Appl Math Optim 41:237–254 (2000)
DOI: 10.1007/s002459910013

Applied Mathematics & Optimization

Lipschitzian Regularity of Minimizers for Optimal Control Problems with Control-Affine Dynamics*

A. V. Sarychev and D. F. M. Torres

Department of Mathematics, University of Aveiro,
3810 Aveiro, Portugal
{ansar,delfim}@mat.ua.pt

Communicated by J. Stoer

Abstract. We study the Lagrange Problem of Optimal Control with a functional $\int_a^b L(t, x(t), u(t))\, dt$ and control-affine dynamics $\dot{x} = f(t, x) + g(t, x)\, u$ and (a priori) unconstrained control $u \in \mathbb{R}^m$. We obtain conditions under which the minimizing controls of the problem are bounded—a fact which is crucial for the applicability of many necessary optimality conditions, like, for example, the Pontryagin Maximum Principle. As a corollary we obtain conditions for the Lipschitzian regularity of minimizers of the Basic Problem of the Calculus of Variations and of the Problem of the Calculus of Variations with higher-order derivatives.

Key Words. Optimal control, Calculus of variations, Pontryagin Maximum Principle, Boundedness of minimizers, Nonlinear control-affine systems, Lipschitzian regularity.

AMS Classification. 49J15, 49J30.

1. Introduction

Under standard hypotheses of the Tonelli existence theory in the Calculus of Variations, the existence of minimizers is guaranteed in the class of absolutely continuous functions possibly with unbounded derivative. As is known, in such cases the optimality

---

* This research was partially presented at the International Conference dedicated to the 90th Anniversary of L. S. Pontryagin, Moscow, September 1998.

conditions—like the Euler–Lagrange equation—may fail. Therefore it is important to try to obtain Lipschitzian regularity conditions under which the minimizers are Lipschitzian. The main part of the results obtained (starting with those of Tonelli) refer to the Basic Problem of the Calculus of Variations (see [1], [2], [6], [8]–[10], [14], and [17]). Less is known for problems with high-order derivatives [11]. For the Lagrange problem and for problems of optimal control, regularity results are a rarity. We are only aware of progress due to Clarke and Vinter [12] for problems associated with linear, autonomous (i.e., time-invariant) dynamics (see also [4]). For this particular class of problems, regularity results are obtained via transformation of the initial problem into a problem of the Calculus of Variations with higher-order derivatives. In this paper we develop a different approach to establishing Lipschitzian regularity (boundedness of minimizing controls) for the Lagrange problems with a functional $\int_a^b L\left(t,\ x\left(t\right),\ u\left(t\right)\right)\ dt$ and control-affine nonautonomous dynamics: $\dot{x} = f\left(t,\ x\right) + g\left(t,\ x\right)u$. This class of systems appears in a wide range of problems relevant to mechanics, sub-Riemannian geometry, etc. We make use of an approach developed by Gamkrelidze [13, Chapter 8]: a reduction of the Lagrange problem to an autonomous time-optimal control problem, with a subsequent compactification of the set of control values. If the Pontryagin Maximum Principle is applicable to the compactified problem, one can use its formulation to derive conditions for boundedness of minimizers of the original Lagrange problem and to determine the bounds for the magnitudes of minimizing controls. The main result is Theorem 1 (Section 3). As its corollaries we obtain (Section 4) results on the Lipschitzian regularity of minimizers in the Calculus of Variations (see [18]).

2. Preliminaries

2.1. *Optimal Control Problems with Control-Affine Dynamics and Unconstrained Controls*

We study minima of the problem

$$
\begin{aligned}
(P)\quad & J\left[x\left(\cdot\right),\ u\left(\cdot\right)\right] = \int_a^b L\left(t,\ x\left(t\right),\ u\left(t\right)\right)\ dt \to \min,\\
& \dot{x}\left(t\right) = f\left(t,\ x\left(t\right)\right) + g\left(t,\ x\left(t\right)\right)u\left(t\right) \qquad \text{a.e. on } [a,\ b]\,,\\
& x\left(a\right) = x_a, \qquad x\left(b\right) = x_b,\\
& x\left(\cdot\right) \in AC\left([a,\ b]\,;\ \mathbb{R}^n\right), \qquad u\left(\cdot\right) \in L_1\left([a,\ b]\,;\ \mathbb{R}^m\right).
\end{aligned}
$$

Here $a, b \in \mathbb{R}$, $a < b$; $L\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $g\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are given functions; $m \le n$; $x_a, x_b \in \mathbb{R}^n$ are given; $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, $g(t,\ x) = (g^1(t,\ x), \dots, g^m(t,\ x))$. The controls $u(\cdot)$ are integrable. The absolute continuous solution $x(\cdot)$ of the differential equation is a *state trajectory* corresponding to the control $u\left(\cdot\right)$. We assume that $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are C^1–functions in $\mathbb{R}^{1+n}$.

2.2. *The Pontryagin Maximum Principle for* (P)

The following first-order necessary optimality condition for the problem (P) is provided by the Pontryagin Maximum Principle. We use $\langle\cdot, \cdot\rangle$ to denote the usual inner product in $\mathbb{R}^n$.

Theorem 2.2.1. *If $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is a minimizer of the problem (P) and the control $\tilde{u}(\cdot) \in L_\infty^m([a, b])$, then there exists a nonzero pair $(\tilde{\psi}_0, \tilde{\psi}(\cdot))$, where $\tilde{\psi}_0 \leq 0$ is a constant and $\tilde{\psi}(\cdot)$ is an absolutely continuous vector-function with domain $[a, b]$, such that the quadruple $(\tilde{x}(\cdot), \tilde{\psi}_0, \tilde{\psi}(\cdot), \tilde{u}(\cdot))$ satisfies*:

(i) *the Hamiltonian system*

$$\dot{x} = \frac{\partial H}{\partial \psi}, \qquad \dot{\psi} = -\frac{\partial H}{\partial x},$$

with the Hamiltonian $H = \psi_0 L(t, x, u) + \langle \psi, f(t, x) \rangle + \langle \psi, g(t, x) u \rangle$;

(ii) *the maximality condition*

$$H\left(t, \tilde{x}(t), \tilde{\psi}_0, \tilde{\psi}(t), \tilde{u}(t)\right) = M\left(t, \tilde{x}(t), \tilde{\psi}_0, \tilde{\psi}(t)\right) = \sup_{u \in \mathbb{R}^m} H\left(t, \tilde{x}(t), \tilde{\psi}_0, \tilde{\psi}(t), u\right)$$

for almost all $t \in [a, b]$.

