Meissner's Mysterious Bodies

BERND KAWOHL AND CHRISTOF WEBER

Certain three-dimensional convex bodies have a counterintuitive property: they are of constant width. In this particular respect they resemble a sphere without being one. Discovered a century ago, Meissner's bodies have often been conjectured to minimize volume among bodies of given constant width. However, this conjecture is still open. We draw attention to this challenging and beautiful open problem by presenting some of its history and recent development.

A Century of Bodies of Constant Width

In §32 of their famous book "Geometry and the Imagination", Hilbert and Cohn-Vossen list eleven properties of the sphere and discuss which of these suffices to determine uniquely the shape of the sphere [22]. One of those properties is called *constant width*: if a sphere is squeezed between two parallel (supporting) planes, it can rotate in any direction and always touch both planes. As the reader may suspect, there are many other convex sets with this property of constant width. To indicate that they have this property in common with spheres, such three-dimensional objects are sometimes called *spheroforms* ([8, p. 135], [36], [7, p. 33]).

Some of the three-dimensional convex sets of constant width have a rotational symmetry. They can be generated by rotating plane sets of constant width with a reflection symmetry about their symmetry line. The drawing in Figure 1 is taken from a catalogue of mathematical models produced by the publisher Martin Schilling in 1911 [34, p. 149]. Under the influence of mathematicians such as Felix Klein, such models were produced for educational purposes, many of them made of plaster. Figure 1 appears to be the earliest drawing showing a nontrivial three-dimensional body of constant width. This body is generated by rotating the Reuleaux triangle around its axis of symmetry. The Schilling catalogue also advertises another rotational as well as a nonrotational body of constant width. The author of its mathematical description is Ernst Meissner, with the help of Friedrich Schilling, not to be confused with Martin Schilling, the editor of the catalogue ([34, p. 106f.], and for a slightly expanded version see [28]). Because Meissner seems to have discovered this body, it is called a *Meissner body*. Although it is obvious that its construction can lead to two noncongruent bodies of constant width, Meissner explicitly describes only one of them, M_V (for details we refer to the paragraph "Identifying the Suspect" below). Because the construction of both bodies follows similar principles, one often speaks of "the" Meissner body.

The earliest printed photograph of a plaster Meissner body, the one described in the Schilling catalogue M_V, can be found in the 1932 German version of "Geometry and the Imagination", shown in Figure 2 [22, p. 216]. Photographs of all three bodies of constant width mentioned by Meissner can be found in more recent publications ([7, p. 64ff.], [16, p. 96–98]). The mathematical models must have sold well, for they can still be found in display cases of many mathematics departments. For instance, they can be found not only at many German universities (for the plaster model of the Meissner body M_V at the Technical University of Halle in Germany, see http://did.mathematik.uni-halle.de/modell/modell.php?Nr=Dg-003) but also at Harvard University in the US and even at the University of Tokyo (http://www.math.harvard.edu/~angelavc/models/locations.html).

Certainly there are many more bodies of constant width than the four mentioned so far. A very nice collection is displayed in the exhibit "Pierres qui roulent" ("Rolling Stones") in the Palais de la Découverte in Paris (see Figure 8 at the end of the paper). In addition to some rotated Reuleaux polygons (two triangles, four pentagons) it shows two Meissner bodies, both of the same type M_F. The exhibit offers the visitor a hands-on, tactile experience of the phenomenon of constant width. Sliding a transparent plate over these bodies of the same constant width causes the bodies to roll, while the plate appears to slide as if lying on balls.

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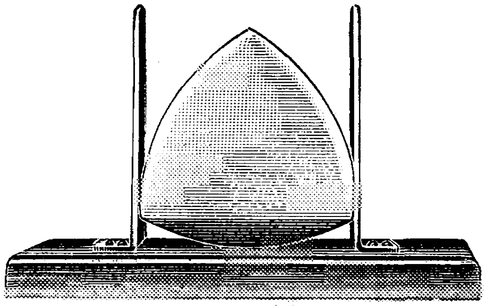

Figure 1. Rotated regular Reuleaux triangle, squeezed between a gauging instrument.

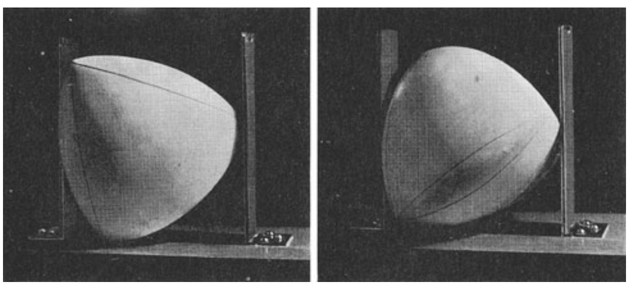

Figure 2. Plaster Model of Meissner body M_V.

