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Towards a statistical theory of solid dry friction

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Abstract. Wearless dry friction of an elastic block of weight N, driven by an external force F over a rigid substrate, is investigated. The slider and substrate surfaces are both microscopically rough, interacting via a repulsive potential that depends on the local overlap. The model reproduces Amontons's laws which state that the friction force is proportional to the normal loading force N and independent of the nominal surface area. In this model, the dynamic friction force decays for large velocities and approaches a finite static friction for small velocities if the surface profiles are self-affine on small length scales.

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1. Introduction

The physics of solid dry friction is an old and fascinating field. Yet, many quite fundamental problems are still subject of debate. The basic phenomenological facts, though, have since long been known as the Coulomb-Amontons's laws of friction: (i) The frictional force is independent of the size of the surfaces in contact, (ii) friction is proportional to the normal load, and (iii) kinetic friction is not (or not much) dependent on the velocity and typically lower than the static friction force [1].

A simple explanation for these laws arises from Bowden and Tabor's adhesion theory of friction, in which plastic deformation of the surfaces accounts for the load dependence of real contact area and friction force [1, 2]. Furthermore, plastic deformation leads to a logarithmic time dependence of the static friction and logarithmic velocity dependence of the kinetic friction [3, 4]. Although plastic flow is assumed to yield the main contribution to solid friction, other mechanisms may play a role as well. In particular, it was noticed long ago that elastical multistability and hysteresis also gives rise to friction [5].

Recently, steps have been taken towards the understanding of wearless friction as a collective phenomenon, dominated by the competition of pinning forces emerging from rough surfaces and bulk elasticity, neglecting plastic deformations [6, 7]. However, a quantitative understanding is still lacking. Attempts in this direction are inspired by studies of the depinning transition of driven charge density waves [8], interfaces in random media [9, 10] and of vortex lines in type-II superconductors [11], where the behaviour near the depinning threshold force turned out to be a non-equilibrium critical phenomenon described by new universal critical exponents and scaling laws. It is tempting to assume that friction is a related phenomenon. Indeed, in a recent investigation by Cule and Hwa [12], a bead-spring model for friction has been considered which exhibits a depinning transition of the universality class of interface depinning. Bead or block chain models do however not account for Amontons's laws. The aim of the present paper is to study a simple statistical model where friction solely arises from hysteretic elastic response, and to find whether it is nevertheless capable of reproducing these fundamental laws. The situation we consider is the weak pinning limit, where elastic multistability arises as a collective effect. The opposite limit of strong pinning, where multistability already emerges on the local scale of single traps, has been considered recently by Caroli and Nozières [6].

2. The model

To be specific, we consider an elastic body of weight N and linear size L, which is pulled over a rigid substrate, cf. Fig. 1. The two surface profiles, separated by a mean distance d, are parameterized by scalar height functions $l(\mathbf{x})$ and $h(\mathbf{x})$, respectively. $\mathbf{x}$ denotes the 2-dimensional position vector in the reference plane parallel to the surface and the substrate. For simplicity, both surfaces are assumed to have the same statistical properties: They independently obey Gaussian distributions with mean zero, characterized by a short-range pair correlation function

$$\langle h(\mathbf{x})h(\mathbf{x}')\rangle \equiv \bar{h}^2\, k((\mathbf{x}-\mathbf{x}')/\sigma), \tag{1}$$

where $\bar{h}$ defines the width of the substrate surface, σ is the typical lateral corrugation length, $k(\mathbf{0}) = 1$ and $k(\mathbf{x}) \approx 0$ for $|\mathbf{x}| \gg 1$. Correspondingly, $\bar{l}^2 \equiv \langle l^2(0)\rangle$ describes the width

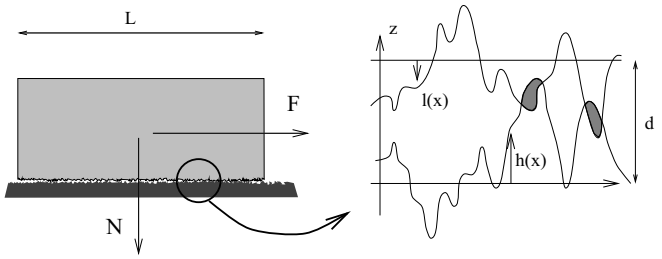

Fig. 1. Cartoon of the model. A block of weight N is pulled over a rigid substrate. The two adjacent surface profiles h and l, separated by a mean distance d, are enlarged. Typical overlap areas are shown

of the slider surface. Unless otherwise stated, we will however assume throughout the paper that both surface profiles obey the same distribution, hence $l = h$.

Short range correlations characterize a macroscopically flat surface. Self-affine surfaces, on the other hand, are characterized, in Fourier space, by a height-height correlator $\langle \tilde{h}_k \tilde{h}_{k'} \rangle \sim \delta(\mathbf{k}+\mathbf{k}')|\mathbf{k}|^{-2\zeta-2}$, where ζ is the roughness exponent. Fracture surfaces, for instance, typically have $\zeta = 0.6 \ldots 0.9$ [13]. The power law behaviour is usually cut off below some wave number $1/\sigma$ [2], so it is reasonable to restrict the study first to short-range correlations.

The lateral elastic properties of the bottom slider surface depend on the shape of the body and are usually quite complicated. In general, the elastic energy can be written as [14]

$$\int d^2x \int d^2x' \gamma_{\alpha\beta\gamma\delta}(\mathbf{x}-\mathbf{x}') \partial_\alpha \mathbf{r}_\beta(\mathbf{x}) \partial'_\gamma \mathbf{r}_\delta(\mathbf{x}'), \tag{2}$$

where $\mathbf{r}(\mathbf{x})$ denotes the local lateral displacement from the equilibrium position. For simplicity, we restrict ourselves to $\gamma_{\alpha\beta\gamma\delta}(\mathbf{x}) = \gamma(\mathbf{x})\delta_{\alpha\gamma}\delta_{\beta\delta}$ and consider only two limiting cases: If the slider is a 2-dimensional object, like a latex membrane pulled over a rod [15], $\gamma(\mathbf{x}) = \gamma\delta(\mathbf{x})$, i.e. the elastic interaction is local. If, on the other hand, the slider is a semi-infinite 3-dimensional object, elastic response is mediated by bulk elasticity and nonlocal in space with $\gamma(\mathbf{x}) \approx \gamma/|\mathbf{x}|$. Both cases can be treated simultaneously by introducing an exponent α, so that the elastic kernel in Fourier space scales like

$$\tilde{\gamma}(\mathbf{k}) \equiv \int d^2x \, \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}} \gamma(\mathbf{x}) \sim k^{\alpha-2}, \tag{3}$$

with $\alpha = 1$ and 2 for bulk and surface elasticity, respectively. Consequently, the dispersion of the elastic energy will behave as k^α. Generalizing (2) to a D-dimensional surface, the real space elastic kernel scales like $\gamma(\mathbf{x}) \sim |\mathbf{x}|^{2-D-\alpha}$.

In the direction perpendicular to the reference plane, the slider surface would strictly also have to be treated as elastic, interacting with the substrate via a hard wall potential (in the absence of adhesion forces). To make the model analytically amenable, however, we allow the surfaces to overlap and introduce a repulsive potential $V(z)$ that depends on the local overlap

$$z(\mathbf{x}) = h(\mathbf{x}+\mathbf{r}(\mathbf{x},t)) + l(\mathbf{x}) - d. \tag{4}$$

This potential is used to mimic vertical elasticity. We choose

$$V(z) = V_0 \, z^n \, \Theta(z), \tag{5}$$

where $\Theta(z)$ is the Heaviside step function, and $n > 1$. With $n = 3/2$, the Hertzian result on the distance dependence of the repulsive force between two elastic spheres [14] is reproduced (this nontrivial dependence being the result of the interplay between Hooke's law and the spherical geometry). As it turns out, the results do not depend very sensitively on the chosen value of n. Note that setting $V(z) \equiv 0$ for $z < 0$ via the Heaviside function is natural and justified, in the absence of adhesion.

