ \qquad +(4a_{40} + b_{31} + \frac{1}{3}b_{11}a_{21} + \frac{1}{6}b_{11}b_{12})z_1^3z_2 + R_6(z_1, z_2), \end{cases} \tag{3.4}$$

where $R_5(z_1, z_2)$ and $R_6(z_1, z_2)$ are smooth functions of their arguments with at least fifth-order terms of (z_1, z_2).

Last we set $x = z_1,\ \ y = z_2 + R_5(z_1, z_2)$, system (3.4) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = b_{11}xy + b_{30}x^3 + (b_{21} + 3a_{30})x^2y + (b_{40} - b_{11}a_{30})x^4 \\ \qquad +(4\,a_{40} + b_{31} + \frac{1}{3}\,b_{11}a_{21} + \frac{1}{6}\,b_{11}b_{12})x^3y + R(x, y), \end{cases} \tag{3.5}$$

which implies that system (3.1) is locally topologically equivalent to system (3.2) in a neighborhood of the origin.

From arguments in Dumortier et al. (1991), we know that the equilibrium $(0, 0)$ of system (3.2) is a degenerate equilibrium of codimension 3 if $5b_{30}(b_{21} + 3a_{30}) - 3b_{11}(b_{40} - b_{11}a_{30}) \neq 0$. And according to Theorem 7.2 in Zhang et al. 1992, p. 152, we obtain the topological classification of $(0, 0)$ of system (3.1) as follows: the equilibrium $(0, 0)$ is a degenerate saddle if $b_{30} > 0$; it is a degenerate focus or center if $b_{30} < 0$ and $b_{11}^2 + 8b_{30} < 0$; and it is a degenerate elliptic if $b_{30} < 0$ and $b_{11}^2 + 8b_{30} \geq 0$. Therefore, the proof is completed. □

Now we discuss the topological classification of the degenerate equilibrium E^* of system (1.1).

Theorem 3.1 *Assume that $P_T = T_u$ and $\frac{\sigma-1}{\sigma} = \frac{r}{3\alpha}(u^2 - u + 1)$. Then system (1.1) has a unique endemic equilibrium $E^* = (x^*, y^*)$, where*

$$x^* = \frac{u+1}{3}, \quad y^* = \frac{r}{27\alpha}(u+1)(1-2u)(2-u).$$

The endemic equilibrium E^* is a unstable (stable) degenerate node of codimension one if $\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T > 0$ ($\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T < 0$, respectively), and E^* is an elliptic equilibrium of codimension 3 if $\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T = 0$.

Proof From Lemma 2.2 we know that system (1.1) has a unique endemic equilibrium $E^* = (x^*, y^*)$ if $P_T = T_u$ and $\frac{\sigma-1}{\sigma} = \frac{r}{3\alpha}(u^2 - u + 1)$.

Moving E^* to the origin, we set $x_1 = x - x^*,\ \ y_1 = y - y^*$. Note that $P_T = T_u$ and $\frac{\sigma-1}{\sigma} = \frac{r}{3\alpha}(u^2 - u + 1)$. Then system (1.1) becomes

$$\begin{cases} \dot{x}_1 = \frac{\alpha(\sigma-1)}{\sigma}x_1 - \alpha y_1 - r x_1^3, \\ \dot{y}_1 = (\frac{u+1}{3} - P_T)\left(\frac{(\sigma-1)^2}{\sigma}x_1 - (\sigma-1)y_1\right) + (\sigma-1)x_1y_1 - \sigma y_1^2, \end{cases} \tag{3.6}$$

We see that the Jacobian matrix at the equilibrium $(0, 0)$ of system (3.6) has two eigenvalues Λ_1 and Λ_2, $\Lambda_1 = 0$ and $\Lambda_2 = \frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T$. Therefore, we distinguish two cases $\Lambda_2 \neq 0$ and $\Lambda_2 = 0$ to discuss the topological classification of the equilibrium $(0, 0)$.

If $\Lambda_2 \neq 0$, i.e. $\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T \neq 0$, then by standard center manifold theory we can obtain that endemic equilibrium E^* of system (1.1) is a unstable (stable) degenerate node if $\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T > 0$ ($\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T < 0$, respectively).

If $\Lambda_2 = 0$, i.e. $\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T = 0$, then system (3.6) can be written as

$$\begin{cases} \dot{x_1} = \frac{\alpha(\sigma-1)}{\sigma}x_1 - \alpha y_1 - r x_1^3, \\ \dot{y_1} = \frac{\alpha(\sigma-1)^2}{\sigma^2}x_1 - \frac{\alpha(\sigma-1)}{\sigma}y_1 + (\sigma-1)x_1 y_1 - \sigma y_1^2. \end{cases} \tag{3.7}$$

Equilibrium (0, 0) of system (3.7) has double zero eigenvalues. In the following we reduce system (3.7) to a normal form of Bogdanov–Takens singularity. Let

$$x_2 = x_1, \quad y_2 = \frac{\alpha(\sigma-1)}{\sigma}x_1 - \alpha y_1.$$

Then system (3.7) becomes

$$\dot{x}_2 = y_2 - r x_2^3, \quad \dot{y}_2 = \frac{\sigma}{\alpha} y_2^2 - (\sigma-1)x_2 y_2 - \frac{\alpha r(\sigma-1)}{\sigma}x_2^3. \tag{3.8}$$

In a small neighborhood of the origin, we make a near-identity topological transformation of coordinates $x_3 = x_2 - \frac{1}{2}\frac{\sigma}{\alpha}x_2^2$, $y_3 = y_2 - \frac{\sigma}{\alpha}x_2 y_2$, which transforms system (3.8) into

$$\begin{cases} \dot{x}_3 = y_3 - \frac{3}{2}\frac{\sigma^2}{\alpha^2}x_3^2 y_3 - r x_3^3 - \frac{r\sigma}{2\alpha}x_3^4 - \frac{\sigma^3}{2\alpha^3}x_3^3 y_3 + O((x_3, y_3)^5), \\ \dot{y}_3 = -(\sigma-1)x_3 y_3 - \frac{\sigma^2}{\alpha^2}x_3 y_3^2 - \frac{\sigma(\sigma-1)}{2\alpha}x_3^2 y_3 - \frac{1}{2}r(\sigma-1)x_3^4 \\ \qquad -\frac{\alpha r(\sigma-1)}{\sigma}x_3^3 + \frac{\sigma(2\alpha r+3\sigma^2-3\sigma)}{2\alpha^2}x_3^3 y_3 - \frac{5}{2}\frac{\sigma^3}{\alpha^3}x_3^2 y_3^2 + O((x_3, y_3)^5). \end{cases} \tag{3.9}$$

From Lemma 3.1, we can reduce system (3.9) to the following system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \sqrt{\frac{\sigma(\sigma-1)}{\alpha r}}xy - x^3 + [\frac{3\sigma}{\alpha(\sigma-1)} - \frac{\sigma^2}{2\alpha^2 r}]x^2 y + \frac{3}{2}\frac{\sigma}{\alpha}\sqrt{\frac{\sigma}{\alpha r(\sigma-1)}}x^4 \\ \qquad +[\frac{2}{3}\frac{\sigma^3}{\alpha^3 r} - \frac{2\sigma^2}{\alpha^2(\sigma-1)}]\sqrt{\frac{\sigma}{\alpha r(\sigma-1)}}x^3 y + O(|x, y|^5). \end{cases} \tag{3.10}$$

Since $-5(\frac{3\sigma}{\alpha(\sigma-1)} - \frac{\sigma^2}{2r\alpha^2}) - 3\sqrt{\frac{\sigma(\sigma-1)}{\alpha r}}\frac{3\sigma}{2\alpha}\sqrt{\frac{\sigma}{\alpha r(\sigma-1)}} = -\frac{\sigma}{\alpha}(\frac{15}{\sigma-1} + \frac{2\sigma}{\alpha r}) \neq 0$, the equilibrium (0, 0) of system (3.10) is a degenerate equilbrium of codimension 3 by Lemma 3.1.

Furthermore, we claim that equilibrium (0, 0) of system (3.10) is an elliptic degenerate equilibrium of codimension 3.