Of course there are also many other, nonrotational bodies of constant width. For their construction see [36], [24], [31], and [2].

In this article we restrict our attention for the most part to the three-dimensional setting. The reader can find more material in excellent surveys on plane and higher-dimensional sets of constant width, for example, in Chakerian & Groemer [12], Heil & Martini [21], or Böhm & Quaisser [7, ch. 2].

As already mentioned, there are two different types of Meissner bodies M_V and M_F (their construction will be described in the following text). They not only have identical volume and surface area, but *are conjectured to minimize volume among all three-dimensional convex bodies of given constant width*.

We could not find a written record by Meissner himself that explicitly states the conjecture, but he seems to have guessed that his bodies are of minimal volume [7, p. 72]. Whereas Hilbert and Cohn-Vossen in their book of 1932 do not comment in this direction, Bonnesen and Fenchel mention the conjecture two years later. In the German edition of their "Theory of Convex Bodies", they write, "es ist anzunehmen" which still reads "it is to be assumed" in the English edition of 1987 [8, p. 144]). Since then the conjecture has been stated again and again. For example, Yaglom and Boltyansky make it in all editions of their book "Convex Figures", from the Russian "predpolagaiut" in 1951, via the German "es ist anzunehmen" in 1956, to the English "we shall assume without proof" in 1961 [39, p. 81].

On the other hand, there was the belief that the body that minimizes volume among all three-dimensional bodies of constant width must have the symmetry group of a regular tetrahedron, a property not displayed by the Meissner bodies. This belief was first expressed by Danzer in the 1970s, as Danzer has confirmed to us in personal communication ([19, p. 261], [13, p. 34] and [7, p. 72]). In 2009 an attempt was made to arrive at a body of full tetrahedral symmetry and minimal volume via a deformation flow argument [17].

Incidentally, the Minkowski sum $\frac{1}{2}M_V \oplus \frac{1}{2}M_F$, which one obtains halfway through the process of morphing M_V into M_F, would provide a body with tetrahedal symmetry (see Figure 7). It actually has the same constant width as M_V and M_F. Its volume, however, is larger than that of the Meissner bodies, due to the *Brunn-Minkowski inequality*. It can be shown that the increase in volume is slightly more than 2% of the volume of the Meissner bodies [32].

Generating Constant-Width Bodies by Rotation

Every two-dimensional convex set can be approximated by convex polygons. Similarly, every two-dimensional convex set of constant width can be approximated by circular arcs

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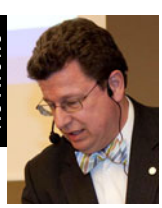

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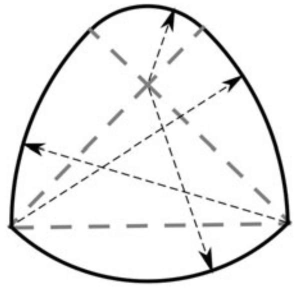

Figure 3. Nonregular Reuleaux tetragon with four circular arcs.

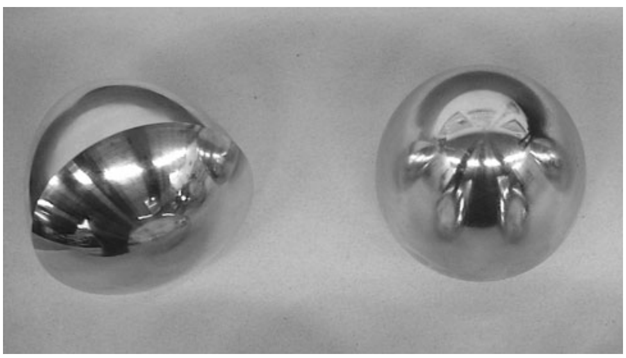

Figure 4. Figure 3 rotated (side and top view).

and thus by *Reuleaux polygons* of constant width. If the arcs are all of the same length, one has regular Reuleaux triangles, pentagons, and so on. But to generate a plane convex set of constant width, it is not necessary that all circular arcs be of the same length. Figure 3 shows a plane set of constant width, a *Reuleaux tetragon*, which is constructed along the lines of [9, p. 192f.]. Note that it is bounded by four circular arcs.

Whenever a plane set of constant width is reflection symmetric with respect to some axis, it can be rotated around that axis to generate a three-dimensional set of constant width. A rotated regular Reuleaux triangle leads to the body shown in Figure 1, and if an appropriate nonregular Reuleaux tetragon or a Reuleaux trapezoid is rotated, one will obtain a body similar to the one in Figure 4. Both are not only bodies of revolution but are three-dimensional sets of constant width [9, p. 196f.].

