Note

The Olympic Medals Ranks, Lexicographic Ordering, and Numerical Infinities

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The International Olympic Committee (IOC) does not produce any official ranking of the countries participating in the Olympic Games. However, it does publish tables showing the medals won by athletes representing each country. The convention used by the IOC to order the countries in this unofficial rank is the following. First, countries are sorted by the number of gold medals won. If the number of gold medals won by two or more countries is the same, the number of silver medals is taken into consideration, and then the number of bronze. If the countries have an equal number of gold, silver, and bronze medals, then equal ranking is given and the countries are listed alphabetically by their IOC country code (for instance, in the 2010 Winter Olympics held in Vancouver, China and Sweden each won 5 gold, 2 silver, and 4 bronze medals; both countries have the 7th place in the rank, but China is higher in the table). Table 1 shows countries sorted by this rank at the Sochi 2014 Olympic Games (the first ten countries). We will call this rank R1.

However, there are several methods for ranking countries (some of them are illustrated in Tables 2 and 3 showing the best 10 countries for each rank; for more countries see, e.g., [4]). First, in many countries ranking by the total number of Olympic medals is very popular. This rank (R2) gives equal ratings to gold, silver, and bronze medals. So, if country A has won g_A gold, s_A silver, and b_A bronze medals, then its rank is the sum

$$R2(A) = g_A + s_A + b_A.$$

Because R2 assigns the same weight to gold, silver, and bronze medals, there have been several proposals to improve this way of counting by introducing weights for medals. For instance, the Fibonacci weighted point system (this method is shown in Table 2 as R3) uses the following weights: gold gets 3 points, silver 2 points, and bronze 1 point; these weights are called the 3:2:1 system. Thus

$$R3(A) = 3g_A + 2s_A + b_A.$$

Table 2 shows that Norway and United States have the same rank R3, but in the rank Norway has a higher position because it has won more golds (the same situation holds for Switzerland and Sweden). To make gold medals more precious, the exponential weighted point system assigns 4 points to gold, 2 points to silver, and 1 point to bronze—the 4:2:1 system. The variation used by the British press during the Olympic Games in London in 1908 used the weights 5:3:1. There exist also systems 5:3:2, 6:2:1, 10:5:1, etc.

Other rankings use completely different ideas. For instance, one method counts all the medals won (weighted or not), counting separately the medals for each individual athlete in team sports. Another uses an improvement rank based on the percentage improvement attained by countries with respect to the previous Games results. There exist ranks built in comparison to expectations. Among them there are predictions based on previous results (in the Games or other competitions) and predictions using economics, population, and a range of other criteria.

Another interesting proposal is to calculate the rank by dividing the number of medals by the population of the country. The column R4 in Table 3 shows the total number of medals won by a country per 10 million people. Whereas criteria R1–R3 yield similar results, criterion R4 puts different countries, mainly those with relatively small populations, at the top. In fact, Norway, with 26 medals and a population of approximately 5 million people, is the best in this ranking. In general, the countries that top the list have small populations in comparison, for instance, with the United States and the Russian Federation. The number of medals per \$100 billion of the gross domestic product (GDP) of the country (this rank is called R5 in Table 2) also favors smaller countries.

In this note, I do not discuss the advantages and disadvantages of various ranks. Instead, we consider a purely mathematical problem regarding a difference between the unofficial International Olympic Committee rank R1 and the other ranks R2–R5. In fact, although ranks R2–R5 produce numerical coefficients for each country that allow one to rank-order the countries, rank R1 does not produce any number that can be used for this purpose. This rank uses the *lexicographic ordering*, used in dictionaries to order words: first words are ordered with respect to the first symbol in the word, then with respect to the second one, and so on. In working with the rank R1 we have words that consist of three symbols g_A, s_A, b_A and, therefore, their length $w = 3$.

I show, however, that there is a procedure for computing rank R1 numerically for each country and for any number of medals. Moreover, the computation can be generalized from words consisting of three symbols to words having a general finite length w and used in situations that require lexicographic ordering.

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How Can We Compute the Rank R1 for Any Number of Medals?

Evidently, in the rank R1, gold medals are more precious than silver ones, which in turn are better than the bronze ones. An interesting issue arises. Let us consider Belarus and Austria, which occupy the 8^{th} and 9^{th} positions, respectively. Belarus has 5 gold medals and Austria only 4. The fact that Austria has 8 silver medals and Belarus has none is not taken into consideration. Austria could have *any* number of silver medals, but the fifth gold medal of Belarus will be more important than all of them.