Indeed, we only need to prove that $\frac{\sigma(\sigma-1)}{\alpha r} - 8 \geq 0$ by Lemma 3.1. From the conditions $P_T = T_u$, $\frac{\sigma-1}{\sigma} = \frac{r}{3\alpha}(u^2 - u + 1)$ and $\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T = 0$, we can solve the three equations and obtain that

$$\begin{aligned} r &= \frac{(\sigma-1)(2-u)(1-2u)(u+1)}{3(u^2-u+1)^2}, \qquad \alpha = \frac{\sigma(2-u)(1-2u)(u+1)}{9(u^2-u+1)}, \\ d &= -\frac{(u+1)}{9(u^2-u+1)^2}\left(\sigma(1+u)^2(u^2-4u+1) + u^4 - 5u^3 + 15u^2 - 5u + 1\right). \end{aligned} \tag{3.11}$$

Obvious, r and α are positive if $\sigma > 1$ and $0 < u < \frac{1}{2}$, and d is positive for all allowance u and σ by biological meaning.

Assume to the contrary that $\frac{\sigma(\sigma-1)}{\alpha r} < 8$. Then $\frac{27(u^2-u+1)^3}{(u+1)^2(u-2)^2(2u-1)^2} - 8 < 0$, which leads to $0 < u < \frac{7}{2} - \frac{3\sqrt{5}}{2}$. We now prove that this implies that $d < 0$.

In fact, from the expression of d we know that $h(u)d < 0$, where

$$h(u) = \sigma(1+u)^2(u^2 - 4u + 1) + u^4 - 5u^3 + 15u^2 - 5u + 1.$$

Note that $\sigma > 1$. If $0 < u < \frac{7}{2} - \frac{3\sqrt{5}}{2}$, we then have

$$\begin{aligned} h(u) &= (\sigma - 1)(u+1)^2(2+\sqrt{3}-u)(2-\sqrt{3}-u) + 2u^4 - 7u^3 + 9u^2 - 7u + 2 \\ &> 2u^4 - 7u^3 + 9u^2 - 7u + 2 \triangleq q(u). \end{aligned}$$

Since $q''(u) = 24u^2 - 42u + 18 = 6(3-4u)(1-u) > 0$, $q'(u) < q'(\frac{7}{2} - \frac{3\sqrt{5}}{2}) < 0$. Hence, $q(u) > q(0) > 0$. This leads to that $h(u) > 0$ and consequently, $d < 0$.

However, we know that d must be positive for biological meaning. So $u \geq \frac{7}{2} - \frac{3\sqrt{5}}{2}$, which implies $\frac{\sigma(\sigma-1)}{\alpha r} \geq 8$. We complete the proof. □

Since $P_T = T_u$, $P_T > u$. From Lemma 2.3 and Theorem 3.1, we obtain the dynamics of system (1.1) with a triple endemic equilibrium as follows.

Theorem 3.2 *Assume that $P_T = T_u$ and $\frac{\sigma-1}{\sigma} = \frac{r}{3\alpha}(u^2 - u + 1)$. Then $u < P_T < 1$ and system (1.1) has a unique endemic equilibrium $E^* = (x^*, y^*)$ and three disease-free equilibria: $O(0,0)$, $A(u,0)$ and $E_0(1,0)$. $O(0,0)$ is a stable node, $A(u,0)$ and $E_0(1,0)$ are hyperbolic saddles. The endemic equilibrium E^* is a degenerate unstable (stable) node if $\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T > 0$ ($\frac{\alpha}{\sigma} - \frac{u+1}{3} + P_T < 0$, respectively), and E^* is an elliptic equilibrium of codimension 3 if $(r, d, \alpha, u, \sigma)$ satisfies (3.11). The dynamics of system (1.1) in this case is shown in Fig. 4.*

As observed in Hadeler and van den Driessche (1997) and Hilker et al. (2009), the basic reproduction number is not the unique indicator for disease persistent. Theorem 3.2 shows if the disease can persist depends on the initial infectious number, Allee threshold and the infectious rate even though the basic reproduction number $R_0 > 1$ (i.e. $0 < P_T < 1$). Disease establishment and bi-stability occur if $u < P_T < 1$ and $\frac{\alpha}{\sigma} \leq \frac{u+1}{3} - P_T$, and the whole population can go extinction if $u < P_T < 1$ and $\frac{\alpha}{\sigma} > \frac{u+1}{3} - P_T$.

In the following we study if system (1.1) can undergo degenerate BT bifurcation of codimension 3 in a small neighborhood of equilibrium $E^*(x^*, y^*)$ as parameters $(r, d, \alpha, u, \sigma)$ varies in a small neighborhood of $(r_0, d_0, \alpha_0, u_0, \sigma_0)$ which satisfies (3.11).

We choose (d, u, σ) as bifurcation parameters. Let

$$d = d_0 + \lambda_1, u = u_0 + \lambda_2, \sigma = \sigma_0 + \lambda_3$$

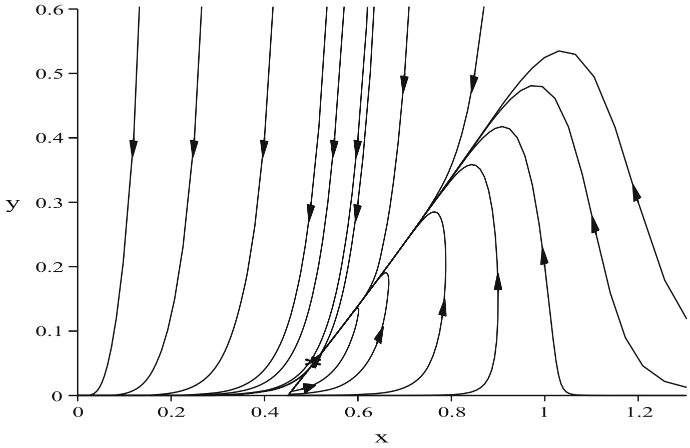


Fig. 4 The dynamics of system (1.1) with an elliptic type endemic equilibrium of codimension 3, where $r = 2, u = 0.45, d = 3.05, \alpha = 0.55$ and $\sigma = 11$. The endemic equilibrium E^* is $(\frac{1}{2}, \frac{1}{22})$, which is represented by $*$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a very small parameter vector $(0 < ||\lambda|| \ll 1)$, and $(d_0, u_0, \sigma_0, \alpha_0, r_0)$ is a set of given parameters which satisfy (3.11). For simplification we denote $(d_0, u_0, \sigma_0, \alpha_0, r_0)$ by $(d, u, \sigma, \alpha, r)$.

Now we rewrite system (1.1) as follows.

$$\begin{cases} \dot{x} = r(1-x)(x-u-\lambda_2)x - \alpha y, \\ \dot{y} = [-\alpha - d - \lambda_1 - r(u+\lambda_2) + (\sigma+\lambda_3-1)x - (\sigma+\lambda_3)y]y. \end{cases} \tag{3.12}$$

Let $x_1 = x - x^*,\ y_1 = y - y^*$. Then by (3.11) system (3.12) becomes

$$\begin{cases} \dot{x}_1 = c_1 + ax_1 + by_1 + p_{11}x_1^2 - rx_1^3, \\ \dot{y}_1 = c_2 + cx_1 + ey_1 + 2q_{12}x_1y_1 + q_{22}y_1^2, \end{cases} \tag{3.13}$$

where

$$\begin{aligned} &a = \tfrac{\alpha(\sigma-1)}{\sigma} + \tfrac{r\lambda_2}{3}(2u-1), \quad b = -\alpha, \quad c = (\sigma+\lambda_3-1)\tfrac{\alpha(\sigma-1)}{\sigma^2}, \\ &c_1 = \tfrac{1}{9}r\lambda_2(u+1)(u-2), \quad c_2 = \left[-\lambda_1 - r\lambda_2 + \left(\tfrac{u+1}{3} - \tfrac{\alpha(\sigma-1)}{\sigma^2}\right)\lambda_3\right]\tfrac{\alpha(\sigma-1)}{\sigma^2}, \\ &e = -\left[\tfrac{\alpha(\sigma-1)}{\sigma} + \lambda_1 + r\lambda_2 - \left(\tfrac{u+1}{3} - \tfrac{2\alpha(\sigma-1)}{\sigma^2}\right)\lambda_3\right], \\ &p_{11} = r\lambda_2, \quad q_{12} = \tfrac{\sigma+\lambda_3-1}{2}, \quad q_{22} = -(\sigma+\lambda_3). \end{aligned}$$

By an affine transformation $x_2 = x_1,\ y_2 = c_1 + ax_1 + by_1$, system (3.13) becomes

$$\begin{cases} \dot{x}_2 = y_2 + p_{11}x_2^2 - rx_2^3, \\ \dot{y}_2 = \left(bc_2 - ec_1 + \tfrac{q_{22}c_1{}^2}{b}\right) + \left(bc - ae - 2q_{12}c_1 + 2\,\tfrac{q_{22}c_1a}{b}\right)x_2 \\ \qquad +(a + e - 2\,\tfrac{q_{22}c_1}{b})y_2 + (ap_{11} - 2aq_{12} + \tfrac{q_{22}a^2}{b})x_2^2 \\ \qquad +\tfrac{q_{22}}{b}y_2^2 + 2\left(q_{12} - \tfrac{q_{22}a}{b}\right)x_2y_2 - arx_2^3. \end{cases} \tag{3.14}$$