According to the *theorem of Blaschke-Lebesgue*, the Reuleaux triangle minimizes area among all plane convex domains of given width. Thus one could expect that the rotated Reuleaux triangle in Figure 1 would minimize volume among all *rotational* bodies of given width. It was not until recently (1996 and 2009) that this long-standing conjecture was confirmed ([10, 25] and [1]).

Identifying the Suspect: Meissner Bodies

The plane Reuleaux triangle of constant width d is constructed as the intersection of three discs of radius d, each centered at a different corner of an equilateral triangle. In an analogous way, a *Reuleaux tetrahedron* R_T can be constructed by intersecting four balls of radius d, each of which is centered at a vertex of a regular tetrahedron with side length d. It consists of four vertices, four pieces of spheres, and six curved edges each of which is an intersection of two spheres.

Whenever this Reuleaux tetrahedron is squeezed between two parallel planes with a vertex touching one plane and the corresponding spherical surface touching the other, their distance is d by construction. However, the distance of the planes must be slightly enlarged by a factor of up to $\sqrt{3} - \frac{\sqrt{2}}{2} \approx 1.025$ when the planes touch two opposite edges of R_T. This means that the width of R_T is not constant but varies depending on its direction up to 2.5%. Incidentally, as Meissner mentioned in [27, p. 49], the ball is the only body of constant width that is bounded only by spherical pieces. Thus R_T, which is bounded only by spherical pieces and is different from a ball, ought not to be of constant width, and indeed it just fails to be.

Nevertheless, R_T can be used as a starting point for a set of constant width. According to Meissner, some edges must be rounded off by the following procedure ([28], [8, p. 144], [39, p. 81], [6, p. 54f.]):

a) Imagine two planes bounding adjacent facets of the underlying tedrahedron. Remove the wedge located between the two planes and containing the curved edge of the Reuleaux tetrahedron R_T (see Figure 5 from [39, p. 81]).
b) The intersection of the planes with R_T contains two circular arcs that meet in the two ends of the wedge. Rotate one of these arcs around the corresponding edge of the tetrahedron. This generates a spindle-shaped surface, a spindle torus.
c) Notice that now the sharp edge has become a differentiable surface even across the boundary between spindle-torus and spherical piece.

After rounding off three edges of R_T that meet in a vertex, according to this procedure, one obtains the first type of Meissner body, M_V (see Figure 6, left). The second Meissner body M_F is obtained by rounding off three edges surrounding one of the faces of R_T (see Figure 6, right). Either Meissner body features four vertices, three circular edges, four spherical surfaces, and three toroidal surfaces. Both bodies have identical volume and surface area, and they are invariant under a rotation of 120° around a suitable

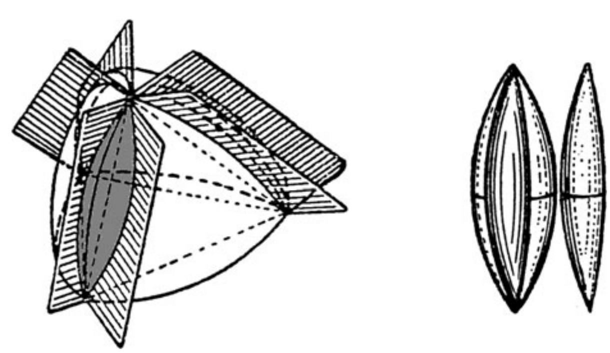

Figure 5. Replacing three wedges (left, one shaded gray) by pieces of spindle tori (right).

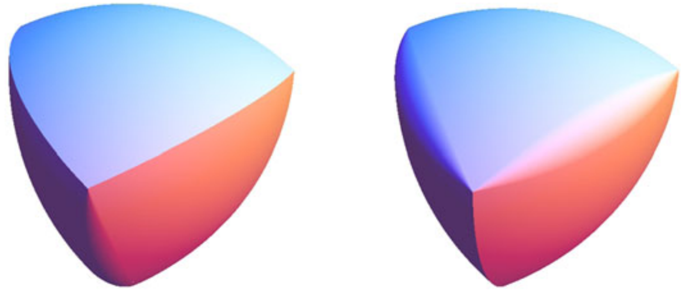

Figure 6. Meissner body M_V with rounded edges meeting in a vertex (left bottom) and Meissner body M_F with rounded edges surrounding a face (right top).

axis. A computer animation showing both bodies M_V and M_F from all sides can be viewed under [38].

Meissner bodies touch two parallel planes between which they are squeezed always in one of two possible ways: either one contact point is located in a vertex and the antipodal contact point is located on a spherical piece of the body, or one contact point is located on a sharp edge and the antipodal contact point is located on a rounded edge of the body.