Definition 1. A quadruple $\left(\tilde{x}(\cdot), \tilde{\psi}_0, \tilde{\psi}(\cdot), \tilde{u}(\cdot)\right)$ in which $\tilde{\psi}_0$, $\tilde{\psi}(\cdot)$ are as in Theorem 2.2.1, $\tilde{u}(\cdot)$ is integrable and satisfies conditions (i) and (ii) of Theorem 2.2.1, is called extremal for the problem (P). The control $\tilde{u}(\cdot)$ is called an extremal control. An extremal $\left(\tilde{x}(\cdot), \tilde{\psi}_0, \tilde{\psi}(\cdot), \tilde{u}(\cdot)\right)$ is called normal if $\tilde{\psi}_0 \neq 0$ and abnormal if $\tilde{\psi}_0 = 0$. We call $\tilde{u}(\cdot)$ an abnormal extremal control if it corresponds to an abnormal extremal $\left(\tilde{x}(\cdot), 0, \tilde{\psi}(\cdot), \tilde{u}(\cdot)\right)$.

2.3. *The Pontryagin Maximum Principle for Autonomous Time Optimal Control Problems and Constrained Controls*

An autonomous time optimal problem is

$$\begin{aligned} & T \to \min \\ & \quad \text{subject to} \\ & \quad \dot{x}(t) = F(x(t), u(t)) \qquad \text{a.e.} \quad t \in [a, T], \\ & \quad x(\cdot) \in AC\left([a, T]; \mathbb{R}^n\right), \qquad u(\cdot) \in L_\infty([a, T]; U), \\ & \quad x(a) = x_a, \qquad x(T) = x_T. \end{aligned} \tag{2.1}$$

Here $F(\cdot, \cdot)$ is a continuous function on $\mathbb{R}^{n+m}$ and has continuous partial derivatives with respect to x. The control $u(\cdot)$, defined on $[a, T]$, takes its values in $U \subset \mathbb{R}^m$, and is a measurable and *bounded* function. The following first-order necessary condition—the Pontryagin Maximum Principle—holds for any minimizer of the problem (details can be found, for example, in [13]). There are plenty of generalizations and modifications of the Pontryagin Maximum Principle. For example, a Pontryagin Maximum Principle under less restrictive assumptions for smoothness of data can be found in [5] or [16].

Theorem 2.3.1. *Let* $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ *be a solution of the time-optimal problem* (2.1). *Then there exists a nonzero absolutely continuous function* $\tilde{\psi}(\cdot)$ *satisfying*:

- *the Hamiltonian system*

$$\dot{x} = \frac{\partial H(x, \psi, u)}{\partial \psi}, \qquad \dot{\psi} = -\frac{\partial H(x, \psi, u)}{\partial x},$$

  with corresponding Hamiltonian $H(x, \psi, u) = \langle \psi, F(x, u) \rangle$;
- *the maximality condition*

$$H\left(\tilde{x}(t), \tilde{\psi}(t), \tilde{u}(t)\right) = M\left(\tilde{x}(t), \tilde{\psi}(t)\right) = \sup_{u \in U}\left\{H\left(\tilde{x}(t), \tilde{\psi}(t), u\right)\right\}$$

  for almost all $t \in [a, T]$;
- *the equality* $M\left(\tilde{x}(t), \tilde{\psi}(t)\right) \equiv \text{const} \geq 0$.

3. Main Result

Theorem 1. *Let* $L(\cdot, \cdot, \cdot) \in C^1([a, b] \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R})$, $f(\cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$, $g(\cdot, \cdot) \in C^1\left(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^{n \times m}\right)$, *and*

$$\varphi(t, x, u) = f(t, x) + g(t, x)\, u.$$

Under the hypotheses:

(H1) **full rank condition:** $g(t, x)$ *has rank* m *for all* $t \in [a, b]$ *and* $x \in \mathbb{R}^n$;

(H2) **coercivity:** *there exists a function* $\theta \colon \mathbb{R} \to \mathbb{R}$ *such that*

$$L(t, x, u) \geq \theta(\|u\|) > \zeta,$$

for all $(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^m$, *and*

$$\lim_{r \to +\infty} \frac{r}{\theta(r)} = 0;$$

(H3) **growth condition:** *there exist constants* γ, β, η, *and* μ, *with* $\gamma > 0$, $\beta < 2$, *and* $\mu \geq \max\{\beta - 2, -2\}$, *such that, for all* $t \in [a, b]$, $x \in \mathbb{R}^n$, *and* $u \in \mathbb{R}^m$, *it holds that*

$$\left(|L_t| + |L_{x^i}| + \|L\,\varphi_t - L_t\,\varphi\| + \|L\,\varphi_{x^i} - L_{x^i}\,\varphi\|\right)\|u\|^{\mu} \leq \gamma\, L^{\beta} + \eta,$$

$$i \in \{1, \ldots, n\};$$

then all minimizers $\tilde{u}(\cdot)$ *of the Lagrange problem* (P) *which are not abnormal extremal controls, are essentially bounded on* $[a, b]$.

Remark 1. Recall that if the dynamics is controllable (which is true for problems of the Calculus of Variations treated in Section 4), then all extremals are normal. A priori minimizer $\tilde{u}(\cdot)$ which is not essentially bounded, may fail to satisfy the Pontryagin Maximum Principle and therefore may cease to be an extremal. As far as essentially bounded minimizers are concerned the Pontryagin Maximum Principle is valid, and unbounded minimizers $\tilde{u}(\cdot)$ (if there are any) are, according to Theorem 1, abnormal extremal controls, then we obtain the following:

Corollary 1. *Under the conditions of Theorem* 1 *all minimizers of the Lagrange problem* (P) *satisfy the Pontryagin Maximum Principle.*

Remark 2. We may impose stronger but technically simpler forms of assumption (H3) in Theorem 1. Under these conditions Theorem 1 loses some generality, but sometimes, for a given problem, these conditions are easier to verify. For example, they can be (in increasing order of simplicity and in decreasing order of generality):

$$[L(\|\varphi_t\| + \|\varphi_{x^i}\|) + \|\varphi\|(|L_t| + |L_{x^i}|)]\,\|u\|^{\mu} \le \gamma L^{\beta} + \eta;$$
$$L(\|\varphi_t\| + \|\varphi_{x^i}\|) + \|\varphi\|(|L_t| + |L_{x^i}|)\,\|u\|^{\mu} \le \gamma L^{\beta} + \eta;$$
$$\|\varphi_t\| + \|\varphi_{x^i}\| + |L_t| + |L_{x^i}| \le \gamma L^{\beta'} + \eta, \qquad \beta' < 1.$$

It is easy to see why (H3) follows from any of these conditions. It is enough to notice that

$$\|L\varphi_t - L_t\varphi\| + \|L\varphi_{x^i} - L_{x^i}\varphi\| \le L(\|\varphi_t\| + \|\varphi_{x^i}\|) + \|\varphi\|(|L_t| + |L_{x^i}|);$$

and that $0 \ge \max\{\beta - 2, -2\}$.