Can we quantify what these words, *more important*, mean? Can we introduce a counter that would allow us to compute a numerical rank of a country using the number of gold, silver, and bronze medals in such a way that the higher resulting number would put the country in the higher position in the rank? In situations when the number of medals that can be won is not known a priori, we want a numerical counter that would work for *any* number of medals.

More formally, I wish introduce a number $n(g_A, s_A, b_A)$, where g_A is the number of gold medals, s_A is the number of silver medals, and b_A is the number of bronze won by a country A. This number should be calculated so that, for countries A and B, we have

$$n(g_A, s_A, b_A) > n(g_B, s_B, b_B), \quad \text{if} \begin{cases} g_A > g_B, \\ g_A = g_B, s_A > s_B, \\ g_A = g_B, s_A = s_B, b_A > b_0. \end{cases} \tag{1}$$

Table 1. The International Olympic Committee Unofficial Medal Rank at Sochi 2014 (the First Ten Countries)

Rank R1	Country	Gold	Silver	Bronze
1	Russian Federation	13	11	9
2	Norway	11	5	10
3	Canada	10	10	5
4	United States	9	7	12
5	Netherlands	8	7	9
6	Germany	8	6	5
7	Switzerland	6	3	2
8	Belarus	5	0	1
9	Austria	4	8	5
10	France	4	4	7

Table 2. Medal Ranks Counting the Total Number of Won Medals per Country and Weighted Total Sum (System 3:2:1)

N	Total medals (R2)		Weighted total medals (R3)	
1	Russian Federation	33	Russian Federation	70
2	United States	28	Canada	55
3	Norway	26	Norway	53
4	Canada	25	United States	53
5	Netherlands	24	Netherlands	47
6	Germany	19	Germany	41
7	Austria	17	Austria	33
8	France	15	France	27
9	Sweden	15	Switzerland	26
10	Switzerland	11	Sweden	26

As mentioned earlier, $n(g_A, s_A, b_A)$ should not depend on the upper bound $K > \max\{g_A, s_A, b_A\}$ for the number of medals of each type that can be won by each country.

As a first try in calculating $n(g_A, s_A, b_A)$, let us assign weights to g_A, s_A, and b_A as is done in the positional numeral system with a base β:

$$n(g_A, s_A, b_A) = g_A\beta^2 + s_A\beta^1 + b_A\beta^0 = g_A s_A b_A \tag{2}$$

For instance, in the decimal positional numeral system with $\beta = 10$, the record

$$n(g_A, s_A, b_A) = g_A 10^2 + s_A 10^1 + b_A 10^0 = g_A s_A b_A \tag{3}$$

provides the rank of the country A. However, we see immediately that this does not solve our problem, because it does not satisfy condition (1). In fact, if a country has more than 11 silver medals, then formula (3) implies that these medals are more important than one gold. For instance, the data

$$g_A = 2, s_A = 0, b_A = 0, \qquad g_B = 1, s_B = 11, b_B = 0. \tag{4}$$

give us

$$n(g_A, s_A, b_A) = 2 \cdot 10^2 + 0 \cdot 10^1 + 0 \cdot 10^0 = 200 <$$
$$n(g_B, s_B, b_B) = 1 \cdot 10^2 + 11 \cdot 10^1 + 0 \cdot 10^0 = 210,$$

that is, condition (1) is not satisfied.

Remember that we wish to construct a numerical counter that works for *any* number of medals: we suppose that countries can win any number of medals and this number is unknown for us. Then it is easy to see that situations can occur where the positional system will not satisfy (1) not only for the base $\beta = 10$ but also for any finite β. This can happen if one of the countries has more than β silver (or bronze) medals.

Thus, the contribution of 1 gold medal in the computation of $n(g_A, s_A, b_A)$ should be larger than the contribution of *any* number, s_A, of silver medals, that is, it should be *infinitely larger*. Analogously, the contribution of 1 silver medal should be infinitely larger than the contribution of any finite number of bronze medals.

Unfortunately, it is difficult to make numerical computations with infinity (symbolic computations can be done with nonstandard analysis, see [11]) because in the traditional calculus ∞ absorbs any finite quantity, and we have, for instance,

$$\infty + 1 = \infty, \qquad \infty + 2 = \infty. \tag{5}$$

A Numerical Calculator of the Rank R1 Involving Infinities

To construct a numerical calculator of a medal ranking involving infinite numbers, let us recall the difference between *numbers* and *numerals*: a *numeral* is a symbol or a group of symbols that represents a *number*. The difference between them is the same as the difference between words and the things to which they refer. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols “7,” “seven,” and “VII” are different numerals, but they all represent the same number.