Their constant width becomes obvious if one intersects a sharp, nonrounded edge opposite the rounded edge with a plane orthogonal to the sharp edge. In this plane the sides of the original tetrahedron form an isosceles triangle similar to the one in Figure 3. The line segment passing from the sharp edge of R_T through the opposite sharp edge of the regular tetrahedron varies in length and is generally shorter than the width d. If its length is extended to d, one arrives at the boundary of the edge that has been rounded off.

Meissner showed the constant width of his bodies using Fourier series [27, p. 47ff.]. Like Hurwitz, he originally studied convex closed curves inscribed in a regular polygon, which remain tangent to all the sides of the polygon during rotations of the curve. Nowadays such curves are called *rotors*. Following Minkowski, Meissner characterized the curves by their support functions (length of the polar tangents). These are periodic and thus can be expanded in Fourier series. Using this technique, he finally succeeded in describing all rotors of regular polygons analytically [26]. With the analogous technique in three dimensions, he was able to determine the rotors of the cube as bodies of constant width. He even proved that non-spherical rotors exist not only for the cube, but also for the regular tetrahedron and octahedron. In contrast, there exist no non-spherical rotors for the regular dodecahedron and icosahedron ([30]; for some mechanical adaptions of Meissner's technique see [9, p. 213ff.]).

Volume and Surface Area of Meissner bodies

In this section we give some numerical results on the volume and surface area of the Meissner body of constant width d. The *volume* V_{M_V} and V_{M_F} of the two Meissner bodies is identical and is given by

$$V_{M_V} = V_{M_F} = \left(\frac{2}{3} - \frac{\sqrt{3}}{4} \cdot \arccos\frac{1}{3}\right) \cdot \pi \cdot d^3 \approx 0.419860 \cdot d^3,$$

see [12, p. 68], [7, p. 71], [35, A137615], [31, p. 40–43]. Therefore we will not distinguish between M_V and M_F. The volume of the Meissner body is approximately 80% of the volume $\pi/6$ of a ball of diameter 1 and it is considerably smaller (by about 6%) than the volume of the rotated Reuleaux triangle R_3, which is given by

$$V_{R_3} = \left(\frac{2}{3} - \frac{\pi}{6}\right) \cdot \pi \cdot d^3 \approx 0.449461 \cdot d^3$$

in [10] and [35, A137617]. As far as we know, the highest lower bound for the volume of a body K of constant width 1 is the one given by Chakerian, et al. in 1966,

$$V_K \geqslant \frac{\pi}{3} \cdot \left(3\sqrt{6} - 7\right) \cdot d^3 \approx 0.364916 \cdot d^3,$$

see [11] and [24].

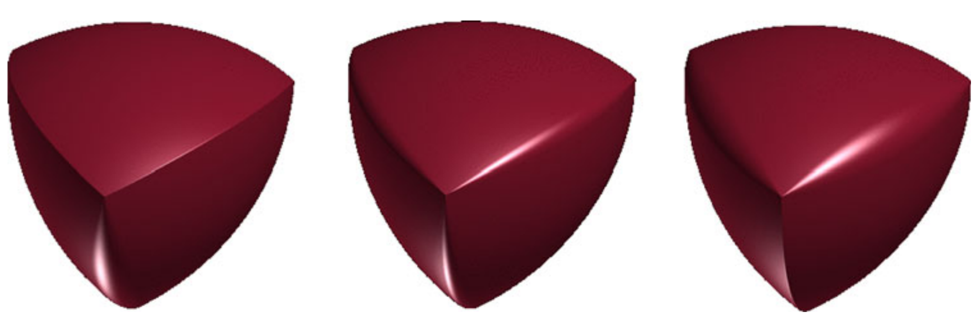

Figure 7. Morphing M_V to M_F including the Minkowski mean (second frame).

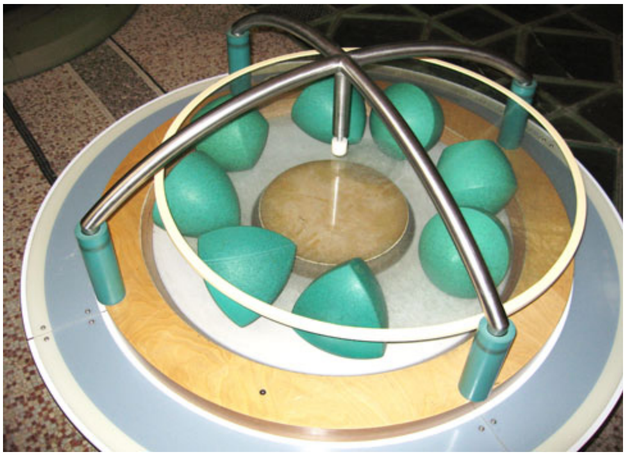

Figure 8. Various Bodies of Constant Width (Palais de la Découverte, Paris).