In a small neighborhood of the origin, we let

$$x_3 = x_2 + \frac{r}{2p_{11}}x_2^2, \quad y_3 = y_2 + p_{11}x_2^2 + \frac{r}{p_{11}}x_2y_2.$$

Then system (3.14) can be converted into

$$\begin{cases} \dot{x}_3 = y_3 + Q_1(x_3, y_3, \lambda), \\ \dot{y}_3 = c_0 + c_{10}x_3 + c_{20}x_3^2 + c_{30}x_3^3 + y_3(c_{01} + c_{11}x_3 + c_{21}x_3^2 + c_{31}x_3^3) \\ \qquad +Q_2(x_3, y_3, \lambda), \end{cases} \tag{3.15}$$

where $Q_i(x_3, y_3, \lambda) = O((|x_3, y_3| + |\lambda|)^4)$, $i = 1, 2$, and c_0 and c_{ij} are smooth functions of $(r, u, \alpha, d, \sigma, \lambda)$. We omit their long expressions here for reasons of space.

Following the steps in Dumortier et al. (1991), system (3.15) can be further transformed to the following canonical unfolding.

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^3 + \mu_2(\lambda)x + \mu_1(\lambda) + y(\mu_3(\lambda) + b(\lambda)x + c(\lambda)x^2 \\ \qquad +x^3h(x, \lambda)) + y^2Q(x, y, \lambda), \end{cases} \tag{3.16}$$

where $\mu_i(\lambda)$, $b(\lambda)$, $c(\lambda)$, $h(x, \lambda)$ and $Q(x, y, \lambda)$ are smooth functions of their arguments, $b(0) = \sqrt{\frac{\sigma(\sigma-1)}{\alpha r}}$, $c(0) = \frac{3\sigma}{\alpha(\sigma-1)} - \frac{\sigma^2}{2\alpha^2 r}$, and the determinant of the Jacobian matrix of $(\mu_1(\lambda), \mu_2(\lambda), \mu_3(\lambda))$ with respect to $(\lambda_1, \lambda_2, \lambda_3)$ at $(0, 0, 0)$ is as follows,

$$D = \left|\frac{\partial(\mu_1,\mu_2,\mu_3)}{\partial(\lambda_1,\lambda_2,\lambda_3)}\right|_{(0,0,0)} = \frac{\alpha^3 r(\sigma-1)^2(u+1)(2-u)\big((1-2u)^2(\sigma-1)+(u+1)(2-u)\big)}{27\sigma^4(u^2-u+1)} \neq 0.$$

Hence, system (3.12) can undergo degenerate BT bifurcation of codimension 3 in the elliptic case by conclusions in Dumortier et al. (1991). Moreover, by blowing up techniques and the integral factor, system (3.15) can be considered as a perturbation of Hamiltonian systems, in which there exist a Hamiltonian system such that the level sets of the Hamiltonian function are ovals surrounding a center. The ovals can be bifurcated at most two limit cycles by property of Abelian integral, and the limit cycles surround only one equilibrium (see Zoladek's work in Dumortier et al. 1991). Therefore, we have the following.

Theorem 3.3 *System (1.1) undergoes degenerate BT bifurcation (elliptic case) of codimension 3 in a small neighborhood of the equilibrium $E^*(x^*, y^*)$ if we fix parameters $u = u_0, \sigma = \sigma_0, d = d_0$ and let parameters (d, u, σ) vary in a small neighborhood of (d_0, u_0, σ_0), where $(d_0, u_0, \sigma_0, r_0, \alpha_0)$ satisfies (3.11). More precisely, there exist parameter values (d, u, σ) such that system (1.1) has three endemic equilibria: one is hyperbolic saddle and the other two are focus or nodes, and a homoclinic loop; there exist other parameter values (d, u, σ) such that system (1.1) has only one endemic equilibrium and at most two limit cycles.*

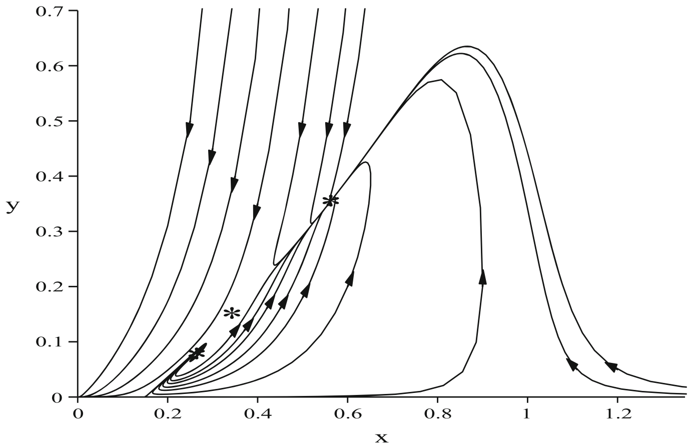


Fig. 5 The dynamics of system (1.1) with three endemic equilibria, where $r = 5$, $u = 0.15$, $d = 0.0813$, $\alpha = 1.441$ and $\sigma = 13.5899$. The three endemic equilibria are approximately $(0.257, 0.071)$, $(0.337, 0.145)$ and $(0.555, 0.347)$, which are represented by $*$

From Theorem 3.3, we know that parameters satisfying condition (3.11) form a surface $\mathcal{DBT}$ in five-dimensional parameter space $(\alpha, d, r, u, \sigma)$, where

$$\mathcal{DBT} = \left\{ (\alpha, d, r, u, \sigma) : \begin{array}{l} \alpha = \frac{\sigma(2-u)(1-2u)(u+1)}{9(u^2-u+1)}, \\ d = -\frac{(u+1)}{9(u^2-u+1)^2}\big(\sigma(1+u)^2(u^2-4u+1) \\ \qquad +u^4 - 5u^3 + 15u^2 - 5u + 1\big) > 0, \\ r = \frac{(\sigma-1)(2-u)(1-2u)(u+1)}{3(u^2-u+1)^2}, \\ \sigma > 1, \;\; \frac{1}{2} > u > 0 \end{array} \right\},$$

which is a degenerate BT bifurcation surface of codimension 3 in the elliptic case. As parameters $(\alpha, d, r, u, \sigma)$ are in $\mathcal{DBT}$, dynamics of system (1.1) is shown in Fig. 4. If parameters $(\alpha, d, r, u, \sigma)$ vary in a small neighborhood of $\mathcal{DBT}$, then system (1.1) has different dynamics depending on parameter values as described in Theorem 3.3. In the case, dynamics of (1.1) depends on not only parameters but also the initial conditions. Biologically, this conclusion shows that the host population with the strong Allee effect is very fragile as parameters $(\alpha, d, r, u, \sigma)$ vary in a small neighborhood of $\mathcal{DBT}$. There are many outcomes for the host population, such as extinction, multiple attractors, disease established or epidemic break out in periodic manner. Thus, system (1.1) is very sensitive to perturbations and control methods. Here we give two phase portraits of (1.1) with one and three endemic equilibria, respectively, to illustrate the complication of dynamics (see Figs. 5, 6).

4 BT bifurcation and Hopf bifurcation of system (1.1)

In this section, we discuss if system (1.1) can undergo BT bifurcation and Hopf bifurcation in a small neighborhood of an endemic equilibrium. We will show that BT

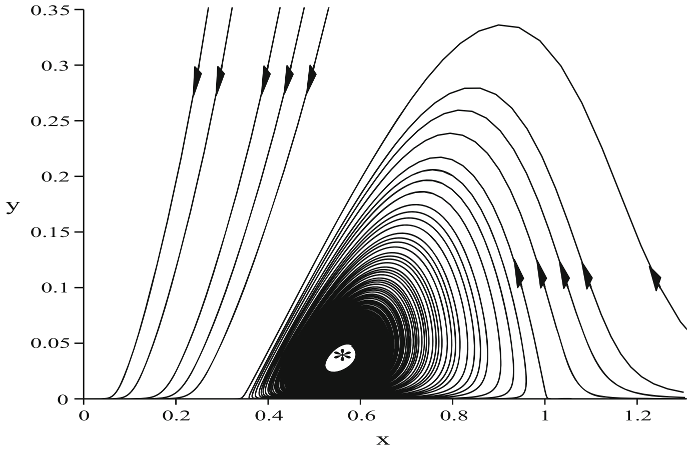


Fig. 6 The dynamics of system (1.1) with one endemic equilibrium, where $r = 1, u = 0.35, d = 1.0813, \alpha = 1.441$ and $\sigma = 6.5899$. The endemic equilibrium is approximately (0.555, 0.035), which is represented by $*$

bifurcation may occur for system (1.1) in a small neighborhood of an endemic equilibrium which is a tangent point of L and C, and Hopf bifurcation may occur for system (1.1) in a small neighborhood of an endemic equilibrium which is a cross point of L and C.

