



# Variations on the Tait–Kneser Theorem

GIL BOR, CONNOR JACKMAN, AND SERGE TABACHNIKOV

*This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on. Contributions are most welcome.*

A smooth plane curve with nonvanishing curvature has at every point an osculating circle that is tangent to the curve at that point and shares its curvature. That is, the osculating circles are second-order tangent to the curve at every point. At some points, the osculating circle may be tangent to higher order. Such points are called vertices, and these are the critical points of the curvature.

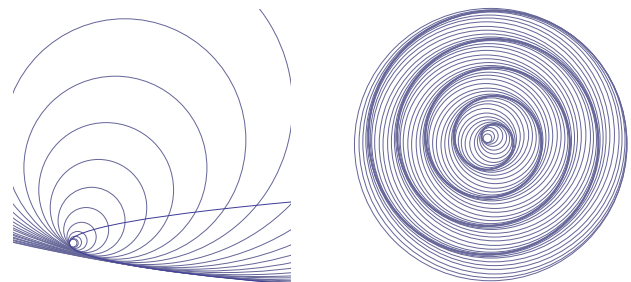
**THEOREM 1.** (Tait–Kneser). *The osculating circles of a vertex-free plane curve with nonvanishing curvature are disjoint and nested, as illustrated in Figure 1.*

This theorem is more than a century old [17]. It has numerous variations and ramifications; see the survey [9].

The osculating circles of a curve with monotone curvature form a foliation of the annulus bounded by the osculating circles at the endpoints of the curve. We leave it to the reader to mull over the seemingly paradoxical property of this foliation: the curve is tangent to the leaves at every point, but it is not contained in a single leaf. (Is it not similar to a nonconstant function having everywhere vanishing derivative?)

Tait’s original proof was very short, just two paragraphs long; it made use of the notions of evolute and involute of a curve. We present a different proof of Theorem 1 that will also work in the variations to follow.

A circle in  $\mathbb{R}^2$  is given by an equation of the form  $(x - a)^2 + (y - b)^2 = r^2$ . Denote by  $\mathbb{R}^{1,2}$  the three-dimensional pseudo-Euclidean space with coordinates  $a, b, r$  equipped with the indefinite quadratic form  $|(a, b, r)|^2 := -a^2 - b^2 + r^2$  (we use this notation even though the form assumes negative values). The space of circles in  $\mathbb{R}^2$  is parameterized by the upper half-space  $\mathbb{R}_+^{1,2} := \{(a, b, r) \mid r > 0\}$ .



**Figure 1.** Osculating nested circles along a curve with monotone nonvanishing curvature: the lower part of a parabola (left) and an Archimedean spiral (right).

➤ Submissions should be uploaded to <http://tmin.edmgr.com> or sent directly to **Sophie Morier-Genoud** ([sophie.morier-genoud@imj-prg.fr](mailto:sophie.morier-genoud@imj-prg.fr)) or **Valentin Ovsienko** ([valentin.ovsienko@univ-reims.fr](mailto:valentin.ovsienko@univ-reims.fr)).

The null cone with vertex  $\mathbf{v}_0 \in \mathbb{R}^{1,2}$  is the set of points  $\mathbf{v} \in \mathbb{R}^{1,2}$  such that  $|\mathbf{v} - \mathbf{v}_0|^2 = 0$ , and its interior consists of points  $\mathbf{v}$  with  $|\mathbf{v} - \mathbf{v}_0|^2 > 0$ . We next describe when two circles are nested, that is, the interior of one of them is contained in that of the other.

**LEMMA 2.** *Given two circles and the corresponding points  $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}_+^{1,2}$ , the circles are nested if and only if  $|\mathbf{v}_1 - \mathbf{v}_2|^2 \geq 0$ , with equality when the circles are nested and tangent.*

That is, the circles are nested when one of the corresponding points in  $\mathbb{R}^{1,2}$  lies in the light cone whose vertex is the other point. See Figure 2.

**PROOF.** Let the radii of the circles be  $R \geq r$ , and let the distance between their centers be  $d$ . The nesting condition is  $d + r \leq R$ , or  $|\mathbf{v}_1 - \mathbf{v}_2|^2 = -d^2 + (R - r)^2 \geq 0$ , with equality if and only if  $d + r = R$ , that is, the circles are tangent.  $\square$

Let  $\gamma$  be a plane curve with nonvanishing curvature and let  $\Gamma \subset \mathbb{R}^{1,2}$  be the curve of osculating circles of  $\gamma$ . The vertices of  $\gamma$  correspond to singular points of  $\Gamma$ , and  $\Gamma$  is regular if  $\gamma$  is vertex-free (see (1) in the proof of the next lemma).

A curve in  $\mathbb{R}^{1,2}$  is said to be null if its tangent vector is tangent to the null cone at every point.