The *surface area* S_{M_V} and S_{M_F} of the two Meissner bodies is identical, as well, and is given by

$$S_{M_V} = S_{M_F} = \left(2 - \frac{\sqrt{3}}{2} \cdot \arccos\frac{1}{3}\right) \cdot \pi \cdot d^2 \approx 2.934115 \cdot d^2,$$

see [12, p. 68], [7, p. 71], [35, A137616]. This follows from the remarkable fact that in three dimensions the volume V_K and surface area S_K of a convex body K of constant width d are related through *Blaschke's identity* ([5, p. 294], [12, p. 66])

$$V_K = \frac{1}{2} \cdot d \cdot S_K - \frac{\pi}{3} \cdot d^3.$$

Since V_K is monotone increasing in S_K, the question of finding the set that minimizes volume is equivalent to finding the set that minimizes surface area (or generalized perimeter) of K among all convex sets of constant width. Incidentally, this is in sharp contrast to the two-dimensional case, in which, according to a *theorem of Barbier*, all sets of constant width d are isoperimetric, that is they have the same perimeter $\pi \cdot d$ ([3], [8, p. 139]).

The major part of the surface of the Meissner body consists of pieces of a sphere of radius d. The rounded edges (or spindle tori) have an angle of rotation of $\arccos(1/3)$, and their smaller principal curvature is constant and has the value $1/d$. Their part of the surface area is

$$S_{Sp} = 3 \cdot \frac{\arccos(\frac{1}{3})}{2\pi} \cdot 2\pi \cdot d^2 \cdot \int_0^1 \left(\sqrt{\frac{3}{4} + x - x^2} - \frac{\sqrt{3}}{2}\right) \cdot \mathrm{d}x$$
$$\approx 0.334523 \cdot d^2.$$

In other words, the non-spherical pieces of the surface of a Meissner body make up about 11% of the total surface area.

Circumstantial Evidence, but No Proof

Why do we believe that Meissner bodies minimize volume among all three-dimensional convex bodies of constant width? There are more than a million different reasons for it. Clearly the fact that the conjecture has remained unsolved for so long shows that a counterexample is hard to come by. But there is more than this one reason supporting the conjecture.

In 2007 Lachand-Robert and Oudet presented a method for constructing a large variety of bodies of constant width in any dimension; see [24]. For plane domains, this construction boils down to a method of Rademacher and Toeplitz from 1930 [33, p. 175f.]. Their algorithm begins with an arbitrary body K_{n-1} of constant width in $(n-1)$ dimensions and arrives at a body K_n of constant width in n dimensions with K_{n-1} as one of its cross-sections. It was used in 2009 to generate randomly one million different three-dimensional bodies of constant width [31]. None of them had a volume as small as that of a Meissner body.

It should be noted, however, that while the algorithm can generate every two-dimensional set of constant width from a one-dimensional interval, it cannot generate all three-dimensional sets of constant width, but only those that have a plane cross-section with the same constant width. In [14], Danzer describes a set K_3 of constant width d for which each of its cross-sections has a width less than d.

Analysts have recently tried to identify the necessary conditions that a convex body M of minimal volume and given constant width must satisfy. The existence of such a body follows from the direct methods in the calculus of variations and the *Blaschke selection theorem*.

Let us mention in passing that the boundary of M cannot be differentiable of class C^2. If it were, one could consider $M_\varepsilon := \{x \in M | \mathrm{dist}(x, \partial M) > \varepsilon > 0\}$, that is the set M with a sufficiently thin ε-layer peeled off and with a volume less than that of M. According to the Steiner formula, its volume V_{M_ε} can be expressed in terms of the volume V_M of M, the mean width d_{M_ε} of M_ε, and the surface area S_{M_ε} of M_ε as follows:

$$V_{M_\varepsilon} = V_M - \varepsilon \cdot S_{M_\varepsilon} - d_{M_\varepsilon} \cdot \varepsilon^2 - \frac{4\pi}{3} \cdot \varepsilon^3.$$