We first study BT bifurcation of system (1.1). From Theorem 2.3, we know that endemic equilibrium $E(x_1, y_1)$ is degenerate if it is a tangent point of L and C. In Sect. 3 we have shown that this equilibrium is a degenerate elliptic equilibrium of codimension 3 if it is a triple tangent point of L and C and it satisfies some conditions. In the following we will show that under some conditions endemic equilibrium $E(x_1, y_1)$ is a cusp of codimension 2 if it is a double tangent point of L and C.

Lemma 4.1 *Suppose that system* (1.1) *has an endemic equilibrium* $E(x_1, y_1)$ *which is a double tangent point of L and C and* $\sigma f(x_1) - \alpha f'(x_1) = 0$. *Then the endemic equilibrium* $E(x_1, y_1)$ *is a cusp of codimension 2 if* $E(x_1, y_1) \in \gamma_2$, *where* γ_2 *be the part of the curve C with* $x^* = \frac{u+1}{3} < x < 1$ (*see Fig.* 1).

Proof Since $E(x_1, y_1)$ is a double tangent point of L and C, and $\sigma f(x_1) - \alpha f'(x_1) = 0$, we obtain that $x_1 = P_T + \frac{\alpha}{\sigma}, \quad y_1 = \frac{\alpha(\sigma-1)}{\sigma^2}$, and

$$P_T + \frac{\alpha}{\sigma} \neq \frac{u+1}{3}, \quad 1 - \sigma = \frac{r\sigma^2}{\alpha^2}\left(P_T + \frac{\alpha}{\sigma}\right)\left(P_T + \frac{\alpha}{\sigma} - u\right)\left(P_T + \frac{\alpha}{\sigma} - 1\right).$$

We now move the endemic equilibrium $E(x_1, y_1)$ to the origin by setting $z_1 = x - x_1,\ z_2 = y - y_1$. Then system (1.1) becomes

$$\begin{cases} \dot{z}_1 = \frac{\alpha(\sigma-1)}{\sigma} z_1 - \alpha z_2 + 3r(-\frac{\alpha}{\sigma} + \frac{u+1}{3} - P_T) z_1^2 - r z_1^3, \\ \dot{z}_2 = \frac{\alpha(\sigma-1)^2}{\sigma^2} z_1 - \frac{\alpha(\sigma-1)}{\sigma} z_2 - \sigma z_2^2 + (\sigma - 1) z_1 z_2. \end{cases} \tag{4.1}$$

Making a linear transformation $X = z_1,\ Y = \frac{\alpha(\sigma-1)}{\sigma} z_1 - \alpha z_2$, system (4.1) is converted to

$$\begin{cases} \dot{X} = Y + 3r(-\frac{\alpha}{\sigma} + \frac{u+1}{3} - P_T)X^2 - rX^3, \\ \dot{Y} = \frac{3\alpha r}{\sigma}(\sigma - 1)(-\frac{\alpha}{\sigma} + \frac{u+1}{3} - P_T)X^2 - (\sigma - 1)XY + \frac{\sigma}{\alpha}Y^2 - \frac{\alpha r(\sigma-1)}{\sigma}X^3. \end{cases} \tag{4.2}$$

In a small neighborhood of the origin, we make a near identity smooth changes

$$x = X - \frac{\sigma}{2\alpha}X^2, \quad y = Y + 3r\left(-\frac{\alpha}{\sigma} + \frac{u+1}{3} - P_T\right)X^2 - \frac{\sigma}{\alpha}XY.$$

Then system (4.2) can be written as

$$\dot{x} = y + O(|x, y|^3), \qquad \dot{y} = d_1x^2 + d_2xy + O(|x, y|^3), \tag{4.3}$$

where $d_1 = \frac{3\alpha r(\sigma-1)}{\sigma}(-\frac{\alpha}{\sigma} + \frac{u+1}{3} - P_T)$ and $d_2 = -(\sigma - 1) + 6r(-\frac{\alpha}{\sigma} + \frac{u+1}{3} - P_T)$.

Since $E(x_1, y_1) \in \gamma_2$, that is $\frac{u+1}{3} < x_1 = \frac{\alpha}{\sigma} + P_T$, one has $d_1 < 0$ and $d_2 < 0$. Thus, $E(x_1, y_1)$ is a cusp of codimension 2. We finish the proof. □

We give an example to show that system (1.1) have a unique endemic equilibrium and in a small neighborhood of this endemic equilibrium, system (1.1) can undergo BT bifurcation.

Theorem 4.1 *If (2.14) holds (i.e. $P_T = u$ and $\frac{\sigma-1}{\sigma} = \frac{r}{4\alpha}$), then system (1.1) has a unique endemic equilibrium $E_{A1} = (\frac{1}{2}, \frac{r}{8\alpha}(1 - 2u))$ and three disease-free equilibria: $O(0, 0)$ is a stable node, $A(u, 0)$ is a saddle-node, and $E_0(1, 0)$ is a saddle. The endemic equilibrium E_{A1} is a cusp of codimension 2 if*

$$(\alpha, \sigma, d) = \left(\frac{r}{4} + \frac{1}{2} - u, \frac{r + 2 - 4u}{2 - 4u}, \frac{(8u^2 - 1)r - 2(2u - 1)^2}{4(1 - 2u)}\right),$$

otherwise E_{A1} is a saddle-node. The global dynamics of system (1.1) with a cusp endemic equilibrium is shown in Fig. 7.

If we choose (α, σ) as bifurcation parameters, then system (1.1) undergoes BT bifurcation in a small neighborhood of endemic equilibrium E_{A1} as (α, σ) varies near $(\frac{r}{4} + \frac{1}{2} - u, \frac{r+2-4u}{2-4u})$. Hence, there exist some parameter values such that system (1.1) has a unstable limit cycle surrounding an endemic equilibrium, and there exist some other parameter values such that system (1.1) has a unstable homoclinic loop.

Proof The first conclusion of the theorem comes from Lemmas 2.2, 2.3 and 4.1 by straighten computation. And the second conclusion of the theorem can be proved by normal form theory. Set

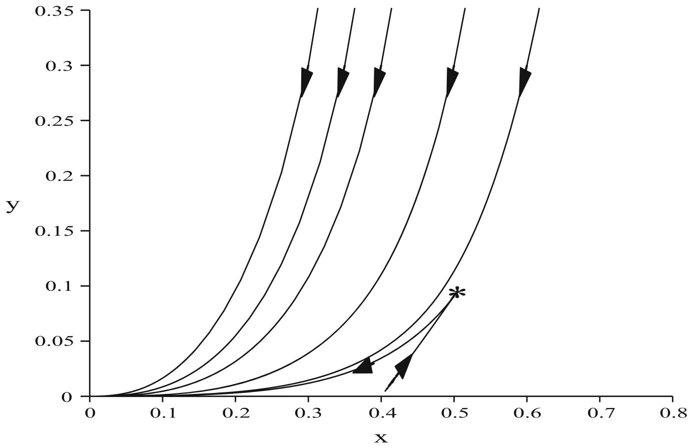


Fig. 7 The phase portrait of system (4.4) at $(\lambda_1, \lambda_2) = (0, 0)$, where $r = 4$, $u = 0.4$, $d = 1.3$, $\alpha = 1.1$ and $\sigma = 11$. The endemic equilibrium is approximately (0.5, 0.09), which is represented by $*$

$$\mathcal{BT} = \left\{ (\alpha, d, r, u, \sigma) : \begin{array}{l} \alpha = \frac{r}{4} + \frac{1}{2} - u, \\ d = \frac{(8u^2-1)r-2(2u-1)^2}{4(1-2u)} > 0, \\ \sigma = \frac{r+2-4u}{2-4u} > 1, \\ r > 0, \ \ \frac{1}{2} > u > 0 \end{array} \right\},$$

which forms a hypersurface in five-dimensional parameter space $(\alpha, d, r, u, \sigma)$.

