J. Math. Biol. 52, 235–276 (2006)
Digital Object Identifier (DOI):
10.1007/s00285-005-0349-z

Mathematical Biology

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Catastrophic shifts in vertical distributions of phytoplankton

The existence of a bifurcation set

Received: 5 February 2004 / Revised version: 31 May 2005
Published online: 29 September 2005 –

Abstract. A model of phytoplankton dynamics within a water column was analyzed with special consideration on the existence of a bifurcation set in the parameter space. We considered two resources, light and a limiting nutrient, for phytoplankton growth and assumed that the water column is separated into two layers by thermal and/or density stratification. It was shown that there exists a bifurcation set in the parameter space when the growth function meets several conditions that are general for growth functions of two essential resources. Specifically, these conditions include that a less abundant of the two resources limits the growth while the effect of the other is sufficiently small. Folded structure with two stable states separated by one unstable state appears in the catastrophe manifold when parameters move to a certain direction with a certain curvature from a point in the bifurcation set. These results suggest that occurrence of discontinuous transition between two alternative vertical patterns is possible nature of phytoplankton dynamics within a stratified water column.

1. Introduction

Phytoplankton, the primary producer of aquatic ecosystems, show strong heterogeneity in time and space, including spring blooms, horizontal patchiness, surface scums, and subsurface maxima. Among them, the heterogeneity in vertical axes has been considered one of the most significant for ecosystem functioning and global biogeochemical cycling and studied extensively for over forty years [12]. The vertical distribution patterns of phytoplankton are formed and maintained by both physical (e.g., sinking of cells; vertical mixing) and biological (e.g., *in situ* growth; grazing; adaptive strategies) factors. While many simulation models have successfully reproduced the development of vertical patterns, there are also studies that focus on qualitative features and mathematical aspects of this subject [4, 11, 14].

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Key words or phrases: Catastrophic shift – Bifurcation set – Phytoplankton model – Vertical distribution

There has been increasing interest in catastrophic shifts in ecological systems in recent years [9, 10]. For the management of ecosystems, it is crucial to examine various systems whether they would include bistability and catastrophic structure both theoretically and empirically. Though several systems have been extensively studied and suggested to have bistable states and catastrophic shifts, examples of such ecological systems are still limited (see reviews in [9, 10]).

Vertical distributions of phytoplankton are classified roughly into two patterns in stratified water columns, surface chlorophyll maxima and subsurface chlorophyll maxima (Figure 1). In [15], it was suggested that these two patterns can exist as bistable states and that a catastrophic shift can occur between these two patterns. This was the first report that suggests that the peak of vertical distributions of phytoplankton can shift discontinuously with gradual changes in environmental conditions and algal blooms may occur in a catastrophic fashion. Results were obtained by numerical experiments of a mathematical model that considers two vertical layers, a surface layer and a deep layer. Phytoplankton growth in the model was assumed to be a function of two resources, light and a limiting nutrient, and the two resources were assumed to be strictly essential, that is, a less abundant resource exclusively limits the growth and the other does not affect the growth. Though this kind of formulation is common in phytoplankton models [5, 13], simultaneous limitations by two or more resources for phytoplankton growth are often observed in natural conditions, implying requirements for other formulations to generalize the results.

In the present paper, we analyze a model of phytoplankton dynamics within two well-mixed vertical layers and show the existence of a bifurcation set in the parameter space. The model is formulated as four simultaneous nonlinear integro-differential equations with general phytoplankton growth function of two resources. Instead of analyzing 4×4 Jacobian directly, we heuristically construct a scalar function K which decreases monotonously with time (lemma 2) and whose critical points correspond exactly to steady states of the model equations (lemma 3). A subset of the model parameter space [$\hat{\mathbf{C}}$, defined in (23)] is shown to be a bifurcation set from which two stable and one unstable steady states bifurcate when parameters move to a certain direction and curvature (theorem 1). The existence of the bifurcation set is shown when the growth function satisfies an additional assumption (theorem 2). All proofs of theorems and lemmas are given in the section 6.

2. Model

2.1. Model equations

We consider phytoplankton population within a water column. The water column is assumed to be separated into two layers, a surface layer and a deep layer, by strong thermal and/or density stratification that is typical in temperate lakes and oceans in summer, or estuaries with fresh water inflow. The vertical position is expressed as the depth from the surface ($z = 0$) to the bottom ($z = z_B$). Both layers are assumed to be completely mixed, that is, all particles are distributed homogeneously. We assume that phytoplankton cells are neutrally buoyant and the growth is expressed as a function of two resources, light and a limiting nutrient, and that the nutrient

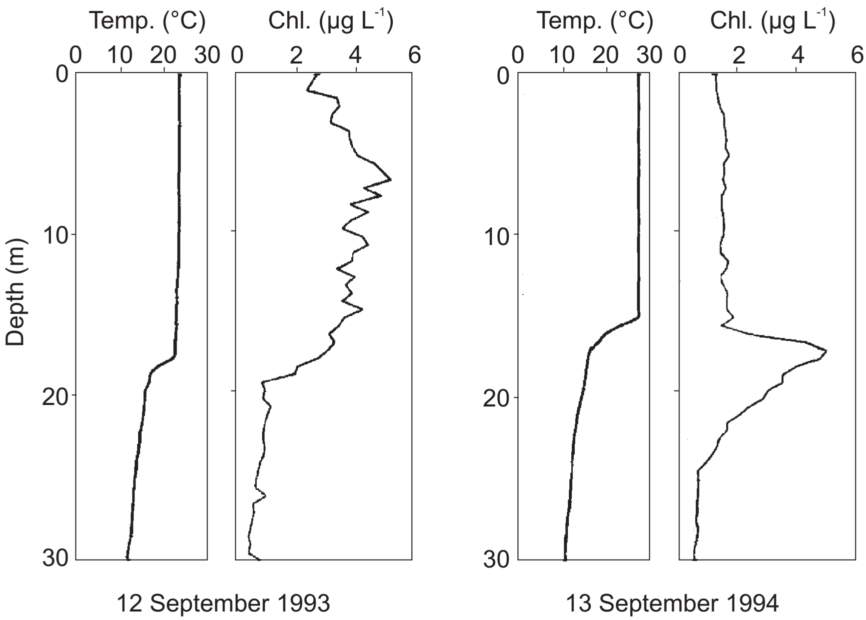

Fig. 1. Examples of surface and subsurface chlorophyll maxima (modified from [7]). Two sets of vertical profiles of water temperature and chlorophyll were taken in September 1993 and 1994 at the same location in Lake Biwa, Japan. Though similar thermal stratification was seen in each year, chlorophyll profiles were significantly different; a surface chlorophyll maximum was observed in 1993 and a subsurface chlorophyll maximum was observed in 1994

diffuses from the bottom sediment of the water column. We ignore the other nutrient sources and nutrient regeneration.

Let P_S and N_S be phytoplankton population density and the limiting nutrient concentration in the surface layer, P_D and N_D, those in the deep layer, and $I(z)$, light intensity at each depth. Model equations for the dynamics of phytoplankton and the nutrient are given by

$$\begin{aligned}
\dot{P}_S &= \frac{1}{z_T}\left\{\gamma D_T(P_D - P_S) + P_S\int_0^{z_T}[f(N_S, I) - \theta]dz\right\},\\
\dot{P}_D &= \frac{1}{z_B - z_T}\left\{-\gamma D_T(P_D - P_S) + P_D\int_{z_T}^{z_B}[f(N_D, I) - \theta]dz\right\},\\
\dot{N}_S &= \frac{1}{z_T}\left[D_T(N_D - N_S) - \alpha P_S\int_0^{z_T} f(N_S, I)dz\right],\\
\dot{N}_D &= \frac{1}{z_B - z_T}\left[D_B(N_B - N_D) - D_T(N_D - N_S) - \alpha P_D\int_{z_T}^{z_B} f(N_D, I)dz\right],
\end{aligned} \tag{1}$$

where the dots over variables stand for time derivatives, $f(N, I)$ is the growth rate of phytoplankton, N_B, the nutrient concentration at the bottom sediment, z_T, the thickness of the surface layer, α, the conversion factor from P to N, θ, the loss rate of phytoplankton, D_T, the diffusion coefficient of nutrient between the two layers, D_B, the sediment-water column permeability, and γ, the ratio of diffusion coefficient of phytoplankton cells to that of nutrient particles. The right-hand sides of (1) are averages of the sum of physical transport and biological reaction (growth/nutrient consumption) in two layers.

In turbulent conditions, the diffusivity of phytoplankton cells and nutrient particles will be the same. In contrast in non-turbulent conditions, the diffusivity of phytoplankton cells is much smaller than nutrient particles due to their particle sizes [6]. In strongly stratified waters, turbulence is segmented by non-turbulent layers and the diffusive transport will be restricted by the non-turbulent layers [1]. Thus the relative diffusivity between surface and deep layers, γ, will change between 0 and 1 depending on the strength of stratification.

The ambient light intensity, $I(z)$, is described as

$$I(z) = \begin{cases} I_0 \exp[-(r + sP_S)z], & \text{for } z \in [0, z_T], \\ I_0 \exp\{-[(r + sP_S)z_T + (r + sP_D)(z - z_T)]\}, & \text{for } z \in (z_T, z_B], \end{cases} \tag{2}$$

where I_0 is the light intensity just below the surface, r, the background light extinction coefficient, and s, the self-shading coefficient of phytoplankton.

By scaling of variables and parameters we have

$$\begin{aligned} \dot{p}_S &= \frac{1}{\zeta_T}\left\{\gamma d_T(p_D - p_S) + p_S\int_0^{\zeta_T}[\varphi(n_S, i) - 1]d\zeta\right\}, \\ \dot{p}_D &= \frac{1}{1-\zeta_T}\left\{-\gamma d_T(p_D - p_S) + p_D\int_{\zeta_T}^{1}[\varphi(n_D, i) - 1]d\zeta\right\}, \\ \dot{n}_S &= \frac{1}{\zeta_T}\left[d_T(n_D - n_S) - \alpha' p_S\int_0^{\zeta_T}\varphi(n_S, i)d\zeta\right], \\ \dot{n}_D &= \frac{1}{1-\zeta_T}\left[d_B(n_B - n_D) - d_T(n_D - n_S) - \alpha' p_D\int_{\zeta_T}^{1}\varphi(n_D, i)d\zeta\right], \end{aligned} \tag{3}$$

where N_S, N_D, I_0, and $I(z)$ are scaled to n_S, n_D, i_0, and $i(\zeta)$ to be dimensionless, $\varphi(n, i) = f(N, I)/\theta$, $\zeta = z/z_B$, $\zeta_T = z_T/z_B$, $p_S = sz_BP_S$, $p_D = sz_BP_D$, $r' = rz_B$, $\alpha' = (\alpha n_S)/(sz_BN_S)$, $d_T = D_T/(\theta z_B)$, and $d_B = D_B/(\theta z_B)$. The scaled light intensity is described as

$$i(\zeta) = \begin{cases} i_0 \exp[-(r' + p_S)\zeta], & \text{for } \zeta \in [0, \zeta_T], \\ i_0 \exp\{-[(r' + p_S)\zeta_T + (r' + p_D)(\zeta - \zeta_T)]\}, & \text{for } \zeta \in (\zeta_T, 1]. \end{cases} \tag{4}$$

In the following, we will omit the primes of parameters in (3) and (4). We define $\mathbf{C}$ as the parameter space of (3) and refer a point in $\mathbf{C}$ as $\mathbf{c} = (c_1, c_2, \ldots)$.

The above equations (3) are rewritten in simplified form

$$\begin{aligned} \dot{p}_S &= \frac{\gamma d_T}{\zeta_T}(p_D - p_S) + (F_S - 1)p_S, \\ \dot{p}_D &= -\frac{\gamma d_T}{1-\zeta_T}(p_D - p_S) + (F_D - 1)p_D, \end{aligned}$$

$$\dot{n}_S = \frac{d_T}{\zeta_T}(n_D - n_S) - \alpha F_S p_S,$$
$$\dot{n}_D = \frac{d_B}{1-\zeta_T}(n_B - n_D) - \frac{d_T}{1-\zeta_T}(n_D - n_S) - \alpha F_D p_D, \tag{5}$$

where

$$F_S = \frac{1}{\zeta_T}\int_0^{\zeta_T} \varphi(n_S, i)d\zeta, \tag{6}$$
$$F_D = \frac{1}{1-\zeta_T}\int_{\zeta_T}^{1} \varphi(n_D, i)d\zeta. \tag{7}$$

From (4) and (6), F_S is a function of p_S, n_S, and $\mathbf{c}$; from (4) and (7), F_D is a function of p_S, p_D, n_D, and $\mathbf{c}$. We specify them as $F_S(p_S, n_S, \mathbf{c})$ and $F_D(p_S, p_D, n_D, \mathbf{c})$, respectively. All parameters are assumed to be within $\mathbf{R}_+ = (0, \infty)$ except γ which is assumed to be within $\bar{\mathbf{R}}_+ = [0, \infty)$.

2.2. *Change of variables*

Here we apply the following changes of variables to (5)

$$u_S = n_S - n_B + \alpha\left[\frac{(d_T + d_B)\zeta_T}{d_T d_B} + \gamma\right] p_S + \alpha\left(\frac{1-\zeta_T}{d_B} - \gamma\right) p_D, \tag{8}$$
$$u_D = n_D - n_B + \frac{\alpha\zeta_T}{d_B} p_S + \frac{\alpha(1-\zeta_T)}{d_B} p_D, \tag{9}$$

and we have

$$\dot{p}_S = \frac{\gamma d_T}{\zeta_T}(p_D - p_S) + (G_S - 1)p_S,$$
$$\dot{p}_D = -\frac{\gamma d_T}{1-\zeta_T}(p_D - p_S) + (G_D - 1)p_D,$$
$$\dot{u}_S = -\frac{d_T}{\zeta_T}u_S + \frac{d_T}{\zeta_T}u_D + \alpha\left[\frac{(d_T + d_B)\zeta_T}{d_T d_B} - 1 + \gamma\right]\dot{p}_S + \alpha\left(\frac{1-\zeta_T}{d_B} - \gamma\right)\dot{p}_D,$$
$$\dot{u}_D = \frac{d_T}{1-\zeta_T}u_S - \frac{d_T + d_B}{1-\zeta_T}u_D + \frac{\alpha\zeta_T}{d_B}\dot{p}_S + \alpha\left(\frac{1-\zeta_T}{d_B} - 1\right)\dot{p}_D, \tag{10}$$

where

$$G_S(p_S, p_D, u_S, \mathbf{c}) = F_S\left\{p_S, u_S + n_B - \alpha\left[\frac{(d_T + d_B)\zeta_T}{d_T d_B} + \gamma\right]p_S - \alpha\left(\frac{1-\zeta_T}{d_B} - \gamma\right)p_D, \mathbf{c}\right\}, \tag{11}$$
$$G_D(p_S, p_D, u_D, \mathbf{c}) = F_D\left[p_S, p_D, u_D + n_B - \frac{\alpha\zeta_T}{d_B}p_S - \frac{\alpha(1-\zeta_T)}{d_B}p_D, \mathbf{c}\right]. \tag{12}$$

By this change of variables, $u_S = u_D = 0$ for any steady states of (10).

2.3. General assumptions for growth function

Here we make several assumptions for the normalized growth function $\varphi(n, i)$ and the derivatives:

(A1) φ is a continuous function for $(n, i) \in \bar{\mathbf{R}}^2_+$ and differentiable for almost all $(n, i) \in \bar{\mathbf{R}}^2_+$,
(A2) $\varphi(n, i) \geq 0$ and $\varphi(0, i) = \varphi(n, 0) = 0$,
(A3) $\partial\varphi/\partial n$, $\partial\varphi/\partial i \geq 0$ and at least one of $\partial\varphi/\partial n$ and $\partial\varphi/\partial i$ is positive for all $(n, i) \in \bar{\mathbf{R}}^2_+$,
(A4) there exists $m_1 > 0$ such that $\partial\varphi/\partial n$, $\partial\varphi/\partial i < m_1$ for all $(n, i) \in \bar{\mathbf{R}}^2_+$,
(A5) there exists $m_2 > 0$ and $\psi > 0$ such that either $\partial\varphi/\partial n > m_2$ or $\partial\varphi/\partial i > m_2$ when $\varphi(n, i) < 1 + \psi$,
(A6) $\partial^2\varphi/\partial n^2$, $\partial^2\varphi/\partial i^2 \leq 0$ for all $(n, i) \in \bar{\mathbf{R}}^2_+$.

All these assumptions are general for any growth function. Easy deduction of (A5) gives

(A5') $\varphi(n, i) > 1$ for $n, i > 1/m_2$.

3. Conditions for a bifurcation set

In this section, we will show conditions for a subset of the parameter space to be a bifurcation set.

Firstly we define $\mathbf{C}^+$ as a subset of the parameter space

$$\mathbf{C}^+ = \left\{ \mathbf{c} \mid G_S(0, 0, 0, \mathbf{c}) > 1 + \frac{\gamma d_T}{\zeta_T} \right\}, \tag{13}$$

$\mathbf{C}^{++}$ as a certain subset of $\mathbf{C}^+$ that satisfies

$$\underline{G_S}(\mathbf{C}^{++}) > 1 + \frac{\gamma d_T}{\zeta_T}, \tag{14}$$

where $\underline{G_S}(\mathbf{C}^{++})$ denotes the maximum lower bound of $G_S(0, 0, 0, \mathbf{c})$ for $\mathbf{c} \in \mathbf{C}^{++}$, and $\Phi_S(p_S, p_D, \mathbf{c})$ as

$$\Phi_S(p_S, p_D, \mathbf{c}) = \frac{\gamma d_T}{\zeta_T}(p_D - p_S) + [G_S(p_S, p_D, 0, \mathbf{c}) - 1]\, p_S. \tag{15}$$

For $\mathbf{C}^+$ and $\mathbf{C}^{++}$, we have the following lemma:

Lemma 1. *For* $\mathbf{c} \in \mathbf{C}^+$, *there is* $\delta_0(\mathbf{c}) > 0$ *such that a map* $\bar{p}_S(p_D, \mathbf{c}) > 0$ *is defined implicitly by*

$$\Phi_S(\bar{p}_S, p_D, \mathbf{c}) = 0,$$

for a region, $[0, \delta_0(\mathbf{c})) \times \mathbf{C}^+$.

For $\mathbf{c} \in \mathbf{C}^{++}$, *there is* $\delta_0^* > 0$ *such that a map* $\bar{p}_S(p_D, \mathbf{c}) > 0$ *is defined implicitly by*

$$\Phi_S(\bar{p}_S, p_D, \mathbf{c}) = 0,$$

for a region, $[0, \delta_0^*) \times \mathbf{C}^{++}$.

The proof will be seen in the section 6.1.

