Tube Formulas for Self-Similar and Graph-Directed Fractals

ALİ DENİZ, ŞAHİN KOÇAK, YUNUS ÖZDEMİR, AND ADEM ERSİN ÜREYEN

In the study of fractals one regularly encounters mathematical objects with fractional dimension. It is surely stranger still to find discussion of complex dimension. What might that mean?

The concept was introduced in the 1990s by Michel Lapidus and has since been extensively developed by him and his collaborators. Our first aim in the present paper is to present an introduction to the theory of Lapidus and Pearse in the original setting of self-similar fractals in $\mathbb{R}^n$. We will go on to give a simple treatment of complex dimensions of graph-directed fractals as investigated by us.

In looking at a fractal, especially a self-similar fractal, one naturally "thickens" it a bit to "make it visible." So one considers, along with any fractal $F \subset \mathbb{R}^n$, its ε-tube: that is, the set of points at distance less than ε from F. To see finer detail, one reduces $\varepsilon > 0$.

Let us begin with one dimension, with the ε-tube C_ε of the middle-third Cantor set $C \subset [0,1]$ in $\mathbb{R}$. Lapidus found the following striking formula for its volume (see [5]):

$$\mathrm{Vol}(C_\varepsilon) = (2\varepsilon)^{1-D}\left(\left(\frac{1}{2}\right)^{\{-\log_3(2\varepsilon)\}} + \left(\frac{3}{2}\right)^{\{-\log_3(2\varepsilon)\}}\right)$$

for all $0 \le \varepsilon \le \frac{1}{2}$; here $D = \log_3 2$ is the Minkowski dimension of the Cantor set and $\{x\} = x - [[x]]$ denotes the fractional part of x.

Manipulating this expression, Lapidus obtained the formula

$$\mathrm{Vol}(C_\varepsilon) = \frac{1}{2\log 3}\sum_{n=-\infty}^{\infty} \frac{(2\varepsilon)^{1-D-\mathbb{i}np}}{(D+\mathbb{i}np)(1-D-\mathbb{i}np)},$$

where $p = \frac{2\pi}{\log 3}$. He coined the term "complex dimensions" for the complex numbers $D + \mathbb{i}np$ occurring here; they are the complex roots of the Moran equation

$$\left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s = 1$$

of the Cantor set.

It was familiar for the dimension to give the rate at which bulk varied with linear dimension, and fractional dimensions were known as an extension of this, as positive solutions of Moran equations of fractals. But until the appearance of the previously mentioned volume formula for the ε-tube of the Cantor set, no attention whatsoever had been given to the complex roots of the Moran equation. Yet now these seemingly meaningless complex roots of the Moran equation of the Cantor set were "controlling" the volume of the ε-tube of the Cantor set! Thus began the theory of complex dimensions of fractals.

A particularly nice and simple class of fractals is the self-similar fractals. These are compact sets $F \subset \mathbb{R}^n$, which satisfy an equation $F = \bigcup_{j=1}^{J} S_j(F)$ with some similarities $S_j : \mathbb{R}^n \to \mathbb{R}^n$ with similarity ratios $0 < r_j < 1$. By similarity, we mean a map $S : \mathbb{R}^n \to \mathbb{R}^n$ of the form $S = r \cdot A + b$, where A is an orthogonal map and $b \in \mathbb{R}^n$. Such a system of similarities is called an iterated function system (IFS) on $\mathbb{R}^n$; and given an arbitrary IFS, there exists a unique compact set F satisfying $F = \bigcup_{j=1}^{J} S_j(F)$, called the attractor of the IFS.

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Figure 1. ε-tube (of the third stage) of the Cantor set.

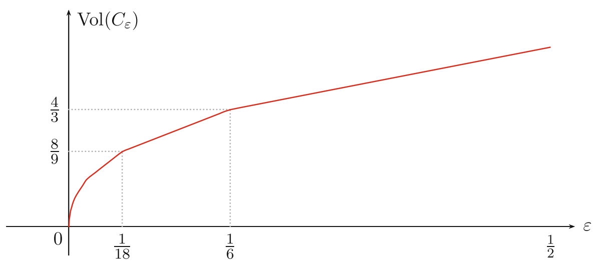

Figure 2. Graph of the volume of the ε-tube of the Cantor set.

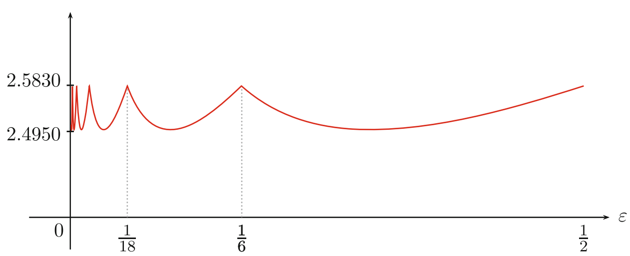

Figure 3. The function $\varepsilon^{D-1} \cdot \mathrm{Vol}(C_\varepsilon)$ oscillates: The Cantor set is not "Minkowski measurable".

Given any ratio list $\{r_1, r_2, \ldots, r_J\}$ with $0 < r_j < 1$, the associated equation

$$r_1^s + r_2^s + \cdots + r_J^s = 1,$$

is called the Moran equation. One ordinarily demands that the system of similarities (S_j) satisfy the so-called open set condition: that there exists an open set $O \subset \mathbb{R}^n$ such that $S_j(O) \subset O$ and $S_i(O) \cap S_j(O) = \emptyset$ for $i \neq j$. In this case, the unique non-negative real number satisfying the Moran equation, the so-called similarity dimension of the ratio list $\{r_1, r_2, \ldots, r_J\}$, coincides with the Hausdorff and Minkowski dimensions of the fractal.

Lapidus and his coworkers (especially van Frankenhuijsen), turning their attention to the complex roots, developed an elaborate theory of complex dimensions in $\mathbb{R}$ as documented in [5]. Their theory provided tube formulas not only for self-similar fractals but also for much more general objects called fractal strings and generalized fractal strings (for fractal strings see [8]). Based on the 1-dimensional theory, Lapidus and Pearse obtained higher-dimensional tube formulas in [6]. Further generalizations are accomplished in [7]. Again, their theory covered not only self-similar fractals but also the more general fractal sprays and generalized fractal sprays, which were originally introduced in [9]. Complex roots of the Moran equation were still contributing to volume formulas of the ε-tubes of fractals, and much more precise information about the underlying fractal geometry is obtained. These roots enable us to express the hidden oscillations in the neighborhood-volumes of fractals, and the frequency spectrum of fractal drums ([5, 8], and references therein); the spectral study is connected to the Riemann zeta function. "In essence, the imaginary parts of the complex dimensions correspond to the frequencies of the oscillations, while the real parts control the amplitudes of the oscillations," as M. Lapidus puts it ([5, 8, 9]). We can give here only a brief introduction to their theory, and we deal only with the information it yields on volumes, not with the oscillations.

We applied the Lapidus-Pearse theory to graph-directed fractals and expressed the volumes of ε-tubes along the lines of the Lapidus-Pearse tube formulas ([1]). For graph-directed fractals there is a counterpart of the Moran equation helping to determine the dimension. A graph-directed fractal has an associated Mauldin-Williams matrix (see later) depending on a complex argument s, and the unique real s-value for which the spectral radius of the MW-matrix is 1 is called the sim-value. This sim-value equals the Hausdorff and Minkowski dimensions under suitable conditions.

In view of the analogy, the natural candidates for complex dimensions for graph-directed fractals would be the set of complex values of s for which the spectral radius of the MW-matrix is 1. Surprisingly, it turns out that all values of s for which 1 is an eigenvalue of the MW-matrix contribute to the

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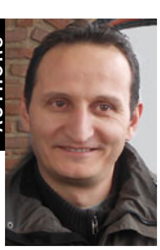

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tube formulas of graph-directed fractals, whether 1 is the spectral radius of the MW-matrix or not. We call these new values—for which the spectral radius of the MW-matrix is not 1, but for which 1 is an eigenvalue of the MW-matrix—the hidden complex dimensions of the graph-directed fractal.

We want to introduce the reader to the whole beautiful theory of complex dimensions and fractal tube formulas. To keep the exposition within reasonable bounds, we will concentrate on these two among many aspects: self-similar and graph-directed fractals.

The Lapidus-Pearse Theory

We will try to explain the essence of the Lapidus-Pearse theory in the following basic setting. Let $G \subset \mathbb{R}^n$ be a nonempty, bounded open set. The inner ε-neighborhood of G is the set $\{x \in G | \mathrm{dist}(x, \partial G) < \varepsilon\}$, and we denote the volume of this set by $V_G(\varepsilon)$. The supremum of the radii of the balls contained in G is called the inradius of G and will be denoted by g. For $\varepsilon \geq g$, obviously, $V_G(\varepsilon)$ does not depend upon ε and equals the volume of G.

What is a spray?

A spray generated by an open set $G \subset \mathbb{R}^n$ is a collection $(G_i)_{i\in\mathbb{N}}$ of pairwise disjoint open sets $G_i \subset \mathbb{R}^n$ such that G_i is a scaled copy of G by some $\lambda_i > 0$ (i.e., G_i is congruent to $\lambda_i G$).

The sequence $(\lambda_i)_{i\in\mathbb{N}}$ is called the associated scaling sequence of the spray.

To make the volume of $\bigcup_{i\in\mathbb{N}} G_i$ finite, we assume $\sum_{i\in\mathbb{N}} \lambda_i^n < \infty$.

In our applications, we will have $\lambda_i < 1$ for $i \geq 1$ and often $\lambda_0 = 1$, $G_0 = G$.