Finally, the total driving force $\mathbf{F}$ is applied homogeneously as a force density $\mathbf{f} = \mathbf{F}/L^2$. This choice appears to be natural in the case of bulk elasticity, whilst for a membrane-like slider the external force should rather be exerted at one border.

Assume that, with the slider lying at rest on the substrate, we turn on the external driving force $\mathbf{F}$. If F is large enough, the slider will start to move, and the interaction between the slider and the substrate will generate a dynamic friction force $F_{fr} \equiv -F$ (as we will show below) which leads, after a transient time of acceleration, to a constant average velocity $\mathbf{v}$ of the center of mass of the slider. In the steady state, the equation of motion of a point $\mathbf{r}(\mathbf{x},t)$ of the slider surface can be written as

$$\eta\,(\dot{\mathbf{r}}(\mathbf{x},t) - \mathbf{v}) = \int d^2x' \gamma(\mathbf{x}-\mathbf{x}') \nabla'^2 \mathbf{r}(\mathbf{x}',t) + \mathbf{f} - \frac{\partial}{\partial \mathbf{r}(\mathbf{x},t)} V[h(\mathbf{x}+\mathbf{r}(\mathbf{x},t)) + l(\mathbf{x}) - d]. \tag{6}$$

The term on the left hand side accounts for surface phononic damping *within* the slider, with a damping coefficient η. Since (6) is written in the laboratory frame, the center of mass velocity $\mathbf{v}$ has to be subtracted from the local velocity $\dot{\mathbf{r}}$. In this way we make sure that rigid sliding, i.e. $\mathbf{r}(\mathbf{x}) \equiv \mathbf{v}t$ with a constant velocity $\mathbf{v}$, is frictionless. The neglect of an inertial term is justified by our primary interest in the depinning region, where kinetic energy is relatively small. If we started, on the other hand, with a theory including an inertial term only, viscous friction would nevertheless be generated by the nonlinear random force after eliminating short wavelength displacement modes.

The mean separation d between slider and substrate obeys a similar equation of motion

$$\Lambda \frac{\partial}{\partial t} d(t) = -N - \int d^2x \, \frac{\partial}{\partial d} V[h(\mathbf{x}+\mathbf{r}(\mathbf{x},t)) + l(\mathbf{x}) - d], \tag{7}$$

which has to be solved simultaneously with (6). Again, overdamped motion is assumed with another friction constant. In the analysis we will take $\Lambda \to \infty$, because in the thermodynamic limit of infinitely large system size L, fluctuations of the mean distance d will vanish, so that it can be treated as a constant parameter that has to be determined self-consistently.

Simple scaling analysis shows that the model is dominated by two dimensionless quantities:

$$\mathcal{E} \equiv V_0 \bar{h}^n / \gamma \sigma^{2-\alpha} \qquad \text{and} \qquad \mathcal{N} \equiv N / V_0 \, \bar{h}^{n-1} \sigma^2, \tag{8}$$

where $\mathcal{E}$ is the ratio between typical overlap and elastic forces on short scales. In this paper, we will restrict ourselves to $\mathcal{E} \ll 1$, corresponding to weak disorder or weak pinning, and $1 \ll \mathcal{N} \ll (L/\sigma)^2$. Under these conditions, $\mathcal{N}$ will turn out to be of the order of the number of contact

points between the two surfaces ($V_0\,\bar{h}^{n-1}\sigma^2$ is the typical overlap force at a single contact), hence the latter condition ensures that the real contact area is smaller than the nominal area L^2 and bigger than the typical size σ^2 of one point of contact. Note that $\mathcal{N} \ll (L/\sigma)^2$ can be rewritten as $p \ll p_0$, where $p = N/L^2$ is the nominal pressure and $p_0 = V_0 h^{n-1}$ a typical pressure at contact points.

3. Static properties

Let us first consider the limit of vanishing surface elasticity $\mathcal{E} \to 0$ (which implies that there is no friction force and hence no pinning at all) and zero external force $\mathbf{F}$. In this limit $\mathbf{r}(\mathbf{x}, t) \equiv 0$, and in (7) the height profiles h and l can be averaged over. We find

$$\mathcal{N} = \frac{1}{V_0 \bar{h}^{n-1}\sigma^2} \int d^2x\, \langle V'[h(\mathbf{x}) + l(\mathbf{x}) - d]\rangle \tag{9}$$

$$= c_n \left(\frac{L}{\sigma}\right)^2 (2\bar{h}/d)^n \mathrm{e}^{-d^2/4\bar{h}^2} \left(1 + \mathcal{O}\left((\bar{h}/d)^2\right)\right), \tag{10}$$

where $c_n \equiv \Gamma(n+1)/\sqrt{4\pi}$. This implicitly determines d as a function of $\mathcal{N}$ and the surface roughness: $d \approx 2\bar{h}\ln^{1/2}(c_n L^2/\mathcal{N}\sigma^2)$. The real contact area

$$A_r \equiv L^2 \langle \Theta(h(\mathbf{x}) + l(\mathbf{x}) - d)\rangle \tag{11}$$

can be calculated in the same limit. Note that, while lateral elasticity is suppressed in the limit $\mathcal{E} \to 0$, the "soft" interaction potential $V(z)$ that mimics vertical elasticity still allows for a finite contact area. Making use of (9), it is given by

$$A_r \approx \frac{\mathcal{N}\sigma^2}{\sqrt{4\pi}\, c_n} \ln^{\frac{4-n}{2n}} \left[c_n L^2/\mathcal{N}\sigma^2\right]. \tag{12}$$

Apart from logarithmic corrections, A_r is thus proportional to the normal load, independent of the total surface area[1]. The dependence on the potential parameter n is weak, influencing only the strength of the logarithmic correction.

The proportionality between load and contact area is a generic feature in our model, as long as the height probability distribution $P(h)$ decreases at least exponentially for heights $|h| \gg \bar{h}$, where $\bar{h}$ is the surface profile width. The only restriction imposed on the choice of the potential $V(z)$ is that it is cut off for $z < 0$, which should be true for any effective, adhesionless interaction, and that it increases slower than exponentially, or, in the case of a Gaussian height probability distribution, at most exponentially; the exact form for $z > 0$ does not matter. Under these assumptions, the expectation value (considered to be a function of d) satisfies

$$\langle V(h-d)\rangle \equiv \int_d^\infty V(h-d)P(h)dh \;\sim\; P(d) \tag{13}$$

to leading order. With the definition in (11) and $N/L^2 = \langle V'(h+l-d)\rangle$, the proportionality between A_r and N is a consequence of this result.

[1] In [16], the proportionality between load and contact area was shown for a purely elastic surface with Gaussian distributed profile. In the model considered there, surface asperities were assumed to be spherically shaped and to obey the Hertzian distance-force relation.

We next assume that the elasticity constant $\mathcal{E}$ is finite and set up a perturbation expansion in the local displacement field $\mathbf{r}(\mathbf{x}, t)$ about an undistorted configuration, first with $F = 0$. For our further calculation it is useful to generalize the model to a D-dimensional slider surface, redefining $\mathcal{N}$ correspondingly to $\mathcal{N} = N/V_0\bar{h}^{n-1}\sigma^D$. The definition of $\mathcal{E}$ is unaffected. The set of equations of motion (6) and (7) can then be solved by iteration, order by order in the strength of the interaction potential V_0, using a diagrammatic technique introduced in [17]. At each step, the random profiles h and l are averaged over. In the following analysis, the mean distance is assumed to be determined self-consistently by (7) in the limit $\Lambda \to \infty$, i.e. it is treated as a constant.