For the region where $\bar{p}_S(p_D, \mathbf{c})$ is defined, $\tilde{G}_S$, $\tilde{G}_D$, and Φ_D are defined as

$$\tilde{G}_S(p_D, \mathbf{c}) = G_S[\bar{p}_S(p_D, \mathbf{c}), p_D, 0, \mathbf{c}], \tag{16}$$

$$\tilde{G}_D(p_D, \mathbf{c}) = G_D[\bar{p}_S(p_D, \mathbf{c}), p_D, 0, \mathbf{c}], \tag{17}$$

$$\Phi_D(p_D, \mathbf{c}) = -\tfrac{\gamma d_T}{1-\zeta_T}[p_D - \bar{p}_S(p_D, \mathbf{c})] + [\tilde{G}_D(p_D, \mathbf{c}) - 1]p_D. \tag{18}$$

We apply another change of variables $\rho = p_S - \bar{p}_S(p_D, \mathbf{c})$ for $(p_D, \mathbf{c}) \in [0, \delta_0^*) \times \mathbf{C}^{++}$ to (10) and get

$$\begin{aligned}
\dot{\rho} &= \frac{\gamma d_T}{\zeta_T}[p_D - (\rho + \bar{p}_S)] + [G_S(\rho + \bar{p}_S, p_D, u_S, \mathbf{c}) - 1](\rho + \bar{p}_S) - \frac{\partial \bar{p}_S}{\partial p_D}\dot{p}_D, \\
\dot{p}_D &= -\frac{\gamma d_T}{1-\zeta_T}[p_D - (\rho + \bar{p}_S)] + \{G_D[(\rho + \bar{p}_S), p_D, u_D, \mathbf{c}] - 1\}p_D, \\
\dot{u}_S &= -\frac{d_T}{\zeta_T}u_S + \frac{d_T}{\zeta_T}u_D + \alpha\left[\frac{(d_T + d_B)\zeta_T}{d_T d_B} - 1 + \gamma\right]\left(\dot{\rho} + \frac{\partial \bar{p}_S}{\partial p_D}\dot{p}_D\right) \\
&\quad + \alpha\left(\frac{1-\zeta_T}{d_B} - \gamma\right)\dot{p}_D, \\
\dot{u}_D &= \frac{d_T}{1-\zeta_T}u_S - \frac{d_T + d_B}{1-\zeta_T}u_D + \frac{\alpha\zeta_T}{d_B}\left(\dot{\rho} + \frac{\partial \bar{p}_S}{\partial p_D}\dot{p}_D\right) + \alpha\left(\frac{1-\zeta_T}{d_B} - 1\right)\dot{p}_D.
\end{aligned} \tag{19}$$

We define $K(\rho, p_D, u_S, u_D, \mathbf{c})$ as

$$K = (\rho \quad u_S \quad u_D)Q\begin{pmatrix}\rho \\ u_S \\ u_D\end{pmatrix} - \kappa\int_0^{p_D}\Phi_D(x, \mathbf{c})dx, \tag{20}$$

where Q is a positive definite matrix and $\kappa > 0$, and a region R_a as

$$R_a = \{(\rho, p_D, u_S, u_D) \mid 0 \le p_D < \delta_1 < \delta_0^*, |\rho|, |u_S|, |u_D| < \delta_2\}, \tag{21}$$

where $\delta_1 > 0$ and $\delta_2 > 0$. We also define another parameter set, $\mathbf{C}^{++}_{\gamma^*}$, that satisfies $\underline{G_S}(\mathbf{C}^{++}_{\gamma^*}) > (1 + \gamma d_T/\zeta_T)$ and $\gamma \in [0, \gamma^*)$ for all $\mathbf{c} \in \mathbf{C}^{++}_{\gamma^*}$.

In the next lemma we will show that K decreases monotonously with time in R_a when γ is sufficiently small.

Lemma 2. *There exist $\delta_1 > 0$, $\delta_2 > 0$, $\kappa > 0$, $\gamma^* > 0$ and a positive definite matrix Q such that $\dot{K} \le 0$ for $(\rho, p_D, u_S, u_D) \in R_a$ and $\mathbf{c} \in \mathbf{C}^{++}_{\gamma^*}$.*

The proof will be seen in the section 6.2

Here we define another region R_b as

$$R_b = \{(\rho, p_D, u_S, u_D) \in R_a \mid K(\rho, p_D, u_S, u_D, \mathbf{c}) < K^*\}, \tag{22}$$

for $\mathbf{c} \in \mathbf{C}^{++}_{\gamma^*}$ and $K^* > 0$.

Lemma 3. *For* $\mathbf{c} \in \mathbf{C}_{\gamma^*}^{++}$, *if there is* $K^* > 0$ *such that the closure of* R_b *is a subset of* R_a, *then all orbits of (19) that passing through points inside* R_b *converge to one of the points that satisfies the equality*

$$\frac{\partial K}{\partial \rho} = \frac{\partial K}{\partial u_S} = \frac{\partial K}{\partial u_D} = \frac{\partial K}{\partial p_D} = 0,$$

and the point is asymptotically stable if

$$\frac{\partial^2 K}{\partial p_D^2} > 0,$$

and unstable if

$$\frac{\partial^2 K}{\partial p_D^2} < 0.$$

The proof will be seen in the section 6.3. According to the lemma 3, the system (19) converges to one of the critical points of K. The critical points are stable if they are local minima of K and unstable if they are local maxima of K.

We define a subset of the parameter space, $\hat{\mathbf{C}} \subset \mathbf{C}^+$ as

$$\mathbf{c} \in \hat{\mathbf{C}} \Leftrightarrow \begin{cases} \gamma = 0, \\ \tilde{G}_D(0, \mathbf{c}) = 1, \\ \frac{\partial \tilde{G}_D(0,\mathbf{c})}{\partial p_D} = 0, \\ \frac{\partial^2 \tilde{G}_D(0,\mathbf{c})}{\partial {p_D}^2} < 0, \\ \frac{\partial \tilde{G}_D(0,\mathbf{c})}{\partial \mathbf{c}} \neq \mathbf{0}, \\ \frac{\partial^2 \tilde{G}_D(0,\mathbf{c})}{\partial p_D \partial \mathbf{c}} \neq \mathbf{0}, \\ \frac{\partial \tilde{G}_D(0,\mathbf{c})}{\partial \mathbf{c}} \not\propto \frac{\partial^2 \tilde{G}_D(0,\mathbf{c})}{\partial p_D \partial \mathbf{c}}, \end{cases} \tag{23}$$

where the symbol $\not\propto$ denotes that the two vectors are linearly independent.

For $\hat{\mathbf{C}}$, we have the following two lemmas:

Lemma 4. $\underline{G_S}(\hat{\mathbf{C}}) > 1 + \gamma d_T/\zeta_T$.

Lemma 5. *There exists* $K^* > 0$ *such that the closure of* R_b *is a subset of* R_a *for a neighborhood of* $\mathbf{c} \in \hat{\mathbf{C}}$.

The proofs will be seen in the sections 6.4 and 6.5. We refer the neighborhood of $\mathbf{c} \in \hat{\mathbf{C}}$ in the lemma 5 as $\nu(\mathbf{c})$ and the sum of $\nu(\mathbf{c})$ as $\hat{\mathbf{C}}'$. According to the lemma 4 and $\gamma = 0$ for $\mathbf{c} \in \hat{\mathbf{C}}$, statements in the lemmas 1–3 can be applied to $\hat{\mathbf{C}}'$.

Finally we have the theorem:

Theorem 1. $\hat{\mathbf{C}}$ *is a bifurcation set from which two stable steady states and one unstable steady state bifurcate.*

The proof will be see in the section 6.6.

For $K(\rho, p_D, u_S, u_D, \mathbf{c})$, a catastrophe manifold [8] is defined as the set of critical points within $R_b \times \nu(\mathbf{c})$ for $\mathbf{c} \in \hat{\mathbf{C}}$:

$$\left(\frac{\partial K}{\partial \rho}, \quad \frac{\partial K}{\partial p_D}, \quad \frac{\partial K}{\partial u_S}, \quad \frac{\partial K}{\partial u_D}\right)\Bigg|_{(\rho, p_D, u_S, u_D, \mathbf{c})} = \mathbf{0}. \tag{24}$$

The top panel of Figure 2 is a sketch of an orthogonal projection of the catastrophe manifold (*catastrophe map*, [8]) to (i_0, n_B) plane based on Eqs. (143)–(146). We can see that two bifurcation sets extend to a same direction with a same curvature from $\hat{\mathbf{c}} \in \hat{\mathbf{C}}$ like a hooked beak. When parameters move along the curve, three critical points, two stable and one unstable, appear in different orders of magnitude (bottom panels of Fig. 2).

4. Existence of the catastrophe bifurcation set

We have shown that $\hat{\mathbf{C}}$, if exists, is a bifurcation set in the previous section. Our next task is to show that the existence of $\hat{\mathbf{C}}$.

First we will give an assumptions for the phytoplankton growth function $\varphi(n, i)$ in addition to the assumptions (A1)–(A6):

(A7) for a function $h(n)$ that satisfies $h(0) = 0$ and $dh/dn > 0$:

$$\frac{\partial \varphi}{\partial i} > 0, \quad \text{for } i < h(n),$$
$$\frac{\partial \varphi}{\partial n} > 0, \quad \text{for } n < h^{-1}(i),$$

there exists $\delta \in [0, 1]$ such that

$$\frac{\partial \varphi}{\partial n} \leq \delta \frac{\partial \varphi}{\partial i}, \quad \text{for } i \leq (1 - \delta)h(n),$$
$$\frac{\partial \varphi}{\partial i} \leq \delta \frac{\partial \varphi}{\partial n}, \quad \text{for } n \leq (1 - \delta)h^{-1}(i).$$

Here we define $\mathcal{F}_\delta$ as a set of functions that satisfy (A1)–(A7) for $\delta \geq 0$. The function $h(n)$ separates (n, i) plane into two parts: a light limited region $i < h(n)$ and a nutrient limited region $n < h^{-1}(i)$. The case $\delta = 0$ in (A7) corresponds to growth functions of two strictly essential resources, that is, only one resource exclusively limits the phytoplankton growth [13].

The next theorem states that there exists $\hat{\mathbf{C}}$ in the parameter space when we take δ sufficiently small.

Theorem 2. *When we take δ sufficiently small, there is the bifurcation set $\hat{\mathbf{C}}$ in the parameter space for $\varphi(n, i) \in \mathcal{F}_\delta$.*

The proof will be seen in the section 6.7

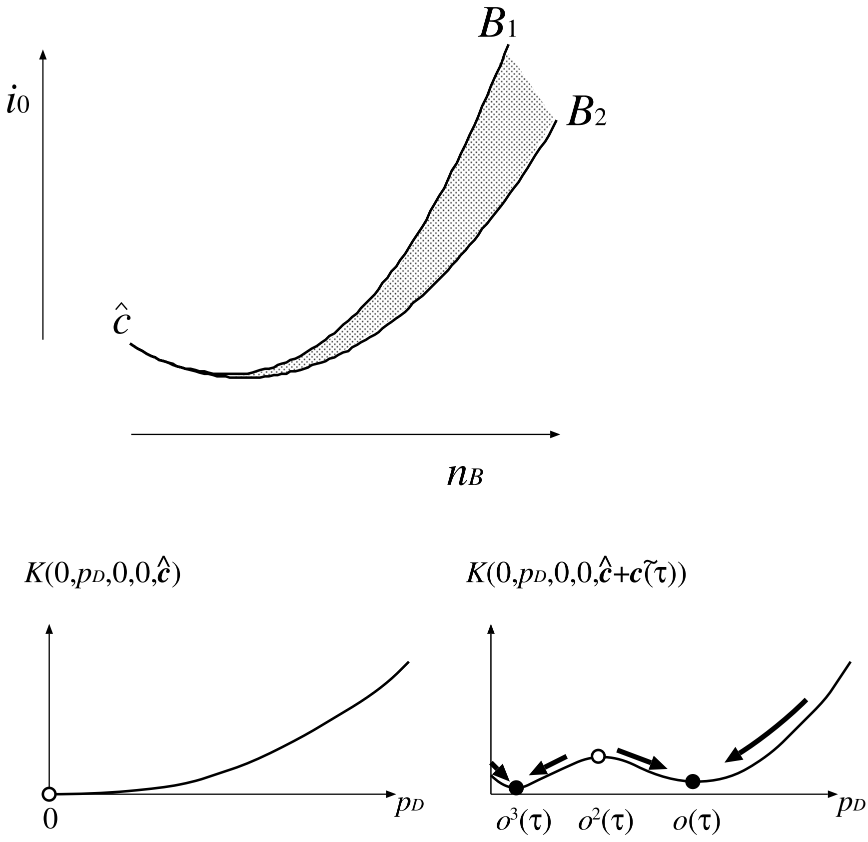

Fig. 2. A sketch of the catastrophe map onto (i_0, n_B) plane (**top**), and the bifurcation pattern at $\hat{\mathbf{c}}$ (**bottom**). Two bifurcation sets, B_1 and B_2, extend from $\hat{\mathbf{c}}$ to the same direction with the same curvature. Within the shaded region between B_1 and B_2, two stable and one unstable states exist. The bottom left panel shows a sketch of the scalar function K at $\rho = u_S = u_D = 0$ and $\mathbf{c} = \hat{\mathbf{c}}$, where one neutrally stable critical point exists at $p_D = 0$. When the parameters move to the right direction and curvature [$\mathbf{c} = \hat{\mathbf{c}} + \tilde{\mathbf{c}}(\tau)$, see section 6.6], bifurcate three positive critical points, whose orders are τ^3 (stable), τ^2 (unstable), and τ (stable) [see Eqs. (148)–(150)]

5. Discussion

We have proved the existence of a bifurcation set in the parameter space of a phytoplankton model when the growth function satisfies several conditions. Instead of analyzing 4×4 Jacobian, which can be extremely complicated, we constructed and analyzed a scalar function $K(\rho, p_D, u_S, u_D)$ that consists of a positive quadratic form of (ρ, u_S, u_D) and a nonlinear term of p_D. Cusp-like structure appears in the catastrophe manifold when parameters move from a point in the bifurcation set to a certain direction with a certain curvature (Fig. 2). The bifurcation pattern is essentially different from cusp catastrophe or pitchfork bifurcation [8]; three steady

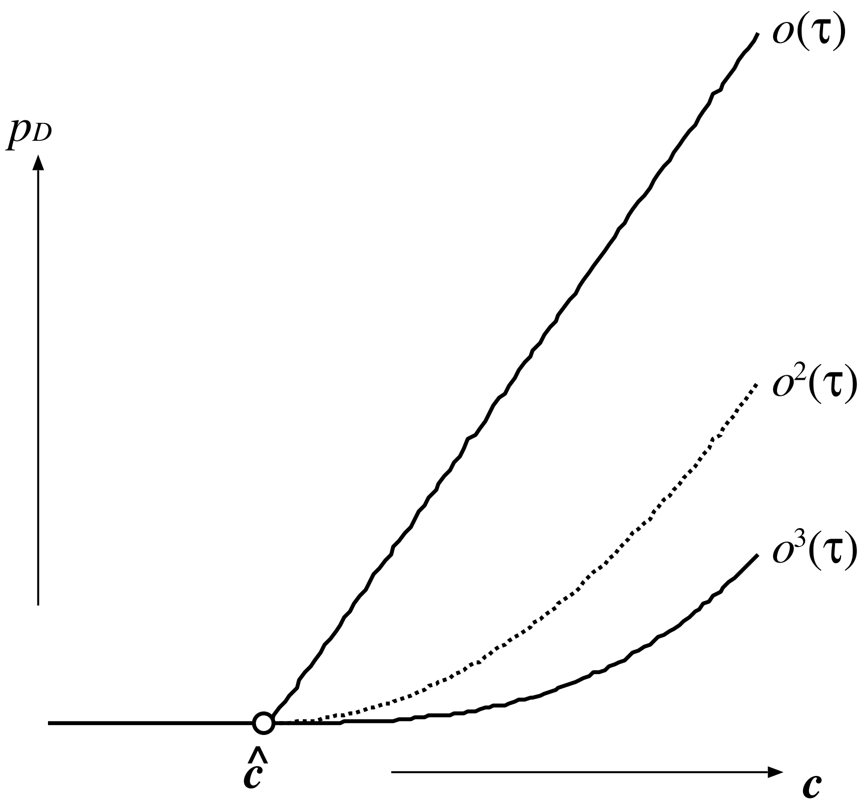

Fig. 3. Bifurcation diagram near the bifurcation point. Two stable (solid lines) and one unstable (dotted line) steady states bifurcate from $\hat{\mathbf{c}}$ in different order of magnitude

states appear on the same side of the original steady state ($p_D = 0$) in different order of magnitude (Figure 3). This bifurcation pattern may be specific to population dynamics or chemical reaction systems where only non-negative solutions are possible. The classification of this bifurcation pattern is not discussed here and still remains to be done.

The conditions (A1)–(A6) are satisfied by most phytoplankton growth functions and the condition (A7) would be satisfied if the two resources are essential [13]. Since light and nutrients are essential resources for phytoplankton growth, these conditions will be satisfied for most natural phytoplankton communities. Growth functions under two essential resources are often formulated as

$$f(x, y) = \min[f_1(x), f_2(y)],$$

for resources x and y. This type corresponds to the case $\delta = 0$ in (A7) and both resource are strictly essential. The next equation

$$f(x, y) = \mu \frac{xy}{(x^q + y^q)^{1/q}},$$

will satisfy (A7) for any small $\delta > 0$ if we take q large enough. According to the theorem 2, the system will have the bifurcation set for these growth functions if other general assumptions are satisfied. For other functions that are often seen in phytoplankton models like

$$f(x, y) = \mu \frac{xy}{x+y} \quad \text{or} \quad \mu \frac{x}{x+X_h} \frac{y}{y+Y_h},$$

we will need extensive analyses and numerical experiments to examine whether the system would have any bifurcation set.

We assumed that phytoplankton and nutrient particles are distributed homogeneously in each layer. This assumption is valid if the order of scaled vertical mixing coefficient in each layer is sufficiently greater than the orders of scaled phytoplankton growth and nutrient consumption terms and fluxes of phytoplankton and nutrient between the two layers. Surface layers in lakes and oceans often meet these conditions, but deep layers are normally heterogeneous. Thus, partial differential equations used in [15] are adequate formulation for the deep layer. Mathematical analysis of the partial differential equations is beyond the scope of this paper, but the hooked beak-like catastrophe map (Figure 2) was also obtained by the numerical analysis of the partial differential equations [15].

The bifurcation set lies on a subset of the parameter space where $\gamma = 0$. Therefore, we have only shown that a catastrophic shift between two stable states occurs when the relative diffusivity is sufficiently small. In weakly stratified layers where turbulent mixing is dominant, the diffusivity of cells and nutrient particles will be the same. We did not answer the question whether there is the bifurcation set when the relative diffusivity is larger or close to 1. This needs to be answered since turbulent mixing is dominant in most cases.

As we noted in previous paper [15], the mechanisms for bistability and the catastrophic shift are positive feedbacks for phytoplankton growth in each layer. The growth in the surface layer suppresses the growth in the deep layer via shading; the growth in the deep layer suppresses the growth in the surface layer via nutrient consumption. Therefore, an increase in biomass in the surface layer enhances nutrient supply from the deep layer by suppressing the growth in the deep layer and produces a positive feedback; an increase in the deep layer enhances light availability by suppressing the growth in the surface layer and produces a positive feedback. The system may be extended to include following factors: other nutrient sources than the bottom sediment, multiple phytoplankton species, heterotrophic bacteria, and glazers. Whether these factors amplify the positive feedbacks or not will be the next investigation.

In spite of the recent increasing concern about catastrophic shifts in ecosystems, most mathematical models are still simple and straightforward to produce positive feedbacks and catastrophic shifts [9]. We have successfully derived a scalar function whose critical points are exactly the steady states of the dynamical system, which enabled us to show the sufficient conditions for the existence of a bifurcation set for a nonlinear system with four variables. Sober mathematical analyses for various ecological systems would be important for the further understanding of positive feedbacks and catastrophic shifts in ecology.