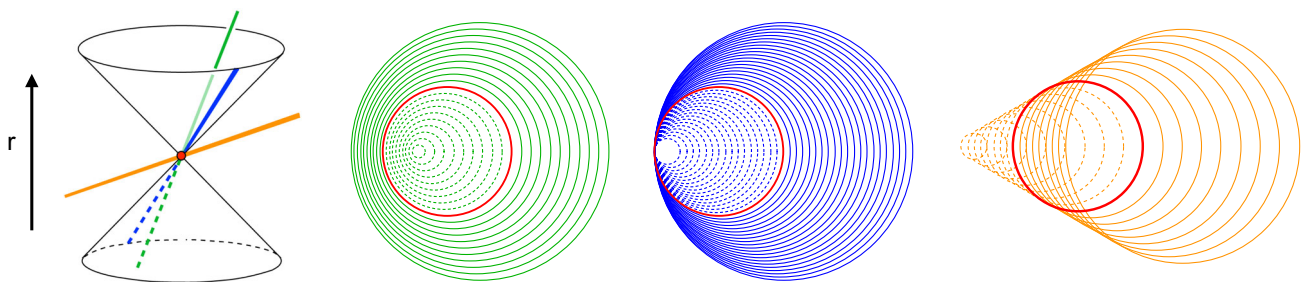
**LEMMA 3.** *The curve  $\Gamma$  is a null curve.*

**PROOF.** Let  $\gamma(t) = (x(t), y(t))$  be an arc-length parameterization of  $\gamma$ . Then  $\kappa = x'y'' - y'x''$  is the curvature of  $\gamma$ , and the osculating circle at a point  $(x, y)$  of  $\gamma$  is given by the equation  $(X - a)^2 + (Y - b)^2 = r^2$  with

$$(a, b) = (x, y) + r(-y', x'), \quad r = \frac{1}{\kappa}.$$

Since  $\Gamma(t) = (a(t), b(t), r(t))$ , one has

$$\begin{aligned} \Gamma' &= -\frac{\kappa'}{\kappa^2}(-y', x', 1) + (x', y', 0) + \frac{1}{\kappa}(-y'', x'', 0) \\ &= -\frac{\kappa'}{\kappa^2}(-y', x', 1), \end{aligned} \quad (1)$$



**Figure 2.** “Lines” of circles. A timelike line (green): nested disjoint circles. A null line (blue): nested circles tangent at a point. A spacelike line (orange): intersecting circles, tangent to a pair of lines.

the last equality due to the equation  $(x'', y'') = \kappa(-y', x')$ . Since  $(x')^2 + (y')^2 = 1$ , one has  $|\Gamma'|^2 = 0$ , as claimed.  $\square$

Here is a “hand-waving” argument that gives the intuition behind the above proof. An osculating circle  $C$  passes through three “consecutive” points of the curve, say  $\gamma(t - \varepsilon), \gamma(t), \gamma(t + \varepsilon)$ . The “next” osculating circle shares two of these points,  $\gamma(t), \gamma(t + \varepsilon)$ , with  $C$ . In the limit  $\varepsilon \rightarrow 0$ , this implies that the curve  $\Gamma$  is tangent to the cone whose vertex is the circle  $C$  and consists of the circles tangent to it.

The last ingredient of our proof of the Tait–Kneser theorem is the next lemma.

**LEMMA 4.** *A regular null curve  $\Gamma : [t_0, t_1] \rightarrow \mathbb{R}^{1,2}$  satisfies  $|\Gamma(t_1) - \Gamma(t_0)|^2 \geq 0$ , with equality if and only if  $\Gamma$  is the null line segment connecting its endpoints.*

**PROOF.** Let  $\Gamma(t) = (a(t), b(t), r(t))$ , and let  $\bar{\Gamma}(t) = (a(t), b(t))$  be the projection on the horizontal plane. The nullity condition  $(r')^2 = (a')^2 + (b')^2$  on  $\Gamma$  implies that the length of  $\bar{\Gamma}$  is  $|r(t_1) - r(t_0)|$ . This length is at least the distance  $|\bar{\Gamma}(t_1) - \bar{\Gamma}(t_0)|$  between the endpoints, with equality if and only if  $\bar{\Gamma}$  is the straight line segment.

It follows that  $|\Gamma(t_1) - \Gamma(t_0)|^2 = -|\bar{\Gamma}(t_1) - \bar{\Gamma}(t_0)|^2 + |r(t_1) - r(t_0)|^2 \geq 0$ , with equality if and only if  $\bar{\Gamma}$  is the line segment. By the nullity condition, this is equivalent to  $\Gamma$  being a null line segment.  $\square$

Theorem 1 follows. Let  $\gamma$  be a plane curve with nonvanishing monotone curvature. Consider two osculating circles  $C_0, C_1$  along  $\gamma$  and the regular null curve  $\Gamma$  connecting the corresponding points  $\mathbf{v}_0, \mathbf{v}_1 \in \mathbb{R}^{1,2}$ . By the above lemmas, either  $|\mathbf{v}_1 - \mathbf{v}_0|^2 > 0$ , in which case  $C_0, C_1$  are nested, or  $|\mathbf{v}_1 - \mathbf{v}_0|^2 = 0$ , in which case  $\Gamma$  is a null segment connecting  $\mathbf{v}_0$  to  $\mathbf{v}_1$ . The latter case corresponds to a family of circles tangent at a point, which is impossible for a family of osculating circles to a curve in  $\mathbb{R}^2$ .