By construction, and because of our smoothness assumption, M_ε is a body of constant width $d - 2\varepsilon$. Therefore its mean width is $d_{M_\varepsilon} = d - 2\varepsilon$. If one blows M_ε up by a linear factor of $d/(d - 2\varepsilon)$ to a set $\tilde{M}$, its volume is given by

$$V_{\tilde{M}} = \left(\frac{d}{d - 2\varepsilon}\right)^3 \cdot V_{M_\varepsilon},$$

and $\tilde{M}$ is of constant width d again. It will now be shown that $V_{\tilde{M}} < V_M$ can occur. In fact, $V_{\tilde{M}} < V_M$ occurs when $(d - 2\varepsilon)^3 \cdot V_{\tilde{M}} = d^3 \cdot V_{M_\varepsilon} < d^3 \cdot V_M$, or equivalently

$$d^3 \cdot \left(V_M - \varepsilon \cdot S_{M_\varepsilon} - (d - 2\varepsilon) \cdot \varepsilon^2 - \frac{4\pi}{3} \cdot \varepsilon^3\right) < d^3 \cdot V_M.$$

Thus for sufficiently small ε the volume $V_{\tilde{M}}$ stays below the original volume V_M of M, contradicting the minimality of M's volume. That is, no body of class C^2 can minimize volume.

In 2007 a stronger result was shown: Any local volume minimizer cannot be simultaneously smooth in any two antipodal (contact) points [4]. In other words, squeezed between two parallel plates, one of the points of contact

Figure 9. Ernst Meissner, 1883–1939 (ETH-Bibliothek Zürich, Bildarchiv).

with the plane must be a vertex or a sharp-edge point. As already pointed out, Meissner bodies have this property. Because rotated Reuleaux polygons possess this property as well, this result also supports the conjecture without proving it. Finally, in 2009, it was shown by variational arguments that a volume-minimizing body of constant width d has the property that any C^2 part of its surface has its smaller principal curvature constant and equal to $1/d$ [1]. Again, Meissner bodies meet this criterion as well, because they consist of spherical and toroidal pieces with exactly this smaller principal curvature.

After this paper was accepted for publication we learned from Qi Guo in personal communication, that Qi Guo and Hailin Jin had just observed another remarkable property of Meissner bodies. It is well known that the inradius r and circumradius R of a body of constant width add up to d. The ratio R/r of these two radii is a measure of asymmetry for a set, and the observation of Guo and Jin is that it is maximized (among all three-dimensional bodies of constant width) by Meissner bodies. For those bodies $R/r = ((3 + 2\sqrt{6})/5) \approx 1.5798$. In fact, R is maximized, given d, by a Meissner body (see, for example, [18]), and so r is minimized and a fortiori R/r is maximized. It is in this sense that the Meissner bodies are more slender and should have less volume than others of constant width d.

All these results seem to suggest that another century will not pass before the conjecture is confirmed.

Appendix: CV of Ernst Meissner

Who was the man who discovered the body that is presumed to minimize volume? Ernst Meissner was born on 1 September 1883 as the son of a manufacturer in Zofingen, Switzerland. He attended secondary school in Aarau, where he had the same teacher in mathematics as Albert Einstein had had earlier, Heinrich Ganter. Ganter's style of teaching is described as follows. "He was a good mathematician but not, in his own reckoning, good enough to pursue a career in higher mathematics. But he could teach, something that many speculative gentlemen cannot do. [...] Ganter never treated us demeaningly, but taught us as men." Meissner

Figure 10. Meissner 1931 teaching students suffering from tuberculosis in the "sanatorium universitaire" in Leysin operated by Swiss universities (Conservatoire Numérique des Arts et Métiers, Bibliothèque du CNAM).

himself described him "as a teacher who, far from transmitting mere information to prepare a pupil for a career, educated the heart and character and truly civilized his charges. If all teachers were like Ganter, [...] there would be no need for school reform." [15, p. 91]

Meissner's own dedication to teaching is evident from a public lecture that he gave on 18 November 1915 in the town hall of Zurich. The renowned newspaper "Neue Zürcher Zeitung" (NZZ) dedicated half a page to his lecture "Why does mathematics appear difficult and boring to some while not to others?" [29]. For Meissner, grasping a mathematical concept is more than passively understanding its logic. In fact, many people are capable of logical thinking without appreciating mathematics. The deeper understanding of mathematics is rather connected to the creation of one's own mental images and concepts. Meissner promotes the idea that mathematical education should not confront pupils with abstract and fully matured facts. Instead it should enable them to construct and connect mental images in several ways. To a great extent his criticism still applies to contemporary teaching.

After graduating from school, Meissner studied from 1902 to 1906 at the Department of Mathematics and Physics of the Swiss Polytechnic, which was later to become the Swiss Federal Institute of Technology (ETH) in Zurich. He was awarded a doctorate there on the basis of a thesis in number theory. After two semesters at the University of Göttingen, where he studied with Klein, Hilbert, and Minkowski, he returned to the ETH. There he qualified as a professor (Habilitation) in 1909 in mathematics and mechanics. A year later he was offered the chair of technical mechanics, which he held until 1938 (see Figures 9 and 10 [37, p.459]). Ernst Meissner died on 17 March 1939 in Zollikon (near Zurich).