6. Proofs

6.1. *Proof of lemma 1*

From the definition (15), we have an equality when $p_D = 0$

$$\Phi_S(p_S, 0, \mathbf{c}) = \left[G_S(p_S, 0, 0, \mathbf{c}) - 1 - \frac{\gamma d_T}{\zeta_T} \right] p_S. \tag{25}$$

Thus when $p_S > 0$, $\Phi(p_S, 0, \mathbf{c}) = 0$ if and only if

$$G_S(p_S, 0, 0, \mathbf{c}) = 1 + \frac{\gamma d_T}{\zeta_T}. \tag{26}$$

From the definition (11), the partial derivative of G_S with respect to p_S is

$$\frac{\partial G_S}{\partial p_S} = \frac{\partial F_S}{\partial p_S} - \left[\frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B} + \alpha\gamma \right] \frac{\partial F_S}{\partial n_S}. \tag{27}$$

From (27) and the definition (6) we have

$$\frac{\partial G_S}{\partial p_S} = \frac{1}{\zeta_T} \int_0^{\zeta_T} \left\{ \frac{\partial i}{\partial p_S} \frac{\partial \varphi}{\partial i} - \left[\frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B} + \alpha\gamma \right] \frac{\partial \varphi}{\partial n} \right\} d\zeta. \tag{28}$$

From the definition (4), we have

$$\frac{\partial i}{\partial p_S} = -i\zeta, \quad \text{for } \zeta \in [0, \zeta_T]. \tag{29}$$

By substituting (29) into (28) we get

$$\frac{\partial G_S}{\partial p_S} = -\frac{1}{\zeta_T} \int_0^{\zeta_T} \left\{ i\zeta \frac{\partial \varphi}{\partial i} + \left[\frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B} + \alpha\gamma \right] \frac{\partial \varphi}{\partial n} \right\} d\zeta. \tag{30}$$

According to (A3) and (A4), the sum of the terms in the braces of (30) is positive and bounded. Hence we have an inequality

$$-\infty < \frac{\partial G_S}{\partial p_S} < 0. \tag{31}$$

From the definitions (13) and (14), we have

$$G_S(0, 0, \mathbf{c}) > 1 + \frac{\gamma d_T}{\zeta_T}, \quad \text{for } \mathbf{c} \in \mathbf{C}^+, \tag{32}$$

$$G_S(0, 0, \mathbf{c}) \geq \underline{G_S}(\mathbf{C}^{++}) > 1 + \frac{\gamma d_T}{\zeta_T}, \quad \text{for } \mathbf{c} \in \mathbf{C}^{++}. \tag{33}$$

As $p_S \to \infty$, the ambient light intensity, $i \to 0$. Thus $\varphi(n, i) \to 0$ as $p_S \to \infty$ according to (A2). Therefore there is $p_S^\dagger > 0$ such that

$$G_S(p_S^\dagger, 0, \mathbf{c}) < 1 + \frac{\gamma d_T}{\zeta_T}, \quad \text{for } \mathbf{c} \in \mathbf{C}^+. \tag{34}$$

From the inequalities (31), (32), and (34), there is a unique $p_S^{\ddagger}(\mathbf{c}) > 0$ for $\mathbf{c} \in \mathbf{C}^+$ such that

$$\Phi_S[p_S^{\ddagger}(\mathbf{c}), 0, \mathbf{c}] = 0. \tag{35}$$

From (33), we have an inequality for $\mathbf{c} \in \mathbf{C}^{++}$

$$\underline{p_S^{\ddagger}}(\mathbf{C}^{++}) > 0, \tag{36}$$

where $\underline{p_S^{\ddagger}}(\mathbf{C}^{++})$ denotes the maximum lower bound of $p_S^{\ddagger}(\mathbf{c})$ for $\mathbf{c} \in \mathbf{C}^{++}$.

The partial derivative of Φ_S with respect to p_S is

$$\frac{\partial \Phi_S}{\partial p_S} = -\frac{\gamma d_T}{\zeta_T} + G_S - 1 + \frac{\partial G_S}{\partial p_S} p_S. \tag{37}$$

From the definition (15), an equality

$$-\frac{\gamma d_T}{\zeta_T} p_D = \left(-\frac{\gamma d_T}{\zeta_T} + G_S - 1\right) p_S, \tag{38}$$

holds when $\Phi_S = 0$. From (37) and (38), when $\Phi_S = 0$ and $p_S \neq 0$, the partial derivative of Φ_S with respect to p_S is

$$\frac{\partial \Phi_S}{\partial p_S} = -\frac{\gamma d_T}{\zeta_T} \frac{p_D}{p_S} + \frac{\partial G_S}{\partial p_S} p_S. \tag{39}$$

From (39) and the inequality (31), we have an inequality

$$-\infty < \frac{\partial \Phi_S}{\partial p_S} < 0, \tag{40}$$

when $\Phi_S = 0$ and $p_S > 0$. It is easy to see that the partial derivative of Φ_S with respect to p_D is bounded, that is,

$$\left| \frac{\partial \Phi_S}{\partial p_D} \right| < \infty. \tag{41}$$

Since the inequality (40) holds when $\Phi_S = 0$ and $p_S > 0$, a map $\bar{p}_S(p_D, \mathbf{c})$ for $p_D \geq 0$ and $\mathbf{c} \in \mathbf{C}^+$ can be defined by

$$\bar{p}_S(0, \mathbf{c}) = p_S^{\ddagger}(\mathbf{c}) \quad \text{and} \quad \Phi_S(\bar{p}_S, p_D, \mathbf{c}) = 0, \tag{42}$$

until $\bar{p}_S > 0$ according to the Implicit Function Theorem. From (40) and (41), the partial derivative of $\bar{p}_S$ with respect to p_D is bounded, that is,

$$\left| \frac{\partial \bar{p}_S}{\partial p_D} \right| = \left| \left(\frac{\partial \Phi_S}{\partial p_D} \right) \left(\frac{\partial \Phi_S}{\partial p_S} \right)^{-1} \right| < \infty. \tag{43}$$

Since $p_S^{\ddagger}(\mathbf{c}) > 0$ for $\mathbf{c} \in \mathbf{C}^+$ and the inequality (43) holds, there is $\delta_0(\mathbf{c})$ such that

$$\bar{p}_S(p_D, \mathbf{c}) > 0, \quad \text{for } p_D \in [0, \delta_0(\mathbf{c})) \text{ and } \mathbf{c} \in \mathbf{C}^+. \tag{44}$$

Since $\underline{p_S}(\mathbf{C}^{++}) > 0$ and the inequality (43) holds, there is δ_0^* such that

$$\bar{p}_S(p_D, \mathbf{c}) > 0, \quad \text{for } p_D \in [0, \delta_0^*) \text{ and } \mathbf{c} \in \mathbf{C}^{++}. \tag{45}$$

Therefore a map $\bar{p}_S(p_D, \mathbf{c}) > 0$ can be defined implicitly by $\Phi_S = 0$ for $p_D \in [0, \delta_0(\mathbf{c}))$ and $\mathbf{c} \in \mathbf{C}^+$; a map $\bar{p}_S(p_D, \mathbf{c}) > 0$ can be defined implicitly by $\Phi_S = 0$ for $p_D \in [0, \delta_0^*)$ and $\mathbf{c} \in \mathbf{C}^{++}$. □

6.2. Proof of lemma 2

For $\mathbf{c} \in \mathbf{C}^{++}$ and $\gamma = 0$, we consider a system of three variables (ρ, u_S, u_D) keeping $p_D = 0$. It is easy to see that this system has a steady state $(\rho, u_S, u_D) = (0, 0, 0)$. By linearizing the system near $(0, 0, 0)$, we have

$$\begin{pmatrix} \dot{\rho} \\ \dot{u}_S \\ \dot{u}_D \end{pmatrix} = M \begin{pmatrix} \rho \\ u_S \\ u_D \end{pmatrix}, \tag{46}$$

where

$$M = \begin{Bmatrix} \frac{\partial G_S}{\partial p_S} \bar{p}_S(0, \mathbf{c}) & \frac{\partial G_S}{\partial u_S} \bar{p}_S(0, \mathbf{c}) & 0 \\ \alpha[\frac{(d_T+d_B)\zeta_T}{d_T d_B} - 1] \frac{\partial G_S}{\partial p_S} \bar{p}_S(0, \mathbf{c}) & -\frac{d_T}{\zeta_T} + \alpha[\frac{(d_T+d_B)\zeta_T}{d_T d_B} - 1] \frac{\partial G_S}{\partial u_S} \bar{p}_S(0, \mathbf{c}) & \frac{d_T}{\zeta_T} \\ \frac{\alpha\zeta_T}{d_B} \frac{\partial G_S}{\partial p_S} \bar{p}_S(0, \mathbf{c}) & \frac{d_T}{1-\zeta_T} + \frac{\alpha\zeta_T}{d_B} \frac{\partial G_S}{\partial u_S} \bar{p}_S(0, \mathbf{c}) & -\frac{d_T+d_B}{1-\zeta_T} \end{Bmatrix}. \tag{47}$$

From the definition (11), we have equalities when $\gamma = 0$

$$\frac{\partial G_S}{\partial p_S} = \frac{\partial F_S}{\partial p_S} - \frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B} \frac{\partial F_S}{\partial n_S}, \tag{48}$$

$$\frac{\partial G_S}{\partial u_S} = \frac{\partial F_S}{\partial n_S}. \tag{49}$$

We substitute (48) and (49) into (47) and have the characteristic equation of (46)

$$\begin{aligned} \lambda^3 &+ \left[\frac{d_T}{\zeta_T} + \frac{d_T + d_B}{1 - \zeta_T} + \alpha \frac{\partial F_S}{\partial n_S} \bar{p}_S(0, \mathbf{c}) - \frac{\partial F_S}{\partial p_S} \bar{p}_S(0, \mathbf{c})\right] \lambda^2 \\ &+ \left[\frac{d_T d_B}{\zeta_T(1 - \zeta_T)} + \alpha \left(\frac{d_T + d_B}{1 - \zeta_T} + 1\right) \frac{\partial F_S}{\partial n_S} \bar{p}_S(0, \mathbf{c}) \right. \\ &\left. - \left(\frac{d_T}{\zeta_T} + \frac{d_T + d_B}{1 - \zeta_T}\right) \frac{\partial F_S}{\partial p_S} \bar{p}_S(0, \mathbf{c})\right] \lambda \\ &+ \frac{\alpha(d_T + d_B)}{1 - \zeta_T} \frac{\partial F_S}{\partial n_S} \bar{p}_S(0, \mathbf{c}) - \frac{d_T d_B}{\zeta_T(1 - \zeta_T)} \frac{\partial F_S}{\partial p_S} \bar{p}_S(0, \mathbf{c}) = 0. \end{aligned} \tag{50}$$

Easy deduction from (A3) gives

$$\frac{\partial F_S}{\partial n_S} \geq 0 \quad \text{and} \quad \frac{\partial F_S}{\partial p_S} \leq 0, \tag{51}$$

and at least one of these is positive. Considering (51), some tedious manipulation yields that each eigenvalue of (50) has a negative real part according to the Routh-Hurwitz Criterion [2]. Therefore there is a positive definite matrix Q such that $-(QM + M^T Q)$ is a positive definite matrix according to the Lyapunov's Theorem [2].

For ρ, u_S, and u_D, we have the following expansions

$$\begin{aligned}
&\frac{\gamma d_T}{\zeta_T}[p_D - (\bar{p}_S + \rho)] + [G_S(\bar{p}_S + \rho, p_D, u_S, \mathbf{c}) - 1](\bar{p}_S + \rho)\\
&\quad = \frac{\gamma d_T}{\zeta_T}(p_D - \bar{p}_S) + [G_S(\bar{p}_S, p_D, 0, \mathbf{c}) - 1]\bar{p}_S + a_p\rho + a_u u_S + o^2(\rho, u_S)\\
&\quad = \Phi_S(\bar{p}_S, p_D, \mathbf{c}) + a_p\rho + a_u u_S + o^2(\rho, u_S)\\
&\quad = a_p\rho + a_u u_S + o^2(\rho, u_S), \qquad (52)\\
&\quad\ -\frac{\gamma d_T}{1-\zeta_T}[p_D - (\bar{p}_S + \rho)] + [G_D(\bar{p}_S + \rho, p_D, u_D, \mathbf{c}) - 1]p_D\\
&\quad = -\frac{\gamma d_T}{1-\zeta_T}(p_D - \bar{p}_S) + [\tilde{G}_D(p_D, \mathbf{c}) - 1]p_D + b_p\rho + b_u u_D + o^2(\rho, u_D)\\
&\quad = \Phi_D(p_D, \mathbf{c}) + b_p\rho + b_u u_D + o^2(\rho, u_D), \qquad (53)
\end{aligned}$$

where

$$a_p = -\frac{\gamma d_T}{\zeta_T} + G_S(\bar{p}_S, p_D, 0, \mathbf{c}) - 1 + \frac{\partial G_S(\bar{p}_S, p_D, 0, \mathbf{c})}{\partial p_S}\bar{p}_S, \qquad (54)$$

$$a_u = \frac{\partial G_S(\bar{p}_S, p_D, 0, \mathbf{c})}{\partial u_S}\bar{p}_S, \qquad (55)$$

$$b_p = \frac{\gamma d_T}{1-\zeta_T} + \frac{\partial G_D(\bar{p}_S, p_D, 0, \mathbf{c})}{\partial p_S}p_D, \qquad (56)$$

$$b_u = \frac{\partial G_D(\bar{p}_S, p_D, 0, \mathbf{c})}{\partial u_D}p_D, \qquad (57)$$

and $o^j(x)$ is a sum of higher order terms than jth order of x.

By applying the expansions (52) and (53), the system (19) is modified to

$$\begin{aligned}
\dot{\rho} &= a_p\rho + a_u u_S - \frac{\partial \bar{p}_S}{\partial p_D}\left(\Phi_D + b_p\rho + b_u u_D\right) + o^2(\rho, u_S, u_D)\\
&= \left(a_p - \frac{\partial \bar{p}_S}{\partial p_D}b_p\right)\rho + a_u u_S - \frac{\partial \bar{p}_S}{\partial p_D}b_u u_D - \frac{\partial \bar{p}_S}{\partial p_D}\Phi_D + o^2(\rho, u_S, u_D),\\
\dot{u}_S &= -\frac{d_T}{\zeta_T}u_S + \frac{d_T}{\zeta_T}u_D\\
&\quad +\alpha\left[\frac{(d_T + d_B)\zeta_T}{d_T d_B} - 1 + \gamma\right]\left[a_p\rho + a_u u_S + o^2(\rho, u_S)\right]\\
&\quad +\alpha\left(\frac{1-\zeta_T}{d_B} - \gamma\right)\left[\Phi_D + b_p\rho + b_u u_D + o^2(\rho, u_D)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \left\{ \left[\frac{(d_T + d_B)\zeta_T}{d_T d_B} - 1 + \gamma \right] a_p + \left(\frac{1-\zeta_T}{d_B} - \gamma \right) b_p \right\} \rho \\
&\quad + \left\{ \alpha \left[\frac{(d_T + d_B)\zeta_T}{d_T d_B} - 1 + \gamma \right] a_u - \frac{d_T}{\zeta_T} \right\} u_S \\
&\quad + \left[\alpha \left(\frac{1-\zeta_T}{d_B} - \gamma \right) b_u + \frac{d_T}{\zeta_T} \right] u_D \\
&\quad + \alpha \left(\frac{1-\zeta_T}{d_B} - \gamma \right) \Phi_D + o^2(\rho, u_S, u_D), \\
\dot{u}_D &= \frac{d_T}{1-\zeta_T} u_S - \frac{d_T + d_B}{1-\zeta_T} u_D + \frac{\alpha \zeta_T}{d_B} \left[a_p \rho + a_u u_S + o^2(\rho, u_S) \right] \\
&\quad + \alpha \left[\frac{(1-\zeta_T)}{d_B} - 1 \right] \left[\Phi_D + b_p \rho + b_u u_D + o^2(\rho, u_D) \right] \\
&= \alpha \left[\frac{\zeta_T}{d_B} a_p + \left(\frac{1-\zeta_T}{d_B} - 1 \right) b_p \right] \rho + \left(\frac{\alpha \zeta_T}{d_B} a_u + \frac{d_T}{1-\zeta_T} \right) u_S \\
&\quad + \left[\alpha \left(\frac{1-\zeta_T}{d_B} - 1 \right) b_u - \frac{d_T + d_B}{1-\zeta_T} \right] u_D + \alpha \left(\frac{1-\zeta_T}{d_B} - 1 \right) \Phi_D \\
&\quad + o^2(\rho, u_S, u_D), \\
\dot{p}_D &= b_p \rho + b_u u_D + \Phi_D + o^2(\rho, u_S, u_D). \qquad (58)
\end{aligned}
$$

Some manipulation of (58) yields

$$
\begin{pmatrix} \dot{\rho} \\ \dot{u}_S \\ \dot{u}_D \\ \dot{p}_D \end{pmatrix} = \begin{bmatrix} \bar{M}(p_D, \mathbf{c}) & \mathbf{v}(p_D, \mathbf{c}) \\ \mathbf{w}^T(p_D, \mathbf{c}) & 1 \end{bmatrix} \begin{pmatrix} \rho \\ u_S \\ u_D \\ \Phi_D \end{pmatrix} + o^2(\rho, u_S, u_D), \qquad (59)
$$

where

$$
\begin{aligned}
&\bar{M}(p_D, \mathbf{c}) \\
&= \left\{ \begin{array}{ccc}
a_p - \frac{\partial \bar{p}_S}{\partial p_D} b_p & a_u & -\frac{\partial \bar{p}_S}{\partial p_D} b_u \\
\begin{array}{c} \alpha \left[\frac{(d_T+d_B)\zeta_T}{d_T d_B} - 1 + \gamma \right] a_p \\ + \alpha \left(\frac{1-\zeta_T}{d_B} - \gamma \right) b_p \end{array} & \alpha \left[\frac{(d_T+d_B)\zeta_T}{d_T d_B} - 1 + \gamma \right] a_u - \frac{d_T}{\zeta_T} & \alpha \left(\frac{1-\zeta_T}{d_B} - \gamma \right) b_u + \frac{d_T}{\zeta_T} \\
\alpha \left[\frac{\zeta_T}{d_B} a_p + (\frac{1-\zeta_T}{d_B} - 1) b_p \right] & \frac{\alpha \zeta_T}{d_B} a_u + \frac{d_T}{1-\zeta_T} & \alpha (\frac{1-\zeta_T}{d_B} - 1) b_u - \frac{d_T + d_B}{1-\zeta_T}
\end{array} \right\},
\end{aligned} \qquad (60)
$$

$$
\mathbf{v}^T(p_D, \mathbf{c}) = \left[-\frac{\partial \bar{p}_S}{\partial p_D} \;\; \alpha \left(\frac{1-\zeta_T}{d_B} - \gamma \right) \;\; \alpha \left(\frac{1-\zeta_T}{d_B} - 1 \right) \right], \qquad (61)
$$

$$
\mathbf{w}^T(p_D, \mathbf{c}) = (b_p \quad 0 \quad b_u). \qquad (62)
$$

From (20) and (59)–(62), the time derivative of K is

$$
\dot{K} = -\left[\rho \;\, u_S \;\, u_D \;\, \Phi_D(p_D, \mathbf{c}) \right] L(p_D, \mathbf{c}) \begin{bmatrix} \rho \\ u_S \\ u_D \\ \Phi_D(p_D, \mathbf{c}) \end{bmatrix} + o^3(\rho, u_S, u_D), \qquad (63)
$$

where

$$L(p_D, \mathbf{c}) = \begin{bmatrix} -(Q\bar{M} + \bar{M}^T Q) & -Q\mathbf{v} + \kappa\mathbf{w}/2 \\ -\mathbf{v}^T Q + \kappa\mathbf{w}^T/2 & \kappa \end{bmatrix}. \tag{64}$$

Since $\bar{M}(0, \mathbf{c}) = M$ and $\mathbf{w}(0, \mathbf{c}) = 0$ when $\gamma = 0$, we have

$$L(0, \mathbf{c}) = \begin{bmatrix} -(QM + M^T Q) & -Q\mathbf{v}(0, \mathbf{c}) \\ -\mathbf{v}^T(0, \mathbf{c})Q & \kappa \end{bmatrix}, \tag{65}$$

when $\gamma = 0$. If we take κ sufficiently large then $L(0, \mathbf{c})$ is a positive definite matrix since the matrix $-(QM + M^T Q)$ is positive definite. Hence there is $\delta_1 > 0$ and $\gamma^* > 0$ such that $L(p_D, \mathbf{c})$ is a positive definite matrix for $p_D \in [0, \delta_1)$ and $\mathbf{c} \in \mathbf{C}^{++}_{\gamma^*}$. The first term of (63) is negative and the second order of ρ, u_S and u_D. The second term of (63) is the third order or higher of ρ, u_S and u_D. Therefore there is $\delta_2 > 0$ such that $\dot{K} \leq 0$ for $|\rho|$, $|u_S|$, $|u_S| < \delta_2$, that is, $\dot{K} \leq 0$ for $(\rho, p_D, u_S, u_D) \in R_a$. □

6.3. Proof of lemma 3

Let $\mathrm{Fr}R_b$ and $\bar{R}_b$ be the boundary of R_b and the closure of R_b, respectively. From the definition (22), $\mathrm{Fr}R_b$ either satisfies $K = K^*$ or is the boundary of R_a. In order that $\bar{R}_b$ will be a subset of R_a, $\mathrm{Fr}R_b$ needs to be a subset of R_a. From the definition (21), R_a does not include its boundary unless $p_D = 0$. Therefore If $\bar{R}_b \subset R_a$, we have

$$\mathrm{Fr}R_b \subset \{(\rho, p_D, u_S, u_D) \in R_a \mid K = K^* \cup (p_D = 0 \cap K < K^*)\}. \tag{66}$$

Since $p_D \geq 0$ for any orbit of the system (19) and K decreases with time in R_b, orbits passing through R_b will stay inside R_b and converge to a point in the limit set $\{(\rho, p_D, u_S, u_D) \in R_b \mid \dot{K} = 0\}$ if $\bar{R}_b \subset R_a$.