**REMARK 5.** The Lorentzian geometry of the space of circles and the relation between the osculating circles of a curve with null curves are investigated in [13].

### Centroaffine Geometry: Hooke Orbits

A smooth plane curve  $\gamma$  is said to be star-shaped with respect to the origin if  $[\gamma, \gamma'] \neq 0$  (the bracket is the determinant made by a pair of vectors). Such a curve can be parameterized so that  $[\gamma(t), \gamma'(t)] = 1$ . Then  $\gamma'' = -p\gamma$ , where the function  $p(t)$  is called the centroaffine curvature of  $\gamma$ . For example, the origin-centered circle of radius  $r$  has  $p = 1/r^4$ .

A central conic  $ax^2 + 2bxy + cy^2 = 1$  has  $p = ac - b^2$ , the determinant of the coefficient matrix  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ . Central conics are the trajectories of mass points subject to Hooke's law: the radial force is proportional to the distance to the origin. If the force is attractive, the trajectory is a central ellipse with  $p > 0$ , and if it is repulsive, the trajectory is a hyperbola with  $p < 0$ . Central conics play the role of circles in centroaffine geometry.

The osculating central conic of a star-shaped curve  $\gamma$  with nonvanishing centroaffine curvature is the central conic tangent to  $\gamma$  and sharing its centroaffine curvature at the tangency point. It coincides with  $\gamma$  to second order at the point of tangency, and the order is higher if  $p' = 0$  at this point.

Here is a centroaffine version of the Tait–Kneser theorem.

**THEOREM 6.** *The osculating central conics of a star-shaped plane curve with monotone nonvanishing centroaffine curvature are disjoint and nested; see Figure 3.*

The proof of Theorem 6 goes along the same lines as our proof of the Tait–Kneser theorem.

The space of central conics is three-dimensional with coordinates  $(a, b, c)$ . This space has a pseudo-Euclidean

metric given by the determinant of the quadratic form  $ac - b^2$ , and we identify it with  $\mathbb{R}^{1,2}$ .

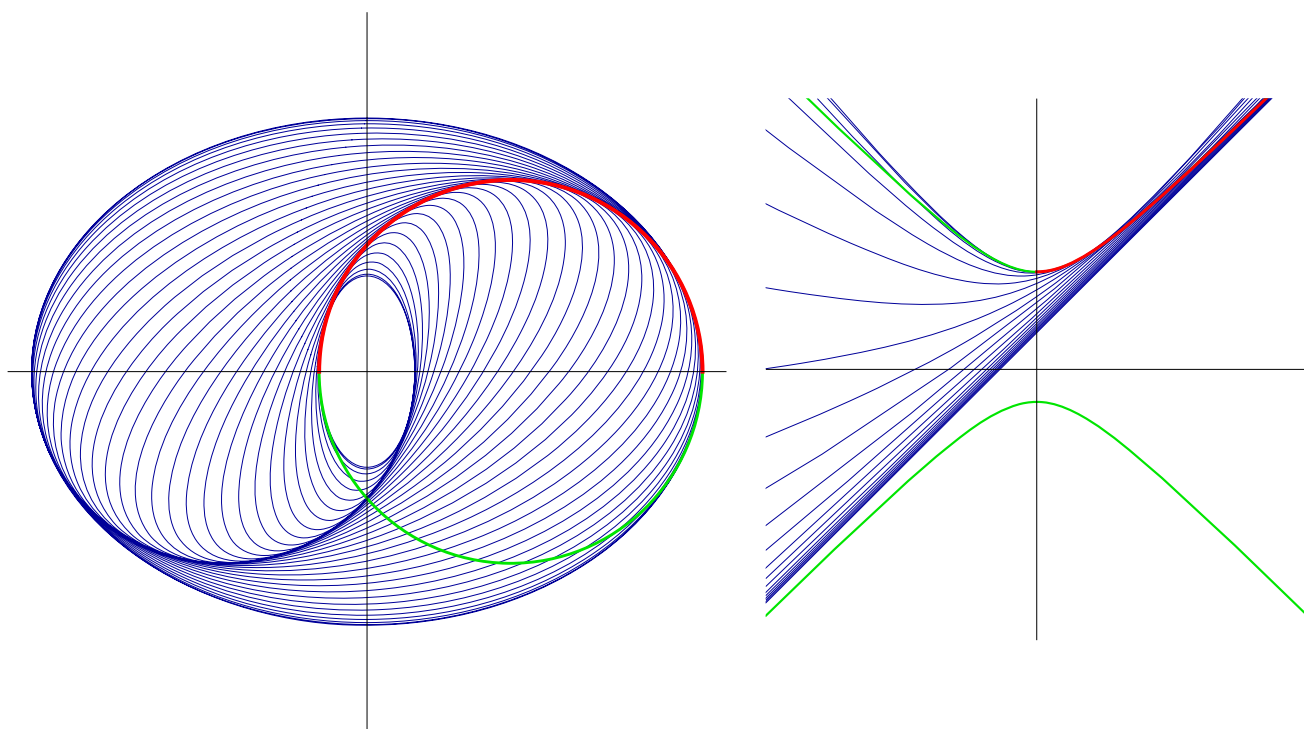
There is an analogue of Lemma 2:

**LEMMA 7.** *Let  $C_1, C_2$  be two central conics of the same type (ellipses or hyperbolas). If  $\det(C_2 - C_1) \geq 0$ , then they are nested, and equality implies that they are tangent. (We denote a conic and the quadratic form defining it by the same letter.)*

**PROOF.** We use the well-known fact from linear algebra that two real quadratic forms, one of which is definite (positive or negative), can be diagonalized simultaneously. In dimension 2, a quadratic form is definite if and only if its determinant is positive.

Now, if  $C_1, C_2$  are ellipses, then the quadratic forms are both positive definite, so without loss of generality, by the above fact, the ellipses are  $x^2 + y^2 = 1$  and  $ax^2 + by^2 = 1$ , with  $a, b > 0$ . The condition  $\det(C_2 - C_1) \geq 0$  is then  $(a - 1)(b - 1) \geq 0$ , which is equivalent to  $a, b \geq 1$  or  $a, b \leq 1$ . In both cases, the ellipses are nested, with equality when they are tangent.

If  $C_1, C_2$  are hyperbolas, suppose that  $\det(C_2 - C_1) > 0$ . Then  $\Delta C := C_2 - C_1$  is a definite quadratic form, and by interchanging the conics if necessary, it is positive. Hence  $\Delta C$  and  $C_1$  can be transformed to  $ax^2 + by^2 = 1$  and  $x^2 - y^2 = 1$ , with  $a, b > 0$ . Then  $C_2$  is  $(a + 1)x^2 - (1 - b)y^2 = 1$ , and since it is a hyperbola, we have  $0 < b < 1$ . Renaming the constants gives that  $C_2$  is



**Figure 3.** Osculating central conics. Left: along the upper part (red) of the circle  $(x - 0.75)^2 + y^2 = 1$  (green). Right: along the upper right part (red) of the hyperbola  $(y - 0.5)^2 - x^2 = 1$  (green).

$(x/a)^2 - (y/b)^2 = 1$ ,  $0 < a < 1 < b$ . It is now easy to see that  $C_1$  is nested in  $C_2$ . See Figure 4.

The case  $\det(\Delta C) = 0$  is the limiting case of  $\det(\Delta C) > 0$ , since being nested is a closed condition.  $\square$

**REMARK 8.** Although we do not use it, we note that the converse of Lemma 7 holds for ellipses (as follows easily from the above proof), but not for hyperbolas. For example, the hyperbolas  $x^2 - y^2 = 1$  and  $(x/3)^2 - (y/2)^2 = 1$  are nested, but their determinants are negative. We do not dwell on the precise algebraic condition for a pair of quadratic forms to determine a pair of nested hyperbolas.

Lemma 3 also has an analogue (with the same hand-waving explanation as before). Again we denote by  $\Gamma$  the curve in the space of Hooke conics that osculate a centroaffine curve  $\gamma$ .

**LEMMA 9.** *The curve  $\Gamma$  is a null curve.*

**PROOF.** Let  $\gamma(t) = (x(t), y(t))$ , where  $xy' - yx' = 1$ , and let  $p(t)$  be the centroaffine curvature. A direct calculation shows that the osculating central conic is given by the equation  $aX^2 + 2bXY + cY^2 = 1$ , with

$$a = py^2 + (y')^2, \quad b = -(pxy + x'y'), \quad c = px^2 + (x')^2$$

(one needs to check that  $ax^2 + 2bxy + cy^2 = 1$ ,  $(ax + by)x' + (bx + cy)y' = 0$ , and  $p = ac - b^2$ ).

Then another calculation shows that

$$a' = p'y^2, \quad b' = -p'xy, \quad c' = p'x^2,$$

and hence  $d'c' - (b')^2 = 0$ , as claimed.  $\square$

Now Lemma 4 applies, both for ellipses and hyperbolas, thereby completing the proof of Theorem 6.