Now the main question is the following: What is the volume of the inner ε-neighborhood of $\bigcup_{i\in\mathbb{N}} G_i$ for a spray $(G_i)_{i\in\mathbb{N}}$? The question is more tricky than it sounds. The knowledge of G and of the scaling sequence should determine the inner ε-neighborhood volume, but there are several technical obstructions to getting a closed expression for it. First, the inner neighborhood volume function of G can be rather complicated. There are famous formulas (e.g., the Steiner formula) for outer neighborhood volumes, but the inner neighborhood behavior of even a planar polygon can be unexpectedly complicated, and there are no available general formulas.

The function $V_G(\varepsilon)$ need not be polynomial. If $V_G(\varepsilon)$ is a polynomial function of ε in the whole range between zero and the inradius of G, then we say that G is monophase polynomial (or simply monophase), see Example 1. In this case we will write

$$V_G(\varepsilon) = \kappa_0(G)\varepsilon^n + \kappa_1(G)\varepsilon^{n-1} + \cdots + \kappa_{n-1}(G)\varepsilon, \qquad \text{for } 0 \leq \varepsilon \leq g.$$

EXAMPLE 1 *Let $G \subset \mathbb{R}^2$ be the parallelogram in Figure 5. Then the inradius is $g = \frac{1}{2\sqrt{2}}$ and $V_G(\varepsilon) = 6\varepsilon - 4\sqrt{2}\varepsilon^2$ for $0 \leq \varepsilon \leq \frac{1}{2\sqrt{2}}$, so its inner ε-neighborhood volume function is monophase polynomial on the range* $[0, g]$.

EXAMPLE 2 *Let $G \subset \mathbb{R}^2$ be the hexagon in Figure 6. Its inner ε-neighborhood volume function displays polynomial and nonpolynomial behavior on different subsegments of* $[0, g]$. *With $h = \frac{\phi-1}{2\phi}$, $h' = \frac{\phi-1}{\sqrt{4\phi+3}}$, $g = \frac{\sqrt{4\phi+3}}{4\phi+2}$, and ϕ the golden ratio*:

$$V_G(\varepsilon) = \begin{cases} 6\varepsilon - \left(\frac{4\sqrt{4\phi+3}}{2\phi-1} - \frac{\pi}{5}\right)\varepsilon^2 & ; \quad 0 \leq \varepsilon \leq h \\ 6\varepsilon - \left(\frac{4\sqrt{4\phi+3}}{2\phi-1} - \frac{\pi}{5}\right)\varepsilon^2 - 2\varepsilon^2 \arccos\left(\frac{\phi-1}{2\phi\varepsilon}\right) + \frac{\phi-1}{\phi}\sqrt{\varepsilon^2 - \frac{2-\phi}{4\phi+4}} & ; \quad h \leq \varepsilon \leq h' \\ \frac{1}{2}\frac{(\phi-2)\sqrt{4\phi+3}}{\phi+1} + 2\varepsilon\sqrt{2\phi+1} - 2\varepsilon^2\frac{8\phi+5}{\sqrt{4\phi+3}} & ; \quad h' \leq \varepsilon \leq g \\ \frac{1}{2}\frac{(2\phi-1)\sqrt{4\phi+3}}{\phi+1} & ; \quad \varepsilon > g = \textit{inradius}, \end{cases}$$

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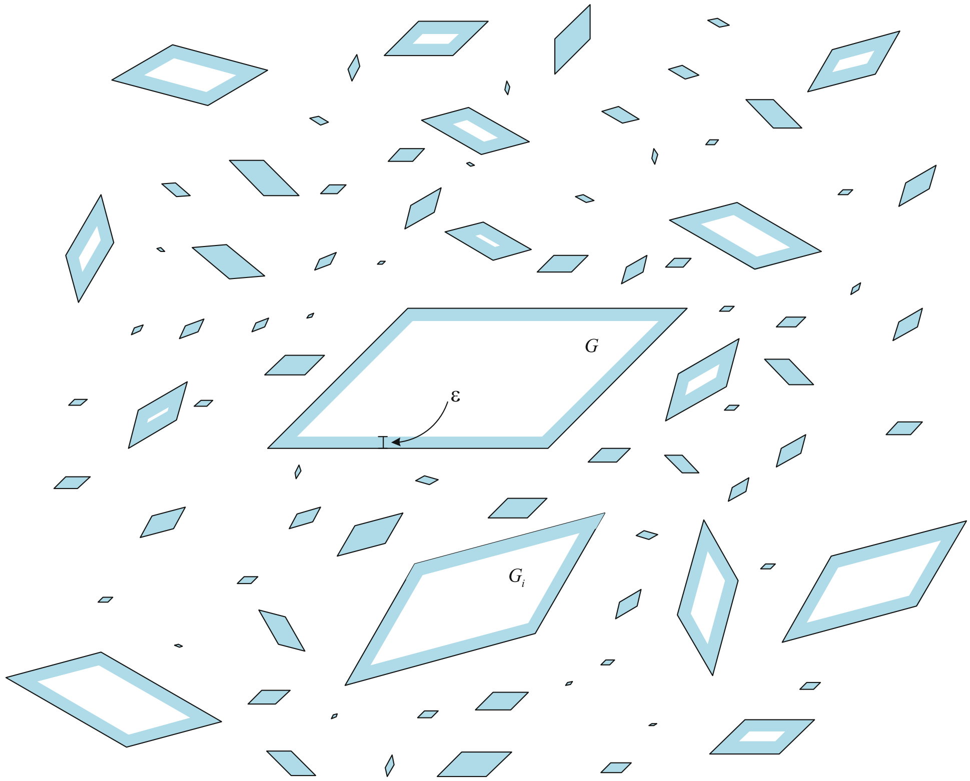

Figure 4. A spray $(G_i)_{i\in\mathbb{N}}$ and its inner ε-neighborhood.

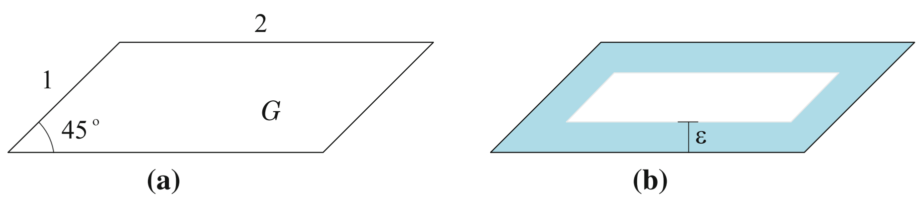

Figure 5. (a) Parallelogram G and (b) inner ε-neighborhood of G.

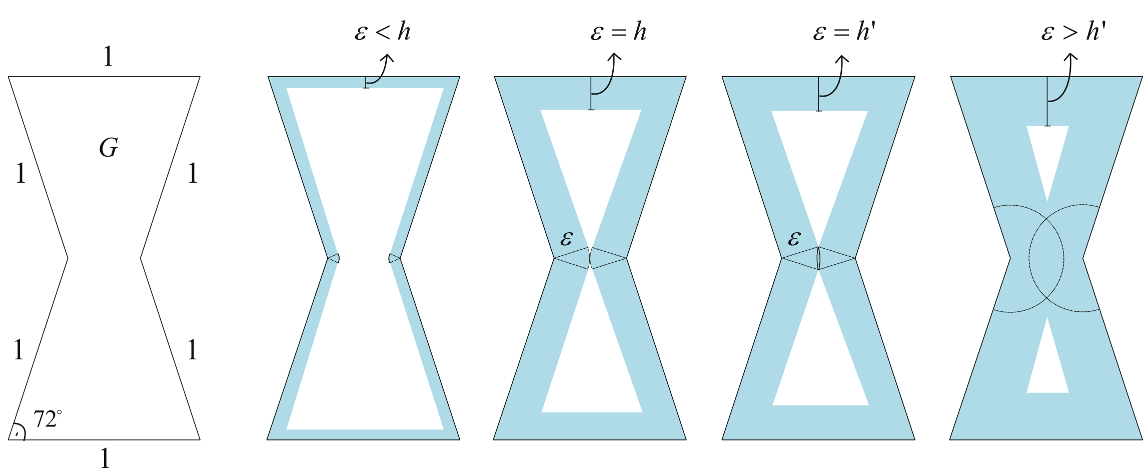

Figure 6. A polygon with nonpolynomial inner ε-neighborhood volume.

If $V_G(\varepsilon)$ is piecewise-polynomial on $[0, g]$, then G is said to be pluriphase-polynomial. Otherwise, G is said to be nonpolynomial. In Example 2, $V_G(\varepsilon)$ shows polynomial and nonpolynomial behavior on different subsegments of $[0, g]$.

Even in the case where $V_G(\varepsilon)$ is nonpolynomial, it has been shown in [7] that an analogue of Theorem 1 below still holds. Our exposition will cover only the monophase case.

The second difficulty for the computation of the volume of the inner ε-neighborhood of $\bigcup G_i$ is related to the nature of the scaling sequence.

What is the scaling ζ-function of a spray?
Given a spray $(G_i)_{i\in\mathbb{N}}$, with the associated scaling sequence $(\lambda_i)_{i\in\mathbb{N}}$, the function $\zeta(s) = \sum_{i=0}^{\infty} \lambda_i^s$ is called the scaling ζ-function of the spray.

A series of the form $\sum_{i=0}^{\infty} \lambda_i^s$, $(\lambda_i > 0,\ \lambda_i \to 0)$ is called a Dirichlet series. It converges and is analytic on a half-plane $\mathrm{Re}(s) > \sigma_0$ $(-\infty \le \sigma_0 \le \infty)$; the number σ_0 is called the abscissa of convergence. The general theory of computing the volume-function $V_{\cup G_i}(\varepsilon)$ depends on technicalities concerning the behavior of the Dirichlet series (see [5] for details). For self-similar and graph-directed fractals, we need only special scaling sequences λ_i, which we describe below, and the corresponding ζ-functions are well behaved.