From (63), $\dot{K} = 0$ if and only if $\rho = u_S = u_D = \Phi_D = 0$ since $L(p_D, \mathbf{c})$ is a positive definite matrix for $p_D \in [0, \delta_1)$ and $\mathbf{c} \in \mathbf{C}^{++}_{\gamma^*}$. If $\rho = u_S = u_D = \Phi_D = 0$ then the following equality holds by some manipulation of the definition (20)

$$\frac{\partial K}{\partial \rho} = \frac{\partial K}{\partial u_S} = \frac{\partial K}{\partial u_D} = \frac{\partial K}{\partial p_D} = 0. \tag{67}$$

Conversely, if (67) holds then $\rho = u_S = u_D = \Phi_D = 0$ by easy deduction. Thus $\rho = u_S = u_D = \Phi_D = 0$ is equivalent to (67). A point that satisfies (67) is a local minimum and asymptotically stable if the matrix

$$\begin{pmatrix} \frac{\partial^2 K}{\partial \rho^2} & \frac{\partial^2 K}{\partial \rho \partial u_S} & \frac{\partial^2 K}{\partial \rho \partial u_D} & \frac{\partial^2 K}{\partial \rho \partial p_D} \\ \frac{\partial^2 K}{\partial u_S \partial \rho} & \frac{\partial^2 K}{\partial u_S^2} & \frac{\partial^2 K}{\partial u_S \partial u_D} & \frac{\partial^2 K}{\partial u_S \partial p_D} \\ \frac{\partial^2 K}{\partial u_D \partial \rho} & \frac{\partial^2 K}{\partial u_D \partial u_S} & \frac{\partial^2 K}{\partial u_D^2} & \frac{\partial^2 K}{\partial u_D \partial p_D} \\ \frac{\partial^2 K}{\partial p_D \partial \rho} & \frac{\partial^2 K}{\partial p_D \partial u_S} & \frac{\partial^2 K}{\partial p_D \partial u_D} & \frac{\partial^2 K}{\partial p_D^2} \end{pmatrix}, \tag{68}$$

is positive definite. From the definition (20), we obtain

$$(68) = \begin{pmatrix} 2Q & 0 \\ 0 & \frac{\partial^2 K}{\partial p_D^2} \end{pmatrix}. \tag{69}$$

Since Q is a positive definite matrix, the matrix (69) is positive definite if and only if $\partial^2 K/\partial p_D^2 > 0$. Therefore the point that satisfies (67) is asymptotically stable if $\partial^2 K/\partial p_D^2 > 0$ and unstable if $\partial^2 K/\partial p_D^2 < 0$. □

6.4. Proof of lemma 4

Suppose an equality

$$\underline{G_S}(\hat{\mathbf{C}}) = 1 + \frac{\gamma d_T}{\zeta_T}, \tag{70}$$

holds for $\hat{\mathbf{C}}$. Then there is $\mathbf{c}^* \in \hat{\mathbf{C}}$ for any $\varepsilon > 0$ such that

$$G_S(0, 0, 0, \mathbf{c}^*) - \left(1 + \frac{\gamma d_T}{\zeta_T}\right) < \varepsilon^2. \tag{71}$$

Since $\gamma = 0$ for $\mathbf{c}^* \in \hat{\mathbf{C}}$, (71) is modified to

$$G_S(0, 0, 0, \mathbf{c}^*) - 1 < \varepsilon^2. \tag{72}$$

From the inequality (72) and the derivation of $\bar{p}_S(0, \mathbf{c})$ in the section 6.1, it is easy to see that

$$\bar{p}_S(0, \mathbf{c}^*) = o^2(\varepsilon). \tag{73}$$

From (12), $\partial \tilde{G}_D(0, \mathbf{c}^*)/\partial p_D$ is written as

$$\frac{\partial \tilde{G}_D(0, \mathbf{c}^*)}{\partial p_D} = \frac{\partial G_D[\bar{p}_S(0, \mathbf{c}^*), 0, 0, \mathbf{c}^*]}{\partial p_D} + \frac{\partial \bar{p}_S(0, \mathbf{c}^*)}{\partial p_D} \frac{\partial G_D[\bar{p}_S(0, \mathbf{c}^*), 0, 0, \mathbf{c}^*]}{\partial p_S}. \tag{74}$$

In the subsequent proof, we take $G_S(p_S, p_D, u_S, \mathbf{c})$, $G_D(p_S, p_D, u_D, \mathbf{c})$, and their derivatives at $(p_S, p_D, u_S, u_D, \mathbf{c}) = [\bar{p}_S(0, \mathbf{c}^*), 0, 0, 0, \mathbf{c}^*]$. From (15) and the definition of $\bar{p}_S$, $\partial \bar{p}_S(0, \mathbf{c}^*)/\partial p_D$ will be

$$\begin{aligned} \frac{\partial \bar{p}_S(0, \mathbf{c}^*)}{\partial p_D} &= -\left(\frac{\partial \Phi_S}{\partial p_D}\right)\left(\frac{\partial \Phi_S}{\partial p_S}\right)^{-1} \\ &= -\left[\frac{\gamma d_T}{\zeta_T} + \frac{\partial G_S}{\partial p_D}\bar{p}_S(0, \mathbf{c}^*)\right]\left[-\frac{\gamma d_T}{\zeta_T} + \frac{\partial G_S}{\partial p_S}\bar{p}_S(0, \mathbf{c}^*) + G_S - 1\right]^{-1}. \end{aligned} \tag{75}$$

Since $\gamma = 0$ for $\mathbf{c}^*$, we have

$$G_S[\bar{p}_S(p_D, \mathbf{c}), p_D, 0, \mathbf{c}] - 1 = 0, \tag{76}$$

and (75) is modified to

$$\frac{\partial \bar{p}_S(0,\mathbf{c}^*)}{\partial p_D} = -\left(\frac{\partial G_S}{\partial p_D}\right)\left(\frac{\partial G_S}{\partial p_S}\right)^{-1}. \tag{77}$$

Since $\partial \tilde{G}_D(0,\mathbf{c})/\partial p_D = 0$ for $\mathbf{c} \in \hat{\mathbf{C}}$, we have an equality for $\mathbf{c}^* \in \hat{\mathbf{C}}$ from (74) and (77)

$$\frac{\partial \tilde{G}_D(0,\mathbf{c}^*)}{\partial p_D} = \left(\frac{\partial G_S}{\partial p_S}\right)^{-1}\left(\frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S}\right) = 0. \tag{78}$$

Since $\partial G_S/\partial p_S \in (-\infty, 0)$ from (31), we have an equality

$$\frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S} = 0, \tag{79}$$

for $\mathbf{c}^* \in \hat{\mathbf{C}}$.

From (11) and (12), partial derivatives of G_S and G_D when $\gamma = 0$ are

$$\frac{\partial G_S}{\partial p_S} = \frac{\partial F_S}{\partial p_S} - \frac{\alpha(d_T+d_B)\zeta_T}{d_T d_B}\frac{\partial F_S}{\partial n_S}, \tag{80}$$

$$\frac{\partial G_S}{\partial p_D} = -\frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_S}{\partial n_S}, \tag{81}$$

$$\frac{\partial G_D}{\partial p_S} = \frac{\partial F_D}{\partial p_S} - \frac{\alpha\zeta_T}{d_B}\frac{\partial F_D}{\partial n_D}, \tag{82}$$

$$\frac{\partial G_D}{\partial p_D} = \frac{\partial F_D}{\partial p_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_D}{\partial n_D}. \tag{83}$$

By substituting (80)–(83) into (79), we get an equality

$$\begin{aligned}
&\frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S}\\
&\quad = \frac{\partial F_S}{\partial p_S}\left[\frac{\partial F_D}{\partial p_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_D}{\partial n_D}\right]\\
&\qquad - \frac{\partial F_S}{\partial n_S}\left[\frac{\alpha(d_T+d_B)\zeta_T}{d_T d_B}\frac{\partial F_D}{\partial p_D} - \frac{\alpha^2\zeta_T(1-\zeta_T)}{d_T d_B}\frac{\partial F_D}{\partial n_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_D}{\partial p_S}\right]\\
&\quad = 0.
\end{aligned} \tag{84}$$

From (11), (12), (23), and the definition of $\bar{p}_S$, we have

$$G_S[\bar{p}_S(0,\mathbf{c}^*), 0, 0, \mathbf{c}^*] = F_S[\bar{p}_S(0,\mathbf{c}^*), \bar{n}_S, \mathbf{c}^*] = 1, \tag{85}$$

$$G_D[\bar{p}_S(0,\mathbf{c}^*), 0, 0, \mathbf{c}^*] = F_D[\bar{p}_S(0,\mathbf{c}^*), 0, \bar{n}_D, \mathbf{c}^*] = 1, \tag{86}$$

where

$$\bar{n}_S = n_B - \frac{\alpha(d_T+d_B)\zeta_T}{d_T d_B}\bar{p}_S(0,\mathbf{c}^*), \tag{87}$$

$$\bar{n}_D = n_B - \frac{\alpha\zeta_T}{d_B}\bar{p}_S(0,\mathbf{c}^*). \tag{88}$$

From (85), (86) and definitions (6) and (7), we have

$$\frac{1}{\zeta_T}\int_0^{\zeta_T} \varphi[\bar{n}_S, i(\zeta)]d\zeta = 1, \tag{89}$$

$$\frac{1}{1-\zeta_T}\int_{\zeta_T}^{1} \varphi[\bar{n}_D, i(\zeta)]d\zeta = 1. \tag{90}$$

Since the equality (89) holds and $\varphi(\bar{n}_S, i)$ decrease with ζ, we have inequalities

$$\varphi[\bar{n}_S, i(0)] \geq 1, \tag{91}$$

$$\varphi[\bar{n}_S, i(\zeta_T)] \leq 1. \tag{92}$$

Similarly, we have

$$\varphi[\bar{n}_D, i(\zeta_T)] \geq 1, \tag{93}$$

$$\varphi[\bar{n}_D, i(1)] \leq 1. \tag{94}$$

For the difference between $\varphi[\bar{n}_S, i(\zeta_T)]$ and $\varphi[\bar{n}_D, i(\zeta_T)]$, we have an equality

$$\varphi[\bar{n}_D, i(\zeta_T)] - \varphi[\bar{n}_S, i(\zeta_T)] = \int_{\bar{n}_S}^{\bar{n}_D} \frac{\partial\varphi}{\partial n} dn. \tag{95}$$

From (A5) and equalities (87) and (88), we have an inequality

$$\varphi[\bar{n}_D, i(\zeta_T)] - \varphi[\bar{n}_S, i(\zeta_T)] < (\bar{n}_D - \bar{n}_S)m_1 = \frac{\alpha\zeta_T}{d_T}\bar{p}_S(0, \mathbf{c}^*)m_1. \tag{96}$$

From (96) and (73), we have

$$\varphi[\bar{n}_D, i(\zeta_T)] - \varphi[\bar{n}_S, i(\zeta_T)] = o^2(\varepsilon). \tag{97}$$

From inequalities (92), (93), and (97), we have inequalities

$$1 - \varphi[\bar{n}_S, i(\zeta_T)] = o^2(\varepsilon), \tag{98}$$

$$\varphi[\bar{n}_D, i(\zeta_T)] - 1 = o^2(\varepsilon). \tag{99}$$

Some tedious manipulation of (89) and (90) with consideration of (A5) yields

$$\varphi[\bar{n}_S, i(0)] - 1 = o^1(\varepsilon), \tag{100}$$

$$1 - \varphi[\bar{n}_D, i(1)] = o^1(\varepsilon). \tag{101}$$

From (98) and (100), we have

$$\varphi[\bar{n}_S, i(0)] - \varphi[\bar{n}_S, i(\zeta_T)] = o^1(\varepsilon). \tag{102}$$

For the difference between $\varphi[\bar{n}_S, i(0)]$ and $\varphi[\bar{n}_S, i(\zeta_T)]$ we have an equality

$$\begin{aligned}\varphi[\bar{n}_S, i(0)] - \varphi[\bar{n}_S, i(\zeta_T)] &= \int_{i(\zeta_T)}^{i(0)} \frac{\partial\varphi}{\partial i} di = \int_{\zeta_T}^{0} \frac{\partial i}{\partial\zeta}\frac{\partial\varphi}{\partial i} d\zeta \\ &= \int_0^{\zeta_T} [r + \bar{p}_S(0, \mathbf{c}^*)]i(\zeta)\frac{\partial\varphi}{\partial i} d\zeta. \end{aligned} \tag{103}$$

From (102) and (103), we have

$$\int_0^{\zeta_T} \frac{\partial \varphi}{\partial i} d\zeta = o^1(\varepsilon). \tag{104}$$

From (98) and (100), when we take ε sufficiently small, $\varphi[\bar{n}_S, i(\zeta)] < 1 + \psi$ for $\zeta \in [0, \zeta_T]$ and $\psi > 0$. Hence either $\partial\varphi[\bar{n}_S, i(\zeta)]/\partial n$ or $\partial\varphi[\bar{n}_S, i(\zeta)]/\partial i$ is greater than m_2 for $\zeta \in [0, \zeta_T]$ when ε is sufficiently small according to (A6). Thus we have

$$\int_0^{\zeta_T} \left(\frac{\partial \varphi}{\partial i} + \frac{\partial \varphi}{\partial n} \right) d\zeta > \zeta_T m_2. \tag{105}$$

From (104) and (105), we have

$$\frac{\partial F_S}{\partial n_S} = \int_0^{\zeta_T} \frac{\partial \varphi}{\partial n} d\zeta > \zeta_T m_2 - o^1(\varepsilon). \tag{106}$$

Similarly, we have

$$\frac{\partial F_D}{\partial n_D} = \int_{\zeta_T}^1 \frac{\partial \varphi}{\partial n} d\zeta > (1 - \zeta_T) m_2 - o^1(\varepsilon). \tag{107}$$

From (6) and (104), the partial derivative of F_S with respect to p_S is

$$\frac{\partial F_S}{\partial p_S} = \frac{1}{\zeta_T} \int_0^{\zeta_T} \frac{\partial \varphi}{\partial i} \frac{\partial i}{\partial p_S} d\zeta = o^1(\varepsilon). \tag{108}$$

Similarly we have

$$\frac{\partial F_D}{\partial p_S} = o^1(\varepsilon), \tag{109}$$

$$\frac{\partial F_D}{\partial p_D} = o^1(\varepsilon). \tag{110}$$

From (106)–(110) and (84), we have

$$\begin{aligned} \frac{\partial G_S}{\partial p_S} \frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D} \frac{\partial G_D}{\partial p_S} &= \frac{\alpha(1 - \zeta_T)}{d_B} \frac{\partial F_S}{\partial n_S} \frac{\partial F_D}{\partial n_D} + o^1(\varepsilon) \\ &= \frac{\alpha \zeta_T (1 - \zeta_T)^2}{d_B} m_2^2 + o^1(\varepsilon) > 0, \end{aligned} \tag{111}$$

for sufficiently small ε. Therefore we have

$$\frac{\partial \tilde{G}_D(0, \mathbf{c}^*)}{\partial p_D} \neq 0. \tag{112}$$

This contradicts the definition of $\hat{\mathbf{C}}$. Consequently, we have $\underline{G_S}(\hat{\mathbf{C}}) > 1 + \gamma d_T / \zeta_T$. $\square$

6.5. Proof of lemma 5

For $p_D \in [0, \delta_1]$ and $\mathbf{c} \in \hat{\mathbf{C}}$, we have an expansion of $(\tilde{G}_D - 1)$ with respect to p_D

$$\tilde{G}_D(p_D, \mathbf{c}) - 1 = \tilde{G}_D(0, \mathbf{c}) - 1 + \frac{\partial \tilde{G}_D(0, \mathbf{c})}{\partial p_D} p_D + \frac{1}{2} \frac{\partial^2 \tilde{G}_D(0, \mathbf{c})}{\partial {p_D}^2} {p_D}^2 + o^3(p_D). \tag{113}$$

Considering the definition (23), we have

$$\tilde{G}_D(p_D, \mathbf{c}) - 1 = \frac{1}{2} \frac{\partial^2 \tilde{G}_D(0, \mathbf{c})}{\partial {p_D}^2} {p_D}^2 + o^3(p_D). \tag{114}$$

Since $\partial \tilde{G}_D(0, \mathbf{c})/\partial {p_D}^2 < 0$ for $\mathbf{c} \in \hat{\mathbf{C}}$ from the definition (23), there is $\delta_1{}' \in (0, \delta_1)$ such that $\tilde{G}_D(p_D, \mathbf{c}) - 1 < 0$ for $p_D \in (0, \delta_1{}']$ and $\mathbf{c} \in \hat{\mathbf{C}}$. We define a closed set $R'_a \subset R_a$ as

$$R'_a = \{(\rho, p_D, u_S, u_D) \mid 0 \le p_D \le \delta'_1, |\rho|, |u_S|, |u_D| \le \delta'_2 < \delta_2\}. \tag{115}$$

Let K_1 be the first term and K_2, the second term of the right-hand side of (20), that is,

$$K_1(\rho \quad u_S \quad u_D) = (\rho \quad u_S \quad u_D) Q \begin{pmatrix} \rho \\ u_S \\ u_D \end{pmatrix}, \tag{116}$$

$$K_2(p_D, \mathbf{c}) = -\kappa \int_0^{p_D} \Phi_D(x, \mathbf{c}) dx. \tag{117}$$

For $\mathbf{c} \in \hat{\mathbf{C}}$, $\gamma = 0$. Thus K_2 will be

$$K_2 = -\kappa \int_0^{p_D} \Phi_D(x, \mathbf{c}) dx = -\kappa \int_0^{p_D} [\tilde{G}_D(x, \mathbf{c}) - 1] x dx, \tag{118}$$

for $\mathbf{c} \in \hat{\mathbf{C}}$. Here we define K_1^* as the minimum of K_1 for the boundary of a cube $\{(\rho, u_S, u_D) \mid |\rho|, |u_S|, |u_D| = \delta'_2\}$, and K_2^* as the maximum of K_2 for $p_D \in [0, \delta'_1]$. Since Q is a positive definite matrix, an ellipsoid

$$E_0 = \{(\rho, u_S, u_D) \mid K_1 \le K_1^*\}, \tag{119}$$

is included in the cube $\{(\rho, u_S, u_D) \mid |\rho|, |u_S|, |u_D| \le \delta'_2\}$. Hence we have a 4-dimensional cylinder-like closed set within R'_a for K_1^*

$$R'_b = \{(\rho, p_D, u_S, u_D) \mid \bigcup_{0 \le p_D \le \delta'_1} E_0\} \subset R'_a. \tag{120}$$

Since $G_D - 1 < 0$ for $p_D \in (0, \delta'_1]$, we have an inequality

$$\frac{\partial K_2}{\partial p_D} = -\kappa [\tilde{G}_D(p_D, \mathbf{c}) - 1] p_D > 0, \tag{121}$$

for $p_D \in (0, \delta_1']$ and $\mathbf{c} \in \hat{\mathbf{C}}$. Therefore $K_2^* = K_2(\delta_1', \mathbf{c})$.