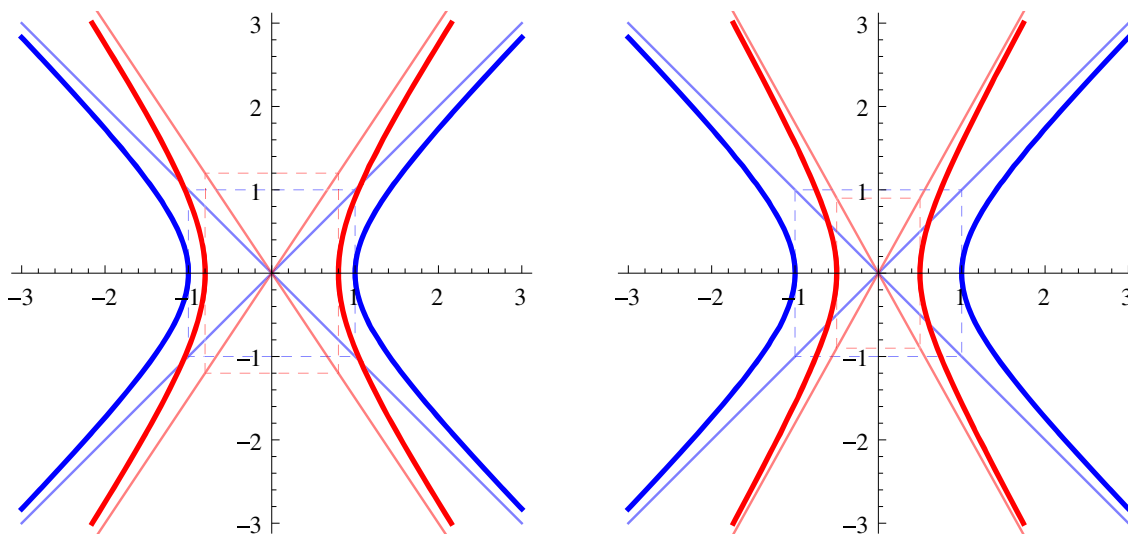
Let us add that the Tait–Kneser theorem is closely related to another classical result, the four-vertex theorem, which, in its simplest form, states that a plane oval has at least four vertices. As the example in Figure 3 shows, its analogue does not hold in centroaffine geometry: the circle of radius 1 centered at  $(0.5, 0)$  has only two “vertices,” that is, hyperosculating central conics.

**REMARK 10.** The Lorentzian geometry of the space of central conics is studied in [16].

### Kepler Orbits

A Kepler orbit is a plane conic (ellipse, parabola, or hyperbola) with a focus at the origin. These are the trajectories of mass points subject to Newton’s inverse-square law (either attractive or repulsive): the radial force is proportional to the inverse square of the distance to the origin. For attractive force, the orbits are ellipses, parabolas, or hyperbola branches bending around the origin. For repulsive force, only hyperbolas appear, the branches left out by the attractive force hyperbolas. See Figure 5.

A Kepler conic is the orthogonal projection on the horizontal plane of the intersection of the cone  $x^2 + y^2 = z^2$  in  $\mathbb{R}^3$  with a plane  $ax + by + cz = 1$ ,  $c > 0$  [8]. Thus the space of Kepler conics is parameterized by the space  $\mathbb{R}_+^{1,2}$  with coordinates  $(a, b, c)$  and the quadratic form  $|(a, b, c)|^2 = -a^2 - b^2 + c^2$ . The null cone  $|(a, b, c)|^2 = 0$  parameterizes Kepler parabolas, its interior  $|(a, b, c)|^2 > 0$



**Figure 4.** For the proof of Lemma 7. A pair of nested hyperbolas is shown in each figure,  $x^2 - y^2 = 1$  (blue) and  $(x/a)^2 - (y/b)^2 = 1$  (red). Left: if the pair has timelike difference,  $\det(C_2 - C_1) > 0$ , then either  $0 < a < 1 < b$  or  $0 < b < 1 < a$ , and the pair is nested. Right: timelike difference is not a necessary nesting condition; if  $a < b < 1$  or  $1 < b < a$ , then the difference is spacelike, but the hyperbolas are still nested.

parameterizes ellipses, and its exterior  $|(a, b, c)|^2 < 0$  parameterizes hyperbolas.

As before, a smooth star-shaped curve with nonvanishing centroaffine curvature can be second-order approximated at every point by its osculating Kepler conic; this osculating conic may hyperosculate, that is, approximate the curve to higher order.

We have the following analogue of the Tait–Kneser theorem.

**THEOREM 11.** *Consider a star-shaped curve, free from hyperosculating Kepler conics. Then its osculating Kepler conics are nested. See Figure 6.*

One can prove Theorem 11 along the same lines as before. We present only an analogue of Lemmas 2 and 7.

**LEMMA 12.** *Two Kepler conics corresponding to points  $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}_+^{1,2}$  are disjoint if and only if  $|\mathbf{v}_1 - \mathbf{v}_2|^2 > 0$ .*

Thus for Kepler conics, being nested and disjoint are equivalent.