Table 3. Medal Ranks Counting Total Medals per 10 Million People and Total Medals per $100 Billion of the Gross Domestic Product

N	Total medals per 10^7 people (R4)		Total medals per $100 billion of GDP (R5)	
1	Norway	51.8	Slovenia	17.7
2	Slovenia	38.9	Latvia	14.1
3	Austria	20.1	Belarus	9.5
4	Latvia	19.8	Norway	5.2
5	Sweden	15.8	Austria	4.3
6	Netherlands	14.3	Czech Republic	4.1
7	Switzerland	13.8	Netherlands	3.1
8	Finland	9.2	Sweden	2.9
9	Czech Republic	7.6	Finland	2.0
10	Canada	7.2	Switzerland	1.7

Different numeral systems can represent different numbers. For instance, the Roman numeral system cannot represent zero and negative numbers. Even weaker numeral systems exist. A study of a numeral system of a tribe living in Amazonia—Pirahã—has been published (see [5]). These people use a very simple numeral system for counting: one, two, many. For Pirahã, all quantities larger than 2 are just "many," and such operations as $2+2$ and $2+1$ yield the same result, that is, "many." Using their weak numeral system, Pirahã are not able to see, for instance, numbers 3, 4, 5, and 6 to execute arithmetical operations with them; and, in general, to say anything about these numbers because in their language there are neither words nor concepts for them. It is important to emphasize that the records $2+1=$ "many" and $2+2=$ "many" are not wrong. They are correct in their language, and if one is satisfied with the accuracy of the answer "many," it can be used (and *is used* by Pirahã) in practice. Note that the result of Pirahã is not wrong, it is just *inaccurate*. Analogously, the answer "many" to the question "How many trees are there in a park?" is correct, but its precision is low.

Thus, if we need a more precise result than "many," it is necessary to introduce a more powerful numeral system that allows us to express the required answer in a more accurate way. By using numeral systems with additional numerals for expressing numbers "three" and "four" we find that within "many" there are several objects, the numbers 3 and 4 among them.

Our great attention to the numeral system of Pirahã is because of the following fact: their numeral "many" gives them such results as

$$\text{"many"}+1=\text{"many"}, \qquad \text{"many"}+2=\text{"many"}, \quad (6)$$

that are very familiar to us, see (5). This comparison shows that we treat infinity in the same way that Pirahã treat quantities larger than 2. Thus, our difficulty in working with infinity is not connected to the *nature of infinity itself* but is just a result of *inadequate numeral systems* that we use to work with infinity.

To avoid such situations as (5) and (6), a new numeral system has been proposed in [12, 14, 20, 24]. It is based on an infinite unit of measure expressed by the numeral ① called *grossone*. Several authors have obtained a number of powerful theoretical and applied results with the new methodology. The new approach has been compared with the panorama of ideas concerning infinity and infinitesimals in [6, 7, 9, 26]. It has been successfully applied in hyperbolic geometry (see [10]), percolation (see [2, 8, 13]), fractals (see [8, 13, 15, 23]), numerical differentiation and optimization (see [1, 16, 21, 29]), infinite series and the Riemann zeta function (see [17, 22, 28]), the first Hilbert problem and Turing machines (see [19, 26, 27]), and cellular automata (see [3]). The use of numerical infinitesimals opens possibilities for creating new numerical methods having an accuracy that is superior to existing algorithms working only with finite numbers (see, e.g., algorithms for solving ordinary differential equations in [25]).

In particular, the Infinity Computer executing numerical computations with infinite and infinitesimal numbers has been patented (see [18]) and its software prototype has been constructed. This computer can be used to calculate the medal rank $n(g_A, s_A, b_A)$ satisfying condition (1) because it works with numbers expressed in the new positional numeral system with the infinite base ①. A number C is subdivided into groups corresponding to powers of ①:

$$C = c_{p_m}①^{p_m} + \ldots + c_{p_1}①^{p_1} + c_{p_0}①^{p_0} + c_{p_{-1}}①^{p_{-1}} + \ldots + c_{p_{-k}}①^{p_{-k}}. \quad (7)$$