Remark 3. The result of Theorem 1 admits a generalization for Lagrange problems with dynamics which are nonlinear in control. This will be addressed in a forthcoming paper.

3.1. *Proof of the Theorem*

We begin with an elementary observation.

Remark 4. It suffices to prove Theorem 1 in the special case in (H2) where we put $\zeta = 0$. Indeed, minimization of

$$\int_a^b L(t, x(t), u(t))\, dt$$

under the conditions in (P) is equivalent to

$$\int_a^b (L(t, x(t), u(t)) - \zeta)\, dt \to \min,$$

since the difference of the integrands is constant.

Remember that everywhere below, the notation $\varphi(t, x, u)$ stands for $f(t, x) + g(t, x)u$.

3.1.1. *Reduction to a Time-Optimal Problem.* The following idea of transforming the variational problem into a time-optimal control problem and subsequent compactification of the control set was used earlier by Gamkrelidze [13, Chapter 8] to prove some existence results. We introduce a new time variable

$$\tau(t) = \int_a^t L(\theta, x(\theta), u(\theta))\, d\theta \qquad t \in [a, b],$$

which is a strictly monotonous absolutely continuous function of t, for any pair $(x(t), u(t))$ satisfying $\dot{x}(t) = \varphi(t, x(t), u(t))$. Obviously $\tau(b) = T$ coincides with the value of the functional of the original problem. As far as

$$\frac{d\tau(t)}{dt} = L(t, x(t), u(t)) > 0$$

holds then $\tau(\cdot)$ admits a monotonous inverse function $t(\cdot)$ defined on $[0, T]$, such that

$$\frac{dt}{d\tau}(\tau) = \frac{1}{L(t(\tau), x(t(\tau)), u(t(\tau)))}.$$

Notice that the inverse function $t(\cdot)$ is also absolutely continuous. Obviously

$$\frac{dx(t(\tau))}{d\tau} = \frac{dx(t(\tau))}{dt}\frac{dt(\tau)}{d\tau} = \frac{\varphi(t(\tau), x(t(\tau)), u(t(\tau)))}{L(t(\tau), x(t(\tau)), u(t(\tau)))}. \tag{3.1}$$

Taking τ as a new time variable, considering $t(\tau)$ and $z(\tau) = x(t(\tau))$ as components of state trajectory, and $v(\tau) = u(t(\tau))$ as the control, we can transform the problem (P) into the following form:

$$T \longrightarrow \min, \tag{3.2}$$

$$\begin{cases} \dot{t}(\tau) = \frac{1}{L(t(\tau), z(\tau), v(\tau))}, \\ \dot{z}(\tau) = \frac{\varphi(t(\tau), z(\tau), v(\tau))}{L(t(\tau), z(\tau), v(\tau))}, \end{cases} \tag{3.3}$$

$$v\colon \mathbb{R} \to \mathbb{R}^m,$$

$$\begin{aligned} t(0) &= a, \qquad t(T) = b, \\ z(0) &= x_a, \qquad z(T) = x_b. \end{aligned} \tag{3.4}$$

3.1.2. *Compactification of the Space of Admissible Controls.* So far the new control variable takes its values in $\mathbb{R}^m$ and the control $v(\tau)$ can be unbounded. This unboundedness is kind of fictitious as the set

$$\left\{\left(\frac{1}{L(t, z, v)}, \frac{\varphi(t, z, v)}{L(t, z, v)}\right) : v \in \mathbb{R}^m\right\}$$

is bounded under the hypotheses (H1) and (H2). This set (for fixed $(t, z) \in \mathbb{R} \times \mathbb{R}^n$) is not closed, but becomes compact if we add to it the point $(0, 0) \in \mathbb{R} \times \mathbb{R}^n$ which corresponds to the "infinite value" of the control $v \in \mathbb{R}^m$. The set

$$E(t, z) = \{(0, 0)\} \cup \left\{\left(\frac{1}{L(t, z, v)}, \frac{\varphi(t, z, v)}{L(t, z, v)}\right) : v \in \mathbb{R}^m\right\}$$

can be represented as a homeomorphic image of the m-dimensional sphere $S^m \approx \overline{\mathbb{R}}^m$. This homeomorphism is defined in a standard way: we fix a point $\hat{w} \in S^m$, called the north pole, and consider the stereographic projection

$$\pi\colon\ S^m \setminus \left\{\hat{w}\right\} \to \mathbb{R}^m. \tag{3.5}$$

Obviously $\pi|_{S^m\setminus\{\hat{w}\}}$ is continuous and $\lim_{w\to\hat{w}} \|\pi\ (w)\| = +\infty$. Functions

$$w \to \frac{1}{L\ (t,\ z,\ \pi(w))} \quad \text{and} \quad w \to \frac{\varphi\ (t,\ \ z,\ \pi(w))}{L\ (t,\ z,\ \pi(w))} \tag{3.6}$$

are continuous on $S^m \setminus \left\{\hat{w}\right\}$, since $L\ (t,\ z,\ \pi(w)) > 0$. Due to hypotheses (H1) and (H2)

$$\lim_{\|v\|\to+\infty} \frac{\|\varphi\ (t,\ z,\ v)\|}{\theta\ (\|v\|)} = 0$$

and therefore

$$\lim_{\|v\|\to+\infty} \frac{\|\varphi\ (t,\ \ z,\ v)\|}{L\ (t,\ z, v)} = 0. \tag{3.7}$$

Hence, one can extend the functions defined by (3.6) up to the functions $\phi^{t,z}(\cdot)$ and $h^{t,z}(\cdot)$, which are continuous on the entire sphere S^m (on the compactified space $\overline{\mathbb{R}}^m$):

$$\phi^{t,z}(w) = \phi(t,\ z,\ w) = \begin{cases} \frac{1}{L(t,z,\pi(w))} & \text{if} \quad w \neq \hat{w}, \\ 0 & \text{if} \quad w = \hat{w}, \end{cases}$$

$$h^{t,z}(w) = h(t,\ z, w) = \begin{cases} \frac{\varphi(t,z,\pi(w))}{L(t,z,\pi(w))} & \text{if} \quad w \neq \hat{w}, \\ 0 & \text{if} \quad w = \hat{w}. \end{cases}$$

Given (H1), the map

$$w \to \left(\phi^{t,z}(w),\ h^{t,z}(w)\right),$$

of S^m onto $E(t,\ z)$, is continuous and one-to-one and therefore a homeomorphism since S^m is compact. Thus, we have come to the autonomous optimal control problem:

$$T \longrightarrow \min, \tag{3.8}$$

$$\begin{cases} \dot{t}(\tau) = \phi\ (t(\tau),\ z(\tau),\ w(\tau))\,, \\ \dot{z}(\tau) = h\ (t(\tau),\ z(\tau),\ w(\tau))\,, \end{cases} \tag{3.9}$$

$$\begin{aligned} & w\colon \mathbb{R} \to S^m, \\ & t(0) = a, \qquad t(T) = b, \\ & z(0) = x_a, \qquad z(T) = x_b, \end{aligned} \tag{3.10}$$

with a compact set S^m of values of control parameters.