**PROOF.** The system of equations

$$a_1x + b_1y + c_1z = 1, \quad a_2x + b_2y + c_2z = 1, \quad x^2 + y^2 = z^2$$

has no solutions if and only if the system

$$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z = 0, \quad x^2 + y^2 = z^2$$

has no nonzero solutions, if and only if the vector  $(a_1 - a_2, b_1 - b_2, c_1 - c_2)$  lies in the interior of the cone  $x^2 + y^2 = z^2$ , which is equivalent to the condition  $|\mathbf{v}_1 - \mathbf{v}_2|^2 > 0$ . See Figure 7.  $\square$

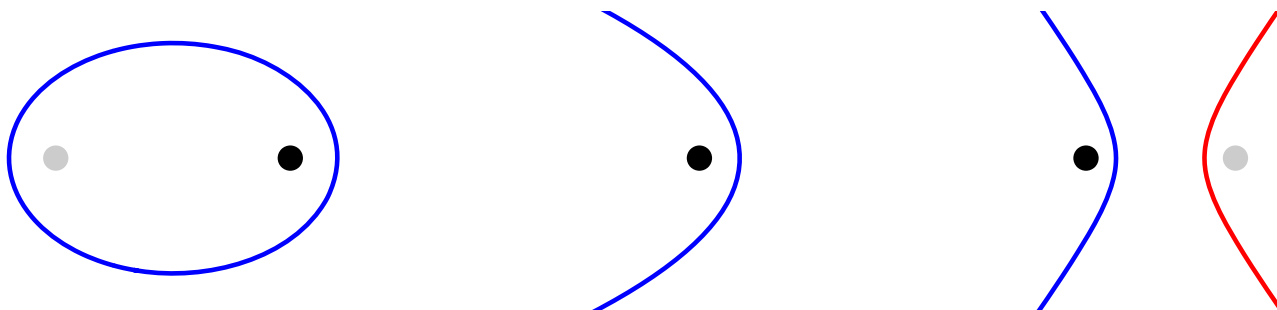
Unlike the situation for Hooke conics, one has a version of the four-vertex theorem for Kepler conics. Let  $\gamma$  be a simple closed star-shaped curve, and let us call a point at which it is approximated by a Kepler conic to third order (higher than usual) a vertex.

**THEOREM 13.** *The curve  $\gamma$  has at least four distinct vertices.*

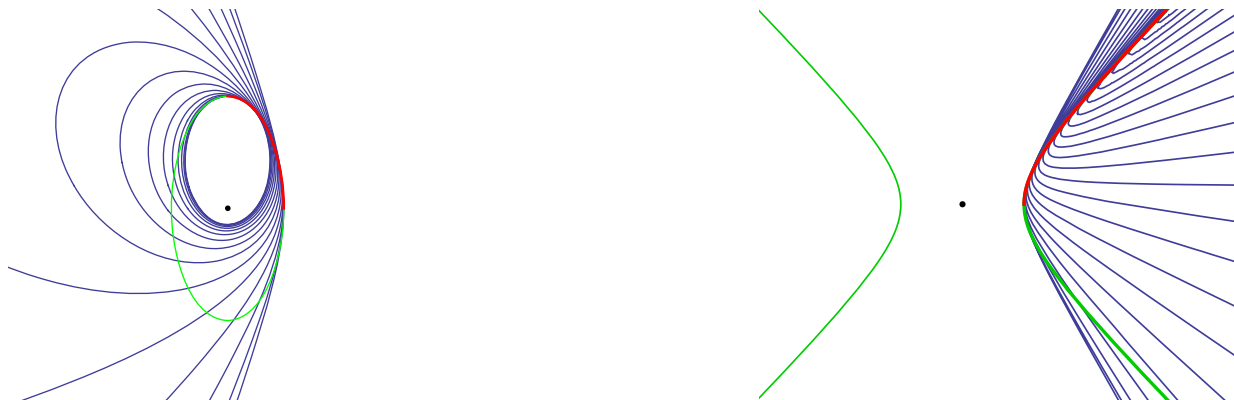
**PROOF.** A Kepler conic, in polar coordinates  $(\alpha, r)$ , is given by the formula

$$r = \frac{c}{1 + e \cos(\alpha + \varphi)},$$

where  $c, e, \varphi$  are constants. Let  $\rho = 1/r$ . Then  $\rho(\alpha)$  is a first harmonic, whence  $\rho''' + \rho' = 0$  for all  $\alpha$ . This equation characterizes Kepler conics.