In order to set up the perturbation theory, we transform the equation of motion (6) to Fourier space:

$$G_0^{-1}(\mathbf{k},\omega)\,\tilde{\mathbf{r}}_{\mathbf{k},\omega} = (2\pi)^{D+1}\delta^D(\mathbf{k})\delta(\omega)\mathbf{f} \tag{14}$$
$$- \int d^Dx\, dt\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x} - \mathrm{i}\omega t} \frac{\partial}{\partial \mathbf{r}(\mathbf{x},t)} V[h(\mathbf{x} + \mathbf{r}(\mathbf{x},t)) + l(\mathbf{x}) - d],$$

where the bare propagator is given by

$$G_0(\mathbf{k},\omega) = (\gamma|\mathbf{k}|^\alpha + \mathrm{i}\eta\omega)^{-1}. \tag{15}$$

Expanding the last term in (14) in powers of $\mathbf{r}(\mathbf{x}, t)$, one gets for the displacement field to first order in V_0

$$\tilde{\mathbf{r}}_{\mathbf{k},\omega} = -G_0(\mathbf{k},\omega) \int d^Dx\, dt\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x} - \mathrm{i}\omega t} \times \tag{16}$$
$$\nabla h(\mathbf{x})\, V'[h(\mathbf{x}) + l(\mathbf{x}) - d] + \mathcal{O}(V_0^2).$$

For the displacement correlation function in lowest non-vanishing order follows

$$\langle \tilde{\mathbf{r}}_{\mathbf{k},\omega}\tilde{\mathbf{r}}_{\mathbf{k}',\omega'}\rangle = -(2\pi)^{D+2}\delta(\omega)\,\delta(\omega')\delta^D(\mathbf{k}+\mathbf{k}') \times \tag{17}$$
$$|G_0(\mathbf{k},\omega)|^2 \int d^Dx\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}} \times$$
$$\nabla^2_{\mathbf{r}}\big|_{\mathbf{r}=0} \underbrace{\langle V[h(\mathbf{x}+\mathbf{r}) + l(\mathbf{x}) - d]V[h(0) + l(0) - d]\rangle}_{\equiv\, C_2(\mathbf{x}+\mathbf{r},\mathbf{x};d)}.$$

Since the surface height profiles are Gaussian distributed, the pair correlator C_2 will be a function of the first (which we have chosen to be zero) and second moments of the distribution only. More specifically, only the sum of the second moments enters the definition of C_2, so we can define

$$\bar{C}_2(K_h(\mathbf{x}_1) + K_l(\mathbf{x}_2); d) \equiv C_2(\mathbf{x}_1, \mathbf{x}_2; d), \tag{18}$$

with $K_h(\mathbf{x}) = \langle h(\mathbf{x})h(0)\rangle$ and $K_l(\mathbf{x}) = \langle l(\mathbf{x})l(0)\rangle$, where the average is taken over the respective height distributions. The exact form of $\bar{C}_2(K, d)$ is analyzed in appendix A. The main result of this analysis is that the value of $\bar{C}_2$ is proportional to the normal load N if $|\mathbf{x}_1|, |\mathbf{x}_2| \lesssim \sigma$. For the real space displacement correlation function follows (with $\sigma \ll |\mathbf{x}| \ll L$)

$$\frac{1}{2}\langle\, [\mathbf{r}(\mathbf{x}) - \mathbf{r}(0)]^2\,\rangle = \int_{\mathbf{k}} (1 - \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}) \langle \tilde{\mathbf{r}}_{\mathbf{k},0}\tilde{\mathbf{r}}_{-\mathbf{k},0}\rangle \tag{19}$$
$$= \int_{\mathbf{k},\mathbf{p}} \frac{1 - \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}}{\gamma^2|\mathbf{k}|^{2\alpha}} \int d^Dx_1 d^Dx_2 \mathbf{p}^2 \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}_2 + \mathrm{i}\mathbf{p}(\mathbf{x}_2 - \mathbf{x}_1)} C_2(\mathbf{x}_1, \mathbf{x}_2; d)$$
$$\sim \sigma^2\, \mathcal{N}\mathcal{E}^2\, (\sigma/L)^D \times \begin{cases} \frac{1}{D_c - D}\left(\frac{|\mathbf{x}|}{\sigma}\right)^{D_c - D} & \text{for } D < D_c \\ \ln(|\mathbf{x}|/\sigma) & \text{for } D = D_c, \end{cases}$$

where $\int_{\mathbf{k}}$ is short hand for $\int \frac{d^D k}{(2\pi)^D}$. In the last equation, we have made use of (49) from App. A, with $\kappa \approx 1$. The (upper) critical dimension is given by

$$D_c = 2\alpha, \tag{20}$$

implying that the physically relevant situation $D = 2$ and $\alpha = 1$ corresponds to the marginal case. Ignoring for a moment the dependence on length scales, the result is proportional to $\mathcal{N}\mathcal{E}^2 \sim NV_0/\gamma^2$; the linear dependence on N stems from replacing $V_0 \mathrm{e}^{-d^2/4\bar{h}^2}$ (being the leading dependence on the mean distance d) by N/L^D via (9).

Like in other cases of collective pinning, one can now determine a Larkin length L_L from the condition $\langle\, [\mathbf{r}(L_L) - \mathbf{r}(0)]^2 \,\rangle \approx \sigma^2$ [18], i.e. L_L is the length scale where typical displacements become of the order of the corrugation length of the substrate. This yields

$$\begin{aligned} L_L &\approx \sigma (L^D \epsilon / \sigma^D \mathcal{N}\mathcal{E}^2)^{1/\epsilon} \\ &= \sigma [\gamma^2 \sigma^{2(2-\alpha)} / V_0 \bar{h}^{n+1} p\,]^{1/\epsilon}, \end{aligned} \tag{21}$$

where $\epsilon = D_c - D$. Note that L_L depends, apart from a combination of intrinsic model parameters, only on the nominal pressure $p = N/L^D$.

What are the consequences for the static friction force, which in our model is identified with the critical force at the depinning transition [10]? L_L will be smaller than the system size only if we choose our model parameters such that $\mathcal{N} \gg \mathcal{E}^{-2}(L/\sigma)^{D-\epsilon}$. In this case, it is possible to estimate the static friction force in the standard way [19, 20, 21]: On small length scales, the elastic energy is dominant compared to the interaction with the random potential, so that adjacent sites move coherently. On larger length scales, the random forces become more relevant until on the scale of the Larkin length L_L, elastic and pinning forces are of the same order of magnitude, so the elastic manifold becomes able to explore the inhomogeneous force field emerging from the randomly distributed contact points on scales larger than L_L. Regions of linear size L_L can hence be assumed to adapt independently to the disorder, each of them giving an independent contribution to the pinning force of the order of the elastic force on this scale. The force *density* in a Larkin region is thus given by $f_{fr}(L_L) \approx \gamma\sigma L_L^{-\alpha}$, which is the product of a typical gradient σ/L_L^2, a typical value of the elastic kernel $\gamma L_L^{-D+2-\alpha}$, and the area L_L^D of the region. The total friction force is thus of the order

$$F_{fr} \approx (L/L_L)^D \, L_L^D f_{fr}(L_L) \approx \gamma L^D \sigma L_L^{-\alpha}, \tag{22}$$

which is proportional to the nominal area L^D and hence violates Amontons's first law.

With $\mathcal{N}(\sigma/L)^D \ll 1$ and $\mathcal{E} \ll 1$, however, L_L will typically be much larger than L. The fact that in solid dry friction the Larkin length is typically $\gg L$, has been recently remarked also by Persson and Tossati [7] and by Caroli and Nozières [6]. In the following, we will restrict ourselves to this situation. In the physical situation $D = 2$ with bulk elasticity $\alpha = 1$, being just at the marginal dimension D_c (i.e. $\epsilon = 0$), L_L is *exponentially* large [19, 7]:

$$L_L \sim \sigma \exp\left(c \, \frac{L^D}{\sigma^D \mathcal{N}\mathcal{E}^2} \right), \tag{23}$$

where c is a dimensionless constant.