1. **Scaling sequences of the first type ("Fullshift" sequences)**
The first type is related to self-similar fractals. Let $\{r_1, r_2, \ldots, r_J\}$ be a ratio list $(0 < r_j < 1, j = 1, 2, \ldots, J)$. Consider the set W_k of words $w = w_1 w_2 \cdots w_k$ of length k with letters from $\{1, 2, \ldots, J\}$ and $W = \bigcup_{k=0}^{\infty} W_k$, where W_0 denotes for convenience the set $\{\emptyset\}$ of the empty word. For $w = w_1 w_2 \cdots w_k$ we define $r_w = r_{w_1} r_{w_2} \cdots r_{w_k}$ (for the empty word we set $r_\emptyset = 1$). Choose an order on W, say, by the lexicographic order on W_k, and consider the sequence $(r_w)_{w\in W}$. This sequence will look like
$$1, r_1, r_2, \ldots, r_J, r_1r_1, r_1r_2, \ldots, r_1r_J, r_2r_1, r_2r_2, r_2r_3, \ldots.$$
Our first type of scaling sequences $(\lambda_i)_{i\in\mathbb{N}}$ will have this form. We will call such a sequence $(\lambda_i)_{i\in\mathbb{N}}$ a sequence associated with a ratio list $\{r_1, r_2, \ldots, r_J\}$.
The relation of these sequences to self-similar fractals is as follows: If $F \subset \mathbb{R}^n$ is a self-similar fractal with $F = \bigcup_{j=1}^{J} S_j(F)$ $(S_j : \mathbb{R}^n \to \mathbb{R}^n$ are similarities with $0 < r_j < 1)$, then for every word $w = w_1 w_2 \cdots w_k$ with letters from $\{1, 2, \ldots, J\}$ there is a scaled copy $S_{w_1} \circ S_{w_2} \circ \cdots \circ S_{w_k}(F)$ of the fractal F inside itself with the scaling ratio $r_w = r_{w_1} r_{w_2} \cdots r_{w_k}$. If the scaling sequence $(\lambda_i)_{i\in\mathbb{N}}$ is of the aforementioned form, then the Dirichlet series $\zeta(s) = \sum_{i=0}^{\infty} \lambda_i^s$ has the abscissa of convergence $\sigma_0 = D$, where D is the similarity dimension of the ratio list $\{r_1, r_2, \ldots, r_J\}$. The sum $\sum_{i=0}^{\infty} \lambda_i^s$ can be easily evaluated to yield ([5, Theorem 2.4])
$$\zeta(s) = \frac{1}{1 - (r_1^s + r_2^s + \cdots + r_J^s)}, \qquad \text{for Re } (s) > D.$$

2. **Scaling sequences of the second type ("Subshift of finite type" sequences)**
The second type is related to graph-directed fractals. We will consider graph-directed fractals later, but anticipating a bit, we can introduce weighted directed graphs and define scaling sequences of the second type.
Let $\mathcal{G} = (V, E, r)$ be a (finite) weighted directed graph, where V is the set of vertices, E is the set of edges, and $r : E \to (0, \infty)$ is the weight function. We assume the weights to lie always in the interval $(0,1)$, in which case the graph is called strictly contracting. Let a transition function θ on $E \times E$ be defined by
$$\theta(e,f) = \begin{cases} 1; & \text{if the terminal vertex of } e \text{ equals the initial vertex of } f \\ 0; & \text{otherwise.} \end{cases}$$
Consider the set W_k^{uv} of words $\alpha = e_1 e_2 \cdots e_k$ of length k with letters from E such that $\theta(e_i, e_{i+1}) = 1$ $(i = 1, \ldots, k-1)$, the initial vertex of e_1 is u, and the terminal vertex of e_k is v. (These words correspond to paths of length k from the vertex u to the vertex v in the graph.) For $k = 0$ we use the conventions $W_0^{uu} = \{\emptyset\}$ and $W_0^{uv} = \emptyset$ for $u \neq v$.
For $\alpha \in W_k^{uv}$ we define $r_\alpha = r_{e_1} r_{e_2} \cdots r_{e_k}$ (where we write r_e for $r(e)$). For the empty word $\alpha \in W_0^{uu}$, we set $r_\alpha = 1$.
Choose an order on $W^{uv} = \bigcup_{k=0}^{\infty} W_k^{uv}$ and consider the sequence $(r_\alpha)_{\alpha\in W^{uv}}$. A sequence of this type will be called a scaling sequence of the second type associated with the graph $\mathcal{G}$ and the vertex pair (u, v). They will emerge naturally in the computation of the tube volumes of graph-directed fractals, and we will discuss their scaling ζ-functions in that connection.

How to compute the volume function of the inner ε-neighborhood of a spray?
Now, let $(G_i)_{i\in\mathbb{N}}$ be a spray with the associated scaling sequence $(\lambda_i)_{i\in\mathbb{N}}$. We have to compute $V_{\cup G_i}(\varepsilon) = \sum_i V_{G_i}(\varepsilon) = \sum_i V_{\lambda_i G}(\varepsilon)$. One of the problems is that the inner ε-neighborhood of a scaled copy of G is not the corresponding scaled copy of the inner ε-neighborhood of G. Obviously, $V_{G_i}(\lambda_i \varepsilon) = \lambda_i^n V_G(\varepsilon)$ for $0 \le \varepsilon \le g$ and
$$V_{G_i}(\varepsilon) = \lambda_i^n V_G(\varepsilon/\lambda_i) \quad \text{for } 0 \le \varepsilon \le \lambda_i g =: g_i,\ g_i \text{ being the inradius of } G_i.$$

In the sum $\sum_i V_{\lambda_i G}(\varepsilon)$, if λ_i is small enough (i.e., if $\lambda_i < \varepsilon/g$), the inner ε-neighborhood will fill G_i, and for such λ_i we will have $V_{G_i}(\varepsilon) = \mathrm{Vol}(G_i) = \lambda_i^n \mathrm{Vol}(G)$. For the other λ_i's, there are genuine ε-neighborhoods not filling the G_i fully (see Fig. 4). Therefore we can write
$$V_{\cup G_i}(\varepsilon) = \sum_{\lambda_i \ge \varepsilon/g} V_{G_i}(\varepsilon) + \sum_{\lambda_i < \varepsilon/g} V_{G_i}(\varepsilon) = \sum_{\lambda_i \ge \varepsilon/g} V_{G_i}(\varepsilon) + \sum_{\lambda_i < \varepsilon/g} \lambda_i^n \mathrm{Vol}(G).$$
It is, however, almost impossible to manipulate this bipartite sum except in simple special cases. (See, however, [7].)

At this point, another approach to handling the sum $V_{\cup G_i}(\varepsilon) = \sum V_{G_i}(\varepsilon)$ proves useful. The strategy will be to take an appropriate Mellin transform and then the inverse Mellin transform to express this sum as a sum of residues of a certain meromorphic function.

Computation for the monophase case with scaling sequence of the first type

We assume $G \subset \mathbb{R}^n$ to be monophase, so that we can write its inner ε-neighborhood volume function as

$$V_G(\varepsilon) = \begin{cases} \sum_{i=0}^{n-1} \kappa_i \varepsilon^{n-i} & ; \quad 0 \le \varepsilon \le g \\ \mathrm{Vol}(G) & ; \quad \varepsilon > g \end{cases} \tag{1}$$

and we assume the scaling sequence $(\lambda_i)_{i\in\mathbb{N}}$ to be of the first type associated with a ratio list $\{r_1, r_2, \ldots, r_J\}$. Let D denote the similarity dimension of the ratio list.

We define the auxiliary function $f_G(x, \varepsilon) := V_{xG}(\varepsilon)$ (for $x, \varepsilon > 0$), which expresses the inner ε-neighborhood volume of a copy of G scaled by $x > 0$. Since $V_{xG}(\varepsilon) = x^n V_G(\varepsilon/x)$,

$$f_G(x, \varepsilon) = \begin{cases} \sum_{i=0}^{n-1} \kappa_i x^i \varepsilon^{n-i} & ; \quad 0 \le \varepsilon \le xg \\ x^n \mathrm{Vol}(G) & ; \quad \varepsilon > xg \end{cases}.$$

The Mellin transform $\mathcal{M}[f; s]$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$\mathcal{M}[f; s] = \widetilde{f}(s) = \int_0^\infty x^{s-1} f(x) dx.$$

For fixed ε, we take the Mellin transform of $f_G(x, \varepsilon)$ as a function of x. We easily obtain

$$\widetilde{f}_G(s, \varepsilon) = \varepsilon^{s+n} \left(\frac{\mathrm{Vol}(G)}{g^{s+n}(s+n)} - \sum_{i=0}^{n-1} \frac{\kappa_i}{g^{s+i}(s+i)} \right)$$
$$\text{for } -n < \mathrm{Re}(s) < -n+1.$$

With the convention $\kappa_n = -\mathrm{Vol}(G)$, this is simply

$$\widetilde{f}_G(s; \varepsilon) = -\varepsilon^{s+n} \sum_{i=0}^{n} \frac{\kappa_i}{g^{s+i}(s+i)}.$$