We claim that every pair $(x\ (\cdot)\ ,\ u\ (\cdot))$ satisfying $\dot{x}\ =\ f\ (t,\ x) + g\ (t,\ x)\, u$ corresponds to a trajectory $(t\ (\tau)\ ,\ z\ (\tau)\ ,\ w\ (\tau))$ of the system (3.9) with $w(\tau) \neq \hat{w}$ for almost all $\tau \in [\,0, T\,]$ and the transfer time T for this latter solution equal to the value $J\,[x(\cdot),\ u\ (\cdot)]$:

$$T\ = J\,[x(\cdot),\ u\ (\cdot)] = \int_a^b L\ (t,\ x(t),\ u(t))\ dt.$$

Indeed we may define $(t(\tau), z(\tau), w(\tau))$ setting

$$t(\tau) \text{ is the inverse function to } \quad \tau(t) = \int_a^t L(\theta, x(\theta), u(\theta))\, d\theta,$$
$$z(\tau) = x(t(\tau)), \qquad w(\tau) = \pi^{-1}(u(t(\tau))),$$
$$0 \le \tau \le T = J[x(\cdot), u(\cdot)],$$

where $\pi^{-1}(\cdot)$ is the mapping inverse to (3.5). Function $z(\cdot)$ is absolutely continuous since it is a composition of an absolutely continuous function with another strictly monotonous absolutely continuous function $t(\tau)$. Function $w(\cdot)$ is measurable because $\pi^{-1}(\cdot)$ is a continuous function, $u(\cdot)$ is measurable, and $t(\cdot)$ is a strictly monotonous absolutely continuous function. We already know that

$$\frac{dt(\tau)}{d\tau} = \frac{1}{L(t(\tau), z(\tau), \pi(w(\tau)))},$$
$$\frac{dz(\tau)}{d\tau} = \frac{\varphi(t(\tau), z(\tau), \pi(w(\tau)))}{L(t(\tau), z(\tau), \pi(w(\tau)))},$$

and $w(\tau) \neq \hat{w}$ for almost all $\tau \in [0, T]$, since $u(\cdot)$ has finite values almost everywhere and $t(\cdot)$ is strictly monotonous.

We shall show now that every solution of (3.9), with $w(\tau) \neq \hat{w}$ a.e., results from this correspondence. Taking the absolutely continuous function $\tau(t)$, $a \le t \le b$, which is the inverse of the strictly monotonous, absolutely continuous function $t(\tau)$, $0 \le \tau \le T$, we set

$$\begin{cases} x(t) = z(\tau(t)), \\ u(t) = \pi(w(\tau(t))), \end{cases} \qquad a \le t \le b.$$

The curve $x(\cdot)$ defined in this way is absolutely continuous (because $z(\cdot)$ and $\tau(\cdot)$ are absolutely continuous and $\tau(\cdot)$ is strictly monotonous) and satisfies the boundary conditions

$$x(a) = x_a \quad \text{and} \quad x(b) = x_b.$$

The function $u(\cdot)$ is measurable because $\pi(\cdot)$ is continuous, $w(\cdot)$ is measurable, and $\tau(\cdot)$ is continuous and monotonous. Also

$$\int_a^b \|u(t)\|\, dt = \int_a^b \|\pi(w(\tau(t)))\|\, dt$$
$$= \int_0^T \left\| \frac{\pi(w(\tau))}{L(t(\tau), z(\tau), \pi(w(\tau)))} \right\| d\tau.$$

As the latter integrand is bounded due to the coercivity condition, we conclude that $u(\cdot)$ is integrable on $[a, b]$. Differentiating $x(t)$ with respect to t, we conclude that

$$\frac{dx(t)}{dt} = \frac{dz(\tau(t))}{dt} = \frac{dz(\tau(t))}{d\tau}\frac{d\tau(t)}{dt} = \varphi(t, x(t), u(t))$$

for almost all $t \in [a, b]$. Integrating

$$\frac{d\tau(t)}{dt} = L(t, z(\tau(t)), \pi(w(\tau(t)))) = L(t, x(t), u(t))$$

one obtains

$$J\,[x(\cdot),\ u\,(\cdot)] = \int_a^b L\,(t,\ x(t),\ u(t))\ dt = \int_a^b \frac{d\tau\,(t)}{dt}\,dt$$
$$= \tau\,(b) - \tau\,(a) = T.$$

3.1.3. *Continuous Differentiability of the Right-Hand Side of* (3.9). The functions $\phi(\cdot,\cdot,\cdot)$ and $h(\cdot,\cdot,\cdot)$ are continuous in

$$\left\{(t,\ z,\ w) :\ t \in [a,\ b]\,,\ z \in \mathbb{R}^n, w \in S^m\right\}.$$

To apply the Pontryagin Maximum Principle (Theorem 2.3.1) to the problem (3.8)–(3.10), we need to assure that the right-hand side of (3.9) is continuously differentiable with respect to t and z. Since $L\,(\cdot,\cdot,\cdot)$, $f\,(\cdot,\cdot)$, $g\,(\cdot,\cdot)$ are C^1, $L\,(t,\ x,\ u) > 0$ for all $(t,\ x,\ u)$, and $\pi(\cdot)$ is continuous, we conclude at once that $\phi_{z^i}\,(t,\ z,\ w)$, $\phi_t\,(t,\ z,\ w)$, $h_{z^i}\,(t,\ z,\ w)$, and $h_t\,(t,\ z,\ w)$ are continuous for $w \neq \hat{w}$. The only problem is the continuous differentiability at $\hat{w}$. Since $\phi\left(\cdot,\cdot,\hat{w}\right) \equiv 0$ and $h\left(\cdot,\cdot,\hat{w}\right) \equiv 0$, we have (from now on, when not indicated, L and φ are evaluated at $(t,\ z,\ \pi(w))$)

$$\phi_{z^i}\,(t,\ z,\ w) = \begin{cases} -\frac{L_{x^i}(t,z,\pi(w))}{L^2(t,z,\pi(w))} & \text{if}\quad w \neq \hat{w}, \\ 0 & \text{if}\quad w = \hat{w}, \end{cases} \qquad i = 1,\ \ldots,\ n,$$

$$h_{z^i}\,(t,\ z,\ w) = \begin{cases} \frac{L\,\varphi_{x^i} - L_{x^i}\,\varphi}{L^2} & \text{if}\quad w \neq \hat{w}, \\ 0 & \text{if}\quad w = \hat{w}, \end{cases} \qquad i = 1,\ \ldots,\ n,$$

$$\phi_t\,(t,\ z,\ w) = \begin{cases} -\frac{L_t(t,z,\pi(w))}{L^2(t,z,\pi(w))} & \text{if}\quad w \neq \hat{w}, \\ 0 & \text{if}\quad w = \hat{w}, \end{cases}$$

$$h_t\,(t,\ z,\ w) = \begin{cases} \frac{L\,\varphi_t - L_t\,\varphi}{L^2} & \text{if}\quad w \neq \hat{w}, \\ 0 & \text{if}\quad w = \hat{w}. \end{cases}$$