The elastic response of the surface also allows for a reduction of the potential energy of the slider by decreasing the mean distance d. Technically, this appears in the form of a correction factor to the r.h.s. of (9). It is calculated by expanding $V'[h(\mathbf{x}+\mathbf{r}(\mathbf{x})) + l(\mathbf{x}) - d]$ in $\mathbf{r}(\mathbf{x})$ and substituting for $\mathbf{r}(\mathbf{x})$ the first order expression (16), yielding

$$\begin{aligned} &\langle V'[h(\mathbf{x}+\mathbf{r}(\mathbf{x})) + l(\mathbf{x}) - d]\rangle = \langle V'[h(0) + l(0) - d]\rangle \\ &+ \int_{\mathbf{k}} G_0(\mathbf{k}, 0) \int d^D x' \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}'} \times \\ &\quad \nabla_{\mathbf{r}}^2\big|_{\mathbf{r}=0} \langle V[h(\mathbf{x}'+\mathbf{r}) + l(\mathbf{x}') - d] V'[h(0) + l(0) - d]\rangle \\ &+ \mathcal{O}(V_0 \bar{h}^{n-1} \mathcal{E}^2). \end{aligned} \tag{24}$$

In this order of perturbation theory, the correction can be represented by a factor $(1 - c\,\mathcal{E}(\bar{h}/d)^n)$ with another dimensionless constant c. Since this correction factor is smaller than 1, the self-consistent determination of the mean distance d via (9) will in turn lead to a slightly decreased value of $d \to d - \Delta d$, so that the enlarged value of the leading factor

$$\sim \mathrm{e}^{-(d-\Delta d)^2/4\bar{h}^2} \tag{25}$$

just compensates the reduction. A similar correction in (12), together with the modified d, leads to a modified contact area $A_r \to A_r + \delta A_r$.

In order to determine the sign of the relative correction $\delta A_r/A_r$, one has to compare the magnitude of the relative corrections to $\langle V'(h+l-d)\rangle$ and to A_r, as defined in (11), respectively. Using the result of (52) in App. A, one finds that the relative correction to A_r is smaller by an approximate factor of $\Gamma(n)\Gamma(n+1)/\Gamma(2n)$ (which is smaller than 1 for $n > 1$). Consequently, the increase in A_r due to $-\Delta d < 0$ in the factor (25) will dominate, so the ratio $\delta A_r/A_r$ (which is of order $\mathcal{E}$) is positive.

If there is a friction force at all, it should depend on δA and vanish for $\delta A = 0$, because a completely rigid surface is never pinned. We can hence give a dimensional argument for a characteristic friction force $\tilde{F}_{fr}$. The simplest way to estimate $\tilde{F}_{fr}$ is to write it as the product of the excess contact area $\delta A_r \approx \sigma^D \mathcal{N}\mathcal{E}$ and a typical lateral force density $V_0\bar{h}^n/\sigma = \gamma\sigma^{1-\alpha}\mathcal{E}$

$$\tilde{F}_{fr} \;\approx\; \gamma \sigma^{D+1-\alpha} \mathcal{N}\mathcal{E}^2. \tag{26}$$

Here, a logarithmic correction of the order $\ln(L^D/\mathcal{N}\sigma^D)$ has been omitted. Defining as usual the friction coefficient $\mu \equiv F_{fr}/N$, this expression corresponds to a value of μ of order $\mathcal{E}h/\sigma$ which depends on the ratio between elastic and repulsive forces on small length scales but not on the load. Below, we will show that this estimate gives indeed the right order of magnitude of F_{fr}.

4. Kinetic and static friction

Next, we consider the case of a moving slider, $v \equiv |\mathbf{v}| \neq 0$, driven by a finite force $\mathbf{F}$. The perturbative expansion is now set up in

$$\mathbf{u}(\mathbf{x}, t) \equiv \mathbf{r}(\mathbf{x}, t) - \mathbf{v}t, \tag{27}$$

relative to steady, rigid sliding. For further simplification, we allow only for displacements in the direction $\mathbf{e}_\parallel \equiv \mathbf{F}/|\mathbf{F}|$, corresponding to a further restriction $\gamma_{\alpha\beta\gamma\delta}(\mathbf{x}) = \gamma(\mathbf{x})\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\alpha 1}$ in (2), where 1 denotes the direction $\mathbf{e}_\parallel$. Hence, $\mathbf{u}(\mathbf{x},t) = u(\mathbf{x},t)\,\mathbf{e}_\parallel$. It has been argued in a closely related context (the depinning of a driven flux line in a random medium) that this restriction will not alter the critical dynamics in the average direction of motion [22]. This approximation will however overestimate the force needed to overcome a repulsive trap, because asperities cannot avoid each other by simply bending away. Recently it has been shown explicitly that this avoiding process completely rules out *local* multistability in the case of isolated isotropic traps, while multistability is re-established for anisotropic traps [23]. Full account of D-dimensional lateral elasticity has been taken in a simpler version of our model, where the random potential energy is proportional to the product of two charge densities on the two elastic manifolds, in [24]; there, however, the case of $L_L \ll L$ was considered.

To find the average dynamic friction force to lowest order in perturbation theory, we follow a procedure used by Feigel'man for driven interfaces [20]. He calculated the first correction to the mobility constant η, treating the average velocity v as a parameter which has to be determined self-consistently. Technically, we proceed as follows: First, replace $\mathbf{r}(\mathbf{x},t)$ by $\mathbf{v}t + \mathbf{u}(\mathbf{x},t)$ in (14), and expand the random potential part in this equation in powers of $\mathbf{u}(\mathbf{x},t)$. Now, insert the Fourier transform of (16), having replaced the l.h.s. of this equation by $\tilde{\mathbf{u}}_{\mathbf{k},\omega}$ and the argument of h on the r.h.s. by $\mathbf{x}+\mathbf{v}t$, into the next order expansion term in (14),

$$\int d^D\!x\, dt\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}-\mathrm{i}\omega t} u(\mathbf{x},t) \frac{\partial^2}{v^2\partial t^2} V[h(\mathbf{x}+\mathbf{v}t) + l(\mathbf{x}) - d], \quad (28)$$

and perform the average over h and l. One finds to second order in V_0

$$f = \frac{V_0^2 \overline{h}^{4n+1}}{d^{2n+1}} \frac{2^n \Gamma(2n+1)}{\sqrt{\pi}} \int_{\mathbf{p},\Omega} G_0(\mathbf{p},\Omega) \times \quad (29)$$

$$\int d^D\!x\, dt\, \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}+\mathrm{i}\Omega t} \frac{1}{v^3}\frac{\partial^3}{\partial t^3}\left[f_2\left(\frac{K_h(\mathbf{x}+\mathbf{v}t)+K_l(\mathbf{x})}{2\overline{h}^2}\right) \exp\left(-\frac{d^2}{2\overline{h}^2 + K_h(\mathbf{x}+\mathbf{v}t)+K_l(\mathbf{x})}\right)\right].$$

Here, the representation of the random potential correlation function obtained in App. A, with the dimensionless function $f_2(\kappa)$ as defined in (52), has been used.

The r.h.s. of (29), multiplied with the full area L_D of the slider, is identified with the total friction force $F_{fr}(v) = L^D f_{fr}(v)$. The resulting expression for $F_{fr}(v)$ can be written in the form

$$F_{fr}(v) = \quad (30)$$
$$L^D \mathcal{E}^2 \gamma \sigma^{1-\alpha} \frac{v}{v_0} \int_{\boldsymbol{\xi},\boldsymbol{\chi}} \frac{\boldsymbol{\xi}_1^4}{|\boldsymbol{\chi}|^{2\alpha} + (\frac{v}{v_0})^2 \boldsymbol{\xi}_1^2} \frac{\tilde{c}_2(\sigma\boldsymbol{\xi}, \sigma(\boldsymbol{\chi}-\boldsymbol{\xi}))}{\sigma^{2D}},$$

where $\tilde{c}_2(\mathbf{p},\mathbf{p}')$, defined in (56) in App. A, is the Fourier transform of the random potential correlator defined in (17). $\boldsymbol{\chi} = \sigma\mathbf{p}$ and $\boldsymbol{\xi} = \sigma\mathbf{q}$ are dimensionless integration variables, and we introduced the velocity scale

$$v_0 \equiv \gamma\sigma^{1-\alpha}/\eta, \quad (31)$$

which is a typical relaxation velocity.