Taking now the inverse Mellin transform of $\widetilde{f}_G(s, \varepsilon)$, we obtain

$$f_G(x, \varepsilon) = \mathcal{M}^{-1}[\widetilde{f}_G(s, \varepsilon)] = \frac{1}{2\pi \mathbb{i}} \int_{-c-\mathbb{i}\infty}^{-c+\mathbb{i}\infty} x^{-s} \widetilde{f}_G(s, \varepsilon) ds$$

for any $n - 1 < c < n$. By the change of variable $s \to -s$,

$$f_G(x, \varepsilon) = \frac{1}{2\pi \mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} x^{s} \widetilde{f}_G(-s, \varepsilon) ds$$

for $n - 1 < c < n$. The assumption $\sum \lambda_i^n < \infty$ implies $D < n$ and we choose c to satisfy also $D < c < n$. Inserting the aforementioned expression into $V_{\cup G_i}(\varepsilon) = \sum_{i=0}^\infty V_{\lambda_i G}(\varepsilon)$, we obtain

$$V_{\cup G_i}(\varepsilon) = \sum_{i=0}^\infty \frac{1}{2\pi \mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} \lambda_i^s \widetilde{f}_G(-s, \varepsilon) ds.$$

Changing the order of the sum and the integral (a justification can be found in [2]), we obtain

$$V_{\cup G_i}(\varepsilon) = \frac{1}{2\pi \mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} \left(\sum_{i=0}^\infty \lambda_i^s \right) \widetilde{f}_G(-s, \varepsilon) ds.$$

As we have noted previously,

$$\zeta(s) = \sum_{i=0}^\infty \lambda_i^s = \frac{1}{1 - (r_1^s + r_2^s + \cdots + r_J^s)}, \qquad \text{for } \mathrm{Re}(s) > D.$$

The right-hand side has a meromorphic extension to the whole complex plane, and from now on we will understand by $\zeta(s)$ this meromorphic extension. It can easily be shown that all the poles of this function lie on a vertical strip $D_\ell \le \mathrm{Re}(s) \le D$ for some $D_\ell \in \mathbb{R}$ (see [5, Theorem 3.6]). We choose D_ℓ to be negative.

What are the complex dimensions of a spray with scaling sequence of the first type?

Let $(G_i)_{i\in\mathbb{N}}$ be a spray with a scaling sequence (λ_i) of the first type associated with a ratio list $\{r_1, r_2, \ldots, r_J\}$. The poles of the (extended) ζ-function

$$\zeta(s) = \frac{1}{1 - (r_1^s + r_2^s + \cdots + r_J^s)}$$

are called the complex dimensions of the spray (G_i). This set of poles will be denoted by $\mathfrak{D}$.

What is the geometric ζ-function of a spray?

Let $(G_i)_{i\in\mathbb{N}}$ be a spray with the associated scaling sequence (λ_i) of either the first or the second type. Let $\zeta(s)$ be the zeta function of the spray (G_i). Assume that the generator G is monophase with the inner ε-neighborhood volume function as given in (1). Then the product

$$\zeta(s)\widetilde{f}_G(-s, \varepsilon) = \zeta(s)\varepsilon^{n-s} \sum_{i=0}^{n} \frac{g^{s-i}}{s-i} \kappa_i$$

is called the geometric zeta function of the spray and is denoted by $\zeta^G_{(\lambda_i)}(s, \varepsilon)$ or $\zeta^G(s, \varepsilon)$.

While the scaling zeta function $\zeta(s)$ depends only on the scaling sequence of the spray, the geometric zeta function depends on the geometry of G also.

The volume of the inner ε-neighborhood of a spray with a scaling sequence of the first type can now be expressed as

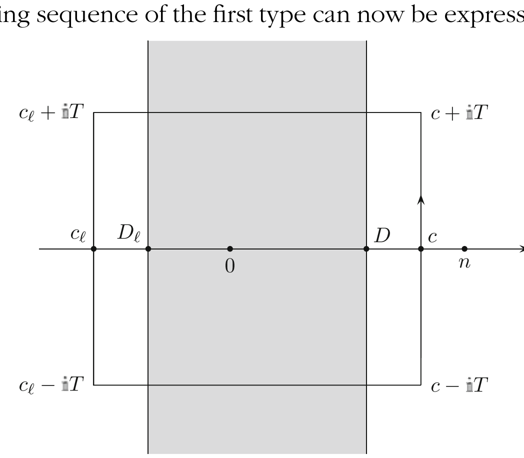

Figure 7. The poles of the geometric ζ-function lie in the vertical strip $D_\ell \le \mathrm{Re}(s) \le D$. To evaluate the integral (2), we apply the residue theorem to the rectangle with corners $c - \mathbb{i}T, c + \mathbb{i}T, c_\ell - \mathbb{i}T, c_\ell + \mathbb{i}T$ and let $T \to \infty$.

$$V_{\cup G_i}(\varepsilon) = \frac{1}{2\pi \mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} \zeta^G(s,\varepsilon)ds. \qquad (2)$$

This integral can be written as a sum of residues of the geometric zeta function $\zeta^G(s,\varepsilon)$ by applying the residue theorem to an appropriate rectangle with a right side edge $[c-\mathbb{i}T, c+\mathbb{i}T]$ and then letting $T\to\infty$ (see Fig. 7; for the details of the proof see [1, 2, 5, 6]).

We thus have given a sketch-proof of the following theorem, which was originally proved in a more general setting and distributionally in [11] and [6]. A pointwise proof for self-similar as well as graph-directed fractals was given in [1] (for a simplified version of the proof see also [2]). For further generalizations and pointwise proofs, see [7] and [5, 2nd ed.].

Theorem 1 *Let $(G_i)_{i\in\mathbb{N}}$ be a spray generated by a monophase open set $G\subset\mathbb{R}^n$ with a scaling sequence $(\lambda_i)_{i\in\mathbb{N}}$ of the first type associated with ratio list $\{r_1, r_2, \ldots, r_J\}$.*

Then the volume $V_{\cup G_i}(\varepsilon)$ of the inner ε-neighborhood of $\bigcup G_i$ is given by the formula

$$V_{\cup G_i}(\varepsilon) = \sum_{\omega\in\mathfrak{D}\cup\{0,1,2,\ldots,n-1\}} \operatorname{res}(\zeta^G(s,\varepsilon);\omega), \quad \text{for } \varepsilon<g$$

where $\zeta^G(s,\varepsilon)$ is the geometric ζ-function, and $\mathfrak{D}$ is the set of the complex dimensions (i.e., the poles of the scaling ζ-function $\zeta(s)$) of the spray (G_i).

Example 3 ***(Recovery of the tube formula of the Cantor set)*** *The middle-third Cantor set $C\subset[0,1]$ is the attractor of an IFS on $\mathbb{R}$ with a ratio list $\{\frac{1}{3},\frac{1}{3}\}$. (Take $S_1(x)=\frac{x}{3}, S_2(x)=\frac{x}{3}+\frac{2}{3}$). The Moran equation is $\frac{1}{3^s}+\frac{1}{3^s}=1$ giving the dimension $D=\log_3 2$. The ε-neighborhood of C consists of two parts: The outer ε-neighborhood of $[0,1]$ with volume 2ε, and the inner ε-neighborhood of the Cantor spray $(G_i)_{i\in\mathbb{N}}$ generated by $G=\left(\frac{1}{3},\frac{2}{3}\right)$. The scaling sequence is of the first type and the scaling ζ-function is $\zeta(s)=\frac{1}{1-2\cdot\frac{1}{3^s}}$. The set of poles of $\zeta(s)$ (i.e., the complex dimensions of the Cantor spray) is $\mathfrak{D}=\{D+\mathbb{i}np \mid n\in\mathbb{Z}\}$ with with $p=\frac{2\pi}{\log 3}$.*

The volume of the inner ε-neighborhood of G is given by

$$V_G(\varepsilon)=\begin{cases} 2\varepsilon & ; \quad 0\le\varepsilon\le\frac{1}{6} \\ \frac{1}{3} & ; \quad \varepsilon>\frac{1}{6}, \end{cases}$$

so that the geometric ζ-function is given by

$$\zeta^G(s,\varepsilon)=\zeta(s)\varepsilon^{1-s}\left(\frac{1}{6^s}\frac{1}{s}\cdot 2-\frac{1}{6^{s-1}}\frac{1}{s-1}\cdot\frac{1}{3}\right).$$

The volume of the inner ε-neighborhood of the spray (G_i) is given by Theorem 1 as follows:

$$\begin{aligned} V_{\cup G_i}(\varepsilon) &= \sum_{\omega\in\mathfrak{D}\cup\{0\}} \operatorname{res}(\zeta^G(s,\varepsilon);\omega) \\ &= \frac{1}{2\log 3}\sum_{n=-\infty}^{\infty}\frac{(2\varepsilon)^{1-D-\mathbb{i}np}}{(D+\mathbb{i}np)(1-D-\mathbb{i}np)}-2\varepsilon. \end{aligned}$$

Adding the volume of the outer ε-neighborhood, we obtain the formula for $\operatorname{Vol}(C_\varepsilon)$ in the introduction.

As in this example of the Cantor set, Theorem 1 can be applied under certain conditions to compute the tube-volumes of IFS-generated fractals. We give a second example in the same spirit.

Example 4 *Consider the IFS on $\mathbb{R}^2$ with ratio list $\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4}\}$ as shown in Figure 9(a),(b). The attractor F of this IFS is indicated in Figure 9(c). The ε-tube of F consists of two parts: the outer ε-neighborhood of the convex hull of the fractal and the inner ε-neighborhood of the spray $(G_i)_{i\in\mathbb{N}}$ generated by the parallelogram G (see Figure 9(d)).*

The Moran equation of the ratio list is

$$3\left(\frac{1}{2}\right)^s+2\left(\frac{1}{4}\right)^s=1,$$

and the similarity dimension is $D=-\log_2\frac{-3+\sqrt{17}}{4}\approx 1.832$. The scaling zeta function is $\zeta(s)=\frac{1}{1-3\cdot 2^{-s}-2\cdot 2^{-2s}}$. The set of poles of $\zeta(s)$ is $\mathfrak{D}=\{D+\mathbb{i}np \mid n\in\mathbb{Z}\}\cup\{D'+\mathbb{i}\left(n+\frac{1}{2}\right)p \mid n\in\mathbb{Z}\}$ with $p=\frac{2\pi}{\log 2}$ and $D'=-\log_2\frac{3+\sqrt{17}}{4}$.