To verify the continuity of the derivatives at $\hat{w}$, we have to prove that, for all $(t_0,\ z_0) \in [a,\ b] \times \mathbb{R}^n$,

$$\lim_{\rho\left((t,z,w),\,\left(t_0,z_0,\hat{w}\right)\right)\to 0} \frac{L_{x^i}\,(t,\ z,\ \pi(w))}{L^2\,(t,\ z,\ \pi(w))} = 0, \tag{3.11}$$

$$\lim_{\rho\left((t,z,w),\,\left(t_0,z_0,\hat{w}\right)\right)\to 0} \frac{L_t\,(t,\ z,\ \pi(w))}{L^2\,(t,\ z,\ \pi(w))} = 0, \tag{3.12}$$

$$\lim_{\rho\left((t,z,w),\,\left(t_0,z_0,\hat{w}\right)\right)\to 0} \frac{\|L\,\varphi_{x^i} - L_{x^i}\,\varphi\|}{L^2} = 0, \tag{3.13}$$

$$\lim_{\rho\left((t,z,w),\,\left(t_0,z_0,\hat{w}\right)\right)\to 0} \frac{\|L\,\varphi_t - L_t\,\varphi\|}{L^2} = 0, \tag{3.14}$$

where $\rho\,(\cdot,\cdot)$ is a distance defined on $[a,\ b] \times \mathbb{R}^n \times S^m$. We shall see that (3.11)–(3.14) are true under our hypotheses.

Let $N\,(t,\ z,\ \pi(w))$ denote any of the numerators in (3.11)–(3.14). From the growth condition (H3), we obtain that, for all $t \in [a,\ b]$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$,

$$\|N\,(t,\ x,\ u)\| \leq \gamma\,L^\beta\,(t,\ x,\ u)\,\|u\|^{-\mu} + \eta\,\|u\|^{-\mu}.$$

As long as $2-\beta>0$, one concludes

$$\frac{\|N(t,\,z,\,\pi(w))\|}{L^2(t,\,z,\,\pi(w))}\leq\gamma\,\frac{\|\pi(w)\|^{-\mu}}{L^{2-\beta}(t,\,z,\,\pi(w))}+\eta\,\frac{\|\pi(w)\|^{-\mu}}{L^2(t,\,z,\,\pi(w))}.$$

Since by (H2)

$$\frac{1}{L(t,\,z,\,\pi(w))}\leq\frac{1}{\theta(\|\pi(w)\|)},$$

then

$$\frac{\|N(t,\,z,\,\pi(w))\|}{L^2(t,\,z,\,\pi(w))}\leq\gamma\,\frac{\|\pi(w)\|^{-\mu}}{\theta^{2-\beta}(\|\pi(w)\|)}+\eta\,\frac{\|\pi(w)\|^{-\mu}}{\theta^2(\|\pi(w)\|)}.$$

If we recall that

$$\lim_{\|\pi(w)\|\to+\infty}\frac{\|\pi(w)\|}{\theta(\|\pi(w)\|)}=0,$$

$-\mu\leq 2-\beta\,\wedge\,-\mu\leq 2\,\wedge\,2-\beta>0$, then we obtain

$$\lim_{\rho\left((t,z,w),\,\left(t_0,z_0,\hat{w}\right)\right)\to 0}\frac{\|N(t,\,z,\,\pi(w))\|}{L^2(t,\,z,\,\pi(w))}=0,$$

which proves equalities (3.11)–(3.14).

3.1.4. *Pontryagin Maximum Principle and Lipschitzian Regularity.* Let $(\tilde{x}(\cdot),\,\tilde{u}(\cdot))$ be a minimizer of the original problem (P) and let $\left(\tilde{t}(\cdot),\,\tilde{z}(\cdot),\,\tilde{w}(\cdot)\right)$ be the correspondent minimizer for the time-optimal problem (3.8)–(3.10) with $\tilde{w}(\tau)\neq\hat{w}$ almost everywhere. Applying Pontryagin's Maximum Principle to the time-minimal problem (3.8)–(3.10), we conclude that there exist absolutely continuous functions on $[0,\,\tilde{T}]$:

$$\tilde{\lambda}\colon\,\mathbb{R}\to\mathbb{R},\qquad\tilde{p}\colon\,\mathbb{R}\to\mathbb{R}^n,$$

where $\tilde{T}$ denotes the minimal time for the problem (3.8)–(3.10) and $\tilde{p}(\tau)$ is a row covector, not vanishing simultaneously, satisfying:

(i) the Hamiltonian system

$$\begin{cases}\dot{\lambda}(\tau)=-H_t\left(t(\tau),z(\tau),\,\lambda(\tau),\,\tilde{p}(\tau),\,w(\tau)\right),\\ \dot{p}(\tau)=-H_z\left(t(\tau),z(\tau),\,\lambda(\tau),\,p(\tau),\,w(\tau)\right),\end{cases}$$

with the Hamiltonian

$$H(t,\,z,\,\lambda,\,p,\,w)=\lambda\,\phi(t,\,z,\,w)+\langle p,\,h(t,\,z,\,w)\rangle\,;$$

(ii) the maximality condition

$$0\leq c\;=\;\sup_{w\in S^m}\left\{\tilde{\lambda}(\tau)\,\phi\left(\tilde{t}(\tau),\tilde{z}(\tau),w\right)+\left\langle\tilde{p}(\tau),h\left(\tilde{t}(\tau),\tilde{z}(\tau),w\right)\right\rangle\right\}\tag{3.15}$$

$$\overset{\text{a.e.}}{=}\tilde{\lambda}(\tau)\,\phi\left(\tilde{t}(\tau),\tilde{z}(\tau),\tilde{w}(\tau)\right)+\left\langle\tilde{p}(\tau),h\left(\tilde{t}(\tau),\tilde{z}(\tau),\tilde{w}(\tau)\right)\right\rangle,\tag{3.16}$$

where c is a constant, $\tilde{z}(\tau)=\tilde{x}\left(\tilde{t}(\tau)\right)$, and $\tilde{w}(\tau)=\pi^{-1}\left(\tilde{u}(\tilde{t}(\tau))\right)$.