Written in this way, the integral on the r.h.s. of (30) is just a number, the velocity dependence entering only via the ratio v/v_0. Hence, we can write the friction force in the form

$$F_{fr}(v) \approx \tilde{F}_{fr}\ \phi\left(\frac{v}{v_0}\right), \quad (32)$$

where $\phi(x)$ is a dimensionless function that depends on the explicit form of $V(z)$ and the statistics of the surface profiles. A detailed discussion of the velocity dependence in $\phi(x)$ is given in App. B. Here, we will only summarize the results.

At $v \approx v_0$, to begin with, $F_{fr}(v)$ is indeed found to be of the order $\tilde{F}_{fr}$, which followed from our naïve estimate (26). The amplitude $\tilde{F}_{fr}$, implying in particular the proportionality between $F_{fr}(v)$ and N, follows for similar reasons as those that led to (20), making use of the random potential correlator calculated in App. A. Thus, our dimensional argument (26) is justified *a posteriori*.

In the high velocity regime $v \gg v_0$, the friction force decays proportional to $1/v$, independent of D and α. Note that as we have not included inertial terms in the equations of motion (6) and (7), the experimentally observed velocity strengthening behaviour of the friction force for high velocities cannot be reproduced by our model.

In the most interesting regime $v \ll v_0$, the behaviour crucially depends on the characterics of the surface profile correlator $k(x)$ and on the dimensionality. If the height-height correlator $k(x)$ is analytic in the origin, the friction force reaches its maximum (of order $\tilde{F}_{fr}$) at $v \approx v_0$, and it decreases for $v \ll v_0$ as $(v/v_0)^{1-\epsilon/\alpha}$. This is the usual contribution to friction from phononic damping, vanishing for $v \to 0$ if $\epsilon/\alpha < 1$. In the marginal dimension $D = 2$, with bulk elasticity, $F_{fr}(v)$ thus depends linearly on v (times a logarithm of v/v_0, cf. (63) in App. B) in the small velocity regime.

In order to find a finite static friction force, however, $F_{fr}(v)$ has to bend towards a non-zero value for $v \to 0$. In related problems like the depinning of driven interfaces and charge density waves, it has been shown that a finite pinning threshold appears due to contributions to $\phi(x)$ on length scales larger than the Larkin length [10]. On these length scales, configurational multistability emerges as a collective effect, leading to collective pinning. This is reflected, in a renormalization group treatment, by the renormalized random force correlator developing a cusp-like singularity at the origin. Since in our case L_L is typically larger than the system size, the collective pinning mechanism is absent. We thus have to find criteria for the existence of local multistability.

As mentioned above, real world surfaces often have self-affine properties spanning several orders of magnitude. A (one dimensional) surface profile, for instance, that is short range correlated on the scale σ, and self-affine with a roughness exponent $\zeta = 1/2$ below this scale, is described, in real space, by the correlator

$$K_h(x-x') = \langle h(x)h(x')\rangle = \overline{h}^2 \mathrm{e}^{-|x-x'|/\sigma}. \quad (33)$$

If the self-affinity covers the length scales from σ down to a microscopic length scale a, $\langle h(x)h(x')\rangle$ will exhibit the cusp

only when considering it on a coarser scale than a; on finer scales, it is analytic again. In Fourier space, the height-height correlator corresponding to such a profile is given by

$$K_h(q) = \frac{2\,\mathrm{e}^{-(qa)^2}}{1+(\sigma q)^2}. \tag{34}$$

A surface with this correlation function can be generated by an Ornstein-Uhlenbeck process [25]: Given a stochastic_spatial noise $\zeta(x)$ with $\langle\zeta(x)\rangle = 0$ and $\langle\zeta(x)\zeta(x')\rangle = (2h^2/\sigma)\delta_a(x-x')$, where δ_a produces short range correlations over a length scale a, the profile $h(x)$ obeying the differential equation

$$\frac{dh}{dx} = -\frac{1}{\sigma}\,h(x) + \zeta(x) \tag{35}$$

has the desired statistical properties.

For such surface profiles, we find that $\phi(v/v_0)$ takes a finite value of order 1 for $v_0(a/\sigma)^{1/\alpha} \ll v \ll v_0$. In this case, the total friction force is indeed of the estimated order $\tilde{F}_{fr}$ and almost constant for $v < v_0$, and decays $\sim 1/v$ for $v \gg v_0$. More generally, a finite static friction force is found if the (unrenormalized) surface profile correlator $k(\mathbf{x})$ has a cusp in the origin (precisely, we need $\lim_{x_\parallel\to 0_+}\partial_{x_\parallel}k(\mathbf{x}) \neq 0$, where $x_\parallel$ is the component of $\mathbf{x}$ parallel to $\mathbf{f}$), which corresponds to the first spatial derivative of $k(x)$ undergoing a jump (of order 1) at $x = 0$. A surface characterized by such a correlator has local slopes that may take arbitrarily high values, eventually leading to multistability even for arbitrarily small roughness $\bar{h}$.

Note that for $a \ll \sigma$, the lower velocity scale $v_0(a/\sigma)^{1/\alpha}$ will typically be so small that it has no significance for a finite sample, since velocity fluctuations will become so large that the slider gets pinned. Consequently, for L finite and hence in any experimental situation the regime $v \ll v_0(a/\sigma)^{1/\alpha}$ where $\phi(v/v_0) \ll 1$ is unlikely to be observable.

5. Strong pinning

Our results have been obtained using the framework of perturbation theory about weak disorder, which is sufficient in the weak pinning limit and which is the natural starting point for a renormalization group analysis that reveals configurational multistability on length scales larger than L_L. In principle however, it is possible to have multistability already locally on the scale of a single trap. This situation corresponds to the strong pinning limit [26], which cannot be treated successfully within finite order perturbation theory.

Such a situation was recently considered by Caroli and Nozières (CN) [6]. They consider two flat surfaces with a sparse distribution of bumps and sinks, where 'active' traps are formed when two adjacent asperities are in contact. Writing the interaction at an active trap as a potential energy $V(\rho)$ that depends on the distance ρ between their centers, they derive a criterion for the existence of local (single-site) multistability which reads

$$\max|\partial_\rho^2 V(\rho)| > E\sigma. \tag{36}$$

E is Young's modulus – which, in $D = 2$ and with $\alpha = 1$, is our γ – and σ a typical length scale of the trap. Similar considerations have been applied for instance in mean-field like descriptions of driven interfaces or charge density waves [27]. The frictional force is then proportional to the typical energy gap at a spinodal jump, multiplied with the density of active traps.

To make contact with the CN model, we can identify the overlapping asperities in our model (which are relatively few in the limit of small normal load) with these active traps. We aim to derive an estimate for the onset of local multistability in the case of short range correlated surface profiles, with a typical corrugation length σ, and a potential $V(z) = V_0 z^2\theta(z)$, in the physical situation $D = 2$ with bulk elasticity ($\alpha = 1$). A typical energy that can be stored in one active trap can roughly be estimated as $V_0\bar{h}^2\sigma^2$, and the second derivative with respect to a lateral displacement will be of order $V_0\bar{h}^2$. Comparing this with the elastic energy term $\gamma\sigma$, we find as condition for the presence of local multistability that

$$\mathcal{E} \geq \mathcal{O}(1), \tag{37}$$

a condition which violates our assumptions.