The volume of the inner ε-neighborhood of the generator G is given by

$$V_G(\varepsilon)=\begin{cases}(1+\sqrt{2})\varepsilon-4\sqrt{2}\varepsilon^2 & ; \quad 0\le\varepsilon\le\frac{1}{4\sqrt{2}} \\ \frac{1}{4} & ; \quad \varepsilon>\frac{1}{4\sqrt{2}}\end{cases}$$

so that the geometric zeta function is given by

$$\zeta^G(s,\varepsilon)=\varepsilon^{2-s}\zeta(s)\left[\left(\frac{1}{4\sqrt{2}}\right)^s\frac{1}{s}(-4\sqrt{2})+\left(\frac{1}{4\sqrt{2}}\right)^{s-1}\times\frac{1}{s-1}(1+\sqrt{2})+\left(\frac{1}{4\sqrt{2}}\right)^{s-2}\frac{1}{s-2}\left(-\frac{1}{4}\right)\right].$$

By Theorem 1, the volume of the inner ε-neighborhood of the spray (G_i) is given as

$$\begin{aligned} V_{\cup G_i}(\varepsilon) &= \sum_{\omega\in\mathfrak{D}\cup\{0,1\}}\operatorname{res}(\zeta^G(\varepsilon,s);\omega) \\ &= \sum_{n\in\mathbb{Z}}\operatorname{res}(\zeta^G(\varepsilon,s);D+\mathbb{i}np)+\sum_{n\in\mathbb{Z}}\operatorname{res}(\zeta^G(\varepsilon,s);D'+\mathbb{i}(n+1/2)p)+\operatorname{res}(\zeta^G(\varepsilon,s);0)+\operatorname{res}(\zeta^G(\varepsilon,s);1) \\ &= \sum_{n\in\mathbb{Z}}\frac{(4\sqrt{2}\varepsilon)^{2-D-\mathbb{i}np}}{2\log 2\,(17-3\sqrt{17})}\left(\frac{-\sqrt{2}}{D+\mathbb{i}np}+\frac{2+\sqrt{2}}{D-1+\mathbb{i}np}-\frac{2}{D-2+\mathbb{i}np}\right) \\ &+ \sum_{n\in\mathbb{Z}}\frac{(4\sqrt{2}\varepsilon)^{2-D'-\mathbb{i}(n+\frac{1}{2})p}}{2\log 2\,(17+3\sqrt{17})}\left(\frac{-\sqrt{2}}{D'+\mathbb{i}(n+\frac{1}{2})p}+\frac{2+\sqrt{2}}{D'-1+\mathbb{i}(n+\frac{1}{2})p}-\frac{2}{D'-2+\mathbb{i}(n+\frac{1}{2})p}\right) \\ &+ \sqrt{2}\varepsilon^2-(1+\sqrt{2})\varepsilon. \end{aligned}$$

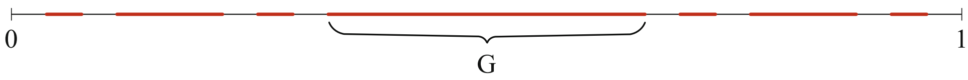

Figure 8. The Cantor spray.

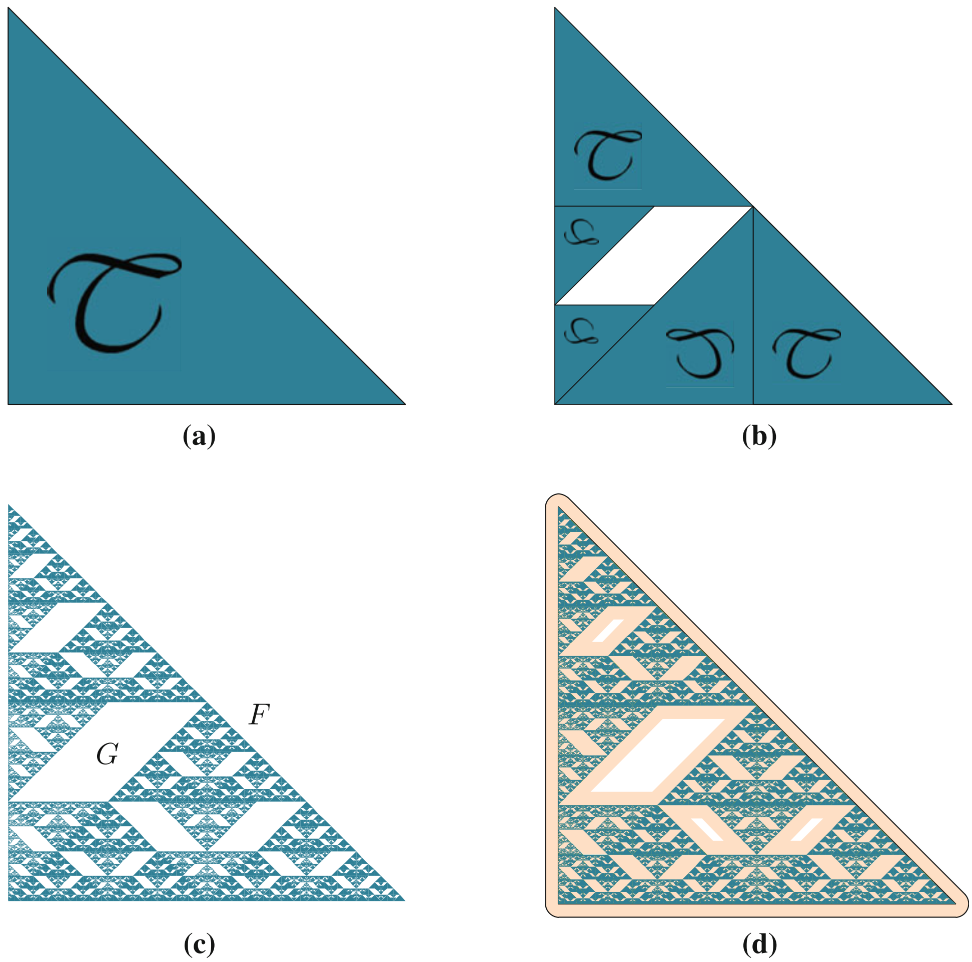

Figure 9. (a) An isosceles right-angled triangle of side-length 2, (b) The similarities $S_1, S_2, \ldots, S_5$, (c) The attractor F, (d) The ε-tube of F.

The volume of the outer ε-neighborhood of the convex hull of the fractal simply equals $\pi\varepsilon^2 + (4 + 2\sqrt{2})\varepsilon$. *Adding these two terms gives the volume of the ε-tube of F.*

How to compute the tube volumes of self-similar fractals

The picture we discussed in the Examples 3 and 4 holds for the attractor F of an iterated function system $\{S_j | j = 1, 2, \ldots, J\}$ under the following fairly general conditions ([13]).

(IFS.1) $\dim C = n$, where C is the convex hull of F
(IFS.2) (Tileset Condition) The open set condition should be satisfied with $O = \mathrm{int}(C)$
(IFS.3) (Nontriviality Condition) $\mathrm{int}(C) \not\subseteq \bigcup_{j=1}^{J} S_j(C)$
(IFS.4) (Pearse-Winter Condition) $\partial C \subset F$.

If these conditions hold, then the ε-tube of the fractal decomposes into a disjoint union of the outer ε-tube of the convex hull C and the inner ε-tube of a spray (G_i) generated by the set $G = \mathrm{int}(C) \setminus \bigcup_{j=1}^{J} S_j(C)$. The scaling factors λ_i of (G_i) are given by the sequence $(\lambda_i)_{i \in \mathbb{N}}$ associated with the ratio list $\{r_1, r_2, \ldots, r_J\}$. If the connected components of G are monophase, then the procedure we described earlier can be applied to compute the volume of the ε-tube of the fractal (see [5, 6, 12, 13]).

As is well-known, the Steiner formula states that the volume $\mathrm{Vol}(A_\varepsilon)$ of the ε-tube of a convex body $A \subset \mathbb{R}^n$ is given by a polynomial

$$\mathrm{Vol}(A_\varepsilon) = \sum_{i=0}^{n} c_i \varepsilon^{n-i}.$$

Under the above conditions (and if all the poles of the geometric ζ-function are simple) the volume of the ε-tube of a fractal can be expressed as a series

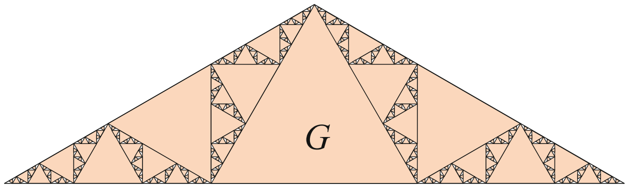

Figure 10. The Koch spray.

$$\mathrm{Vol}(F_\varepsilon) = \sum_{\omega \in \mathfrak{D} \cup \{0,1,\ldots,n-1\}} c_\omega \varepsilon^{n-\omega},$$

where $\mathfrak{D}$ is the set of complex dimensions of the fractal. In this sense, the fractal tube formula is a generalization of the Steiner formula. This analogy shows the significance of the notion of complex dimensions.

We now recall a famous example, which illustrates these notions but which does not satisfy the Pearse-Winter condition.