Here we take $K^* < \min(K_1^*, K_2^*)$ and define R_b as (22). Then there is $\delta_1'' \in (0, \delta_1')$ such that $K_2(\delta_1'', \mathbf{c}) = K^*$ and $K_2(p_D, \mathbf{c}) < K^*$ for $p_D \in [0, \delta_1'')$. We define another ellipsoid $E(p_D)$ as

$$E(p_D) = \{(\rho, u_S, u_D) \mid K_1 < K^* - K_2(p_D, \mathbf{c})\}, \tag{122}$$

for $p_D \in [0, \delta_1'')$. Then we have

$$\begin{aligned}\{(\rho, p_D, u_S, u_D) \mid \bigcup_{0 \le p_D < \delta_1''} E(p_D)\} &= \{(\rho, p_D, u_S, u_D) \mid K = K_1 + K_2 < K^*\} \\ &= R_b. \end{aligned}\tag{123}$$

Since $E(p_D) \subset E_0$ for all $p_D \in [0, \delta_1'')$, R_b is a subset of R_b' that is a subset of the closed set R_a'. Therefore $\bar{R}_b$ is a subset of R_a.

The set of points $\{(\rho, p_D, u_s, u_D) \mid K(\rho, p_D, u_S, u_D, \mathbf{c}) = K^*\}$ will change with the change in the parameters. The change of these points will be continuous with the change in the parameters if all of $\partial K/\partial \rho$, $\partial K/\partial p_D$, $\partial K/\partial u_S$, and $\partial K/\partial u_D$ are not equal to 0. This condition will be satisfied anywhere in R_b except $(\rho, p_D, u_S, u_D) = (0, 0, 0, 0)$ for $\mathbf{c} \in \hat{\mathbf{C}}$. Since $K(0, 0, 0, 0, \mathbf{c}) = 0$, $\{(\rho, p_D, u_S, u_D) \mid K = K^*\}$ do not include $(0, 0, 0, 0)$ for $K^* > 0$. Therefore there is a neighborhood of $\mathbf{c} \in \hat{\mathbf{C}}$ for which $\bar{R}_b$ is still a subset of R_a. □

6.6. *Proof of theorem 1*

Here we consider $\hat{\mathbf{c}} \in \hat{\mathbf{C}}$ and check multiple steady states would appear in the neighborhood. Let $\tilde{\mathbf{c}} = \mathbf{c} - \hat{\mathbf{c}}$ which is expressed by a continuously differentiable map of a parameter τ as $\tilde{\mathbf{c}}(\tau) = [\tilde{c}_1(\tau), \tilde{c}_2(\tau), \ldots]$ such that $\tilde{\mathbf{c}}(0) = 0$ and $\gamma = \varsigma\tau^6 + o^7(\tau)$.

By expanding $\Phi_D(p_D, \mathbf{c})$ in the neighborhood of $(p_D, \mathbf{c}) = (0, \hat{\mathbf{c}})$, we get

$$\begin{aligned}&\Phi_D[p_D, \hat{\mathbf{c}} + \tilde{\mathbf{c}}(\tau)] \\ &= \left[\Phi_D(0, \hat{\mathbf{c}}) + \frac{\partial \Phi_D(0, \hat{\mathbf{c}})}{\partial \tau}\tau + \frac{1}{2}\frac{\partial^2 \Phi_D(0, \hat{\mathbf{c}})}{\partial \tau^2}\tau^2 + \cdots\right] \\ &\quad + \left[\frac{\partial \Phi_D(0, \hat{\mathbf{c}})}{\partial p_D} + \frac{\partial^2 \Phi_D(0, \hat{\mathbf{c}})}{\partial p_D \partial \tau}\tau + \frac{1}{2}\frac{\partial^3 \Phi_D(0, \hat{\mathbf{c}})}{\partial p_D \partial \tau^2}\tau^2 + \frac{1}{6}\frac{\partial^4 \Phi_D(0, \hat{\mathbf{c}})}{\partial p_D \partial \tau^3}\tau^3 + \cdots\right] p_D \\ &\quad + \left[\frac{1}{2}\frac{\partial^2 \Phi_D(0, \hat{\mathbf{c}})}{\partial {p_D}^2} + \frac{1}{2}\frac{\partial^3 \Phi_D(0, \hat{\mathbf{c}})}{\partial {p_D}^2 \partial \tau}\tau + \frac{1}{4}\frac{\partial^4 \Phi_D(0, \hat{\mathbf{c}})}{\partial {p_D}^2 \partial \tau^2}\tau^2 + \cdots\right] {p_D}^2 \\ &\quad + \left[\frac{\partial^3 \Phi_D(0, \hat{\mathbf{c}})}{\partial {p_D}^3} + o(\tau)\right] {p_D}^3 + o^4(p_D). \end{aligned}\tag{124}$$

The following equalities hold for $(p_D, \mathbf{c}) = (0, \hat{\mathbf{c}})$ according to the definition of $\hat{\mathbf{C}}$

$$\Phi_D(0, \hat{\mathbf{c}}) = 0, \tag{125}$$

$$\frac{\partial \Phi_D(0, \hat{\mathbf{c}})}{\partial \tau}\tau = \left\{-\left[\frac{d_T}{1 - \zeta_T}(p_D - \bar{p}_S)\right]\frac{d\gamma}{d\tau} + (\cdots)\gamma + \frac{\partial \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial \tau} p_D\right\}\tau$$

$$= \left(\frac{d_T}{1-\zeta_T}\bar{p}_S\right)\frac{d\gamma}{d\tau}\tau = 6\varsigma\left(\frac{d_T}{1-\zeta_T}\bar{p}_S\right)\tau^6 + o^7(\tau), \tag{126}$$

$$\begin{aligned}
\frac{\partial^2\Phi_D(0,\hat{\mathbf{c}})}{\partial\tau^2}\tau^2 &= \left\{-\left[\frac{d_T}{1-\zeta_T}(p_D-\bar{p}_S)\right]\frac{d^2\gamma}{d\tau^2}\right.\\
&\quad\left.+(\cdots)\frac{d\gamma}{d\tau}+(\cdots)\gamma+\frac{\partial^2\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial\tau^2}p_D\right\}\tau^2\\
&= \left[\left(\frac{d_T}{1-\zeta_T}\bar{p}_S\right)\frac{d^2\gamma}{d\tau^2}+(\cdots)\frac{d\gamma}{d\tau}\right]\tau^2\\
&= 30\varsigma\left(\frac{d_T}{1-\zeta_T}\bar{p}_S\right)\tau^6+o^7(\tau),
\end{aligned} \tag{127}$$

$$\vdots$$

$$\frac{\partial^6\Phi_D(0,\hat{\mathbf{c}})}{\partial\tau^6}\tau^6 = 6!\varsigma\left(\frac{d_T}{1-\zeta_T}\bar{p}_S\right)\tau^6+o^7(\tau), \tag{128}$$

$$\frac{\partial^7\Phi_D(0,\hat{\mathbf{c}})}{\partial\tau^7}\tau^7 = o^7(\tau). \tag{129}$$

From (125)–(129), the sum of terms in the first pair of brackets in the right-hand side of (124) will be

$$\begin{aligned}
&\left[\Phi_D(0,\hat{\mathbf{c}})+\frac{\partial\Phi_D(0,\hat{\mathbf{c}})}{\partial\tau}\tau+\frac{1}{2}\frac{\partial^2\Phi_D(0,\hat{\mathbf{c}})}{\partial\tau^2}\tau^2+\cdots\right]\\
&\quad= A\varsigma\left(\frac{d_T}{1-\zeta_T}\bar{p}_S\right)\tau^6+o^7(\tau),
\end{aligned} \tag{130}$$

where $A = 6+30+120+\cdots+6!$ Similarly, the following equalities hold

$$\begin{aligned}
\frac{\partial\Phi_D(0,\hat{\mathbf{c}})}{\partial p_D} &= -\frac{\gamma d_T}{1-\zeta_T}\left[1-\frac{\partial\bar{p}_S(0,\hat{\mathbf{c}})}{\partial p_D}\right]\\
&\quad+\frac{\partial\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D}p_D+G_D(0,\hat{\mathbf{c}})-1=0,
\end{aligned} \tag{131}$$

$$\begin{aligned}
\frac{\partial^2\Phi_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau &= \left(-\left\{\frac{d_T}{1-\zeta_T}\left[1-\frac{\partial\bar{p}_S(0,\hat{\mathbf{c}})}{\partial p_D}\right]\right\}\frac{d\gamma}{d\tau}+(\cdots)\gamma\right.\\
&\quad\left.+\frac{\partial^2\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}p_D+\frac{\partial\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial\tau}\right)\tau\\
&= \frac{\partial\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial\tau}\tau+o^6(\tau),
\end{aligned} \tag{132}$$

$$\frac{\partial^3\Phi_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau^2}\tau^2 = \frac{\partial^2\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial\tau^2}\tau^2+o^6(\tau), \tag{133}$$

$$\frac{\partial^4\Phi_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau^3}\tau^3 = \frac{\partial^3\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial\tau^3}\tau^3+o^6(\tau), \tag{134}$$

$$\frac{\partial^5 \Phi_D(0,\hat{\mathbf{c}})}{\partial p_D \partial \tau^4}\tau^4 = o^4(\tau), \tag{135}$$

$$\begin{aligned}\frac{\partial^2 \Phi_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2} &= \frac{\gamma d_T}{1-\zeta_T}\frac{\partial^2 \bar{p}_S(0,\hat{\mathbf{c}})}{\partial p_D{}^2} + \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2} p_D + 2\frac{\partial \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D} \\ &= 2\frac{\partial \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D} = 0,\end{aligned} \tag{136}$$

$$\begin{aligned}\frac{\partial^3 \Phi_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2 \partial \tau}\tau &= \left\{\left[\frac{d_T}{1-\zeta_T}\frac{\partial^2 \bar{p}_S(0,\hat{\mathbf{c}})}{\partial p_D{}^2}\right]\frac{d\gamma}{d\tau}\right. \\ &\quad \left. + (\cdots)\,\gamma + \frac{\partial^3 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2 \partial \tau} p_D + 2\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D \partial \tau}\right\}\tau \\ &= 2\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D \partial \tau}\tau + o^6(\tau),\end{aligned} \tag{137}$$

$$\begin{aligned}\frac{\partial^3 \Phi_D(0,\hat{\mathbf{c}})}{\partial p_D{}^3} &= \frac{\gamma d_T}{1-\zeta_T}\frac{\partial^3 \bar{p}_S(0,\hat{\mathbf{c}})}{\partial p_D{}^3} + \frac{\partial^3 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^3} p_D \\ &\quad + \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2} + 2\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2} \\ &= 3\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2}.\end{aligned} \tag{138}$$

We apply (131)–(138) to (124) and have

$$\begin{aligned}\Phi_D[p_D, \hat{\mathbf{c}} + \tilde{\mathbf{c}}(\tau)] &= \left\{A\varsigma\left[\frac{d_T}{1-\zeta_T}\bar{p}_S(0,\hat{\mathbf{c}})\right]\tau^6 + o^7(\tau)\right\} \\ &\quad + \left[\frac{\partial \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial \tau}\tau + \frac{1}{2}\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial \tau^2}\tau^2 + \frac{1}{6}\frac{\partial^3 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial \tau^3}\tau^3 + o^4(\tau)\right] p_D \\ &\quad + \left[\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D \partial \tau}\tau + o^2(\tau)\right] p_D{}^2 + \left[\frac{1}{2}\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2} + o(\tau)\right] p_D{}^3 \\ &\quad + o^4(p_D).\end{aligned} \tag{139}$$

Some manipulation for $\partial \tilde{G}_D/\partial \tau$, $\partial^2 \tilde{G}_D/\partial \tau^2$ and $\partial^3 \tilde{G}_D/\partial \tau^3$ gives

$$\frac{\partial \tilde{G}_D}{\partial \tau} = \frac{\partial \tilde{G}_D}{\partial \mathbf{c}}\frac{d\tilde{\mathbf{c}}}{d\tau} = \sum_i \frac{\partial \tilde{G}_D}{\partial c_i}\frac{d\tilde{c}_i}{d\tau}, \tag{140}$$

$$\begin{aligned}\frac{\partial^2 \tilde{G}_D}{\partial \tau^2} &= \sum_i \left[\frac{\partial}{\partial \tau}\left(\frac{\partial \tilde{G}_D}{\partial c_i}\right)\frac{d\tilde{c}_i}{d\tau} + \frac{\partial \tilde{G}_D}{\partial c_i}\frac{d^2 \tilde{c}_i}{d\tau^2}\right] \\ &= \sum_{i,j}\frac{\partial^2 \tilde{G}_D}{\partial c_i \partial c_j}\frac{d\tilde{c}_i}{d\tau}\frac{d\tilde{c}_j}{d\tau} + \sum_i \frac{\partial \tilde{G}_D}{\partial c_i}\frac{d^2 \tilde{c}_i}{d\tau^2},\end{aligned} \tag{141}$$

$$\frac{\partial^3 \tilde{G}_D}{\partial \tau^3} = \sum_{i,j} \left[\frac{\partial}{\partial \tau} \left(\frac{\partial^2 \tilde{G}_D}{\partial c_i \partial c_j} \right) \frac{d\tilde{c}_i}{d\tau} \frac{d\tilde{c}_j}{d\tau} + \frac{\partial^2 \tilde{G}_D}{\partial c_i \partial c_j} \frac{d^2 \tilde{c}_i}{d\tau^2} \frac{d\tilde{c}_j}{d\tau} + \frac{\partial^2 \tilde{G}_D}{\partial c_i \partial c_j} \frac{d\tilde{c}_i}{d\tau} \frac{d^2 \tilde{c}_j}{d\tau^2} \right]$$
$$+ \sum_i \left[\frac{\partial}{\partial \tau} \left(\frac{\partial \tilde{G}_D}{\partial c_i} \right) \frac{d^2 \tilde{c}_i}{d\tau^2} + \frac{\partial \tilde{G}_D}{\partial c_i} \frac{d^3 \tilde{c}_i}{d\tau^3} \right]$$
$$= \sum_{i,j,k} \frac{\partial^3 \tilde{G}_D}{\partial c_i \partial c_j \partial c_k} \frac{d\tilde{c}_i}{d\tau} \frac{d\tilde{c}_j}{d\tau} \frac{d\tilde{c}_k}{d\tau} + 3 \sum_{i,j} \frac{\partial^2 \tilde{G}_D}{\partial c_i \partial c_j} \frac{d^2 \tilde{c}_i}{d\tau^2} \frac{d\tilde{c}_j}{d\tau} + \sum_i \frac{\partial \tilde{G}_D}{\partial c_i} \frac{d^3 \tilde{c}_i}{d\tau^3}. \tag{142}$$

Since $\partial \tilde{G}_D(0, \hat{\mathbf{c}})/\partial \mathbf{c} \neq \mathbf{0}$ according to the definition of $\hat{\mathbf{C}}$, we can take $d\tilde{\mathbf{c}}(0)/d\tau \neq \mathbf{0}$ to satisfy

$$\frac{\partial \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial \tau} = \sum_i \frac{\partial \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial c_i} \frac{d\tilde{c}_i(0)}{d\tau} = 0. \tag{143}$$

In this case, the following relationship holds

$$\frac{\partial^2 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial p_D \partial \tau} = \sum_i \frac{\partial^2 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial p_D \partial c_i} \frac{d\tilde{c}_i(0)}{d\tau} \neq 0, \tag{144}$$

since $\partial \tilde{G}_D(0, \hat{\mathbf{c}})/\partial \mathbf{c}$ and $\partial^2 \tilde{G}_D(0, \hat{\mathbf{c}})/(\partial p_D \partial \mathbf{c})$ are independent according to the definition of $\hat{\mathbf{C}}$. Similarly, we can take $d^2 \tilde{\mathbf{c}}(0)/d\tau^2$ to satisfy

$$\frac{\partial^2 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial \tau^2} = \sum_{i,j} \frac{\partial^2 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial c_i \partial c_j} \frac{d\tilde{c}_i(0)}{d\tau} \frac{d\tilde{c}_j(0)}{d\tau} + \sum_i \frac{\partial \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial c_i} \frac{d^2 \tilde{c}_i(0)}{d\tau^2} = 0, \tag{145}$$

and $d^3 \tilde{\mathbf{c}}(0)/d\tau^3$ to satisfy

$$\frac{\partial^3 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial \tau^3} = \sum_{i,j,k} \frac{\partial^3 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial c_i \partial c_j \partial c_k} \frac{d\tilde{c}_i(0)}{d\tau} \frac{d\tilde{c}_j}{d\tau} \frac{d\tilde{c}_k(0)}{d\tau} + 3 \sum_{i,j} \frac{\partial^2 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial c_i \partial c_j} \frac{d^2 \tilde{c}_i(0)}{d\tau^2} \frac{d\tilde{c}_j}{d\tau}$$
$$+ \sum_i \frac{\partial \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial c_i} \frac{d^3 \tilde{c}_i(0)}{d\tau^3} \neq 0. \tag{146}$$