In a small neighborhood of the endemic equilibrium E_{A1}, we consider a perturbation system of (1.1)

$$\begin{cases} \frac{dx}{dt} = r(1-x)(x-u)x - (\alpha_0 + \lambda_1)y, \\ \frac{dy}{dt} = (-(\alpha_0 + \lambda_1) - d_0 - ru + (\sigma_0 + \lambda_2 - 1)x - (\sigma_0 + \lambda_2)y)y, \end{cases} \tag{4.4}$$

where $(\alpha_0, \sigma_0, d_0) \in \mathcal{BT}$, λ_1 and λ_2 are very small parameters. When $\lambda_1 = \lambda_2 = 0$, system (4.4) has a cusp of codimension 2 at E_{A1} (see Fig. 7).

Following the process of deriving normal form, system (4.4) can be reduced to

$$\begin{cases} \frac{dx}{ds} = y, \\ \frac{dy}{ds} = \nu_1(\lambda_1, \lambda_2) + \nu_2(\lambda_1, \lambda_2)y + x^2 - (\frac{2\sqrt{2}(8u^2-8u+3)}{(1-2u)\sqrt{1-2u}} + O(\lambda))xy \\ \qquad +R(x, y, \lambda), \end{cases} \tag{4.5}$$

where $s = -kt, k > 0$, $\mu_i(\cdot)$, $O(\cdot)$ and $R(\cdot)$ are smooth functions with respective to their arguments. $O(0) = 0$, $R(x, y, \lambda)$ have Taylor expansions in (x, y) starting with at least cubic terms, and the Jacobian of (ν_1, ν_2) with respect to (λ_1, λ_2) at the origin is

$$J_\nu = \left| \frac{\partial(\nu_1, \nu_2)}{\partial(\lambda_1, \lambda_2)} \right|_{(0,0)} = \frac{4((4u-3)(5u-1)(4u-1)r + 3(1-2u)^2(8u^2-8u+1))}{r^3(r+2-4u)(1-2u)^3}.$$

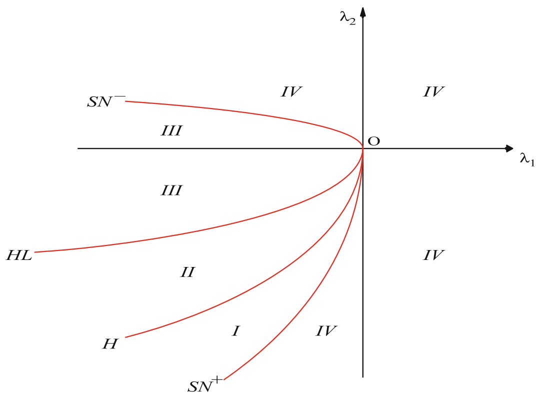


Fig. 8 The BT bifurcation diagram of system (4.4), where λ_1 and λ_2 are bifurcation parameters, $SN^{\pm}$ are saddle-node bifurcations, H is Hopf bifurcation and HL is homoclinic bifurcation

Therefore, there are values of (r, u), for instance, $(4u-3)(5u-1)(4u-1)r+3(1-2u)^2(8u^2-8u+1) \neq 0$, such that $J_\nu \neq 0$. From the classic results in Bogdanov (1981) and Takens (1974), system (4.5) undergoes BT bifurcation in a small neighborhood of endemic equilibrium E_{A1} as (λ_1, λ_2) varies in a small neighborhood of the origin. Therefore, there exist some parameter values of (λ_1, λ_2) such that system (4.5) has a unstable limit cycle surrounding an endemic equilibrium, and there exist some other parameter values such that system (4.5) has a unstable homoclinic loop. This leads to the second conclusion of the theorem. □

From Theorem 4.1, we can see that almost all orbits of system (4.4) go to the origin if $\lambda_1 = \lambda_2 = 0$, see Fig. 7. Hence, the disease can not establish and almost all host population goes to extinction even though $R_0 > 1$ and there exists an endemic equilibrium. However, as λ_1 and λ_2 vary in a small neighborhood of the origin (see Fig. 8), system (4.4) has different dynamical behaviors, for example, as λ_1 and λ_2 cross the curve SN^+, system (4.4) undergoes saddle-node bifurcation. When parameters λ_1 and λ_2 lie the region I, system (4.4) has two endemic equilibria, one is a hyperbolic saddle and the other is a unstable node or a focus. When parameters λ_1 and λ_2 cross the curve H into the region II (i.e., the region between H and H), the unstable focus becomes stable, so an unstable limit cycle appears. When parameters λ_1 and λ_2 lie on the curve HL,the limit cycle meet the saddle, change into homoclini loop. And as parameters λ_1 and λ_2 cross the curve H into the region III, system (4.4) has two endemic equilibria, one is a hyperbolic saddle and the other is a stable node or a focus (see Fig. 9). Last when parameters λ_1 and λ_2 cross the curve SN^- into the region IV, system (4.4) has no endemic equilibria. Biologically, this conclusion shows that bi-stable phenomenon occurs and the disease can establish in the host population as (λ_1, λ_2) are in the range $I \cup II \cup III$.

We now discuss Hopf bifurcation of system (1.1). From Theorem 2.3, we know that endemic equilibrium $E(x_0, y_0)$ may be a weak focus if it is a cross point of L and C, and it satisfies that $\frac{\sigma-1}{\sigma} - f'(x_0) > 0$ and $\sigma f(x_0) - \alpha f'(x_0) = 0$. In the general case we first give the existence of weak focus of system (1.1).

Lemma 4.2 *Suppose that system* (1.1) *has an endemic equilibrium* $E(x_0, y_0)$ *which satisfies* $\frac{\sigma-1}{\sigma} - f'(x_0) > 0$ *and* $\sigma f(x_0) - \alpha f'(x_0) = 0$. *Then the endemic equilibrium*

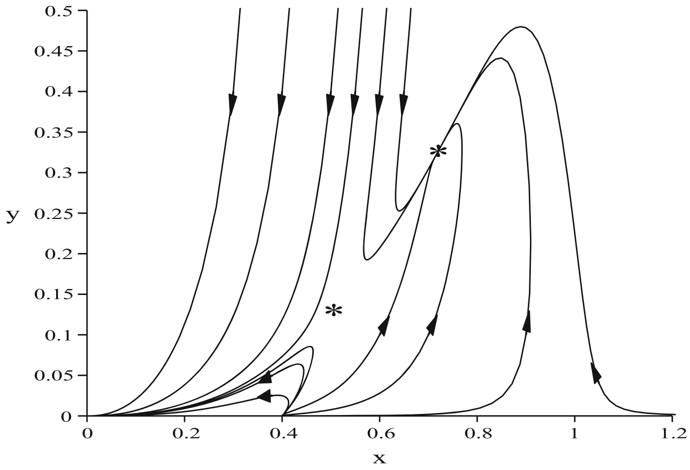


Fig. 9 The phase portrait of system (4.4) at $(\lambda_1, \lambda_2) = (0.2, -0.3)$, where $r = 4$, $u = 0.4$, $d = 1.3$, $\alpha = 0.8$ and $\sigma = 11.2$. The two endemic equilibria are approximately $(0.5, 0.125)$ and $(0.715, 0.321)$, which are represented by $*$

$E(x_0, y_0)$ is a stable (respectively unstable) weak focus of order one if $\sigma > \sigma_1$ (respectively $1 < \sigma < \sigma_1$), and $E(x_0, y_0)$ may be a weak focus of order two if $\sigma = \sigma_1$, where $\sigma_1 = \frac{B+\sqrt{B^2+12\alpha(u+1)}}{2(u+1)}$, $B = 2r + ru + 3d + 1 + u + 2ru^2$.