We shall prove now that, under the hypotheses of the theorem, c must be positive. Recall that

$$\phi(t,\ z, w) = \frac{1}{L\ (t,\ z,\ \pi(w))}, \qquad h(t,\ z, w) = \frac{\varphi\ (t,\ z,\ \pi(w))}{L\ (t,\ z,\ \pi(w))}$$

almost everywhere. In fact, if $c = 0$, then (3.15) implies (after a substitution $v = \pi\ (w)$) that

$$\sup_{v\in\mathbb{R}^m}\left\{\frac{\tilde{\lambda}\ (\tau) + \left\langle \tilde{p}\ (\tau)\ ,\ f\ \left(\tilde{t}\ (\tau)\ , \tilde{z}\ (\tau)\right)\right\rangle + \left\langle \tilde{p}\ (\tau)\ \ g\ \left(\tilde{t}\ (\tau)\ , \tilde{z}\ (\tau)\right), v\right\rangle}{L\ \left(\tilde{t}\ (\tau)\ , \tilde{z}\ (\tau)\ , v\right)}\right\}$$

vanishes for almost all $\tau \in \left[0,\ \tilde{T}\right]$. This obviously implies

$$\tilde{p}\ (\tau)\ g\ \left(\tilde{t}\ (\tau)\ , \tilde{z}\ (\tau)\right) \equiv 0. \tag{3.17}$$

At the same time, since L is positive, $\tilde{\lambda}\ (\tau) + \left\langle \tilde{p}\ (\tau)\ ,\ \ f\ \left(\tilde{t}\ (\tau)\ ,\ \tilde{z}\ (\tau)\right)\right\rangle$ must be nonpositive. Then, since a finite value for v implies a finite value for $L\ \left(\tilde{t}\ (\tau)\ ,\ \tilde{z}\ (\tau)\ ,\ v\right)$, one concludes

$$\tilde{\lambda}\ (\tau) + \left\langle \tilde{p}\ (\tau)\ ,\ f\ \left(\tilde{t}\ (\tau)\ , \tilde{z}\ (\tau)\right)\right\rangle = 0 \quad \text{a.e.} \tag{3.18}$$

The following proposition tell us that (3.17) and (3.18) imply that $\tilde{u}\ (\cdot)$ is an *abnormal extremal control for* (P).

Proposition 1. *If the equalities* (3.17) *and* (3.18) *hold, then the quadruple* $(\tilde{x}(\cdot), \tilde{\psi}_0, \tilde{\psi}(\cdot), \tilde{u}(\cdot))$ *defined as*

$$(\tilde{z}\ (\tilde{\tau}\ (\cdot))\ , 0,\ \tilde{p}\ (\tilde{\tau}\ (\cdot))\ , \tilde{v}\ (\tilde{\tau}\ (\cdot)))\ ,$$

where $\tilde{\tau}\ (\cdot)$ *is the inverse function of* $\tilde{t}\ (\cdot)$, *is an abnormal extremal for the problem* (P).

Proof. The respective Hamiltonian for the problem (3.8)–(3.10) equals

$$H\ (t,\ z,\ \lambda,\ p,\ v) = \frac{\lambda + \langle p,\ \ f\ (t,\ z)\rangle + \langle p\, g\ (t,\ z)\ , v\rangle}{L\ (t,\ z,\ v)}.$$

We have to verify conditions (i) and (ii) of Theorem 2.2.1 for the quadruple $\left(\tilde{x}\ (\cdot)\ , 0, \tilde{\psi}\ (\cdot)\ , \tilde{u}\ (\cdot)\right)$ defined in the formulation of the proposition. We introduce the abnormal Hamiltonian

$$\overline{H} = \langle \psi,\ \ f\ (t,\ x)\rangle + \langle \psi,\ g\ (t,\ x)\ \ u\rangle\ .$$

Direct computation shows

$$\dot{\tilde{x}}\ (t) = \frac{d}{dt}\ \{\tilde{z}\ (\tilde{\tau}\ (t))\} = \dot{\tilde{\tau}}\ (t)\, \dot{\tilde{z}}\ (\tilde{\tau}\ (t))\ ,$$

$$\dot{\tilde{z}}\ (\tilde{\tau}\ (t)) = \frac{f\ (t, \tilde{x}\ (t)) + g\ (t, \tilde{x}\ (t))\ \tilde{u}\ (t)}{L\ (t, \tilde{x}\ (t)\ , \tilde{u}\ (t))},$$

$$\dot{\tilde{\tau}}\ (t) = L\ (t, \tilde{x}\ (t)\ , \tilde{u}\ (t))\ ,$$

and hence $\dot{\tilde{x}}(t) = f(t, \tilde{x}(t)) + g(t, \tilde{x}(t))\, \tilde{u}(t)$. Also

$$\dot{\tilde{\psi}}^i(t) = \frac{d}{dt}\tilde{p}^i(\tilde{\tau}(t)) = \dot{\tilde{\tau}}(t)\, \dot{\tilde{p}}^i(\tilde{\tau}(t)).$$

Obviously $\tilde{\psi}(\cdot)$ is absolutely continuous as a composition of the absolutely continuous function $\tilde{p}(\cdot)$ with the absolutely continuous monotonous function $\tilde{\tau}(\cdot)$. For all $i = 1, \ldots, n$,

$$\begin{aligned}\dot{\tilde{p}}^i(\tau) &= -\frac{\partial H}{\partial z^i} \\ &= -\frac{\left\langle \tilde{p}(\tau),\ f_{x^i}\left(\tilde{t}(\tau),\ \tilde{z}(\tau)\right)\right\rangle + \left\langle \tilde{p}(\tau),\ g_{x^i}\left(\tilde{t}(\tau),\ \tilde{z}(\tau)\right)\ \tilde{v}(\tau)\right\rangle}{L\left(\tilde{t}(\tau),\ \tilde{z}(\tau),\ \tilde{v}(\tau)\right)} \\ &\quad + \frac{\left(\lambda(\tau) + \left\langle \tilde{p}(\tau),\ f\left(\tilde{t}(\tau),\ \tilde{z}(\tau)\right)\right\rangle\right) L_{x^i}(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau))}{L^2\left(\tilde{t}(\tau),\ \tilde{z}(\tau),\ \tilde{v}(\tau)\right)} \\ &\quad + \frac{\left\langle \tilde{p}(\tau),\ g\left(\tilde{t}(\tau),\ \tilde{z}(\tau)\right)\tilde{v}(\tau)\right\rangle L_{x^i}\left(\tilde{t}(\tau),\ \tilde{z}(\tau),\ \tilde{v}(\tau)\right)}{L^2\left(\tilde{t}(\tau),\ \tilde{z}(\tau),\ \tilde{v}(\tau)\right)}.\end{aligned}$$