By taking $d^3 \tilde{\mathbf{c}}(0)/d\tau^3$ adequately, $\partial^3 \tilde{G}_D(0, \hat{\mathbf{c}})/\partial \tau^3$ will be either positive or negative. Hence if we take $\tilde{\mathbf{c}}(\tau)$ to satisfy (143)–(146), (139) is written as

$$\Phi_D[p_D, \hat{\mathbf{c}} + \tilde{\mathbf{c}}(\tau)] = \left\{ A\varsigma \left[\frac{d_T}{1 - \zeta_T} \bar{p}_S(0, \hat{\mathbf{c}}) \right] \tau^6 + o^7(\tau) \right\}$$
$$+ \left[\frac{1}{6} \frac{\partial^3 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial \tau^3} \tau^3 + o^4(\tau) \right] p_D$$
$$+ \left[\frac{\partial^2 \tilde{G}_D(0, \hat{\mathbf{c}})}{\partial p_D \partial \tau} \tau + o^2(\tau) \right] {p_D}^2$$

$$+\left[\frac{1}{2}\frac{\partial^2\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial {p_D}^2}+o(\tau)\right]{p_D}^3$$
$$+o^4(p_D). \tag{147}$$

Solving $\Phi_D = 0$ with respect to p_D by the expansion method [3], we obtain three solutions of the first order of τ or higher

$$p_{D1} = -2\left[\frac{\partial^2\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\right]\left[\frac{\partial^2\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial {p_D}^2}\right]^{-1}\tau + o^2(\tau) = o(\tau), \tag{148}$$

$$p_{D2} = -\frac{1}{6}\left[\frac{\partial^3\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial\tau^3}\right]\left[\frac{\partial^2\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\right]^{-1}\tau^2 + o^3(\tau) = o^2(\tau), \tag{149}$$

$$p_{D3} = -\frac{A\varsigma d_T}{1-\zeta_T}\bar{p}_S(0,\hat{\mathbf{c}})\left[\frac{\partial^3\tilde{G}_D(0,\hat{\mathbf{c}})}{\partial\tau^3}\right]^{-1}\tau^3 + o^4(\tau) = o^3(\tau). \tag{150}$$

As in the proof of the lemma 4, if $\Phi_D(p_D, \mathbf{c}) = 0$ for $p_D \geq 0$ then $(\rho, p_D, u_S, u_D) = (0, p_D, 0, 0)$ is a steady state of the system. Since $\partial^2\tilde{G}_D(p_D, \hat{\mathbf{c}})/\partial {p_D}^2 < 0$ according to the definition of $\hat{\mathbf{C}}$, p_{D1} will be positive if

$$\frac{\partial^2\tilde{G}_D(p_D,\mathbf{c})}{\partial p_D\partial\tau}\tau > 0, \tag{151}$$

holds; p_{D2} will be positive if (151) and

$$\frac{\partial^3\tilde{G}_D(p_D,\mathbf{c})}{\partial\tau^3}\tau < 0, \tag{152}$$

hold; p_{D3} will be positive if $\varsigma > 0$ and (152) hold when τ is sufficiently small. The inequality (151) will be satisfied if we take the direction of $d\tilde{\mathbf{c}}(0)/d\tau$ adequately; (152) will be satisfied if we take $d^3\tilde{\mathbf{c}}(0)/d\tau^3$ adequately.

Consequently, we have three steady states within R_b

1. $(\rho, p_D, u_S, u_D) = (0, p_{D1}, 0, 0)$,
2. $(\rho, p_D, u_S, u_D) = (0, p_{D2}, 0, 0)$,
3. $(\rho, p_D, u_S, u_D) = (0, p_{D3}, 0, 0)$,

for $\hat{\mathbf{c}}+\tilde{\mathbf{c}}(\tau)$ when τ is sufficiently small and the derivatives of $\tilde{\mathbf{c}}(\tau)$ are appropriate.

Next, we will examine the stability of each steady state. From the lemma 3, steady states are stable if $\partial^2K/\partial {p_D}^2 > 0$ and unstable if $\partial^2K/\partial {p_D}^2 < 0$.

The second partial derivative of K with respect to p_D is

$$\frac{\partial^2K}{\partial {p_D}^2} = -\kappa\left[-\frac{\gamma d_T}{1-\zeta_T}\left(1-\frac{\partial\bar{p}_S}{\partial p_D}\right) + G_D - 1 + \frac{\partial\tilde{G}_D}{\partial p_D}p_D\right]. \tag{153}$$

Since $\Phi_D = 0$ for the three steady states and $p_D \neq 0$

$$G_D - 1 = \frac{\gamma d_T}{1-\zeta_T}\left(1-\frac{\bar{p}_S}{p_D}\right). \tag{154}$$

Therefore we have

$$\frac{\partial^2 K}{\partial p_D{}^2} = -\kappa\left[\frac{\gamma d_T}{1-\zeta_T}\frac{\partial \bar{p}_S}{\partial p_D} - \frac{\gamma d_T}{1-\zeta_T}\frac{\bar{p}_S}{p_D} + \frac{\partial \tilde{G}_D}{\partial p_D}p_D\right]$$
$$= -\kappa\left[\frac{\partial \tilde{G}_D}{\partial p_D}p_D - \frac{\gamma d_T}{1-\zeta_T}\frac{\bar{p}_S}{p_D} + o^6(\tau)\right]. \tag{155}$$

For $\partial\tilde{G}_D/\partial p_D$, we have an expansion

$$\frac{\partial \tilde{G}_D(p_D,\hat{\mathbf{c}}+\tilde{\mathbf{c}})}{\partial p_D} = \frac{\partial \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D} + \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2}p_D + \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau + o^2(p_D) + o^2(\tau)$$
$$= \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2}p_D + \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau + o^2(p_D) + o^2(\tau). \tag{156}$$

We substitute (148)–(150) and (156) into (155) and have the inequalities for the three steady states

$$\left.\frac{\partial^2 K}{\partial p_D{}^2}\right|_{p_D=p_{D1}} = -\kappa\left\{\left[\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2}p_{D1} + \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau + o^2(\tau) + o^2(p_{D1})\right]p_{D1} - \frac{\gamma d_T}{1-\zeta_T}\frac{\bar{p}_S(0,\hat{\mathbf{c}})}{p_{D1}} + o^6(\tau)\right\}$$
$$= -\kappa\left\{\left[-\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau + o^2(\tau) + o^2(p_{D1})\right]p_{D1} + o^5(\tau)\right\}$$
$$= \kappa\left[\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau p_{D1} + o^3(\tau)\right] > 0, \tag{157}$$

$$\left.\frac{\partial^2 K}{\partial p_D{}^2}\right|_{p_D=p_{D2}} = -\kappa\left\{\left[\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2}p_{D2} + \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau + o^2(\tau) + o^2(p_{D2})\right]p_{D2} - \frac{\gamma d_T}{1-\zeta_T}\frac{\bar{p}_S(0,\hat{\mathbf{c}})}{p_{D2}} + o^6(\tau)\right\}$$
$$= -\kappa\left[\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau p_{D2} + o^4(\tau)\right] < 0, \tag{158}$$

$$\left.\frac{\partial^2 K}{\partial p_D{}^2}\right|_{p_D=p_{D3}} = -\kappa\left\{\left[\frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D{}^2}p_{D3} + \frac{\partial^2 \tilde{G}_D(0,\hat{\mathbf{c}})}{\partial p_D\partial\tau}\tau + o^2(\tau) + o^2(p_{D3})\right]p_{D3} - \frac{\gamma d_T}{1-\zeta_T}\frac{\bar{p}_S(0,\hat{\mathbf{c}})}{p_{D3}} + o^6(\tau)\right\} = \kappa\left[\frac{\gamma d_T}{1-\zeta_T}\frac{\bar{p}_S(0,\hat{\mathbf{c}})}{p_{D3}} + o^4(\tau)\right]$$
$$= \kappa\left[\frac{\varsigma\tau^6}{p_{D3}}\frac{d_T\bar{p}_S(0,\hat{\mathbf{c}})}{1-\zeta_T} + o^4(\tau)\right] > 0. \tag{159}$$

Therefore $(0, p_{D1}, 0, 0)$ and $(0, p_{D3}, 0, 0)$ are asymptotically stable steady states and $(0, p_{D2}, 0, 0)$ is an unstable steady state.

Consequently, as we move parameters adequately to a certain direction $[d\tilde{c}(0)/d\tau$ as in (143)] with a certain curvature $[d^2\tilde{c}(0)/d\tau^2$ as in (145)] from $\hat{\mathbf{c}} \in \hat{\mathbf{C}}$, a bifurcation occurs and three steady states appear. Any orbit will converge to one of the steady states in R_b without having other solutions as limit cycles, etc. Two of the three steady states are asymptotically stable and are separated by an unstable steady state. □

6.7. *Proof of theorem 2*

Let $h(n) = n$ without loss of generality in this proof. Anyone can reconstruct the proof for general $h(n)$ though it will be more complicated.

Let $\bar{n}_S(p_D, \mathbf{c})$ and $\bar{n}_D(p_D, \mathbf{c})$ be the nutrient concentrations that correspond to a steady state, $(\rho, p_D, u_S, u_D) = (0, p_D, 0, 0)$ for $\mathbf{c}$. From (8) and (9), $\bar{n}_S(p_D, \mathbf{c})$ and $\bar{n}_D(p_D, \mathbf{c})$ when $\gamma = 0$ are

$$\bar{n}_S(p_D, \mathbf{c}) = n_B - \frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B}\bar{p}_S(p_D, \mathbf{c}) - \frac{\alpha(1-\zeta_T)}{d_B}p_D, \quad (160)$$

$$\bar{n}_D(p_D, \mathbf{c}) = n_B - \frac{\alpha\zeta_T}{d_B}\bar{p}_S(p_D, \mathbf{c}) - \frac{\alpha(1-\zeta_T)}{d_B}p_D. \quad (161)$$

First we prepare the following lemma:

Lemma 6. *For a positive constant ξ and $\varphi(n, i)$, there is $\mathbf{c}'$ such that $\gamma = 0$, $\bar{n}_D(0, \mathbf{c}') = \xi i(\zeta_T)$ and $\tilde{G}_D(0, \mathbf{c}') = 1$.*

Proof. For $\xi > 0$ and $\zeta_T \in (0, 1)$, we take $r > 0$ to satisfy

$$\xi > \exp[-r(1-\zeta_T)] \quad \text{and} \quad \xi^2 > \exp[-r(1-\zeta_T)]. \quad (162)$$

According to (A3), we have the inequality for $i_T > 0$

$$\frac{\partial\varphi\{\xi i_T, i_T\exp[-r(\zeta - \zeta_T)]\}}{\partial i} \geq 0, \quad \text{for} \quad \zeta \in (\zeta_T, 1]. \quad (163)$$

From (162), we have

$$\xi i_T > i_T\exp[-r(1-\zeta_T)]. \quad (164)$$

From (164), there is an interval within $(\zeta_T, 1]$ for any $\xi > 0$ such that

$$\frac{\partial\varphi\{\xi i_T, i_T\exp[-r(\zeta - \zeta_T)]\}}{\partial i} > 0, \quad (165)$$

according to (A7).

Let

$$\tilde{F}_D(i_T) = \frac{1}{1-\zeta_T}\int_{\zeta_T}^{1}\varphi\{\xi i_T, i_T\exp[-r(\zeta - \zeta_T)]\}d\zeta. \quad (166)$$

By differentiating $\tilde{F}_D$ we obtain

$$\frac{d\tilde{F}_D}{di_T} = \frac{1}{1-\zeta_T}\int_{\zeta_T}^{1}\left\{\xi\frac{\partial\varphi}{\partial n} + \frac{\partial\varphi}{\partial i}\exp[-r(\zeta-\zeta_T)]\right\}d\zeta. \tag{167}$$

Since the sum of terms in braces of (167) is positive according to (A3), we have $d\tilde{F}_D/di_T > 0$. It is easy to see $\tilde{F}_D(0) = 0$ according to (A2) and there is $i_T^{\dagger} > 0$ such that $\tilde{F}_D(i_T^{\dagger}) > 1$ according to (A5'). Hence there is $i_T^* > 0$ such that $\tilde{F}_D(i_T^*) = 1$. Easy deduction from (163) and (165) gives

$$\varphi(\xi i_T^*, i_T^*) = \frac{1}{1-\zeta_T}\int_{\zeta_T}^{1}\varphi(\xi i_T^*, i_T^*)d\zeta > F_D(i_T^*) = 1, \tag{168}$$

$$\begin{aligned}\varphi\{\xi i_T^*, i_T^*\exp[-r(1-\zeta_T)]\} = \frac{1}{1-\zeta_T}&\int_{\zeta_T}^{1}\varphi\{\xi i_T^*, i_T^*\exp[-r(1-\zeta_T)]\}d\zeta\\ &< F_D(i_T^*) = 1,\end{aligned} \tag{169}$$

thus we have inequalities

$$\varphi(\xi i_T^*, i_T^*) > 1, \tag{170}$$

$$\varphi\{\xi i_T^*, i_T^*\exp[-r(1-\zeta_T)]\} < 1. \tag{171}$$

Let

$$\tilde{F}_S(p_S) = \frac{1}{\zeta_T}\int_0^{\zeta_T}\varphi\{(\xi i_T^* - \frac{\alpha\zeta_T}{d_T}p_S), i_T^*\exp[(r+p_S)(\zeta_T-\zeta)]\}d\zeta, \tag{172}$$

for $\alpha > 0$ and $d_T > 0$. For $p_S = \xi i_T^* d_T/(\alpha\zeta_T)$, we have

$$\tilde{F}_S(\frac{\xi i_T^* d_T}{\alpha\zeta_T}) = \frac{1}{\zeta_T}\int_0^{\zeta_T}\varphi\{0, i_T^*\exp[(r+p_S)(\zeta_T-\zeta)]\}d\zeta = 0. \tag{173}$$

From (172) and (170), we have an inequality

$$\tilde{F}_S(0) = \frac{1}{\zeta_T}\int_0^{\zeta_T}\varphi\{\xi i_T^*, i_T^*\exp[r(\zeta_T-\zeta)]\}d\zeta \geq \frac{1}{\zeta_T}\int_0^{\zeta_T}\varphi(\xi i_T^*, i_T^*)d\zeta > 1. \tag{174}$$

Thus there is p_S^* such that $\tilde{F}_S(p_S^*) = 1$.

Take i_0 as

$$i_0 = i_T^*\exp[(r+p_S^*)\zeta_T], \tag{175}$$

n_B for $d_B > 0$ as

$$n_B = \left(\xi i_T^* + \frac{\alpha\zeta_T}{d_B}p_S^*\right), \tag{176}$$

and $\gamma = 0$. Let the parameters decided above be $\mathbf{c}'$.

For $\mathbf{c}'$, $\bar{p}_S(0, \mathbf{c}')$ is defined implicitly by

$$G_S(\bar{p}_S, 0, 0, \mathbf{c}') = 1, \tag{177}$$

from the lemma 1. From (11) and (6), $G_S(p_S^*, 0, 0, \mathbf{c}')$ is written as,

$$\begin{aligned} G_S(p_S^*, 0, 0, \mathbf{c}') &= F_S(p_S^*, n_B - \frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B} p_S^*, \mathbf{c}') \\ &= \frac{1}{\zeta_T} \int_0^{\zeta_T} \varphi(n_B - \frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B} p_S^*, i_0 \exp[-(r + p_S^*)\zeta])d\zeta. \end{aligned} \tag{178}$$

By Substituting (175) and (176) into (178), we have

$$\begin{aligned} G_S(p_S^*, 0, 0, \mathbf{c}') &= \frac{1}{\zeta_T} \int_0^{\zeta_T} \varphi(\xi i_T^* - \frac{\alpha\zeta_T}{d_T} p_S^*, i_0 \exp[-(r + p_S^*)(\zeta_T - \zeta)])d\zeta \\ &= \tilde{F}_S(p_S^*) = 1. \end{aligned} \tag{179}$$

Since $\bar{p}_S(0, \mathbf{c})$ is unique for $\mathbf{c}$, we have

$$p_S^* = \bar{p}_S(0, \mathbf{c}'). \tag{180}$$

Similar manipulation gives

$$\tilde{G}_D(0, \mathbf{c}') = \tilde{F}_D(i_T^*) = 1. \tag{181}$$

From the definition (161), $\bar{n}_D(0, \mathbf{c}')$ is written as

$$\begin{aligned} \bar{n}_D(0, \mathbf{c}') &= n_B - \frac{\alpha\zeta_T}{d_B} \bar{p}_S(0, \mathbf{c}') = \xi i_T^* = \xi i_0 \exp\{-[r + \bar{p}_S(0, \mathbf{c}')]\zeta_T\} \\ &= \xi i(\zeta_T). \end{aligned} \tag{182}$$

□

In relation to the above lemma, we have the following note;

Note 1. While constructing $\mathbf{c}'$, d_B can be taken independently of ξ, $\varphi(n, i)$ and parameters other than n_B.

In the followings, we will consider the parameter set $\mathbf{c}'$ for $\xi = 1 - \delta$ and $\varphi(n, i) \in \mathcal{F}_\delta$.

For $\mathbf{c}'$, $\bar{n}_D(0, \mathbf{c}') = (1 - \delta)i(\zeta_T)$. Thus we have an inequality

$$\bar{n}_S(0, \mathbf{c}') < \bar{n}_D(0, \mathbf{c}') \le (1 - \delta)i(\zeta), \tag{183}$$

for $\zeta \in [0, \zeta_T]$. Hence we have an inequality according to (A7) and (A4)

$$\frac{\partial\varphi[\bar{n}_S(0, \mathbf{c}'), i(\zeta)]}{\partial i} \le \delta \frac{\partial\varphi[\bar{n}_S(0, \mathbf{c}'), i(\zeta)]}{\partial n} < \delta m_1, \tag{184}$$

for $\zeta \in [0, \zeta_T)$. From (162), we have

$$(1-\delta)^2 > \exp[-r(1-\zeta_T)]. \tag{185}$$

Therefore there is $\zeta^\dagger \in (\zeta_T, 1)$ such that

$$(1-\delta)^2 = \exp[-r(\zeta^\dagger - \zeta_T)], \tag{186}$$

$$(1-\delta)^2 > \exp[-r(\zeta - \zeta_T)], \quad \text{for } \zeta \in (\zeta^\dagger, 1]. \tag{187}$$

By multiplying both sides of the inequality (187) by $\bar{n}_D(0, \mathbf{c}')/(1-\delta) = i(\zeta_T)$, we have

$$(1-\delta)\bar{n}_D(0, \mathbf{c}') > i(\zeta), \quad \text{for } \zeta \in (\zeta^\dagger, 1]. \tag{188}$$

Hence we have an inequality according to (A7) and (A4)

$$\frac{\partial \varphi[\bar{n}_D(0, \mathbf{c}'), i(\zeta)]}{\partial n} \leq \delta \frac{\partial \varphi[\bar{n}_D(0, \mathbf{c}'), i(\zeta)]}{\partial i} < \delta m_1, \tag{189}$$

for $\zeta \in (\zeta^\dagger, 1]$. From (186), we obtain

$$\zeta^\dagger - \zeta_T = -\frac{2\log(1-\delta)}{r} = \frac{2\delta}{r} + o^2(\delta) = o^1_+(\delta), \tag{190}$$

where $o^1_+(\delta)$ is a sum of higher order terms than the first order of δ and is positive.