This restriction can be relaxed to some extent if one looks for the occurance of the *first* appearance of a multistable site in a finite sample when tuning, for example, the potential amplitude V_0. The largest asperity in a sample of linear size L will have a height $h_{\max}$ of order $2\bar{h}\ln^{1/2}((L/\sigma)^2)$, and with the expression for the mean distance d given after (9) follows $h_{\max} - d \approx \frac{2\bar{h}^2}{d}\ln\mathcal{N}$. Using further that the second derivative of a Gaussian correlated surface profile will typically take a value of order $h_{\max}/\sigma^2$ at this maximum, we arrive at the criterion

$$\mathcal{E}\ln\mathcal{N} \geq \mathcal{O}(1) \tag{38}$$

where $1 \ll \mathcal{N} \ll (L/\sigma)^2$ has been used.

6. Conclusion

Expressions (26) and (32) are the main results of this paper. They describe a friction force which depends linearly on the weight N (up to logarithmic corrections), but not on the nominal contact area L^D and hence fulfills Amontons's laws. The dependence on the potential parameter n is weak. It is noteworthy that this result was obtained in the limit $A_r \ll L^D$, where statistics are dominated by *rare* events.

Trying other forms for the overlap interaction potential $V(z)$, it turned out that the crucial ingredient leading to $F_{fr} \sim N$ is the cutoff below $z = 0$. If one abandons this cutoff and uses, e.g., an exponential potential $V(z) = V_0\,\mathrm{e}^{z/z_0}$, the proportionality is no longer valid; for the exponential potential, for example, one finds instead to leading order that $F_{fr} \sim N^2$.

To summarize, we have introduced a new stochastic model that incorporates the interplay between bulk elasticity and surface roughness in solid dry friction. To our knowledge, it is the first purely elastic model that treats solid dry friction as a collective phenomenon *and* reproduces the correct load dependence of the friction force F_{fr}, known as the Coulomb-Amontons's laws. For high velocities v, F_{fr} decays like $1/v$, while the behaviour for small v depends on

the surface profile statistics: For a smooth surface, the static friction force vanishes in the weak pinning case, while it is finite if the surface is characterized by a non-analytical height-height correlator. We have also given an estimate for the appearence of local multistability in the case of smooth interfaces.

We have not considered thermal effects in our study, because in most situations where temperature changes do not strongly affect the mechanical strengths of the sliding bodies, the friction coefficient is found to be basically insensitive to temperature variations [28]. An example where friction does however strongly depend on the temperature is the case of rubber sliding over hard surfaces, where the friction properties are intimately connected with the temperature dependent visco-elastic properties of the rubber material [29]. In that specific case, the friction coefficient η in our model would become temperature dependent, modifying the velocity scale v_0 (cf. (31)) while leaving the friction force amplitude $\tilde{F}_{fr}$ (26) constant.

The present study can be extended in many directions. First, let us shortly focus on the damping term in (6). In general, also η is non-local in space (and time): For long wavelengths, it behaves like $\eta(k) \sim k^s$, where the precise value of s depends on the damping mechanism under consideration. Attenuation rates of surface waves for a semi-infinite elastic body with a rough surface lead to $s = 3$ or 4 in the long wavelength regime [30]. In this case, the dynamics are dominated by a modified dynamical critical dimension $D_{c,d} = 2\alpha - s$, while the static properties are unaffected.

In a situation where the Larkin length is smaller than the system size, the perturbative results hold only in the limit of large velocities; for this situation, the critical dynamics close to the threshold force remain to be analyzed. Finally, inertial terms can be included to properly describe the high velocity regime.

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A. Random potential correlation function

Let $v_n(z) \equiv \Theta(z) z^n$, cf. (5). We want to calculate the pair correlator

$$\bar{C}_2(K_h(\mathbf{x}) + K_l(\mathbf{x}'); d) \equiv \langle V[h(\mathbf{x}) + l(\mathbf{x}') - d] V[h(0) + l(0) - d] \rangle, \tag{39}$$

cf. (17), for fixed $\mathbf{x}$ and $\mathbf{x}'$. The first step is to rewrite (39) using the Fourier transform of $V(z) = \int_q \mathrm{e}^{\mathrm{i}qz} \tilde{V}_q$, followed by performing the disorder average over the Gaussian height fields h and l. This results in

$$\int_{q,q'} \tilde{V}_q \tilde{V}_{q'} \mathrm{e}^{-\frac{1}{2}(q^2+q'^2)(\bar{h}^2+\bar{l}^2) - qq'(K_h(\mathbf{x})+K_l(\mathbf{x}')) - \mathrm{i}(q+q')d}. \tag{40}$$

V_q and $V_{q'}$ are Fourier transformed back to $V(z)$ and $V(z')$, and the Gaussian integral over q and q' is carried out. Before writing it down, we go over to dimensionless functions and variables. We choose

$$l_0 \equiv \sqrt{\langle h^2 \rangle + \langle l^2 \rangle} \tag{41}$$

as the length scale in the direction perpendicular to the surface plane. Let

$$\delta \equiv \frac{d}{l_0}, \qquad \kappa \equiv \frac{K_h(\mathbf{x}) + K_l(\mathbf{x}')}{l_0^2}, \tag{42}$$

so κ ranges between 0 and 1, with $\kappa = 1$ for $\mathbf{x} = \mathbf{x}' = 0$ and $\kappa \to 0$ in the opposite limit $|\mathbf{x}|, |\mathbf{x}'| \gg \sigma$. Finally, let us define the dimensionless function

$$c_2(\kappa, \delta) \equiv \frac{\bar{C}_2(\kappa l_0^2, \delta l_0)}{V_0^2 h^{2n}}. \tag{43}$$

The resulting expression is then

$$c_2(\kappa, \delta) = \int \frac{d\zeta d\zeta' \, v_n(\zeta) v_m(\zeta')}{2\pi\sqrt{1-\kappa^2}} \times \exp\left(-\frac{(\zeta - \zeta')^2 + 2(1-\kappa)(\zeta + \delta)(\zeta' + \delta)}{2(1-\kappa^2)} \right). \tag{44}$$

This is a general expression for an arbitrary interaction potential; from now on we will make use of the specific form of $v_n(\zeta)$. Let us first consider the limiting cases $\kappa = 0$ and $\kappa = 1$. First, for $\kappa = 0$ one gets

$$c_2(0, \delta) = \prod_{k \in \{m,n\}} \int_0^\infty \frac{d\zeta}{\sqrt{2\pi}} \zeta^k \, \mathrm{e}^{-(\zeta+\delta)^2/2} \tag{45}$$

$$\overset{(\delta \gg 1)}{\approx} \frac{\Gamma(n+1)\Gamma(m+1)}{2\pi \, \delta^{m+n+2}} \, e^{-\delta^2}. \tag{46}$$

In the opposite limit $\kappa \to 1$,

$$c_2(1, \delta) = \int_0^\infty \frac{d\zeta}{\sqrt{2\pi}} \, \zeta^{m+n} \, \mathrm{e}^{-(\zeta+\delta)^2/2} \tag{47}$$

$$\overset{(\delta \gg 1)}{\approx} \frac{\Gamma(m+n+1)}{\sqrt{2\pi} \, \delta^{m+n+1}} \, \mathrm{e}^{-\delta^2/2}. \tag{48}$$

Comparing the δ-dependence in (45) and (47) with the dependence of the mean distance d on the normal load, (9), – remember that we consider d to be a function of N – one finds to leading order

$$c_2(\kappa, \delta) \sim \begin{cases} \left(\frac{\sigma}{L}\right)^D \mathcal{N} & \text{for } \kappa \approx 1 \\ \left(\left(\frac{\sigma}{L}\right)^D \mathcal{N}\right)^2 & \text{for } \kappa \approx 0, \end{cases} \tag{49}$$

where δ is to be considered a function of $\mathcal{N}/L^D$.