Example 5 *(Koch Spray)* *The standard Koch curve K can be realized as the attractor of an IFS on the complex plane $\mathbb{C}(=\mathbb{R}^2)$ with the pair of similarities $S_1(z) = \xi\bar{z}$ and $S_2(z) = (1-\xi)(\bar{z}-1)+1$ with $\xi = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$. We have the ratio list $\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$ yielding the Moran equation $\left(\frac{1}{\sqrt{3}}\right)^s + \left(\frac{1}{\sqrt{3}}\right)^s = 1$, giving the dimension $D = \log_3 4$. Consider the convex hull of the Koch curve K and subtract from the interior of this convex hull the union $S_1(K) \cup S_2(K)$. The remaining open set G is monophase and generates a spray $(G_i)_{i\in\mathbb{N}}$ under successive iterations of the given similarities. This spray has a scaling sequence of the first type with the scaling ζ-function $\zeta(s) = \frac{1}{1-2\cdot 3^{-s/2}}$. The complex dimensions are $\mathfrak{D} = \{D + \mathbb{i}np \mid n \in \mathbb{Z}\}$ with $p = \frac{2\pi}{\log 3}$. Computing the volume of the inner ε-neighborhood of the generator G and the geometric ζ-function of the spray, one can obtain the volume of the inner ε-neighborhood of the spray by applying Theorem* 1. *(For the details see* [5, 6].*)*

Tubes of Graph-Directed Fractals

Graph-directed fractals, introduced by Mauldin and Williams ([10]), are interesting generalizations of classical fractals. We will give a brief introduction to them and then discuss their tubes.

Let $\mathcal{G} = (V, E, r)$ be a weighted directed graph with weights $r: E \to (0,1)$. For vertices $u, v \in V$, the set of edges from u to v is denoted by E_{uv} and the set of all edges with initial vertex u is denoted by $E_u := \bigcup_{v\in V} E_{uv}$. If $E_u \neq \emptyset$ for all $u \in V$, then the graph $\mathcal{G}$ is called a Mauldin-Williams (MW) graph. A MW-graph is called strongly connected if there is a directed path from any vertex to any other vertex.

A graph-directed iterated function system (GIFS) realizing the MW-graph consists of a set of complete metric spaces (X_v, ρ_v), one for each vertex $v \in V$, and similarities $S_e : X_v \to X_u$, one for each edge $e \in E_{uv}$, with similarity ratio $r_e := r(e)$. We also say then that $(S_e)_{e\in E}$ realizes the graph in complete metric spaces X_v. If for all $v \in V, X_v = \mathbb{R}^n$ for some fixed n, then the realization is called a Euclidean realization. In the special case that the vertex set is a singleton, the GIFS reduces to the classical self-similar IFS. The following result ([3]) is the graph-directed counterpart of the IFS theorem of Hutchinson ([4]).

What is a graph-directed fractal?
Let (V, E, r) be an MW-graph and let $(S_e)_{e\in E}$ realize the graph in complete metric spaces X_v. Then there exist unique nonempty compact sets $K_u \subset X_u$ such that

$$K_u = \bigcup_{v\in V} \bigcup_{e\in E_{uv}} S_e(K_v)$$

for all $u \in V$. The compact metric spaces K_u are called the attractors of the graph-directed iterated function system realizing the MW-graph, or simply, graph-directed fractals.

Here is an example of a GIFS.

Example 6 *Consider the MW-graph in Figure 11 with three vertices. In Figure 12 we give a Euclidean GIFS realizing this MW-graph and in Figure 13 we indicate the attractors of the GIFS.*

If a Euclidean realization (S_e) of a strongly connected MW-graph (V, E, r) satisfies the open set condition, then the dimension of the attractors of the GIFS can be computed by a theorem of Mauldin and Williams ([10]). We recall what is meant by the open set condition in the graph-directed case ([3, p. 205]):

There should exist nonempty open sets $O_v \subset \mathbb{R}^n$ such that

i) $S_e(O_v) \subseteq O_u$ for all u, v and $e \in E_{uv}$.
ii) $S_e(O_v) \cap S_{e'}(O_{v'}) = \emptyset$ for all $u, v, v' \in V, e \in E_{uv}, e' \in E_{uv'}$ with $e \neq e'$.

Now let (V, E, r) be a strongly connected MW-graph, where we assume $V = \{1, 2, \ldots, N\}$. The matrix $A(s) = [A_{uv}(s)]$, where

$$A_{uv}(s) = \sum_{e\in E_{uv}} r_e^s$$

for $1 \leq u,v \leq N$, is called the MW-matrix of the graph. For graph-directed fractals, this matrix in a sense plays the role of the Moran equation. For $s \in \mathbb{R}, A(s)$ is a non-negative, irreducible matrix (by strong connectedness of the graph), and by Perron-Frobenius theory the spectral radius $\rho(A(s))$

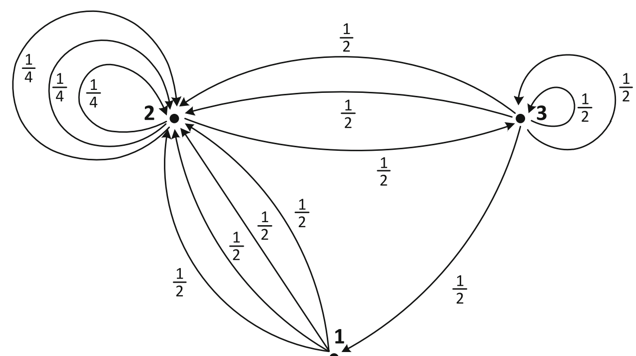

Figure 11. A strongly connected Mauldin-Williams graph.

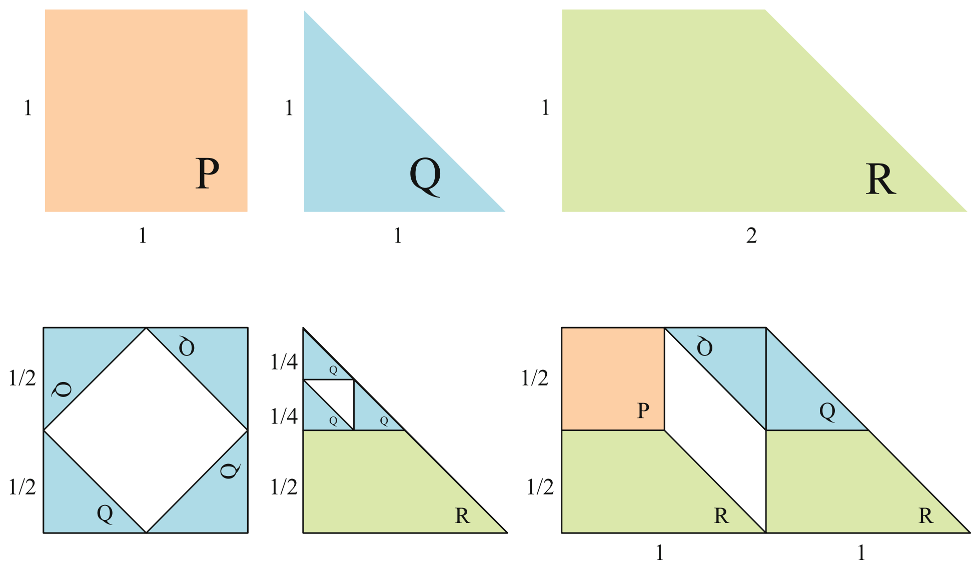

Figure 12. The similarities realizing the MW-graph (in Figure 11) in $\mathbb{R}^2$.

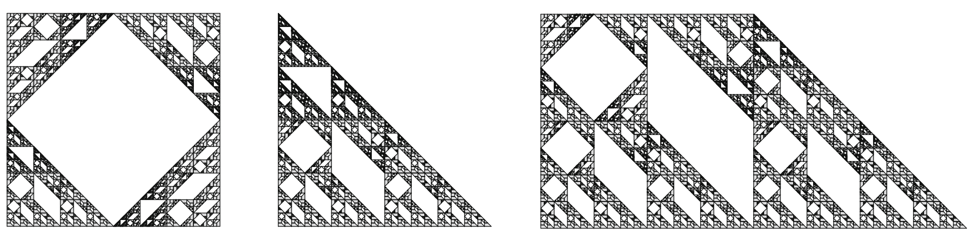

Figure 13. The attractors of the GIFS in Figure 12.

is decreasing (see [3]). The unique non-negative real number s_0 for which the spectral radius $\rho(A(s))$ of the matrix $A(s)$ is 1 is called the sim-value of the graph. Let (S_e) be a Euclidean realization of (V, E, r) satisfying the open set condition. Then the Hausdorff, Minkowski, and packing dimensions of the attractors coincide and equal the sim-value s_0.

For our purposes it will be convenient to consider s as a complex variable.

EXAMPLE 6 *(continued)* *The MW-matrix of the graph in Figure 11 is*

$$A(s) = \begin{pmatrix} 0 & 4(\frac{1}{2})^s & 0 \\ 0 & 3(\frac{1}{4})^s & (\frac{1}{2})^s \\ (\frac{1}{2})^s & 2(\frac{1}{2})^s & 2(\frac{1}{2})^s \end{pmatrix}.$$

Because $\det(I - A(s)) = 2 \cdot 2^{-3s} - 5 \cdot 2^{-2s} - 2 \cdot 2^{-s} + 1$, *the complex numbers s for which* 1 *is an eigenvalue of* $A(s)$ *are*

$$a_0 + \mathbb{i}np, \quad a_1 + \mathbb{i}\left(n + \frac{1}{2}\right)p, \quad a_2 + \mathbb{i}np \qquad (n \in \mathbb{Z}) \tag{3}$$

where $p = \frac{2\pi}{\log 2}$, *and* a_0, a_1, a_2 *are approximately* 1.73, 0.76, *and* -1.48 *respectively. We see that the sim-value is* $s_0 = a_0 \approx 1.73$, *since this is the only non-negative real s for which* 1 *is an eigenvalue of* $A(s)$. *The spectral radius of* $A(s)$ *must in fact be* 1 *for this value. The other values in this list will soon play unexpected roles.*

How to compute the tubes of graph-directed fractals

Let $\mathcal{G} = (V, E, r)$ be a strongly connected MW-graph, $(S_e)_{e\in E}$ be a Euclidean realization, and $(K_u)_{u\in V}$ be the attractors of the graph-directed system. Let C_u be the convex hull of K_u in $\mathbb{R}^n$. We assume the following conditions:

(GIFS.1) $\dim(C_u) = n$

(GIFS.2) (Tileset condition) The open set condition should be satisfied with $O_v = \mathrm{int}(C_v)$

(GIFS.3) (Nontriviality condition) $\mathrm{int}(C_u) \not\subseteq \bigcup_{e\in E_u} S_e(C_v)$

(GIFS.4) (Pearse-Winter Condition) $\partial C_u \subset K_u$.