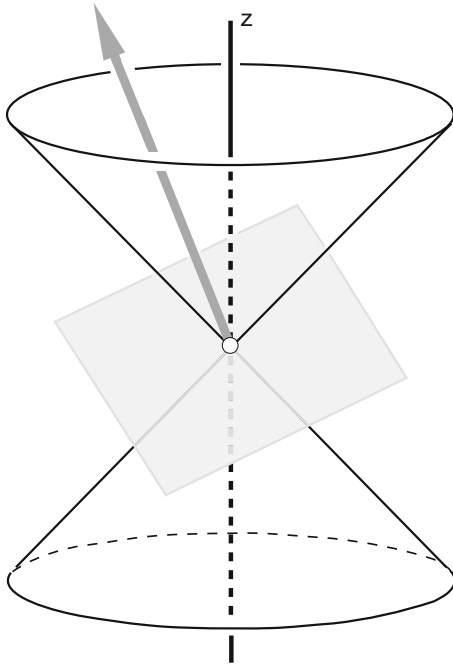


**Figure 5.** Kepler orbits share a focus at the origin (the black dot; the gray dot is the other focus, not shared). The blue curves (ellipse, parabola, hyperbola) bend around the origin, due to an attractive force. The red curve (hyperbola) bends away from the origin, due to a repulsive force.



**Figure 6.** Osculating Kepler conics (blue) along a vertex-free arc (red) of a Hooke conic (green). Left: along a Hooke ellipse. Right: along a Hooke hyperbola.





**Figure 7.** Proof of Lemma 12. The plane  $ax + by + cz = 0$  intersects the cone  $x^2 + y^2 = z^2$  only at its vertex whenever the normal vector  $(a, b, c)$  lies inside the cone.

The curve  $\gamma$  is given by its function  $\rho(x)$ , and its vertices are precisely the points where  $\rho''' + \rho' = 0$ . It remains to use the fact that such an equation has at least four distinct roots for every periodic function  $\rho$ , a particular case of the Sturm–Hurwitz theorem, which asserts that a smooth  $2\pi$ -periodic function has at least  $2n$  roots, where  $n$  is the number of the first harmonic in its Fourier expansion (see, e.g., [15, Appendix 1]). The case at hand is  $n = 2$ : the function  $\rho''' + \rho'$  is free from the constant term and the first harmonics.  $\square$

### Odds and Ends

Identifying the plane with  $\mathbb{C}$ , consider the square map  $z \mapsto z^2$ . This map takes Hooke conics to Kepler conics [1, 3]. It thus provides a direct connection between the results of the above sections on Hooke and Kepler conics.

A more general statement relates the trajectories of mass points subject to radial forces proportional to powers of distances to the origin; see [1] for a modern treatment.

**THEOREM 14.** (Bohlin–Kasner [3, 11]). *Consider two central force laws in the plane, with the force proportional to  $r^a$  and to  $r^b$ , where  $r$  is the distance to the origin. Let  $(a + 3)(b + 3) = 4$ . Then the map  $z \mapsto z^{(a+3)/2}$  takes the trajectories of motion in the first field to those in the second field.*

The Hooke and Newton attraction laws are respectively  $a = 1$  and  $b = -2$ . These cases are distinguished among central force laws.

**THEOREM 15.** (Bertrand [2]). *Assume that all the trajectories of a mass point subject to a central force that depends on the distance to the origin and whose energy does not exceed a certain limit are closed. Then the law of attraction is either Hooke's or Newton's.*

One notes that the family of circles of Hooke conics and that of Kepler conics each depend on three parameters, which is why they can approximate smooth curves to second order.

More generally, given a field of forces in the plane, the trajectory of a mass point depends on its initial position and velocity, and hence the trajectories form a three-parameter family. Which three-parameter families of plane curves are obtained in this way? This problem was thoroughly studied by Edward Kasner, who obtained a complete answer to this question [10, 11].

Other examples of three-parameter families of curves for which a version of the Tait–Kneser theorem holds are parabolas  $y = ax^2 + bx + c$ , the graphs of quadratic polynomials; and hyperbolas  $(x - a)(y - b) = c^2$ , the graphs of linear fractional transformations (see [9]). Our proof works in these cases as well, with the metrics given by  $ds^2 = db^2 - 4(da)(dc)$  and  $ds^2 = dc^2 - (da)(db)$ , respectively.

One can describe the graphs of three-parameter families of functions  $y(x)$  by third-order differential equations  $y''' = F(x, y, y', y'')$ . For example, the equation  $y''' = 0$  describes the vertical parabolas, and the graphs of linear fractional transformations are described by the vanishing of the Schwarzian derivative:

$$\frac{y'''}{y'} - \frac{3}{2} \left( \frac{y''}{y'} \right)^2 = 0.$$

Given such a three-parameter family of curves  $\mathcal{F}$ , one defines null cones whose rulings consist of the curves that are tangent to a fixed line at a fixed point. A smooth plane curve defines an associated curve in  $\mathcal{F}$ , and it is still true that this associated curve is null. However, for a general family  $\mathcal{F}$ , these cones may fail to be quadratic.

The families  $\mathcal{F}$  for which the nullity condition is quadratic, and hence defines a conformal Lorentzian metric on  $\mathcal{F}$ , are characterized by a complicated nonlinear partial differential equation on the function  $F(x, y, y', y'')$  that defines this family. This condition was studied by Karl Wünschmann [18, pp. 6–13] and later by Élie Cartan [5] and Shiing-Shen Chern [6, 7], using the method of equivalence. For recent presentations of this deep result, see [12, 14].