Meissner's scientific achievements were extraordinarily diverse (for a list of his publications see [23, p. 294f.]). In his earlier works, he dealt with questions in pure mathematics (geometry, number theory). Not only his dissertation but also his investigations on sets of constant width fall within this period. During the years between 1910 and 1920 he turned increasingly toward applied mathematics (graphic integration of differential equations, graphic determination of Fourier coefficients), and then to mechanics (geophysics, seismology, theory of oscillations). It is in these applied papers that Meissners true scientific achievements lie because, like Franz Reuleaux, the originator of theoretical kinematics, before him, he always sought his models in pure, strict mathematics.

In an obituary from 1939, Meissner is depicted as a person who not only expected much from himself but also from those around him. "Ernst Meissner demanded the most from himself and others. His intense sense of duty and professional ethics made him seem strict and reserved. However, those who knew him better, his nearest friends and his students, were allowed the unforgettable experience of his extraordinarily comprehensive knowledge and his deep perception, a truly classical appreciation of beauty, touching kindness and finely honed wit." [40].

Note added in Proof

Recently we learned from Chris Sangwin that bodies similar to the ones depicted in Figure 4 can be purchased via the Internet under http://www.grand-illusions.com/acatalog/Solids_of_Constant_Width.html.

REFERENCES

[1] H. Anciaux and N. Georgiou, The Blaschke-Lebesgue problem for constant width bodies of revolution, Preprint, March 2009, 14 pages; http://arxiv.org/abs/0903.4284.

[2] H. Anciaux and B. Guilfoyle, On the three-dimensional Blaschke-Lebesgue problem, Preprint, June 2009, 10 pages; http://arxiv.org/abs/0906.3217.

[3] E. Barbier, Note sur le probèlme de l'aiguille et le jeu du joint couvert, *J. Math. Pures Appl. Ser II*, 5 (1860), 273–286.

[4] T. Bayen, T. Lachand-Robert and É. Oudet, Analytic parametrization of three-dimensional bodies of constant width, *Arch. Ration. Mech. Anal*., 186 (2007), 225–249.

[5] W. Blaschke, Einige Bemerkungen über Kurven und Flächen von konstanter Breite, *Ber. Verh. Sächs. Akad. Leipzig*, 67 (1915), 290–297.

[6] J. Böhm, Convex Bodies of Constant Width, in: *Mathematical Models—Commentary*, Fischer, G., (ed.), Vieweg, Braunschweig, 1986, 49–56.

[7] J. Böhm and E. Quaisser, *Schönheit und Harmonie geometrischer Formen – Sphäroformen und symmetrische Körper*, Akademie Verlag, Berlin, 1991.

[8] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow ID, 1987, 135–149 (transl. from the German: *Theorie der konvexen Körper*, Springer, Berlin, 1934, §15).

[9] J. Bryant and C. Sangwin, *How Round Is Your Circle? Where engineering and mathematics meet*, Princeton University Press, Princeton, NJ, 2008.

[10] S. Campi, A. Colesanti and P. Gronchi, Minimum problems for volumes of constant bodies, in: *Partial Differential Equations and Applications*, Marcellini, P., Talenti, G., and Visintin, E., (eds.), Marcel-Dekker, New York, 1996, 43–55.

[11] G. D. Chakerian, Sets of Constant Width, *Pacific J. Math*., 19 (1966), 13–21.

[12] G. D. Chakerian and H. Groemer, Convex bodies of constant width, in: *Convexity and its Applications*, Gruber, P. M., and Wills, J. M., (eds.), Birkhäuser, Basel, 1983, 49–96.

[13] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved Problems in Geometry*, Springer Verlag, New York, 1991.

[14] L. Danzer, Über die maximale Dicke der ebenen Schnitte eines konvexen Körpers, *Archiv der Mathematik*, 8 (1957), p. 314–316.

[15] K. Datta, The early life of Albert Einstein: Seeking the mature Einstein in his youth, *Resonance*, 10 (2005), 85–96.

[16] G. Fischer, *Mathematical Models—Photograph Volume*, Vieweg, Braunschweig, 1986.

[17] B. Guilfoyle and W. Klingenberg, On C^2-smooth surfaces of constant width, *Tbil. Math. J*., 2 (2009), 1–17.

[18] P. Gritzmann and V. Klee, Inner and outer j-radii of convex bodies in finite-dimensional normed spaces. *Discrete Comput. Geom*., 7 (1992), 255-280.