Now we define $G_u := \mathrm{int}(C_u) \setminus \bigcup_{e\in E_u} S_e(C_v)$. The "hollow space" $H_u = \mathrm{int}(C_u) \setminus K_u$ is a union of certain sprays generated by the G_v's $(v \in V)$. To be more specific,

$$H_u = \bigcup_{v\in V} \bigcup_{O\in spray_u(G_v)} O, \tag{4}$$

where $spray_u(G_v)$ is the spray generated by G_v consisting of the collection of all open sets $(S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k})(G_v)$, where $e_1e_2\cdots e_k$ is a path from u to v in the graph $\mathcal{G}$.

The scaling factor of $S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k}$ is $r_{e_1}r_{e_2}\cdots r_{e_k}$, the weight of the path $e_1e_2\cdots e_k$. We see that the open sets of the spray $spray_u(G_v)$ have the scaling ratios $\lambda_\alpha = r_{e_1}r_{e_2}\cdots r_{e_k}$, where $\alpha = e_1e_2\cdots e_k$ is any path from u to v in the graph $\mathcal{G}$.

This shows that the scaling sequence of this spray is a sequence of the second type.

This consideration reduces the computation of the volume of the inner ε-neighborhood of the hollow space H_u to the computation of the volumes of the inner ε-neighborhoods of the sprays $spray_u(G_v)$ for $v \in V$. So the problem becomes to get control of the associated sequence of weights λ_α for all paths from u to v.

What is a scaling ζ-matrix function?

To encode the "path weights" $\lambda_\alpha = r_{e_1} r_{e_2} \cdots r_{e_k}$ for paths $\alpha = e_1 e_2 \ldots e_k$ from a vertex u to a vertex v in a weighted directed graph $\mathcal{G}$, we define the following scaling ζ-matrix function $\zeta(s) = [\, \zeta_{uv}(s) \,]$:

$$\zeta_{uv}(s) = \sum_{k=0}^{\infty} \sum_{\alpha \in W_k^{uv}} \lambda_\alpha^s,$$

where W_k^{uv} is the set of paths $\alpha = e_1 e_2 \ldots e_k$ of length k from u to v and $\lambda_\alpha = r_{e_1} r_{e_2} \ldots r_{e_k}$. Note that $\zeta_{uv}(s) = \sum_\alpha \lambda_\alpha^s$ for all paths α from u to v.

LEMMA 1 $\zeta(s) = (I - A(s))^{-1}$ *for* $\mathrm{Re}(s) > s_0$, *where* s_0 *is the sim-value of the graph.*

PROOF. $[A^k(s)]_{uv}$ is the sum of λ_α^s's for paths of length k from u to v. Thus $\left[\sum_{k=0}^{\infty} A^k(s)\right]_{uv} = \sum_\alpha \lambda_\alpha^s$ for all paths α from u to v. If s is real and greater than s_0, then $\rho(A(s))$ is less than 1. Therefore $\sum_{k=0}^{\infty} A^k(s)$ converges and equals $(I - A(s))^{-1}$ for $s \in \mathbb{R}, s > s_0$. Hence the series $\sum_\alpha \lambda_\alpha^s$ is convergent for $s \in \mathbb{R}, s > s_0$, and, being a Dirichlet series, it is convergent and analytic in the half plane $\mathrm{Re}(s) > s_0$.

On the other hand, $[(I - A(s))^{-1}]_{uv} = \frac{[\mathrm{adj}(I - A(s))]_{uv}}{\det(I - A(s))}$ is meromorphic on $\mathbb{C}$. Combining the three facts

i) $\sum_\alpha \lambda_\alpha^s$ is analytic on $\mathrm{Re}(s) > s_0$,
ii) $\left[(I - A(s))^{-1}\right]_{uv}$ is meromorphic on $\mathrm{Re}(s) > s_0$,
iii) $\sum_\alpha \lambda_\alpha^s = \left[(I - A(s))^{-1}\right]_{uv}$ for $s \in \mathbb{R}, s > s_0$

we conclude, by the unicity theorem, that

$$\left[(I - A(s))^{-1}\right]_{uv} = \sum_\alpha \lambda_\alpha^s \quad \text{for all } \mathrm{Re}(s) > s_0.$$

The proof of this lemma shows that $\frac{[\mathrm{adj}(I - A(s))]_{uv}}{\det(I - A(s))}$ is the meromorphic extension of $\zeta_{uv}(s)$ to the whole complex plane $\mathbb{C}$. From now on, by $\zeta_{uv}(s)$ we will understand this extension.

What are the complex dimensions of a spray with scaling sequence of the second type?

Let $(G_i)_{i \in \mathbb{N}}$ be a spray with a scaling sequence (λ_i) of the second type associated with a graph $\mathcal{G}$ and vertex pair (u, v). The poles of the (extended) ζ-function

$$\zeta_{uv}(s) = \frac{[\mathrm{adj}(I - A(s))]_{uv}}{\det(I - A(s))}$$

are called the complex dimensions of the spray (G_i). This set of poles will be denoted by $\mathfrak{D}_{uv}$.

Computation of the inner ε-neighborhood volume of a spray with monophase generator and scaling sequence of the second type

The following is the counterpart of Theorem 1 for the volume of the inner ε-neighborhood of a spray $(G_i)_{i \in \mathbb{N}}$, when the associated scaling sequence is of the second type. (The proof of this theorem is similar to the proof of Theorem 1, for details see [1].)

THEOREM 2 *Let* $(G_i)_{i \in \mathbb{N}}$ *be a spray generated by a monophase open set* $G \subset \mathbb{R}^n$ *with a scaling sequence* $(\lambda_i)_{i \in \mathbb{N}}$ *of the second type associated with a graph* $\mathcal{G}$ *and the vertex pair* (u, v). *Then the volume* $V_{\cup G_i}(\varepsilon)$ *of the inner ε-neighborhood of* $\bigcup G_i$ *is given by the formula*

$$V_{\cup G_i}(\varepsilon) = \sum_{\omega \in \mathfrak{D}_{uv} \cup \{0,1,2,\ldots,n-1\}} \mathrm{res}(\zeta_{uv}^G(s, \varepsilon); \omega),$$

where ζ_{uv}^G *is the geometric ζ-function, and* $\mathfrak{D}_{uv}$ *is the set of the complex dimensions (i.e., the poles of the scaling ζ-function* $\zeta_{uv}(s)$*) of the spray* (G_i).

Finally, we are in a position to compute the tube volume of graph-directed fractals. Under the conditions (GIFS.1)-(GIFS.4) we had noted (see (4)) that the hollow space H_u of the graph-directed fractal K_u could be expressed as a union of elements from the sprays $spray_u(G_v)$.

Now, the associated scaling sequence of each spray $spray_u(G_v)$ is of the second type and, if we assume that the components of the G_v's are monophase, we can apply Theorem 2 and compute the volume of the inner ε-neighborhood of the hollow space H_u. To obtain the full tube volume of the fractal K_u, we need only to add the volume of the outer ε-neighborhood of the convex hull of K_u.

EXAMPLE 6 *(continued)*

$$(I - A(s))^{-1} = \frac{1}{2 \cdot 2^{-3s} - 5 \cdot 2^{-2s} - 2 \cdot 2^{-s} + 1} \times \begin{bmatrix} 6 \cdot 2^{-3s} - 5 \cdot 2^{-2s} - 2 \cdot 2^{-s} + 1 & 4 \cdot 2^{-s} - 8 \cdot 2^{-s} & 4 \cdot 2^{-2s} \\ 2^{-2s} & 1 - 2 \cdot 2^{-s} & 2^{-s} \\ 2^{-s} - 3 \cdot 2^{-3s} & 4 \cdot 2^{-2s} + 2 \cdot 2^{-s} & 1 - 3 \cdot 2^{-2s} \end{bmatrix}$$

The zeta function $\zeta_{uv}(s)$ is the uv-entry of the above matrix. For example

$$\zeta_{2,3}(s) = \frac{2^{-s}}{2 \cdot 2^{-3s} - 5 \cdot 2^{-2s} - 2 \cdot 2^{-s} + 1}.$$

As can be seen from Figure 12, each attractor K_u has only one generator G_u. As an example, we examine the $spray_2(G_3)$, the spray of copies of G_3 that appear in the hollow space H_2. Since

$$V_{G_3}(\varepsilon) = \begin{cases} (1 + \sqrt{2})\varepsilon - 4\sqrt{2}\varepsilon^2 & ; \quad 0 \le \varepsilon \le \frac{1}{4\sqrt{2}} \\ \frac{1}{4} & ; \quad \varepsilon > \frac{1}{4\sqrt{2}}, \end{cases}$$

the geometric zeta function $\zeta_{2,3}^{G_3}(s)$ is given by

$$\begin{aligned}\zeta_{2,3}^{G_3}(s) &= \varepsilon^{2-s}\frac{2^{-s}}{2\cdot 2^{-3s}-5\cdot 2^{-2s}-2\cdot 2^{-s}+1}\\ &\times\left(\frac{1}{s}\left(\frac{1}{4\sqrt{2}}\right)^{s}(-4\sqrt{2})+\frac{1}{s-1}\left(\frac{1}{4\sqrt{2}}\right)^{s-1}(1+\sqrt{2})\right.\\ &\left.-\frac{1}{s-2}\left(\frac{1}{4\sqrt{2}}\right)^{s-2}\frac{1}{4}\right).\end{aligned}$$

As computed before (see (3)), the set of poles of the scaling zeta function $\zeta_{2,3}(s)$ is

$$\begin{aligned}\mathfrak{D}_{2,3} &= \{a_0+\mathrm{i}np \mid n\in\mathbb{Z}\}\cup\{a_1+\mathrm{i}(n+1/2)p \mid n\in\mathbb{Z}\}\\ &\cup\{a_2+\mathrm{i}np \mid n\in\mathbb{Z}\}.\end{aligned}$$

What are the hidden complex dimensions of graph-directed fractals?
As we see in the previous example, not only the complex values of s for which the spectral radius of the Mauldin-Williams matrix $A(s)$ is 1 contribute to the volume formula. The other complex values of s for which 1 is an eigenvalue of $A(s)$ do also contribute, by giving the set of poles $\mathfrak{D}_{uv}$ of the scaling zeta function. These values of s are what we call the hidden complex dimensions of graph-directed fractals.