Proof Since $\sigma f(x_0) - \alpha f'(x_0) = 0$ and $E(x_0, y_0)$ is an endemic equilibrium of system (1.1), we can find that

$$x_0 = \frac{1}{6r}\left(2r(u+1) - (\sigma - 1) + \sqrt{(2r(u+1) - (\sigma - 1))^2 + 12r(\alpha + d)}\right),$$
$$y_0 = \frac{\sigma - 1}{\sigma}(x_0 - P_T) > 0.$$

We first move the endemic equilibrium $E(x_0, y_0)$ to the origin. Let $z_1 = x - x_0$, $z_2 = y - y_0$. Then by $\sigma f(x_0) - \alpha f'(x_0) = 0$, system (1.1) becomes

$$\begin{cases} \dot{z}_1 = (\sigma - 1)(x_0 - P_T)z_1 - \alpha z_2 + r(u + 1 - 3x_0)z_1^2 - r z_1^3, \\ \dot{z}_2 = \frac{(\sigma-1)^2}{\sigma}(x_0 - P_T)z_1 - (\sigma - 1)(x_0 - P_T)z_2 + (\sigma - 1)z_1 z_2 - \sigma z_2^2. \end{cases} \tag{4.6}$$

Note that $\sigma - 1 > 0$, $x_0 - P_T > 0$ and $\frac{\sigma-1}{\sigma} - f'(x_0) > 0$. This leads to

$$\beta = \sigma\,(x_0 - P_T)\,(\alpha - \sigma x_0 + \sigma P_T) > 0. \tag{4.7}$$

We make the following linear transformation of state variables and time variable, respectively,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{(x_0 - P_T)\sigma}{\sqrt{\beta}} \\ 0 & \frac{(x_0 - P_T)(\sigma - 1)}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \tau = \frac{\sqrt{\beta}\,(\sigma - 1)}{\sigma} t.$$

Then system (4.6) becomes

$$\begin{cases} \frac{dx}{d\tau} = -y + a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{30}x^3 + a_{03}y^3 + a_{21}x^2y + a_{12}xy^2, \\ \frac{dy}{d\tau} = x + b_{11}xy, \end{cases} \tag{4.8}$$

where

$$\begin{array}{l} a_{11} = \frac{\sigma(1+2r+2ru-\sigma-6rx_0)}{(\sigma-1)(\alpha-\sigma x_0+\sigma P_T)}; \quad a_{02} = \frac{r\sigma(u+1-3x_0)\sqrt{\beta}}{(\sigma-1)(\alpha-\sigma x_0+\sigma P_T)^2}; \quad a_{20} = \frac{r\sigma(u+1-3x_0)}{(\sigma-1)\sqrt{\beta}}; \\ a_{12} = -\frac{3r\sigma\sqrt{\beta}}{(\sigma-1)(\alpha-\sigma x_0+\sigma P_T)^2}; \; a_{21} = -\frac{3r\sigma}{(\sigma-1)(\alpha-\sigma x_0+\sigma P_T)}; \; a_{30} = -\frac{r\sigma}{(\sigma-1)\sqrt{\beta}}; \\ a_{03} = -\frac{r\sigma^2(x_0-P_T)}{(\sigma-1)(\alpha-\sigma x_0+\sigma P_T)^2}; \; b_{11} = \frac{\sigma}{\sqrt{\beta}}. \end{array}$$

We finally determine if the equilibrium (0, 0) of system (4.8) is a weak focus by successor function. It is convenient to introduce polar coordinates (ρ, θ) and to rewrite system (4.8) in polar coordinates by $x = \rho\cos\theta, \; y = \rho\sin\theta$. It is clear that in a small neighborhood of the origin the successor function $D(c_0)$ of system (4.8) can be expressed by

$$D(c_0) = \rho(2\pi, c_0) - \rho(0, c_0),$$

where $\rho(\theta, c_0)$ is the solution of the following system

$$\begin{cases} \frac{d\rho}{d\theta} = R_2(\theta)\rho^2 + R_3(\theta)\rho^3 + R_4(\theta)\rho^4 + R_5(\theta)\rho^5 + \cdots, \\ \rho(0) = c_0, \quad 0 < |c_0| \ll 1, \end{cases}$$

where

$$\begin{array}{l} R_2(\theta) = \cos\theta(a_{02} + b_{11} + a_{11}\cos\theta\sin\theta + \cos^2\theta(a_{20} - a_{02} - b_{11})), \\ R_3(\theta) = \cos\theta(a_{11}^2\cos^2\theta\sin\theta - 2a_{02}^2\cos^2\theta\sin\theta + 2a_{02}a_{11}\cos^5\theta + \\ a_{20}b_{11}\cos^2\theta\sin\theta - 2a_{20}a_{02}\cos^4\theta\sin\theta + 2a_{02}b_{11}\cos^4\theta\sin\theta + a_{02}^2\sin\theta \\ \quad +2a_{20}a_{02}\cos^2\theta\sin\theta - 2a_{20}b_{11}\cos^4\theta\sin\theta - 3a_{02}b_{11}\cos^2\theta\sin\theta \\ \quad +a_{12}\cos\theta - 2a_{20}a_{11}\cos^5\theta + b_{11}^2\cos^4\theta\sin\theta + b_{11}a_{02}\sin\theta + 2a_{20}a_{11}\cos^3\theta \\ \quad +a_{03}\sin\theta + 2a_{02}a_{11}\cos\theta + a_{20}^2\cos^4\theta\sin\theta - b_{11}^2\cos^2\theta\sin\theta - 3b_{11}a_{11}\cos^3\theta \\ \quad -a_{03}\sin\theta\cos^2\theta - a_{11}^2\cos^4\theta\sin\theta + 2b_{11}a_{11}\cos^5\theta + a_{21}\cos^2\theta\sin\theta \\ \quad b_{11}a_{11}\cos\theta - a_{12}\cos^3\theta - 4a_{02}a_{11}\cos^3\theta + a_{02}^2\cos^4\theta\sin\theta + a_{30}\cos^3\theta), \end{array}$$

and $R_i(\theta)$ is a polynomial of $(\sin\theta, \cos\theta)$, $i = 4, 5, \ldots$, whose coefficients can be expressed by the coefficients of system (4.8). We omit them here since the expressions are too long.

From the method of successor function in Andronov et al. (1971) and Zhang et al. (1992), we can obtain the first Liapunov number of the equilibrium (0, 0) of system (4.8)

$$C_1 = \frac{1}{8}(a_{30} + a_{12} + a_{11}(a_{20} + a_{02})) = \frac{r\alpha\sigma S}{8(\sigma-1)^2(\alpha-\sigma x_0+\sigma P_T)\sqrt{\beta}},$$

where $S = -(u+1)\sigma^2 + \left(2r + ru + 3d + 1 + u + 2ru^2\right)\sigma + 3\alpha$ and β is as in (4.7).

Obviously, C_1 has the same sign of S. Let us discuss the sign of S. Set $B = 2r + ru + 3d + 1 + u + 2ru^2$. Then

$$S = -(u+1)(\sigma - \sigma_1)(\sigma - \sigma_2),$$

where $\sigma_1 = \frac{B + \sqrt{B^2 + 12\alpha(u+1)}}{2(u+1)} > 1$ and $\sigma_2 = \frac{B - \sqrt{B^2 + 12\alpha(u+1)}}{2(u+1)} < 0$.

Therefore, the endemic equilibrium $E(x_0, y_0)$ is a unstable (stable) weak focus of order one if $1 < \sigma < \sigma_1$ ($\sigma > \sigma_1$, respectively).

However, if $\sigma = \sigma_1$, then $C_1 = 0$. We further compute the second Liapunov number of equilibrium (0, 0) of system (4.8) as $C_1 = 0$ and obtain

$$\begin{aligned} C_3 = {} & \frac{1}{6}a_{11}a_{02}a_{03} + \frac{1}{24}a_{02}a_{21}a_{11} + \frac{1}{16}a_{20}a_{11}a_{21} + \frac{1}{48}a_{12}a_{21} \\ & + \frac{1}{72}a_{12}a_{11}^2 + \frac{1}{16}b_{11}a_{11}a_{20}a_{02} + \frac{1}{6}a_{02}^2a_{11}a_{20} + \frac{1}{24}b_{11}a_{11}a_{03} \\ & + \frac{7}{48}a_{20}a_{11}a_{03} + \frac{1}{16}a_{12}b_{11}a_{02} + \frac{1}{16}a_{20}^2a_{02}a_{11} + \frac{1}{16}a_{12}b_{11}a_{20} \\ & + \frac{1}{16}b_{11}a_{11}a_{02}^2 + \frac{1}{6}a_{12}a_{20}a_{02} + \frac{1}{72}a_{02}a_{11}^3 + \frac{5}{48}a_{12}a_{02}^2 \\ & + \frac{1}{16}a_{12}a_{03} + \frac{1}{16}a_{12}a_{20}^2 + \frac{5}{48}a_{02}^3a_{11} + \frac{1}{72}a_{20}a_{11}^3, \end{aligned}$$

which can be expressed by parameters (r, u, α, d) of system (4.8). However, it is very long so we omit it.

If $C_3 \neq 0$, then endemic equilibrium $E(x_0, y_0)$ is a weak focus of order two, and endemic equilibrium $E(x_0, y_0)$ is stable (unstable) as $C_3 < 0$ ($C_3 > 0$, respectively). Thus, we complete the proof. □

We now provide an example to show that system (1.1) has an endemic equilibrium which is a weak focus of order two. And under a small perturbation, the system (1.1) can undergo Hopf bifurcation and degenerate Hopf bifurcation, and produce at least two limit cycles.