To make progress with (44) for arbitrary κ, one can introduce polar coordinates $\zeta \equiv r\cos\phi$, $\zeta' \equiv r\sin\phi$. The r-integration can be carried out exactly, leaving the one dimensional angular integral

$$c_2(\kappa, \delta) = \Gamma(\nu+1)(1-\kappa^2)^{\nu/2} \, \mathrm{e}^{-\frac{\delta^2}{1+\kappa}} \times \int_0^{\pi/2} \frac{d\phi}{2\pi} \frac{\cos^n\phi \, \sin^m\phi}{(1 - \kappa \sin 2\phi)^{\frac{\nu+1}{2}}} \, \mathrm{e}^{y^2(\phi)/4} D_{-(\nu+1)}(y(\phi)), \tag{50}$$

where

$$y^2(\phi) = \frac{1-\kappa}{1+\kappa} \frac{1 + \sin 2\phi}{1 - \kappa \sin 2\phi} \delta^2, \tag{51}$$

and $D_{-(\nu+1)}(y)$ with $\nu = m + n + 1$ is a parabolic cylinder function. Equation (50) can easily be calculated numerically. For our present purpose it suffices however to write

$$c_2(\kappa,\delta) = \frac{\Gamma(m+n+1)}{\sqrt{2\pi}\,\delta^{m+n+1}}\; f_2(\kappa)\; \mathrm{e}^{-\delta^2/(1+\kappa)}, \tag{52}$$

where $f_2(\kappa)$ only weakly depends on κ ('weakly' compared to the leading exponential dependence). It is a monotonically growing function with

$$f_2(\kappa) \approx \begin{cases} \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(m+n+1)}\,\frac{1}{\sqrt{2\pi}\delta} & \text{for } \kappa = 0 \\ 1 & \text{for } \kappa = 1. \end{cases} \tag{53}$$

For large enough δ, we can neglect the dependence on $f_2(\kappa)$: The ratio $\mathrm{e}^{-\delta^2/2}/f_2(0)$ is $\approx 1/50$ for $\delta = 4$, $\approx 3\times 10^{-4}$ for $\delta = 5$ etc., with $n = m = 2$.

Since we will primarily need the Fourier transform of the potential correlation function, we define

$$\tilde{C}_2(\mathbf{p},\mathbf{p}') \equiv \int d^D\!x\, d^D\!x'\; \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}+\mathrm{i}\mathbf{p}'\mathbf{x}'}\; C_2(\mathbf{x},\mathbf{x}') \tag{54}$$

and analyze its behaviour for large δ such that $\mathrm{e}^{-\delta^2/2} \ll 1$. Under this constraint, it is a good approximation to consider only the leading δ- and κ-dependence of $c_2(\kappa,\delta)$, i.e. $c_2(\kappa,\delta) \simeq \mathrm{e}^{-\delta^2/(1+\kappa)}$, making an error of $\mathcal{O}(1)$. For sake of simplicity, let us assume henceforth that both surface profiles obey the same distribution $K_h(\mathbf{x})$ with

$$\frac{K_h(\mathbf{x})}{l_0^2} \approx \begin{cases} 1/2 & \text{for } |\mathbf{x}| \lesssim \sigma \\ 0 & \text{for } |\mathbf{x}| \gg \sigma. \end{cases} \tag{55}$$

Let

$$\tilde{c}_2(\mathbf{p},\mathbf{p}') \equiv \tilde{C}_2(\mathbf{p},\mathbf{p}')\,/\,V_0^2\bar{h}^{2n}. \tag{56}$$

Because $C_2(\mathbf{x},\mathbf{x}')$ decays to a nonzero constant for $|\mathbf{x}| \gg \sigma$, it is convenient to write $\tilde{c}_2(\mathbf{p},\mathbf{p}')$ as the sum of four terms:

$$\tilde{c}_2^{\alpha}\delta_{\mathbf{p}}\delta_{\mathbf{p}'} + \tilde{c}_2^{\beta}(\mathbf{p})\delta_{\mathbf{p}'} + \tilde{c}_2^{\beta}(\mathbf{p}')\delta_{\mathbf{p}} + \tilde{c}_2^{\gamma}(\mathbf{p},\mathbf{p}'), \tag{57}$$

only the last two of which will contribute in the integrals $\tilde{C}_2$ appears in (cf. for example (60)). The other two terms will be eliminated due to the $\delta_{\mathbf{p}}$–term being multiplied with a power of p_1.

Now, first consider Gaussian correlated surface profiles $k(x) = \mathrm{e}^{-x^2/2}$ (cf. the definition in (1)). Performing a saddle-point expansion about small $\mathbf{x}$ and $\mathbf{x}'$ in $C_2(\mathbf{x},\mathbf{x}')$ and then Fourier transforming, $\tilde{c}_2^{\beta}(\mathbf{p})$ is found to be proportional to $(\sigma/\delta)^D\mathrm{e}^{-2\delta^2/3}$ for $|\mathbf{p}| \lesssim \delta/\sigma$ and exponentially damped for larger $\mathbf{p}$. Correspondingly, $\tilde{c}_2^{\gamma}(\mathbf{p},\mathbf{p}') \sim (\sigma/\delta)^{2D}\mathrm{e}^{-\delta^2/2}$ for $|\mathbf{p}|, |\mathbf{p}'| \lesssim \delta/\sigma$, with exponential suppression for $|\mathbf{p}|$ or $|\mathbf{p}'| \gtrsim \delta/\sigma$.

If the surface correlation function is taken to have a cusp in the x_1-direction, the only – and important – difference arises in the large p_1-behaviour of $\tilde{c}_2^{\beta}$ and $\tilde{c}_2^{\gamma}$. With the modification that

$$k(x_1) = \mathrm{e}^{-|x_1|}, \tag{58}$$

leaving the dependence on the perpendicular spatial coordinates unmodified, we find

$$\tilde{c}_2^{\beta}(p_1) \sim \sigma_b/(1+p_1^2\sigma_b^2) \tag{59}$$

with $\sigma_b = 9\sigma/2\delta^2$, and a similar result for $\tilde{c}_2^{\gamma}(p_1,p_1')$. In the perpendicular direction $\mathbf{p}_\perp$, c_2^{β} and c_2^{γ} are again exponentially suppressed.

B. Dynamic friction force

We want to analyze the expression (29) that gives the kinetic friction force density at velocity v in first non-vanishing order perturbation theory. Two generic scenarios will be considered, assuming in both cases that the two surface profiles obey identical distributions. First, we consider the case of short range correlated profiles, $k(x) = \mathrm{e}^{-x^2/2}$, and second, profiles that are self-affine in the direction of $\mathbf{F}$. Let us rewrite (29) as

$$f = \int_{\mathbf{p}}\int_{\mathbf{q}} G_0(\mathbf{p}, q_1 v)\; \mathrm{i}q_1^3\; \tilde{C}_2(\mathbf{q}, \mathbf{p}-\mathbf{q}), \tag{60}$$

with $\tilde{C}_2(\mathbf{p},\mathbf{p}')$ as defined in (54). Only the imaginary part of the propagator

$$\Im\,(G_0(\mathbf{p}, q_1 v)) = -\frac{\eta q_1 v}{\gamma^2|\mathbf{p}|^{2\alpha} + (\eta q_1 v)^2} \tag{61}$$

gives a contribution to the integral, the integral over the real part vanishes by symmetry. Now, $\tilde{C}_2(\mathbf{p},\mathbf{p}')$ is of order

$$V_0^2\bar{h}^{2n}\mathrm{e}^{-\delta^2/2} \sim \frac{\gamma\sigma^{3-\alpha}}{L^D}\;\tilde{F}_{fr} \quad \text{for } 1/L < |\mathbf{p}^{(\prime)}| < 1/\sigma, \tag{62}$$