[19] P. M. Gruber and R. Schneider, Problems in Geometric Convexity, in: *Contributions to Geometry* (Proc. Geom. Sympos. Siegen), Tölke, J., and Wills, J. M., (eds.), Birkhäuser, Basel, 1979, 255–278.

[20] Q. Guo, personal communication, March 2011.

[21] E. Heil and H. Martini, Special convex bodies, in: *Handbook of Convex Geometry*, Gruber, P. M., and Wills, J. M., (eds.), Elsevier, Amsterdam, 1993, 363–368.

[22] D. Hilbert and St. Cohn-Vossen, *Geometry and the Imagination*, AMS Chelsea, Providence, R.I., 1952 (transl. from the German: *Anschauliche Geometrie*, Springer, Berlin, 1932).

[23] L. Kollros, Prof. Dr. Ernst Meissner, *Verh. Schweiz. nat.forsch. Ges*., 1939, 290–296.

[24] T. Lachand-Robert and É. Oudet, Bodies of constant width in arbitrary dimension, *Mathematische Nachrichten*, 280 (2007), 740–750.

[25] F. Malagoli, An Optimal Control Theory Approach to the Blaschke Lebesgue Theorem, *J. Convex Anal.*, 16 (2009), 391–407.

[26] E. Meissner, Über die Anwendung der Fourier-Reihen auf einige Aufgaben der Geometrie und Kinematik, *Vierteljahrsschr. Nat. forsch. Ges. Zür*., 54 (1909), 309–329. (http://www.archive.org/stream/vierteljahrsschr54natu#page/308/mode/2up).

[27] E. Meissner, Über Punktmengen konstanter Breite, *Vierteljahrsschr. Nat.forsch. Ges. Zür*., 56 (1911), 42–50. (http://www.archive.org/stream/vierteljahrsschr56natu#page/n53/mode/2up).

[28] E. Meissner and Fr. Schilling, Drei Gipsmodelle von Flächen konstanter Breite, *Zeitschrift für angewandte Mathematik und Physik*, 60 (1912), 92–94.

[29] E. Meissner, Warum erscheint Mathematik einigen schwer und langweilig und anderen nicht? *Neue Zürcher Zeitung*, 1596 (26.11.1915), 1–2.

[30] E. Meissner, Über die durch reguläre Polyeder nicht stützbaren Körper, *Vierteljahrsschr. Nat.forsch. Ges. Zür.*, 63 (1918), 544–551.

[31] M. Müller, *Konvexe Körper konstanter Breite unter besonderer Berücksichtigung des Meissner-Tetraeders*. Diplomarbeit, Universität zu Köln, 2009.

[32] É. Oudet, *A convex Minkowski-combination of M_F and M_V and its volume* (2010) (http://www.lama.univ-savoie.fr/~oudet/Illustrations/Meissner_files/meissner1to2.gif and http://www.lama.univ-savoie.fr/~oudet/Illustrations/Meissner_files/meissner1to2.png).

[33] H. Rademacher and O. Toeplitz, *The Enjoyment of Mathematics*, Princeton University Press, Princeton, NJ, 1994. (transl. from the second German edition: *Von Zahlen und Figuren*, 1933).

[34] M. Schilling, *Catalog mathematischer Modelle für den höheren mathematischen Unterricht*, Leipzig, 1911. (http://uihistoriesproject.chass.illinois.edu/cgi-bin//rview?REPOSID=8&ID=7970).

[35] N. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A137615–A137618, 2008; http://oeis.org/.

[36] G. Tiercy, Sur le surfaces sphériformes, *Tôhoku Math. J.*, 19 (1921), 149–163.

[37] L. Vauthier, Du Sanatorium universitaire suisse au Sanatorium universitaire international, *Bulletin de la Société d'Encouragement pour l'Industrie Nationale*, 1931, 453–471. (http://cnum.cnam.fr/fSYN/BSPI.145.html).

[38] Chr. Weber, *Gleichdick – Körper konstanter Breite*, ''Film – Erster Meissnerscher Körper'' and ''Film – Zweiter Meissnerscher Körper'', 2007; http://www.swisseduc.ch/mathematik/material/gleichdick/index.html.

[39] I. M. Yaglom and V. G. Boltyansky, *Convex Figures*, Holt, Rinehart and Winston, New York, 1961 (transl. from the German: Jaglom and Boltjanski, *Konvexe Figuren*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956, transl. from the Russian: *Vypuklye Figury*, Gosudarstvennoe izdatel'stvo tekhniko-teoreticheskoi literatury, Moscow and Leningrad, 1951).

[40] H. Ziegler, Ernst Meißner, *Z. angew. Math. Mech.*, 19 (1939), 192.