The second addend vanishes by virtue of (3.17)–(3.18) and therefore

$$\begin{aligned}\dot{\tilde{\psi}}^i(t) &= -\left\langle \tilde{\psi}(t),\ f_{x^i}(t,\ \tilde{x}(t))\right\rangle - \left\langle \tilde{\psi}(t),\ g_{x^i}(t,\ \tilde{x}(t))\ \tilde{u}(t)\right\rangle \\ &= -\frac{\partial \overline{H}}{\partial x^i},\end{aligned}$$

so that (i) is fulfilled. On the other hand,

$$\left\langle \tilde{\psi}(t),\ g(t,\ \tilde{x}(t))\ u(t)\right\rangle = \langle \tilde{p}(\tilde{\tau}(t)),\ g(t,\ \tilde{z}(\tilde{\tau}(t)))\ u(t)\rangle \equiv 0,$$

and therefore

$$\left\langle \tilde{\psi}(t),\ f(t,\ \tilde{x}(t)) + g(t,\ \tilde{x}(t))\ u(t)\right\rangle = \left\langle \tilde{\psi}(t),\ f(t,\ \tilde{x}(t))\right\rangle$$

does not depend on u, so the maximality condition (ii) is fulfilled trivially (or abnormally). □

This proves that vanishing c corresponds to an abnormal extremal control of the problem (P). Thus, for minimizers which are not abnormal extremal controls, it holds that $c > 0$. From (3.15) we obtain

$$\begin{aligned}0 < c &\overset{\text{a.e.}}{=} \frac{\tilde{\lambda}(\tau) + \left\langle \tilde{p}(\tau),\ \varphi\left(\tilde{t}(\tau),\ \tilde{z}(\tau),\ \tilde{v}(\tau)\right)\right\rangle}{L\left(\tilde{t}(\tau),\ \tilde{z}(\tau),\ \tilde{v}(\tau)\right)} \\ &\Rightarrow \quad L\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right) = c^{-1}\left(\tilde{\lambda}(\tau) + \left\langle \tilde{p}(\tau), \varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)\right\rangle\right).\end{aligned}$$

Let

$$\left|\tilde{\lambda}(\tau)\right| \le M \quad \text{and} \quad \|\tilde{p}(\tau)\| \le M \qquad \text{on} \left[0,\ \tilde{T}\right].$$

Then, for any fixed $\tau \in [0, \tilde{T}]$,

$$L\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right) \le c^{-1}M\left(1 + \left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)\right\|\right)$$

and hence

$$\begin{aligned} \frac{\theta\left(\|\tilde{v}(\tau)\|\right)}{\left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)\right\|} &\le \frac{L\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)}{\left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)\right\|} \\ &\le c^{-1}M\,\frac{1 + \left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)\right\|}{\left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)\right\|} \end{aligned} \tag{3.19}$$

for almost all $\tau \in [0, \tilde{T}]$. The last term of this inequality can be majorized by $2c^{-1}M$ if $\left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)\right\| \ge 1$. From the growth condition (H2), and from the fact that $g(t, x)$ has full column rank, it follows that $\lim_{\|u\|\to+\infty} \|\varphi(t, x, u)\| = +\infty$ and, from the linearity of $\varphi(t, x, u)$ with respect to u,

$$\lim_{\|\tilde{v}(\tau)\|\to+\infty} \frac{\theta\left(\|\tilde{v}(\tau)\|\right)}{\left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)\right)\right\|} = +\infty.$$

Hence one can find r_0 such that, $\forall r \ge r_0$,

$$\left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), r\right)\right\| \ge 1 \quad \text{and} \quad \frac{\theta(r)}{\left\|\varphi\left(\tilde{t}(\tau), \tilde{z}(\tau), r\right)\right\|} \ge 2c^{-1}M.$$

Therefore for (3.19) to be satisfied there must be

$$\|\tilde{v}(\tau)\| \le r_0.$$

The proof is now complete: $\tilde{v}(\cdot) = \pi(\tilde{w}(\cdot))$ must be essentially bounded (by r_0) on $[0, \tilde{T}]$, that is, $\tilde{u}(\cdot) = \tilde{v}(\tilde{\tau}(\cdot))$ is essentially bounded on $[a, b]$. □

4. Applications to the Calculus of Variations

4.1. *Basic Problem of the Calculus of Variations*

The following result is an immediate corollary of Theorem 1.

Theorem 2. *Let $L(\cdot,\cdot,\cdot)$ be continuously differentiable on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, $a, b \in \mathbb{R}$ $(a < b)$, and $x_a, x_b \in \mathbb{R}^n$. Consider the Basic Problem of the Calculus of Variations*:

$$\begin{aligned} &J[x(\cdot)] = \int_a^b L(t, x(t), \dot{x}(t))\,dt \longrightarrow \min, \\ &x(a) = x_a, \qquad x(b) = x_b. \end{aligned} \tag{4.1}$$

Under the hypotheses:

(H1) **coercivity:** *there is a function θ: $\mathbb{R} \to \mathbb{R}$ such that*

$$L(t, x, u) \ge \theta(\|u\|) > \zeta, \qquad \zeta \in \mathbb{R},$$

for all $(t, x, u) \in \mathbb{R}^{1+n+n}$, *and*

$$\lim_{r \to +\infty} \frac{r}{\theta(r)} = 0;$$

(H2) **growth condition:** *there are constants* γ, β, η, *and* μ, *with* $\gamma > 0$, $\beta < 2$, *and* $\mu \geq \max\{\beta - 1, -1\}$, *such that, for all* $t \in [a, b]$, *and* $x, u \in \mathbb{R}^n$,

$$\left(|L_t(t, x, u)| + |L_{x^i}(t, x, u)|\right) \|u\|^{\mu} \leq \gamma L^{\beta}(t, x, u) + \eta,$$
$$i \in \{1, \ldots, n\};$$

any minimizer of the problem in the class of absolutely continuous functions is Lipschitzian on $[a, b]$.

Remark 5. There are no abnormal extremals in the Basic Problem.

Below we provide an example, which shows that this result of Lipschitzian regularity, is not covered by the previously obtained conditions, we are aware of. First we formulate

Tonelli's Existence Theorem. *If the Lagrangian* $L(\cdot, \cdot, \cdot)$ *is* C^2 *and the following conditions hold*:

(T1) $L(\cdot, \cdot, \cdot)$ *is coercive, i.e., there exist constants* $a, b > 0$ *and* $c \in \mathbb{R}$ *such that*

$$L(t, x, v) \geq a|v|^{1+b} + c \qquad \text{for all} \quad (t, x, v);$$

(T2) $L_{vv}(t, x, v) \geq 0$ *for all* (t, x, v);

then a solution to (4.1) *exists in the class of absolutely continuous functions*.

The following regularity results are due to Clarke and Vinter (see [9] and [7]) and are proven under weaker hypotheses than those we are considering here. Namely, they are valid when L is nonsmooth. Since nonsmoothness is not a phenomenon we study, we restrict ourselves to the differentiable case.