Some manipulation of the partial derivatives of F_S and F_D at $(p_S, p_D, n_S, n_D) = [\bar{p}_S(0, \mathbf{c}'), 0, \bar{n}_S(0, \mathbf{c}'), \bar{n}_D(0, \mathbf{c}')]$ gives equalities

$$\begin{aligned}
\frac{\partial F_D}{\partial p_S} &= \frac{1}{1-\zeta_T}\int_{\zeta_T}^1 \frac{\partial \varphi}{\partial i}\frac{\partial i}{\partial p_S} d\zeta = \frac{1}{1-\zeta_T}\int_{\zeta_T}^1 \frac{\partial \varphi}{\partial i}(-i\zeta_T) d\zeta \\
&= \frac{\zeta_T}{r(1-\zeta_T)}\int_{\zeta_T}^1 \frac{\partial \varphi}{\partial i}(-ir) d\zeta = \frac{\zeta_T}{r(1-\zeta_T)}\int_{\zeta_T}^1 \frac{\partial \varphi}{\partial i}\frac{\partial i}{\partial \zeta} d\zeta \\
&= \frac{\zeta_T}{r(1-\zeta_T)}\int_{\zeta_T}^1 \frac{d\varphi}{d\zeta} d\zeta = \frac{\zeta_T}{r(1-\zeta_T)}\{\varphi[\bar{n}_D(0, \mathbf{c}'), i(1)] - \varphi[\bar{n}_D(0, \mathbf{c}'), i(\zeta_T)]\}, \\
&= \frac{\zeta_T}{r(1-\zeta_T)}\Big\{\varphi[(1-\delta)i(\zeta_T), i(1)] - \varphi[(1-\delta)i(\zeta_T), i(\zeta_T)]\Big\},
\end{aligned} \tag{191}$$

$$\begin{aligned}
\frac{\partial F_D}{\partial p_D} &= \frac{1}{1-\zeta_T}\int_{\zeta_T}^1 \frac{\partial \varphi}{\partial i}\frac{\partial i}{\partial p_D} d\zeta = \frac{1}{1-\zeta_T}\int_{\zeta_T}^1 \frac{\partial \varphi}{\partial i}[-i(\zeta - \zeta_T)] d\zeta \\
&= \frac{1}{r(1-\zeta_T)}\int_{\zeta_T}^1 \frac{d\varphi}{d\zeta}(\zeta - \zeta_T) d\zeta \\
&= \frac{1}{r(1-\zeta_T)}\left\{(1-\zeta_T)\varphi[\bar{n}_D(0, \mathbf{c}'), i(1)] - \int_{\zeta_T}^1 \varphi d\zeta\right\} \\
&= \frac{1}{r}\left\{\varphi[\bar{n}_D(0, \mathbf{c}'), i(1)] - \tilde{G}_D(0, \mathbf{c}')\right\} = \frac{1}{r}\{\varphi[(1-\delta)i(\zeta_T), i(1)] - 1\},
\end{aligned} \tag{192}$$

and inequalities

$$\left|\frac{\partial F_S}{\partial p_S}\right| = -\frac{\partial F_S}{\partial p_S} = -\frac{1}{\zeta_T}\int_0^{\zeta_T}\frac{\partial \varphi}{\partial i}\frac{\partial i}{\partial p_S}d\zeta < \frac{1}{\zeta_T}\int_0^{\zeta_T}\delta\frac{\partial \varphi}{\partial n}i\zeta d\zeta < \delta i_0\zeta_T\frac{\partial F_S}{\partial n_S}, \tag{193}$$

$$\begin{aligned}\frac{\partial F_D}{\partial n_D} &= \frac{1}{1-\zeta_T}\int_{\zeta_T}^1\frac{\partial \varphi}{\partial n}d\zeta = \frac{1}{1-\zeta_T}\left(\int_{\zeta_T}^{\zeta^\dagger}\frac{\partial \varphi}{\partial n}d\zeta + \int_{\zeta^\dagger}^1\frac{\partial \varphi}{\partial n}d\zeta\right)\\ &< \left[(\zeta^\dagger - \zeta_T)m_1 + (1-\zeta_T)\delta m_1\right].\end{aligned} \tag{194}$$

From (193) and (194), we have

$$-\left(\frac{\partial F_S}{\partial p_S}\right)\left(\frac{\partial F_S}{\partial n_S}\right)^{-1} = o_+^1(\delta), \tag{195}$$

$$\frac{\partial F_D}{\partial n_D} = o_+^1(\delta). \tag{196}$$

The same procedure as (74)–(84) in the section 6.4 leads to

$$\frac{\partial \tilde{G}_D(0,\mathbf{c}')}{\partial p_D} = 0 \quad \Leftrightarrow \quad \frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S} = 0, \tag{197}$$

and yields an equality

$$\begin{aligned}\frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S} &= \frac{\partial F_S}{\partial p_S}\left[\frac{\partial F_D}{\partial p_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_D}{\partial n_D}\right]\\ &-\frac{\partial F_S}{\partial n_S}\left[\frac{\alpha\zeta_T(d_T+d_B)}{d_Td_B}\frac{\partial F_D}{\partial p_D} - \frac{\alpha^2\zeta_T(1-\zeta_T)}{d_Td_B}\frac{\partial F_D}{\partial n_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_D}{\partial p_S}\right],\end{aligned} \tag{198}$$

for $\mathbf{c}'$ and $(p_S, p_D, u_S, u_D) = [\bar{p}_S(0,\mathbf{c}'), 0, 0, 0]$. We substitute (191), (192), (195), and (196) into the right-hand side of (198), and have an equality

$$\begin{aligned}&\frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S}\\ &= -\frac{1}{d_B}\frac{\partial F_S}{\partial n_S}\Bigg\{o_+^1(\delta)\left[d_B\frac{\partial F_D}{\partial p_D} - \alpha(1-\zeta_T)o_+^1(\delta)\right]\\ &\quad+\frac{\alpha\zeta_T}{rd_T}\Big(d_B\{\varphi[(1-\delta)i(\zeta_T), i(1)] - 1\} + d_T\{\varphi[(1-\delta)i(\zeta_T), i(\zeta_T)] - 1\}\Big)\\ &\quad-\frac{\alpha^2\zeta_T(1-\zeta_T)}{d_T}o_+^1(\delta)\Bigg\}\\ &= -\frac{1}{d_B}\frac{\partial F_S}{\partial n_S}\Bigg\{d_B\left[\frac{\partial F_D}{\partial p_D}o_+^1(\delta) + \frac{\alpha\zeta_T}{rd_T}\big\{\varphi[(1-\delta)i(\zeta_T), i(1)] - 1\big\}\right]\\ &\quad+\frac{\alpha\zeta_T}{r}\{\varphi[(1-\delta)i(\zeta_T), i(\zeta_T)]-1\} - \alpha(1-\zeta_T)o_+^1(\delta) - \frac{\alpha^2\zeta_T(1-\zeta_T)}{d_T}o_+^1(\delta)\Bigg\}.\end{aligned} \tag{199}$$

As we mentioned in the *Note 1*, we can take d_B independent of parameters except n_B. Terms in the right-hand side of (199) are independent of n_B, thus d_B is independent of these terms except itself. Let $G'_D(d_B)$ be the terms in the braces of the right-hand side of (199)

$$\begin{aligned}G'_D(d_B) = d_B &\left[\frac{\partial F_D}{\partial p_D} o^1_+(\delta) + \frac{\alpha\zeta_T}{r d_T}\left\{\varphi[(1-\delta)i(\zeta_T), i(1)] - 1\right\}\right] \\ &+ \frac{\alpha\zeta_T}{r}\left\{\varphi[(1-\delta)i(\zeta_T), i(\zeta_T)] - 1\right\} \\ &-\alpha(1-\zeta_T)o^1_+(\delta) - \frac{\alpha^2\zeta_T(1-\zeta_T)}{d_T} o^1_+(\delta).\end{aligned} \tag{200}$$

Since $\varphi[(1-\delta)i(\zeta_T), i(\zeta_T)] > 1$ from (170), $\varphi[(1-\delta)i(\zeta_T), i(1)] < 1$ from (171), and $\partial F_D/\partial p_D \leq 0$ for $(\bar{p}_S, 0, 0, 0, \mathbf{c}')$, we have the two inequalities

$$\left[\frac{\partial F_D}{\partial p_D} o^1_+(\delta) + \frac{\alpha\zeta_T}{r d_T}(\varphi[(1-\delta)i(\zeta_T), i(\zeta_T)] - 1)\right] < 0, \tag{201}$$

$$\frac{\alpha\zeta_T}{r}\left\{\varphi[(1-\delta)i(\zeta_T), i(1)] - 1\right\} > 0. \tag{202}$$

From (201) and (202), there is $d_B > 0$ such that $G'_D(d_B) = 0$ for sufficiently small δ. Consequently, there is $\mathbf{c}''$ such that $\gamma = 0$, $\tilde{G}_D(0, \mathbf{c}'') = 1$ and $\partial\tilde{G}_D(0, \mathbf{c}'')/\partial p_D = 0$ for $\varphi \in \mathcal{F}_\delta$ when δ is sufficiently small.

Our next task is to examine the sign of $\partial^2\tilde{G}_D(0, \mathbf{c}'')/\partial {p_D}^2$. $\partial^2\tilde{G}_D(0, \mathbf{c}'')/\partial {p_D}^2$ is written as

$$\begin{aligned}&\frac{\partial^2\tilde{G}_D(0, \mathbf{c}'')}{\partial {p_D}^2} \\ &= \left(\frac{\partial G_S}{\partial p_S}\right)^{-1}\left(\frac{\partial G_S}{\partial p_S}\frac{\partial}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial}{\partial p_S}\right) \\ &\quad\times\left[\left(\frac{\partial G_S}{\partial p_S}\right)^{-1}\left(\frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S}\right)\right] \\ &= \left(\frac{\partial G_S}{\partial p_S}\right)^{-1}\left[\left(\frac{\partial G_S}{\partial p_S}\frac{\partial}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial}{\partial p_S}\right)\left(\frac{\partial G_S}{\partial p_S}\right)^{-1}\right] \\ &\quad\times\left(\frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S}\right) \\ &\quad+\left(\frac{\partial G_S}{\partial p_S}\right)^{-2}\left[\left(\frac{\partial G_S}{\partial p_S}\frac{\partial}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial}{\partial p_S}\right)\right. \\ &\quad\left.\times\left(\frac{\partial G_S}{\partial p_S}\frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial G_D}{\partial p_S}\right)\right].\end{aligned} \tag{203}$$

For $\mathbf{c}''$, the equality (197) holds, so we have

$$\frac{\partial^2\tilde{G}_D(0, \mathbf{c}'')}{\partial {p_D}^2} = \left(\frac{\partial G_S}{\partial p_S}\right)^{-2}\left[\left(\frac{\partial G_S}{\partial p_S}\frac{\partial}{\partial p_D} - \frac{\partial G_S}{\partial p_D}\frac{\partial}{\partial p_S}\right)\right.$$

$$\times \left(\frac{\partial G_S}{\partial p_S} \frac{\partial G_D}{\partial p_D} - \frac{\partial G_S}{\partial p_D} \frac{\partial G_D}{\partial p_S} \right) \Bigg]. \tag{204}$$

Further Manipulation of (204) yields an equality

$$\begin{aligned}
\frac{\partial^2 \tilde{G}_D(0, \mathbf{c}'')}{\partial {p_D}^2} &= \left(\frac{\partial G_S}{\partial p_S} \right)^{-2} \Bigg[- \frac{\partial G_S}{\partial p_D} \frac{\partial G_D}{\partial p_D} \frac{\partial^2 G_S}{\partial {p_S}^2} \\
&\quad + \left(\frac{\partial G_S}{\partial p_S} \frac{\partial G_D}{\partial p_D} + \frac{\partial G_S}{\partial p_D} \frac{\partial G_D}{\partial p_S} \right) \frac{\partial^2 G_S}{\partial p_S \partial p_D} - \frac{\partial G_S}{\partial p_S} \frac{\partial G_D}{\partial p_S} \frac{\partial^2 G_S}{\partial {p_D}^2} \\
&\quad + \left(\frac{\partial G_S}{\partial p_D} \right)^2 \frac{\partial^2 G_D}{\partial {p_S}^2} - 2 \frac{\partial G_S}{\partial p_S} \frac{\partial G_S}{\partial p_D} \frac{\partial^2 G_D}{\partial p_S \partial p_D} + \left(\frac{\partial G_S}{\partial p_S} \right)^2 \frac{\partial^2 G_D}{\partial {p_D}^2} \Bigg] \\
&= \left(\frac{\partial G_S}{\partial p_S} \right)^{-2} \Bigg[- \frac{\partial G_S}{\partial p_D} \frac{\partial G_D}{\partial p_D} \frac{\partial^2 G_S}{\partial {p_S}^2} \\
&\quad + 2 \frac{\partial G_S}{\partial p_S} \frac{\partial G_D}{\partial p_D} \frac{\partial^2 G_S}{\partial p_S \partial p_D} - \frac{\partial G_S}{\partial p_S} \frac{\partial G_D}{\partial p_S} \frac{\partial^2 G_S}{\partial {p_D}^2} \\
&\quad + \left(\frac{\partial G_S}{\partial p_D} \right)^2 \frac{\partial^2 G_D}{\partial {p_S}^2} - 2 \frac{\partial G_S}{\partial p_S} \frac{\partial G_S}{\partial p_D} \frac{\partial^2 G_D}{\partial p_S \partial p_D} + \left(\frac{\partial G_S}{\partial p_S} \right)^2 \frac{\partial^2 G_D}{\partial {p_D}^2} \Bigg].
\end{aligned} \tag{205}$$

From (80)–(83), (195), and (196), we have equalities

$$\begin{aligned}
\left(\frac{\partial G_S}{\partial p_D} \right) \left(\frac{\partial G_S}{\partial p_S} \right)^{-1} &= \left[- \frac{\alpha(1 - \zeta_T)}{d_B} \frac{\partial F_S}{\partial n_S} \right] \left[\frac{\partial F_S}{\partial p_S} - \frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B} \frac{\partial F_S}{\partial n_S} \right]^{-1} \\
&= \frac{d_T(1 - \zeta_T)}{(d_T + d_B)\zeta_T} + o^1(\delta),
\end{aligned} \tag{206}$$

$$\frac{\partial G_D}{\partial p_S} = \frac{\partial F_D}{\partial p_S} + o^1(\delta), \tag{207}$$

$$\frac{\partial G_D}{\partial p_D} = \frac{\partial F_D}{\partial p_D} + o^1(\delta). \tag{208}$$

Applying (206)–(208) to (205) we have

$$\begin{aligned}
\frac{\partial^2 \tilde{G}_D(0, \mathbf{c}'')}{\partial {p_D}^2} &= \left(\frac{\partial G_S}{\partial p_S} \right)^{-1} \\
&\quad \left[- \frac{d_T(1 - \zeta_T)}{(d_T + d_B)\zeta_T} \frac{\partial F_D}{\partial p_D} \frac{\partial^2 G_S}{\partial {p_S}^2} + 2 \frac{\partial F_D}{\partial p_D} \frac{\partial^2 G_S}{\partial p_S \partial p_D} - \frac{\partial F_D}{\partial p_S} \frac{\partial^2 G_S}{\partial {p_D}^2} \right] \\
&\quad + \left[\frac{d_T(1 - \zeta_T)}{(d_T + d_B)\zeta_T} \right]^2 \frac{\partial^2 G_D}{\partial {p_S}^2} - 2 \frac{d_T(1 - \zeta_T)}{(d_T + d_B)\zeta_T} \frac{\partial^2 G_D}{\partial p_S \partial p_D} + \frac{\partial^2 G_D}{\partial {p_D}^2} + o^1(\delta).
\end{aligned} \tag{209}$$

The second partial derivatives of G_S with respect to p_S and p_D are written as

$$\frac{\partial^2 G_S}{\partial p_S{}^2} = \frac{\partial^2 F_S}{\partial p_S{}^2} - \frac{2\alpha(d_T + d_B)\zeta_T}{d_T d_B}\frac{\partial^2 F_S}{\partial p_S \partial n_S} + \left[\frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B}\right]^2 \frac{\partial^2 F_S}{\partial n_S{}^2}, \tag{210}$$

$$\frac{\partial^2 G_S}{\partial p_S \partial p_D} = -\frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial^2 F_S}{\partial p_S \partial n_S} + \left[\frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B}\right]\left[\frac{\alpha(1-\zeta_T)}{d_B}\right]\frac{\partial^2 F_S}{\partial n_S{}^2}, \tag{211}$$

$$\frac{\partial^2 G_S}{\partial p_D{}^2} = \left[\frac{\alpha(1-\zeta_T)}{d_B}\right]^2 \frac{\partial^2 F_S}{\partial n_S{}^2}, \tag{212}$$

and the second partial derivative of F_S with respect to p_S is written as,

$$\frac{\partial^2 F_S}{\partial p_S{}^2} = \frac{1}{\zeta_T}\int_0^{\zeta_T}\left[\frac{\partial^2 \varphi}{\partial i^2}\left(\frac{\partial i}{\partial p_S}\right)^2 + \frac{\partial \varphi}{\partial i}\frac{\partial^2 i}{\partial p_S{}^2}\right]d\zeta. \tag{213}$$

Since $\partial i/\partial p_S = -\zeta i$ and $\partial i/\partial \zeta = -(r + p_S)i$ for $\zeta \in [0, \zeta_T)$, we have

$$\begin{aligned}\frac{\partial^2 F_S}{\partial p_S{}^2} &= \frac{1}{(r+p_S)^2\zeta_T}\int_0^{\zeta_T}\left[\frac{\partial^2 \varphi}{\partial i^2}\left(\frac{\partial i}{\partial \zeta}\right)^2 + \frac{\partial \varphi}{\partial i}\frac{\partial^2 i}{\partial \zeta^2}\right]\zeta^2 d\zeta \\ &= \frac{1}{(r+p_S)^2\zeta_T}\int_0^{\zeta_T}\frac{d^2\varphi}{d\zeta^2}\zeta^2 d\zeta \\ &= \frac{1}{(r+p_S)^2\zeta_T}\left\{\frac{d\varphi[\bar{n}_S(0,\mathbf{c}''), i(\zeta_T)]}{d\zeta}\zeta_T - 2\int_0^{\zeta_T}\frac{d\varphi}{d\zeta}\zeta d\zeta\right\}.\end{aligned} \tag{214}$$

Since $\partial\varphi[\bar{n}_S(0,\mathbf{c}''), i(\zeta)]/\partial i = o^1(\delta)$ for $\zeta \in [0, \zeta_T]$ from (184), we have

$$\frac{d\varphi}{d\zeta} = \frac{\partial \varphi}{\partial i}\frac{\partial i}{\partial \zeta} = o^1(\delta), \quad \text{for} \quad \zeta \in [0, \zeta_T]. \tag{215}$$