with δ as defined in App. A and $\tilde{F}_{fr}$ from (26), and vanishes rapidly for $|\mathbf{p}^{(\prime)}| > 1/\sigma$. This is the contribution from $\tilde{c}_2^{\gamma}(\mathbf{p},\mathbf{p}')$, cf. (57). The second term in the denominator in (61) serves as an infrared cutoff for the $\mathbf{p}$-integration at $p \approx \sigma^{-1}(v/v_0)^{1/\alpha}$, giving rise to a velocity dependence

$$\phi(v/v_0) \sim \begin{cases} \frac{1}{D-2\alpha}\left(\frac{v}{v_0}\right)^{D/\alpha-1} & \text{for } D < 2\alpha \\ \frac{v}{v_0}\ln(v/v_0) & \text{for } D = 2\alpha, \end{cases} \tag{63}$$

hence confirming the discussion after (32) for velocities $v \ll v_0$. For $v \gg v_0$, the $1/v$-behaviour follows from the dominance of the second term in the denominator of (61), regardless of the dimension D and the value of α. The term $\tilde{c}_2^{\beta}(\mathbf{q})\delta_{(\mathbf{p}-\mathbf{q})}$ gives an independent contribution to the integral in (60) proportional to

$$\int_{\mathbf{p}} \frac{\mathrm{i}p_1^3}{\gamma|\mathbf{p}|^{\alpha} + \mathrm{i}\eta v p_1}\;\tilde{c}_2^{\beta}(\mathbf{p}). \tag{64}$$

For small $v \ll v_0$, it is linear in v, and, apart from the velocity dependence, it is smaller by a factor of $\mathrm{e}^{-\delta^2/6}$ compared to the leading term, so we can neglect it here.

Now consider surface profiles that are characterized by a correlation function modified via (58), having a cusp in the direction of the applied force, and being analytic in directions perpendicular to it. In contrast to the previous situation, $\tilde{C}_2(\mathbf{q}, \mathbf{p}-\mathbf{q})$ is exponentially damped now only in the direction perpendicular to $\mathbf{F}$, while the large p-behaviour of c_2^{γ} in the sliding direction is $\sim (p_1-q_1)^{-2}q_1^{-2}$, and $\sim p_1^{-2}$ for c_2^{β}. In the contribution to (60) containing the term c_2^{γ}, performing the q_1-integration leads to

$$\frac{\pi}{2\gamma\sigma^4}\int_{\mathbf{p},\mathbf{q}_\perp}\frac{2|\mathbf{p}|^{\alpha}+\tilde{v}}{(|\mathbf{p}|^{\alpha}+\tilde{v})^2}\;\tilde{C}_2(\mathbf{q}_\perp, \mathbf{p}_\perp - \mathbf{q}_\perp), \tag{65}$$

where $\tilde{v} = v/v_0\sigma^{\alpha}$. The remaining integrals can now easily be carried out, yielding a result which, provided that $D > \alpha$, is *independent* of v for $v \ll v_0$. It is of order

$$\frac{V_0^2 \bar{h}^{2n} \mathrm{e}^{-\delta^2/2}}{(D-\alpha)\gamma\sigma^{3-\alpha}} \approx \frac{1}{(D-\alpha)\, L^D}\ \tilde{F}_{fr}. \tag{66}$$

For $\alpha = 1$, this result is multiplied with a factor of $\ln(\sigma/a)$, where $1/a$ is the UV-cutoff of the p_1-integration, a setting the smallest length scale down to which the self-affinity of the surface profile holds, cf. (34). The result for the friction force in this case is independent of v in the regime $(a/\sigma)^{1/\alpha} v_0 \ll v \ll v_0$. For even lower velocities, the fact that the correlator in (34) is analytic on length scales smaller than a comes into play again, leading to the same v-dependence of $\phi(v/v_0)$ in this regime as given by (63).

Again, there is an independent contribution from $\tilde{c}_2^{\beta}(\mathbf{q})\delta_{(\mathbf{p}-\mathbf{q})}$ of the form (64). This term has a dependence on the mean distance $\sim \mathrm{e}^{-2\delta^2/3}$ and is hence proportional to $((\sigma/L)^D \mathcal{N})^{4/3}$. For $\alpha = 2$, this contribution is negligible, but for $\alpha = 1$, it suffers from a linear divergence in the UV-cutoff $\sim \sigma/a$ due to the large p_1-behaviour of $\tilde{c}_2^{\beta}$. We have checked that in higher orders perturbation theory no divergences in higher order of σ/a appear, so this divergence will be compensated by the δ-dependence of this term for moderately large δ.

References

1. Bowden F.P., Tabor D., The Friction and Lubrication of Solids, Oxford: Clarenden Press 1950
2. Scholz C.H., The Mechanics of Earthquakes and Faulting, Cambridge University Press 1990
3. Dieterich J.H., PAGEOPH **116**, 790 (1978); Rice J.R., PAGEOPH **121**, 443 (1983)
4. Heslot F., Baumberger T., Perrin B., Caroli B., Caroli C., Phys. Rev. E **49**, 4973 (1994)
5. Tomlinson G.A., Philos. Mag. **7**, 905 (1929)
6. Caroli C., Nozières P., Dry friction as a hysteretic elastic response, in: Physics of Sliding Friction, ed. by Persson and Tosatti, Dordrecht: Kluwer Academic 1996
7. Persson B.N.J. and Tossati E., Theory of friction: elastic coherence length and earthquake dynamics, ibid
8. Efetov K.B., Larkin A.I., Sov. Phys. JETP **45** (6), 1236 (1977); Fisher D.S., Phys. Rev. Lett. **50**, 1486 (1983)
9. Koplik J., Levine H., Phys. Rev. B **32**, 280 (1985)
10. Nattermann T., Stepanow S., Tang L.H., Leschhorn H., J. Phys. II France **2**, 1483 (1992)
11. Blatter G. et. al, Rev. Mod. Phys. **66**, 1125 (1994)
12. Cule D., Hwa T., Phys. Rev. Lett. **77**, 278 (1996)
13. Zhang X. et. al, Europhys. Lett. **34** (2), 121 (1996)
14. Landau L.D., Lifshitz E.M., Theory of Elasticity, New York: Pergamon Press 1975
15. Vallette D.P., Gollub J.P., Phys. Rev. E **47**, 820 (1993)
16. Greenwood J.A., Williamson J.B.P., Proc. Roy. Soc. London **295**, 300 (1966)
17. Efetov K.B., Larkin A.I., Sov. Phys. JETP **39**, 1129 (1974)
18. Larkin A.I., Ovchinnikov Yu.N., J. Low Temp. Phys. **34**, 409 (1979)
19. Larkin A.I., Khmelnitski D.E., preprint Landau-Institute (June 1979)
20. Feigel'man M.V., Sov. Phys. JETP **58** (5), 1076 (1983)
21. Bruinsma R., Aeppli G., Phys. Rev. Lett. **52**, 1547 (1984)
22. Ertaş D., Kardar M., Phys. Rev. Lett. **73**, 1703 (1994)
23. Tanguy A., Nozières P., J. de Phys. I **6**, 1251 (1996)
24. Samokhin K., JETP Lett. **64** (1996) 580, and JETP Lett. **64**, 857 (1996)
25. Gardiner C.W., Handbook of Stochastic Methods, Berlin: Springer 1983
26. Fukuyama H., Lee P.A., Phys. Rev. B **17**, 535 (1978)
27. Leschhorn H., Nattermann T., Stepanow S., Tang L.H., Ann. Physik **6**, 1 (1997)
28. Rabinowicz E., Friction and Wear of Materials, New York: John Wiley and Sons 1965
29. Grosch K.A., Proc. R. Soc. London, Ser. A **274**, 21 (1963)
30. Huang X., Maradudin A.A., Phys. Rev. B **36**, 7827 (1987)