Theorem 4.2 *Suppose that* $(H2)$ *holds. Then in the parameters space* $(\alpha, d, r, u, \sigma)$ *there exists a hypersurface* $\mathcal{H}_0$ *and a surface* $\mathcal{H}_1 \subset \mathcal{H}_0$*, where*

$$\mathcal{H}_0 = \left\{ (\alpha, d, r, u, \sigma) : \begin{array}{c} \alpha = \frac{2}{9}\sigma(1+2u)(1-u), \\ d = \frac{1}{9}(1+2u)((2u+1)\sigma - 3r - 3), \\ \frac{8(1-u)(1+2u)(\sigma-1)}{3(4-4u+3u^2)} > r > 0, \ \sigma > 1, \\ \frac{1}{2} > u > \frac{(37+9\sqrt{17})^{\frac{2}{3}} - 2 - 2(37+9\sqrt{17})^{\frac{1}{3}}}{6(37+9\sqrt{17})^{\frac{1}{3}}} \end{array} \right\},$$

$$\mathcal{H}_1 = \left\{ (\alpha, d, r, u, \sigma) : \begin{array}{c} \alpha = \frac{2}{9}\sigma(1+2u)(1-u), \\ d = \frac{(1+2u)((4u^3+4u^2+2u-1)\sigma+2u-1-10u^2)}{9(2u^2-u+1)}, \\ r = \frac{(2-u-4u^2)(\sigma-1)}{3(1-u+2u^2)}, \ \sigma > 1, \\ \frac{1}{2} > u > \frac{(37+9\sqrt{17})^{\frac{2}{3}}-2-2(37+9\sqrt{17})^{\frac{1}{3}}}{6(37+9\sqrt{17})^{\frac{1}{3}}} \end{array} \right\},$$

such that system (1.1) *has a unique endemic equilibrium* $E(x_0, y_0)$ *as* $(\alpha, d, r, u, \sigma) \in \mathcal{H}_0$*, where* $x_0 = \frac{1+2u}{3}$*,* $y_0 = \frac{r(1-u)}{3\sigma}$*. And* $E(x_0, y_0)$ *is a weak focus of order one if* $(\alpha, d, r, u, \sigma) \in \mathcal{H}_0 \backslash \mathcal{H}_1$*, and* $E(x_0, y_0)$ *is a unstable weak focus of order two if* $(\alpha, d, r, u, \sigma) \in \mathcal{H}_1$*.*

Moreover, as parameters $(\alpha, d, r, u, \sigma)$ *vary in the hypersurface* $\mathcal{H}_0$ *and pass through the surface* $\mathcal{H}_1$*, system* (1.1) *undergoes degenerate Hopf bifurcation and produce a unstable limit cycle near* $E(x_0, y_0)$*. And as parameters* $(\alpha, d, r, u, \sigma)$ *further vary in the parameters space and pass through the hypersurface* $\mathcal{H}_0$*, system* (1.1) *undergoes Hopf bifurcation again and produce a stable limit cycle near* $E(x_0, y_0)$*. Therefore, there are parameters values such that system* (1.1) *has two limit cycles, the attracting cycle is surrounded by a repelling cycle.*

Proof If $(\alpha, d, r, u, \sigma) \in \mathcal{H}_0$, then system (1.1) becomes

$$\begin{cases} \frac{dx}{dt} = \frac{9rx(1-x)(x-u)}{2\sigma(1+2u)(1-u)} - \alpha y, \\ \frac{dy}{dt} = \sigma y \left(\frac{3(\sigma-1)x+r+1-\sigma-2u\sigma-ru+2u}{3\sigma} - y \right). \end{cases} \tag{4.9}$$

Since $0 < r < \frac{8(1-u)(1+2u)(\sigma-1)}{3(4-4u+3u^2)}$, by calculation we know that

$$\begin{cases} \frac{9rx(1-x)(x-u)}{2\sigma(1+2u)(1-u)} - \alpha y = 0, \\ \frac{3(\sigma-1)x+r+1-\sigma-2u\sigma-ru+2u}{3\sigma} - y = 0 \end{cases} \tag{4.10}$$

has only a pair of real simple root (x_0, y_0). Thus, system (4.9) has a unique endemic equilibrium $E(x_0, y_0)$, where $x_0 = \frac{1+2u}{3}$, $y_0 = \frac{r(1-u)}{3\sigma}$.

If $(\alpha, d, r, u, \sigma)$ is in a small neighborhood of $\mathcal{H}_0$, then system (4.9) has still a unique endemic equilibrium by the continuity of roots with respect to coefficients.

We now study the local dynamics of equilibrium $E(x_0, y_0)$ if $(\alpha, d, r, u, \sigma) \in \mathcal{H}_0$. Moving $E(x_0, y_0)$ to the origin, we obtain the following system

$$\begin{cases} \frac{dx}{dt} = \frac{1}{3}r(1-u)x - \frac{2}{9}\sigma(1+2u)(1-u)y - rux^2 - rx^3, \\ \frac{dy}{dt} = \frac{r}{3\sigma}(\sigma-1)(1-u)x - \frac{1}{3}r(1-u)y + (\sigma-1)xy - \sigma y^2. \end{cases} \tag{4.11}$$

It is clear that the eigenvalues of system (4.11) at the origin are a pair of purely imaginary numbers. Therefore, $E(x_0, y_0)$ is a center or weak focus.

We further claim that it is a weak focus. We first make linear transformations of state variables (x, y) and time variable t for system (4.11) respectively,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{\frac{3r}{\tilde{\beta}}} \\ 0 & \frac{\sigma-1}{\sigma}\sqrt{\frac{3r}{\tilde{\beta}}} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad \tau = \frac{1-u}{9}\sqrt{3r\tilde{\beta}}\, t,$$

where we have denoted $\tilde{\beta} = 2\sigma - 2 - 3r + 4u\sigma - 4u$ to simplify notations. We obtain the following system

$$\begin{cases} \frac{dX}{d\tau} = -Y + a_{20}X^2 + a_{11}XY + a_{02}Y^2 + a_{30}X^3 + a_{21}X^2Y \\ \qquad\quad + a_{12}XY^2 + a_{03}Y^3, \\ \frac{dY}{d\tau} = X + b_{11}XY, \end{cases} \tag{4.12}$$

where

$$\begin{aligned} a_{20} &= \frac{3u}{u-1}\sqrt{\frac{3r}{\tilde{\beta}}}, \quad a_{11} = \frac{9(2ru+\sigma-1)}{(u-1)\tilde{\beta}}, \quad a_{02} = \frac{3ra_{20}}{\tilde{\beta}}, \quad a_{21} = \frac{3ra_{11}}{2ru+\sigma-1}, \\ a_{12} &= \frac{9ra_{20}}{u\tilde{\beta}}, \quad a_{03} = \frac{ra_{21}}{\tilde{\beta}}, \quad a_{30} = \frac{a_{20}}{u}, \quad b_{11} = -\frac{a_{20}(\sigma-1)}{ru}. \end{aligned}$$

By the method of successor function, we let $X = \rho\cos\theta,\ Y = \rho\sin\theta$. Then in a small neighborhood of the origin, system (4.12) can be modified to

$$\frac{d\rho}{d\theta} = R_2(\theta)\rho^2 + R_3(\theta)\rho^3 + R_4(\theta)\rho^4 + R_5(\theta)\rho^5 + O(\rho^6), \tag{4.13}$$

where $R_i(\theta)$ is a polynomial of $(\sin\theta, \cos\theta)$, for $i = 1, 2, \ldots$.