Then, the record

$$C = c_{p_m}①^{p_m} \ldots c_{p_1}①^{p_1} c_{p_0}①^{p_0} c_{p_{-1}}①^{p_{-1}} \ldots c_{p_{-k}}①^{p_{-k}} \quad (8)$$

represents the number C. The numerals $c_i \neq 0$ can be positive or negative and belong to a traditional numeral system; they are called *grossdigits*. They show how many corresponding units $①^{p_i}$ should be added to or subtracted from the number C. Obviously, because all c_i are finite, it follows that

$$① > c_i. \quad (9)$$

The numbers p_i in (8) are called *grosspowers*. They are sorted in the decreasing order

$$p_m > p_{m-1} > \ldots > p_1 > p_0 > p_{-1} > \ldots p_{-(k-1)} > p_{-k}$$

with $p_0 = 0$, and, in general, can be finite, infinite, and infinitesimal. Hereinafter we consider only finite values of

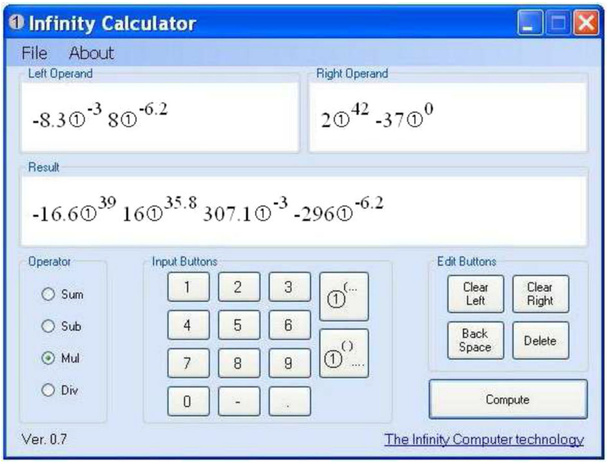

Figure 1. Operation of multiplication executed at the Infinity Calculator. The left operand has two infinitesimal parts, the right operand has an infinite part and a finite one; the result has two infinite and two infinitesimal parts.

p_i. Under this assumption, *infinite numbers* are expressed by numerals having at least one $p_i > 0$. They can have several infinite parts, a finite part, and several infinitesimal ones. *Finite numbers* are represented by numerals having only one grosspower $p_0 = 0$. In this case $C = c_0①^0 = c_0$, where c_0 is a conventional finite number expressed in a traditional finite numeral system. *Infinitesimals* are represented by numerals C having only negative grosspowers. The simplest infinitesimal is $①^{-1} = \frac{1}{①}$ being the inverse element with respect to multiplication for ①:

$$\frac{1}{①} \cdot ① = ① \cdot \frac{1}{①} = 1 \,. \qquad (10)$$

Note that all infinitesimals are not equal to zero. In particular, $\frac{1}{①} > 0$ because it is a result of the division of two positive numbers. Fig. 1 shows the Infinity Calculator built using the Infinity Computer technology.

It becomes very easy to calculate $n(g_A, s_A, b_A)$ using records (7), (8), that is, putting ① instead of a finite base β in (2). Then the number

$$n(g_A, s_A, b_A) = g_A①^2 + s_A①^1 + b_A①^0 = g_A①^2 s_A①^1 b_A①^0 \qquad (11)$$

provides the rank of the country satisfying condition (1). Let us consider as an example the data (4). Because ① is larger than any finite number (see (9)), it follows from (11) that

$$n(g_A, s_A, b_A) = 2 \cdot ①^2 + 0 \cdot ①^1 + 0 \cdot ①^0 = 2①^2 >$$

$$n(g_B, s_B, b_B) = 1 \cdot ①^2 + 11 \cdot ①^1 + 0 \cdot ①^0 = 1①^2 11①^1$$

because

$$2①^2 - 1①^2 11①^1 = 1①^2 - 11①^1 = ①(① - 11) > 0.$$

Thus, we can easily calculate the rank R1 for the data from Table 1 as follows

$$13①^2 11①^1 9①^0 > 11①^2 5①^1 10①^0 > 10①^2 10①^1 5①^0 > 9①^2 7①^1 12①^0 >$$
$$8①^2 7①^1 9①^0 > 8①^2 6①^1 5①^0 > 6①^2 3①^1 2①^0 > 5①^2 0①^1 1①^0 >$$
$$4①^2 8①^1 5①^0 > 4①^2 4①^1 7①^0.$$

The calculator can be used for computing the unofficial International Olympic Committee rank R1 numerically. It can also be applied in all situations that require lexicographic ordering, not only for words with three characters as for the rank R1, but for words having any finite number of characters as well.

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