We end with the observation that the Tait–Kneser and four-vertex theorems for Kepler conics (Theorems 11 and 13) can be derived from their Euclidean analogues via projective duality. Here is a sketch (more details will appear in [4]). The equation  $ax + by = 1$  associates to a point in the  $ab$ -plane a line in the  $xy$ -plane and vice versa. To a curve  $C$  in one plane corresponds its dual curve  $C^*$  in the other plane, whose points parameterize the lines tangent to  $C$ . One then shows that the dual of a Kepler conic is a circle and that duality preserves nesting and order of contacts of curves. It follows that the dual of the osculating

Kepler conic to  $C$  is the osculating circle to  $C^*$ , and the same holds for hyperosculating conics. Thus duality interchanges Euclidean and Keplerian vertices, reducing the Kepler version of the Tait–Kneser and four-vertex theorems to their Euclidean analogues.

#### ACKNOWLEDGMENTS

After the first version of this article was posted, Rui Pacheco and Marcos Salvai brought their relevant papers [13, 16] to our attention, for which we are grateful. We thank Anton Izosimov for interesting discussions and the referee for useful suggestions. GB was supported by CONACYT Grant A1-S-4588. ST was supported by NSF grant DMS-2005444.

Gil Bor  
 CIMAT, A.P. 402  
 36000 Guanajuato, Gto.  
 Mexico  
 e-mail: gil@cimat.mx

Connor Jackman  
 CIMAT, A.P. 402  
 36000 Guanajuato, Gto.  
 Mexico  
 e-mail: connor.jackman@cimat.mx

Serge Tabachnikov  
 Department of Mathematics  
 Penn State University  
 University Park, PA 16802  
 USA  
 e-mail: tabachni@math.psu.edu

#### REFERENCES

- [1] V. I. Arnold. *Huygens and Barrow, Newton and Hooke: Pioneers in Mathematical Analysis and Catastrophe Theory from Evolvents to Quasicrystals*. Birkhäuser, 1990.
- [2] J. Bertrand. Théorème relatif au mouvement d'un point attiré vers un centre fixe. *C. R. Acad. Sci.* 77 (1873), 849–853.

- [3] M. K. Bohlin. Note sur le problème des deux corps et sur une intégration nouvelle dans le problème des trois corps. *Bull. Astron.* 28 (1911), 113–119.
- [4] G. Bor and C. Jackman. Revisiting Kepler: new symmetries of an old problem. Preprint available at [arXiv:2106.02823](https://arxiv.org/abs/2106.02823).
- [5] É. Cartan. La geometría de las ecuaciones diferenciales de tercer orden. *Rev. Mat. Hispano-Amer.* 4 (1941), 1–31.
- [6] S.-S. Chern. Sur la géométrie d'une équation différentielle du troisième ordre. *C. R. Acad. Sci., Paris* 204 (1937), 1227–1229.
- [7] S.-S. Chern. The geometry of the differential equations  $y''' = F(x, y, y', y'')$ . *Sci. Rep. Nat. Tsing Hua Univ.* 4 (1940), 97–111.
- [8] A. Givental. Kepler's laws and conic sections. *Arnold Math J.* 2 (2016), 139–148.
- [9] E. Ghys, S. Tabachnikov, and V. Timorin. Osculating curves: around the Tait–Kneser theorem. *Math. Intelligencer* 35:1 (2013), 61–66.
- [10] E. Kasner. The trajectories of dynamics. *Trans. Amer. Math. Soc.* 7 (1906), 401–424.
- [11] E. Kasner. *Differential-Geometric Aspects of Dynamics*, Princeton Colloquium, vol. 3. AMS, 1913.
- [12] S. Frittelli, C. Kozameh, and E. T. Newman. Differential geometry from differential equations. *Comm. in Math. Phys.* 223:2 (2001), 383–408.
- [13] B. Nolasco and R. Pacheco. Evolutes of plane curves and null curves in Minkowski 3-space. *J. Geom.* 108 (2017), 195–214.
- [14] P. Nurowski. Differential equations and conformal structures. *J. Geom. Phys.* 55:1 (2005), 19–49.
- [15] V. Ovsienko and S. Tabachnikov. *Projective Differential Geometry Old and New: From the Schwarzian Derivative to the Cohomology of Diffeomorphism Groups*. Cambridge Univ. Press, 2005.
- [16] M. Salvai. Centro-affine invariants and the canonical Lorentz metric on the space of centered ellipses. *Kodai Math. J.* 40 (2017), 21–30.
- [17] P. G. Tait. Note on the circles of curvature of a plane curve. *Proc. Edinburgh Math. Soc.* 14 (1896), 403.
- [18] K. Wünschmann. *Über Berührungsbedingungen bei Integralkurven von Differentialgleichungen*. Inauguraldissertation, Leipzig, Teubner, 1905.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.