This set is plotted in Figure 14. By Theorem 2, the volume of the inner ε-neighborhood of $spray_2(G_3)$ is

$$\begin{aligned}V_{spray_2(G_3)}(\varepsilon) &= \sum_{\omega\in\mathfrak{D}_{2,3}\cup\{0,1\}}\mathrm{res}(\zeta_{2,3}^{G_3}(s,\varepsilon);\omega)\\ &= \sum_{\omega\in\{a_0+\mathrm{i}np\,|\,n\in\mathbb{Z}\}}+\sum_{\omega\in\{a_1+\mathrm{i}(n+1/2)p\,|\,n\in\mathbb{Z}\}}+\sum_{\omega\in\{a_2+\mathrm{i}np\,|\,n\in\mathbb{Z}\}}+\sum_{\omega\in\{0,1\}}\mathrm{res}(\zeta_{2,3}^{G_3}(s,\varepsilon);\omega)\\ &=: \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4\end{aligned}$$

where

$$\begin{aligned}\Sigma_1 &= \sum_{n\in\mathbb{Z}}\frac{(4\sqrt{2}\varepsilon)^{2-a_0-\mathrm{i}np}}{16\log 2\,(3\cdot 2^{-2a_0}-5\cdot 2^{-a_0}-1)}\left(\frac{\sqrt{2}}{a_0+\mathrm{i}np}-\frac{2+\sqrt{2}}{a_0-1+\mathrm{i}np}+\frac{2}{a_0-2+\mathrm{i}np}\right),\\ \Sigma_2 &= \sum_{n\in\mathbb{Z}}\frac{(4\sqrt{2}\varepsilon)^{2-a_1-\mathrm{i}(n+\frac{1}{2})p}}{16\log 2\,(3\cdot 2^{-2a_1}+5\cdot 2^{-a_1}-1)}\\ &\times\left(\frac{\sqrt{2}}{a_1+\mathrm{i}(n+\frac{1}{2})p}-\frac{2+\sqrt{2}}{a_1-1+\mathrm{i}(n+\frac{1}{2})p}+\frac{2}{a_1-2+\mathrm{i}(n+\frac{1}{2})p}\right),\\ \Sigma_3 &= \sum_{n\in\mathbb{Z}}\frac{(4\sqrt{2}\varepsilon)^{2-a_2-\mathrm{i}np}}{16\log 2\,(3\cdot 2^{-2a_2}-5\cdot 2^{-a_2}-1)}\left(\frac{\sqrt{2}}{a_2+\mathrm{i}np}-\frac{2+\sqrt{2}}{a_2-1+\mathrm{i}np}+\frac{2}{a_2-2+\mathrm{i}np}\right),\\ \Sigma_4 &= \sqrt{2}\varepsilon^2-\frac{1+\sqrt{2}}{2}\varepsilon.\end{aligned}$$

We note that the volume of the whole inner ε-neighborhood of the hollow space H_2 can be computed by additionally applying the spray formula to the sprays $spray_2(G_1)$ and $spray_2(G_2)$ and adding to the already computed sum. One can then obtain the volume of the whole ε-tube of the second fractal by adding the volume of the outer ε-neighborhood of the convex hull of the second fractal. This recipe works generally for graph-directed fractals under the conditions (GIFS.1)-(GIFS.4).

Noncancellation phenomenon

A last issue to discuss is the so-called noncancellation phenomenon. As $\zeta_{uv}(s)$ is given by

$$[(I-A(s))^{-1}]_{uv}=\frac{[\mathrm{adj}(I-A(s))]_{uv}}{\det(I-A(s))},$$

it can happen that some roots of $\det(I-A(s))$ might be cancelled by the roots of $[\mathrm{adj}(I-A(s))]_{uv}$. We give below

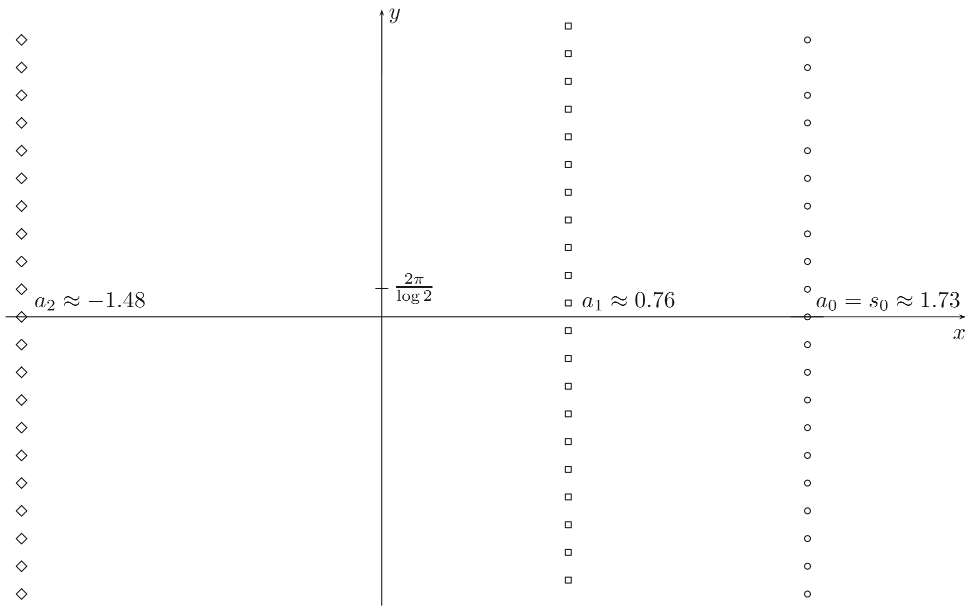

Figure 14. Complex dimensions $\mathfrak{D}_{2,3}$ of the spray $spray_2(G_3)$ of Example 6. The spectral radius of $A(s)$ is 1 if s is a complex dimension on the "right line." For the other complex dimensions (lying on the middle and left lines) 1 is an eigenvalue of $A(s)$, although not the spectral radius.

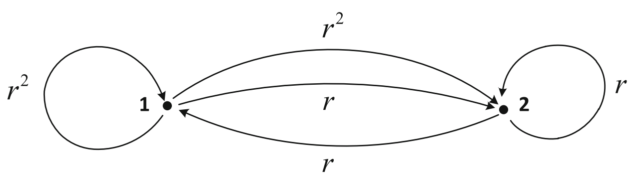

Figure 15. A MW-graph.

an example where this happens. It might seem possible, then, for some roots of $\det(I - A(s))$ not to appear as a pole of any $\zeta_{uv}(s)$.

Example 7 *Consider the graph in Figure 15, where r is some fixed number, $0 < r < 1$. The Mauldin-Williams matrix is* $A(s) = \begin{bmatrix} r^{2s} & r^s + r^{2s} \\ r^s & r^s \end{bmatrix}$, *so that* $\det(I - A(s)) =$

$1 - r^s - r^{2s} = (1 + r^s)(1 - 2r^s)$ *and*

$$(I - A(s))^{-1} = \frac{1}{1 - r^s - r^{2s}} \begin{bmatrix} r^{2s} & r^s + r^{2s} \\ r^s & r^s \end{bmatrix}.$$

The set of complex dimensions of the system is

$$\mathfrak{D} = \{-\log_r 2 + \mathbb{i}np \mid n \in \mathbb{Z}\} \cup \{\mathbb{i}(n + 1/2)p \mid n \in \mathbb{Z}\},$$

where $p = \frac{2\pi}{\log r}$; *whereas, for example,* $\mathfrak{D}_{1,2} = \{-\log_r 2 + \mathbb{i}np \mid n \in \mathbb{Z}\}$ *by cancellation in*

$$\zeta_{1,2} = [(I - A(s))^{-1}]_{12} = \frac{r^s + r^{2s}}{(1 + r^s)(1 - 2r^s)} = \frac{r^s}{1 - 2r^s}.$$

So only some of the values of s occurring in $\mathfrak{D}$ occur as values in $\mathfrak{D}_{1,2}$. Nevertheless one shows by linear algebra that, in general, all of them must occur in $\mathfrak{D}_{uv}$ for some u, v.

Note added in proof: We are informed by the referee that Stephen Muir has introduced and studied a geometric zeta function for graph-directed fractals under suitable conditions in a yet unpublished work. His work, however does not involve the study of tube formulas.

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