From (214) and (215), we have

$$\frac{\partial^2 F_S}{\partial p_S{}^2} = o^1(\delta). \tag{216}$$

We substitute (210)–(212) into the terms in the first pair of brackets of (209), apply (216) and then we get an equality

$$\begin{aligned}&\left[-\frac{d_T(1-\zeta_T)}{(d_T+d_B)\zeta_T}\frac{\partial F_D}{\partial p_D}\frac{\partial^2 G_S}{\partial p_S{}^2} + 2\frac{\partial F_D}{\partial p_D}\frac{\partial^2 G_S}{\partial p_S \partial p_D} - \frac{\partial F_D}{\partial p_S}\frac{\partial^2 G_S}{\partial p_D{}^2}\right] \\ &\quad = \left[\frac{\alpha(1-\zeta_T)}{d_B}\right]\left[\frac{\alpha(d_T+d_B)\zeta_T}{d_T d_B}\frac{\partial F_D}{\partial p_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_D}{\partial p_S}\right]\frac{\partial^2 F_S}{\partial n_S{}^2} + o^1(\delta).\end{aligned} \tag{217}$$

Since $\partial \tilde{G}_D(0, \mathbf{c}'')/\partial p_D = 0$, slight manipulation of (198) gives an equality

$$\begin{aligned}
&\frac{\alpha(d_T + d_B)\zeta_T}{d_T d_B}\frac{\partial F_D}{\partial p_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_D}{\partial p_S} \\
&= \frac{\alpha^2\zeta_T(1-\zeta_T)}{d_T d_B}\frac{\partial F_D}{\partial n_D} + \left(\frac{\partial F_S}{\partial p_S}\right)\left(\frac{\partial F_S}{\partial n_S}\right)^{-1}\left[\frac{\partial F_D}{\partial p_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial F_D}{\partial n_D}\right] \\
&= o^1(\delta).
\end{aligned} \tag{218}$$

From (217) and (218), we have

$$\left[-\frac{d_T(1-\zeta_T)}{(d_T+d_B)\zeta_T}\frac{\partial F_D}{\partial p_D}\frac{\partial^2 G_S}{\partial {p_S}^2} + 2\frac{\partial F_D}{\partial p_D}\frac{\partial^2 G_S}{\partial p_S \partial p_D} - \frac{\partial F_D}{\partial p_S}\frac{\partial^2 G_S}{\partial {p_D}^2}\right] = o^1(\delta). \tag{219}$$

The second partial derivatives of G_D with respect to p_S and p_D are written as

$$\frac{\partial^2 G_D}{\partial {p_S}^2} = \frac{\partial^2 F_D}{\partial {p_S}^2} - \frac{2\alpha\zeta_T}{d_B}\frac{\partial^2 F_D}{\partial p_S \partial n_D} + \left(\frac{\alpha\zeta_T}{d_B}\right)^2\frac{\partial^2 F_D}{\partial {n_D}^2}, \tag{220}$$

$$\begin{aligned}
\frac{\partial^2 G_D}{\partial p_S \partial p_D} &= \frac{\partial^2 F_D}{\partial p_S \partial p_D} - \frac{\alpha(1-\zeta_T)}{d_B}\frac{\partial^2 F_D}{\partial p_S \partial n_D} \\
&\quad - \frac{\alpha\zeta_T}{d_B}\frac{\partial^2 F_D}{\partial p_D \partial n_D} + \frac{\alpha^2\zeta_T(1-\zeta_T)}{{d_B}^2}\frac{\partial^2 F_D}{\partial {n_D}^2},
\end{aligned} \tag{221}$$

$$\frac{\partial^2 G_D}{\partial {p_D}^2} = \frac{\partial^2 F_D}{\partial {p_D}^2} - \frac{2\alpha(1-\zeta_T)}{d_B}\frac{\partial^2 F_D}{\partial p_D \partial n_D} + \left[\frac{\alpha(1-\zeta_T)}{d_B}\right]^2\frac{\partial^2 F_D}{\partial {n_D}^2}. \tag{222}$$

Applying (220)–(222) to (209), some manipulation yields an equality

$$\begin{aligned}
\frac{\partial^2 \tilde{G}_D(0, \mathbf{c}'')}{\partial {p_D}^2} &= \left[\frac{d_T(1-\zeta_T)}{(d_T+d_B)\zeta_T}\right]^2\frac{\partial^2 F_D}{\partial {p_S}^2} - \frac{2d_T(1-\zeta_T)}{(d_T+d_B)\zeta_T}\frac{\partial^2 F_D}{\partial p_S \partial p_D} + \frac{\partial^2 F_D}{\partial {p_D}^2} \\
&\quad + \frac{2\alpha d_T(1-\zeta_T)^2}{(d_T+d_B)^2\zeta_T}\frac{\partial^2 F_D}{\partial p_S \partial n_D} - \frac{2\alpha(1-\zeta_T)}{d_T+d_B}\frac{\partial^2 F_D}{\partial p_D \partial n_D} \\
&\quad + \left[\frac{\alpha(1-\zeta_T)}{d_T+d_B}\right]^2\frac{\partial^2 F_D}{\partial {n_D}^2} + o^1(\delta)
\end{aligned} \tag{223}$$

The second derivatives of F_D with respect to p_S, p_D and n_D are written as

$$\frac{\partial^2 F_D}{\partial {p_S}^2} = \frac{1}{1-\zeta_T}\int_{\zeta_T}^1\left[\frac{\partial^2\varphi}{\partial i^2}\left(\frac{\partial i}{\partial p_S}\right)^2 + \frac{\partial\varphi}{\partial i}\frac{\partial^2 i}{\partial {p_S}^2}\right]d\zeta, \tag{224}$$

$$\frac{\partial^2 F_D}{\partial p_S \partial p_D} = \frac{1}{1-\zeta_T}\int_{\zeta_T}^1\left[\frac{\partial^2\varphi}{\partial i^2}\frac{\partial i}{\partial p_S}\frac{\partial i}{\partial p_D} + \frac{\partial\varphi}{\partial i}\frac{\partial^2 i}{\partial {p_D}^2}\right]d\zeta, \tag{225}$$

$$\frac{\partial^2 F_D}{\partial {p_D}^2} = \frac{1}{1-\zeta_T}\int_{\zeta_T}^1\left[\frac{\partial^2\varphi}{\partial i^2}\left(\frac{\partial i}{\partial p_D}\right)^2 + \frac{\partial\varphi}{\partial i}\frac{\partial^2 i}{\partial {p_D}^2}\right]d\zeta, \tag{226}$$

$$\frac{\partial^2 F_D}{\partial p_S \partial n_D} = \frac{1}{1-\zeta_T}\int_{\zeta_T}^1\frac{\partial^2\varphi}{\partial i \partial n}\frac{\partial i}{\partial p_S}d\zeta, \tag{227}$$

$$\frac{\partial^2 F_D}{\partial p_D \partial n_D} = \frac{1}{1-\zeta_T}\int_{\zeta_T}^{1} \frac{\partial^2 \varphi}{\partial i \partial n}\frac{\partial i}{\partial p_D} d\zeta, \tag{228}$$

$$\frac{\partial^2 F_D}{\partial {n_D}^2} = \frac{1}{1-\zeta_T}\int_{\zeta_T}^{1} \frac{\partial^2 \varphi}{\partial n^2} d\zeta. \tag{229}$$

Applying $[1/(1-\zeta_T)]\int_{\zeta_T}^{1} \varphi d\zeta = 1$, $\partial i/\partial p_S = -\zeta_T i$, $\partial i/\partial p_D = -(\zeta - \zeta_T)i$ and $\partial i/\partial \zeta = -ri$ for $\zeta \in (\zeta_T, 1)$, we decompose (224)–(228) and get

$$\begin{aligned}
\frac{\partial^2 F_D}{\partial {p_S}^2} &= \frac{{\zeta_T}^2}{r^2(1-\zeta_T)}\int_{\zeta_T}^{1}\left[\frac{\partial^2\varphi}{\partial i^2}\left(\frac{\partial i}{\partial \zeta}\right)^2 + \frac{\partial \varphi}{\partial i}\frac{\partial^2 i}{\partial \zeta^2}\right] d\zeta \\
&= \frac{{\zeta_T}^2}{r^2(1-\zeta_T)}\int_{\zeta_T}^{1}\frac{d^2\varphi}{d\zeta^2} d\zeta \\
&= \frac{{\zeta_T}^2}{r^2(1-\zeta_T)}\left\{\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta} - \frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{d\zeta}\right\}, \quad (230) \\
\frac{\partial^2 F_D}{\partial p_S \partial p_D} &= \frac{\zeta_T}{r^2(1-\zeta_T)}\int_{\zeta_T}^{1}\left[\frac{\partial^2\varphi}{\partial i^2}\left(\frac{\partial i}{\partial \zeta}\right)^2 + \frac{\partial \varphi}{\partial i}\frac{\partial^2 i}{\partial \zeta^2}\right](\zeta-\zeta_T) d\zeta \\
&= \frac{\zeta_T}{r^2(1-\zeta_T)}\int_{\zeta_T}^{1}\frac{d^2\varphi}{d\zeta^2}(\zeta-\zeta_T) d\zeta \\
&= \frac{\zeta_T}{r^2(1-\zeta_T)}\left\{\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta}(1-\zeta_T) - \int_{\zeta_T}^{1}\frac{d\varphi}{d\zeta}d\zeta\right\} \\
&= \frac{\zeta_T}{r^2(1-\zeta_T)}\left\{\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta}(1-\zeta_T)\right. \\
&\qquad \left. - \varphi[\bar{n}_D(0,\mathbf{c}''), i(1)] + \varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]\right\}, \quad (231) \\
\frac{\partial^2 F_D}{\partial {p_D}^2} &= \frac{1}{r^2(1-\zeta_T)}\int_{\zeta_T}^{1}\left[\frac{\partial^2\varphi}{\partial i^2}\left(\frac{\partial i}{\partial \zeta}\right)^2 + \frac{\partial \varphi}{\partial i}\frac{\partial^2 i}{\partial \zeta^2}\right](\zeta-\zeta_T)^2 d\zeta \\
&= \frac{1}{r^2(1-\zeta_T)}\int_{\zeta_T}^{1}\frac{d^2\varphi}{d\zeta^2}(\zeta-\zeta_T)^2 d\zeta \\
&= \frac{1}{r^2(1-\zeta_T)}\left\{\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta}(1-\zeta_T)^2 - 2\int_{\zeta_T}^{1}\frac{d\varphi}{d\zeta}(\zeta-\zeta_T)d\zeta\right\} \\
&= \frac{1}{r^2(1-\zeta_T)}\left\{\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta}(1-\zeta_T)^2\right. \\
&\qquad \left. - 2\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)](1-\zeta_T) + 2\int_{\zeta_T}^{1}\varphi d\zeta\right\} \\
&= \frac{1}{r^2}\left\{\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta}(1-\zeta_T) - 2\left(\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)] - 1\right)\right\}, \quad (232) \\
\frac{\partial^2 F_D}{\partial p_S \partial n_D} &= \frac{\zeta_T}{r(1-\zeta_T)}\int_{\zeta_T}^{1}\frac{\partial^2\varphi}{\partial i \partial n}\frac{\partial i}{\partial \zeta} d\zeta = \frac{\zeta_T}{r(1-\zeta_T)}\int_{\zeta_T}^{1}\frac{d}{d\zeta}\left(\frac{\partial\varphi}{\partial n}\right) d\zeta
\end{aligned}$$

$$= \frac{\zeta_T}{r(1-\zeta_T)} \left\{ \frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{\partial n} - \frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{\partial n} \right\}, \quad (233)$$

$$\begin{aligned}
\frac{\partial^2 F_D}{\partial p_D \partial n_D} &= \frac{1}{r(1-\zeta_T)} \int_{\zeta_T}^{1} \frac{\partial^2\varphi}{\partial i \partial n} \frac{\partial i}{\partial \zeta} (\zeta - \zeta_T) d\zeta \\
&= \frac{1}{r(1-\zeta_T)} \int_{\zeta_T}^{1} \frac{d}{d\zeta} \left(\frac{\partial\varphi}{\partial n} \right) (\zeta - \zeta_T) d\zeta \\
&= \frac{1}{r(1-\zeta_T)} \left\{ \frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{\partial n} (1-\zeta_T) - \int_{\zeta_T}^{1} \frac{\partial\varphi}{\partial n} d\zeta \right\} \\
&= \frac{1}{r} \left\{ \frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{\partial n} - \frac{\partial F_D}{\partial n_D} \right\}. \quad (234)
\end{aligned}$$

For (229), we have the inequality according to (A6)

$$\frac{\partial^2 F_D}{\partial {n_D}^2} \leq 0. \quad (235)$$

Since $\partial F_D/\partial n_D = o^1(\delta)$ and $\partial\varphi/\partial n = o^1(\delta)$ for $(n, i) = [\bar{n}_D(0,\mathbf{c}''), i(1)]$, (233) and (234) are modified to

$$\frac{\partial^2 F_D}{\partial p_S \partial n_D} = -\frac{\zeta_T}{r(1-\zeta_T)} \frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{\partial n} + o^1(\delta) \quad (236)$$

$$\frac{\partial^2 F_D}{\partial p_D \partial n_D} = o^1(\delta). \quad (237)$$

We substitute (230)–(232), (236), and (237) into (223) and have an equality

$$\begin{aligned}
\frac{\partial^2 \tilde{G}_D(0,\mathbf{c}'')}{\partial {p_D}^2} &= \frac{{d_T}^2(1-\zeta_T)}{(d_T+d_B)^2 r^2} \left\{ \frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta} - \frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{d\zeta} \right\} \\
&- \frac{2d_T}{(d_T+d_B) r^2} \left\{ \frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta} (1-\zeta_T) - \varphi[\bar{n}_D(0,\mathbf{c}''), i(1)] \right. \\
&\left. + \varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)] \right\} \\
&+ \frac{1}{r^2} \left\{ \frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta} (1-\zeta_T) - 2\left(\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)] - 1\right) \right\} \\
&- \frac{2\alpha d_T(1-\zeta_T)}{(d_T+d_B)^2 r} \frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{\partial n} + \left[\frac{\alpha(1-\zeta_T)}{d_T+d_B} \right]^2 \frac{\partial^2 F_D}{\partial {n_D}^2} + o^1(\delta).
\end{aligned} \quad (238)$$

Some manipulation of (238) yields an equality

$$\frac{\partial^2 \tilde{G}_D(0,\mathbf{c}'')}{\partial {p_D}^2} = \frac{1}{(d_T+d_B)^2 r^2} \left\{ {d_B}^2 (1-\zeta_T) \frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta} \right.$$

$$
\begin{aligned}
&-{d_T}^2(1-\zeta_T)\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{d\zeta}\Bigg\} \\
&-\frac{2}{(d_T+d_B)r^2}\left(d_B\{\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]-1\}+d_T\{\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]-1\}\right) \\
&-\frac{2\alpha d_T(1-\zeta_T)}{(d_T+d_B)^2 r}\frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{\partial n}+\left[\frac{\alpha(1-\zeta_T)}{d_T+d_B}\right]^2\frac{\partial^2 F_D}{\partial {n_D}^2}+o^1(\delta).
\end{aligned}
\tag{239}
$$

Considering $\partial\tilde{G}_D(0,\mathbf{c}'')/\partial p_D=0$, slight manipulation of (199) gives an equality

$$
\begin{aligned}
&\left[d_B(\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]-1)+d_T(\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]-1)\right] \\
&=\frac{rd_T}{\alpha\zeta_T}\left\{\frac{\alpha^2\zeta_T(1-\zeta_T)}{d_T}o^1(\delta)-\left[d_B\frac{\partial F_D}{\partial p_D}-\alpha(1-\zeta_T)o^1(\delta)\right]o^1(\delta)\right\} \\
&=o^1(\delta).
\end{aligned}
\tag{240}
$$

Since $\bar{n}_D(0,\mathbf{c}'')=(1-\delta)i(\zeta_T)$, $\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]/\partial i=o^1(\delta)$. Hence we have

$$
\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{d\zeta}=\frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{\partial i}\frac{\partial i}{\partial\zeta}=o^1(\delta)
\tag{241}
$$

We apply (240) and (241) to (239) and obtain

$$
\begin{aligned}
\frac{\partial^2\tilde{G}_D(0,\mathbf{c}'')}{\partial {p_D}^2}&=\frac{{d_B}^2(1-\zeta_T)}{(d_T+d_B)^2r^2}\frac{d\varphi[\bar{n}_D(0,\mathbf{c}''), i(1)]}{d\zeta} \\
&-\frac{2\alpha d_T(1-\zeta_T)}{(d_T+d_B)^2 r}\frac{\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]}{\partial n}+\left[\frac{\alpha(1-\zeta_T)}{d_T+d_B}\right]^2\frac{\partial^2 F_D}{\partial {n_D}^2}+o^1(\delta).
\end{aligned}
\tag{242}
$$

Since inequalities $d\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]/d\zeta<0$, $\partial\varphi[\bar{n}_D(0,\mathbf{c}''), i(\zeta_T)]/\partial n>0$ and $\partial^2 F_D/\partial {n_D}^2\le 0$ hold, we have an inequality from (242)

$$
\frac{\partial^2\tilde{G}_D(0,\mathbf{c}'')}{\partial {p_D}^2}<0,
\tag{243}
$$

for sufficiently small δ.

Essentially the same tedious manipulation as we have done for $\partial\tilde{G}_D(0,\mathbf{c}'')/\partial p_D$ and $\partial^2\tilde{G}_D(0,\mathbf{c}'')/\partial {p_D}^2$ yields the following inequalities

$$
\left[\frac{\partial\tilde{G}_D(0,\mathbf{c}'')}{\partial i_0},\quad\frac{\partial\tilde{G}_D(0,\mathbf{c}'')}{\partial n_B}\right]\neq\mathbf{0},
\tag{244}
$$

$$
\left[\frac{\partial^2\tilde{G}_D(0,\mathbf{c}'')}{\partial p_D\partial i_0},\quad\frac{\partial^2\tilde{G}_D(0,\mathbf{c}'')}{\partial p_D\partial n_B}\right]\neq\mathbf{0},
\tag{245}
$$

and

$$
\left[\frac{\partial\tilde{G}_D(0,\mathbf{c}'')}{\partial i_0},\quad\frac{\partial\tilde{G}_D(0,\mathbf{c}'')}{\partial n_B}\right]\not\propto\left[\frac{\partial^2\tilde{G}_D(0,\mathbf{c}'')}{\partial p_D\partial i_0},\quad\frac{\partial^2\tilde{G}_D(0,\mathbf{c}'')}{\partial p_D\partial n_B}\right].
\tag{246}
$$

(244)–(245) leads to

$$\frac{\partial \tilde{G}_D(0, \mathbf{c}'')}{\partial \mathbf{c}} \neq \mathbf{0}, \tag{247}$$

$$\frac{\partial^2 \tilde{G}_D(0, \mathbf{c}'')}{\partial p_D \partial \mathbf{c}} \neq \mathbf{0}, \tag{248}$$

$$\frac{\partial \tilde{G}_D(0, \mathbf{c}'')}{\partial \mathbf{c}} \not\propto \frac{\partial^2 \tilde{G}_D(0, \mathbf{c}'')}{\partial p_D \partial \mathbf{c}}. \tag{249}$$

Therefore $\mathbf{c}'' \in \hat{\mathbf{C}}$, a member of the bifurcation set. □

Acknowledgements. We would like to thank A. Hastings and reviewers for their helpful comments. K. Y. wishes to thank J. H. Sharp for his support during the stay at University of Delaware. This work was partially supported by U. S. Environmental Protection Agency and the 21st Century COE program of Kyoto University (A14).

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