From Lemma 4.2, we obtain the first Liapunov number of equilibrium $(0, 0)$ of system (4.12) to be,

$$C_1 = \frac{9\sqrt{3r\tilde{\beta}}(\sigma-1)(1+2u)(4u^2\sigma + 6ru^2 - 4u^2 + u\sigma - u - 3ru - 2\sigma + 2 + 3r)}{(1-u)^2\tilde{\beta}^3}.$$

It is clear that $C_1 > 0$ if $r > \frac{(2-u-4u^2)(\sigma-1)}{3(1-u+2u^2)}$, $C_1 = 0$ if $r = \frac{(2-u-4u^2)(\sigma-1)}{3(1-u+2u^2)}$ and $C_1 < 0$ if $r < \frac{(2-u-4u^2)(\sigma-1)}{3(1-u+2u^2)}$. Hence, the equilibrium $E(x_0, y_0)$ is a unstable weak focus of order one if the parameters $(\alpha, d, r, u, \sigma) \in \mathcal{H}_0^+$, where

$$\mathcal{H}_0^+ = \left\{ (\alpha, d, r, u, \sigma) : \begin{array}{c} \alpha = \frac{2}{9}\sigma(1+2u)(1-u), \\ d = \frac{1}{9}(1+2u)((2u+1)\sigma - 3r - 3), \\ \frac{8(1-u)(1+2u)(\sigma-1)}{3(4-4u+3u^2)} > r > \frac{(2-u-4u^2)(\sigma-1)}{3(1-u+2u^2)}, \\ \sigma > 1,\ \frac{1}{2} > u > \frac{(37+9\sqrt{17})^{\frac{2}{3}} - 2 - 2(37+9\sqrt{17})^{\frac{1}{3}}}{6(37+9\sqrt{17})^{\frac{1}{3}}} \end{array} \right\},$$

and the equilibrium $E(x_0, y_0)$ is a stable weak focus of order one if the parameters $(\alpha, d, r, u, \sigma) \in \mathcal{H}_0^-$, where

$$\mathcal{H}_0^- = \left\{ (\alpha, d, r, u, \sigma) : \begin{array}{l} \alpha = \frac{2}{9}\sigma(1+2u)(1-u), \\ d = \frac{1}{9}(1+2u)((2u+1)\sigma - 3r - 3), \\ \frac{(2-u-4u^2)(\sigma-1)}{3(1-u+2u^2)} > r > 0,\ \sigma > 1, \\ \frac{1}{2} > u > \frac{(37+9\sqrt{17})^{\frac{2}{3}} - 2 - 2(37+9\sqrt{17})^{\frac{1}{3}}}{6(37+9\sqrt{17})^{\frac{1}{3}}} \end{array} \right\}.$$

However, $C_1 = 0$ in the case $r = \frac{(2-u-4u^2)(\sigma-1)}{3(1-u+2u^2)}$, which is equivalent to that $(\alpha, d, r, u, \sigma) \in \mathcal{H}_1$. To determine the dynamics of the equilibrium $E(x_0, y_0)$, we have to compute the second Liapunov number of equilibrium $(0, 0)$ of system (4.12) and we obtain $C_3 = \frac{9s_1(1+2u)(1-u+2u^2)^3\sqrt{u(3+4u+8u^2)(2-u-4u^2)}}{4u^4(1-u+2u^2)(1-u)^2(3+4u+8u^2)^4}$, where

$$s_1 = 40u^3 + 54u^2 - 3u - 1.$$

Since $\frac{1}{2} > u > \frac{(37+9\sqrt{17})^{\frac{2}{3}} - 2 - 2(37+9\sqrt{17})^{\frac{1}{3}}}{6(37+9\sqrt{17})^{\frac{1}{3}}}$, one has $s_1 > 0$. Consequently, $C_3 > 0$ if $(\alpha, d, r, u, \sigma) \in \mathcal{H}_1$. So the equilibrium $E(x_0, y_0)$ is a unstable weak focus of order two if $(\alpha, d, r, u, \sigma) \in \mathcal{H}_1$. Thus, we complete the proof of the first conclusion.

Let us consider the Hopf bifurcation of system (4.9). We fix three parameters (d, σ, u) and choose (α, r) as bifurcation parameters. As parameters (α, r) vary in the hypersurface $\mathcal{H}_0$ and parameter r pass through the surface $\mathcal{H}_1$ from $\mathcal{H}_0^+$ to $\mathcal{H}_0^-$, equilibrium $E(x_0, y_0)$ changes the stability from unstable weak focus of order two to stable weak focus of order one, and it is easy to check that the transversal condition holds. Hence, system (4.9) undergoes degenerate Hopf bifurcation and produce a unstable limit cycle. As parameters (α, r) vary in the parameters space and parameter α passes through the hypersurface $\mathcal{H}_0$ from $\alpha < \frac{2}{9}\sigma(1+2u)(1-u)$ to $\alpha > \frac{2}{9}\sigma(1+2u)(1-u)$, the equilibrium $E(x_0, y_0)$ changes the stability again from stable weak focus of order one to unstable hyperbolic focus and the transversal condition of eigenvalues with respect to α holds. Thus, system (1.1) undergoes Hopf bifurcation and produce a stable limit cycle. Hence, in parameters space $(\alpha, d, r, u, \sigma)$, $\mathcal{H}_0$ is the Hopf bifurcation hypersurface and $\mathcal{H}_1$ is the degenerate Hopf bifurcation surface.

Therefore, there exist values of the parameters such that system (1.1) has a unique endemic equilibrium and two limit cycles surrounding this equilibrium (see Fig. 10). □

From Theorem 4.2, we can see that system (1.1) has a unique endemic equilibrium, a stable limit cycle and a unstable limit cycle for some values of parameters. Thus, tristable phenomenon occurs and epidemic can be observed to break out in a periodic manner for some initial values of the host population.

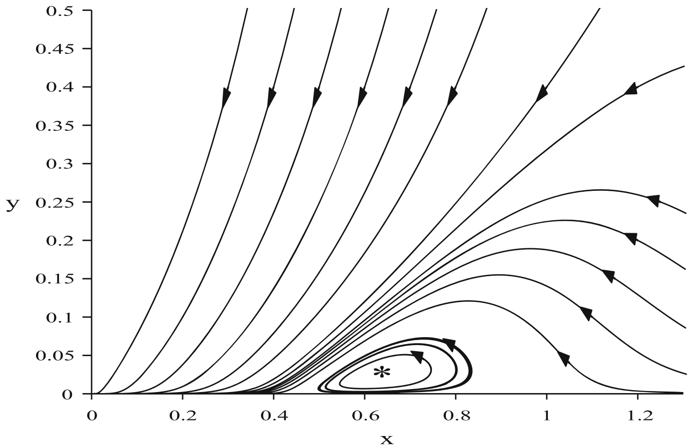


Fig. 10 System (1.1) has two limit cycles, where $r = 0.2582897033$, $u = 0.45$, $d = 0.0053054102$, $\alpha = 0.4644444444$ and $\sigma = 2$. The endemic equilibrium is approximately (0.63333, 0.02367), which is represented by $*$

5 Discussion and conclusion

In this paper we completely study the bifurcations and dynamical behavior of an epidemiological model with strong Allee effect. The qualitative conclusions presented for the model support the numerical bifurcation analyses and conjectures in Hilker et al. (2009). Hence, we highlight the impact of both disease and strong Allee effect on population persistence observed in Hilker et al. (2009). Moreover, we gain new bifurcations phenomena such as pitchfork bifurcation, BT bifurcation of codimension two, degenerate Hopf bifurcation and degenerated BT bifurcation of codimension three in elliptic case. These rich bifurcations exhibit complicated dynamical behavior of the model, for example, the multiple attractors, homoclinic loop, one limit cycle, two limit cycles and etc.. When parameters $(\alpha, d, r, u, \sigma)$ vary in a small neighborhood of pitchfork bifurcation surface, model (1.1) has either an endemic equilibrium or three endemic equilibria. Hence, in this case the disease can establish in the host population. When parameters $(\alpha, d, r, u, \sigma)$ vary in a small neighborhood of BT bifurcation surface, model (1.1) has two endemic equilibria and a unstable limit cycle or homoclinic loop for corresponding values of parameters, which implies that one endemic equilibrium becomes an attractor. Thus, the infected population will tend to a constant for the initial number of host populations in an open set, and sustained oscillation can not be observed. And when parameters $(\alpha, d, r, u, \sigma)$ vary in a small neighborhood of degenerate Hopf bifurcation surface, model (1.1) has two limit cycles for some values of parameters, one of which is stable. This leads that epidemic can be observed to break out in a rather reproducible periodic manner in the host population. These conclusions reveal that the dynamics of model (1.1) is very sensitive to parameters perturbation. Hence, biologically our results have important consequences for disease control and biological conservation. And it also provides, at least in theory, a key to understanding the impact of diseases and Allee effect for the threatened populations or the success of re-introductions, which may help us to prevent both epidemics and Allee effects driven extinctions.

Mathematically, in this paper we provide an approach to seek possible equilibria at which bifurcations occur. These equilibria are tangent points of two nullclines for two dimensional mathematical models. Moreover, we give explicit smooth transformations to reduce a system with an nilpotent equilibrium into a normal form of codimension 3, which is very useful for analysis of degenerate BT bifurcation. And the method to study degenerate Hopf bifurcation of codimension 2 of system (1.1) has its generality, which can also be applied to investigate the other related mathematical models. Therefore, the methods developed in our paper are some approaches to study bifurcations and dynamics of two dimensional mathematical models with multi-parameters.

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