Regularity Results. Let $L(\cdot, \cdot, \cdot)$, in addition to the hypotheses of Tonelli's existence theorem, satisfy any of the conditions (C1)–(C5):

(C1) Lagrangian is autonomous (i.e., does not depend on t);
(C2) there are $k_0 \in \mathbb{R}$ and $k_1(\cdot)$ integrable such that, $\forall (t, x, v) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$|L_t(t, x, v)| \leq k_0 |L(t, x, v)| + k_1(t);$$

(C3) there are $k_0 \in \mathbb{R}$ and $k_1(\cdot)$, $k_2(\cdot)$ integrable such that, $\forall (t, x, v) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\|L_x(t, x, v)\| \leq k_0 |L(t, x, v)| + k_1(t) \|L_v(t, x, v)\| + k_2(t);$$

(C4) for each fixed t, the function $(x, v) \to L(t, x, v)$ is convex;

(C5) $L_{vv}(t, x, v) > 0$ and there exists a constant k_0 such that, $\forall (t, x, v) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\left\| L_{vv}^{-1}(t, x, v)\left(L_x(t, x, v) - L_{vt}(t, x, v) - L_{vx}(t, x, v)\, v\right)\right\|$$
$$\leq k_0 \left(\|v\|^{2+b} + 1\right),$$

where b is the positive constant that appears in the coercivity condition (T1) of Tonelli's existence theorem;

then every minimizer of the Basic Problem of the Calculus of Variations (4.1) is Lipschitzian.

Notice that (C1) is a particular case of condition (C2). The growth conditions (C3) and (C5) are generalizations of classical conditions obtained respectively by Tonelli–Morrey and Bernstein (loc. cit.).

Now we provide an example of a functional which possesses a minimizer, for which not one of conditions (C1)–(C5) is applicable, while Theorem 2 is.

Example. We consider the following problem ($n = 1$):

$$J[x(\cdot)] = \int_0^1 \left[\left(\dot{x}^4 + 1\right)^3 e^{\left(\dot{x}^4+1\right)(t+\pi/2-\arctan x)}\right] dt \to \min,$$
$$x(0) = x_0, \qquad x(1) = x_1. \tag{4.2}$$

Denoting $L(t, x, v) = \left(v^4 + 1\right)^3 e^{\left(v^4+1\right)(t+\pi/2-\arctan x)}$, we conclude that

$$L_{vv}(t, x, v) = \left[\frac{132v^6 + 36v^2}{\left(v^4+1\right)^2} + \left(\frac{96v^6}{v^4+1} + 12v^2\right)\left(t + \frac{\pi}{2} - \arctan x\right) + 16v^6\left(t + \frac{\pi}{2} - \arctan x\right)^2\right] L(t, x, v).$$

Tonelli's theorem guarantees the existence of a minimizer $\hat{x}(\cdot) \in AC$ for the problem (4.2) as long as:

- $L(\cdot, \cdot, \cdot) \in C^2$;
- for all $(t, x, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$ we have

  $$L(t, x, v) > \left(v^4 + 1\right)^3 \geq v^4 + 1 > 0;$$
- $L_{vv}(t, x, v) \geq 0$.

The assumptions of Theorem 2 are also verifiable with $\theta(r) = r^4 + 1$, $\beta = \frac{3}{2}$, $\mu = \frac{1}{2}$, $\gamma = 2$, and $\eta = 0$. Indeed,

$$L_t(t, x, v) = \left(v^4 + 1\right) L(t, x, v), \qquad L_x(t, x, v) = -\frac{v^4 + 1}{1 + x^2} L(t, x, v),$$
$$\left(|L_x(t, x, v)| + |L_t(t, x, v)|\right)|v|^{1/2}$$
$$\leq 2\left(|v|^4\right)^{1/8}\left(v^4 + 1\right) L(t, x, v) < 2\left(v^4 + 1\right)^{9/8} L(t, x, v)$$

$$= 2\left(v^4+1\right)^{33/8} e^{\left(v^4+1\right)(t+\pi/2-\arctan x)}$$
$$< 2\left(v^4+1\right)^{36/8} e^{3/2\left(v^4+1\right)(t+\pi/2-\arctan x)} = 2L^{3/2}(t,x,v)\,.$$

Theorem 2 guarantees, then, that all the minimizers of this functional are Lipschitzian.

As we shall see now, none of conditions (C1)–(C5) is applicable to this example. Indeed:

1. The Lagrangian depends explicitly on t and so condition (C1) fails.
2. If (C2) were true, one might conclude $L_t(t,x,v)/L(t,x,v) \le k_0 + k_1(t)/L(t,x,v)$, which in our case implies
$$v^4+1 \le k_0 + \frac{k_1(t)}{L(t,x,v)},$$
an inequality which fails for v sufficiently large.
3. For (C3) to hold, we should have, for $v>0$, $x=0$, and some t,
$$\frac{|L_x(t,0,v)|}{v^{7/2}L(t,0,v)} \le \frac{k_0}{v^{7/2}} + k_1(t)\,\frac{|L_v(t,0,v)|}{v^{7/2}L(t,0,v)} + \frac{k_2(t)}{v^{7/2}L(t,0,v)};$$
as far as
$$L_v(t,x,v) = \left[\frac{12\,v^3}{v^4+1} + 4v^3\left(t+\frac{\pi}{2}-\arctan x\right)\right]L(t,x,v)$$
holds one obtains the inequality
$$\frac{v^4+1}{v^{7/2}} \le \frac{k_0}{v^{7/2}} + k_1(t)\,\frac{12\,v^3/(v^4+1) + 4\,v^3(t+\pi/2)}{v^{7/2}} + \frac{k_2(t)}{v^{7/2}L(t,0,v)},$$
which fails for v sufficiently large.
4. Condition (C4) is not true either. If we fix t and v we come to the function $x \longrightarrow C^3 e^{C(B-\arctan x)}$ which is not convex: its second derivative equals
$$\frac{2x+C}{\left(1+x^2\right)^2}\,C^4 e^{C(B-\arctan x)},$$
which is not sign-definite.
5. Finally, (C5) fails, since we have $L_{vv}(t,x,v)=0$, for example, for $v=0$.

4.2. *Variational Problems with Higher-Order Derivatives*

We now consider the problem of the Calculus of Variations with higher-order derivatives:

$$\int_a^b L\left(t,\, x(t),\, \dot{x}(t),\, \ldots,\, x^{(m)}(t)\right) dt \longrightarrow \min, \tag{4.3}$$

$$\begin{array}{ll} x(a) = x_a^0, & x(b) = x_b^0, \\ \vdots & \vdots \\ x^{(m-1)}(a) = x_a^{m-1}, & x^{(m-1)}(b) = x_b^{m-1}. \end{array} \tag{4.4}$$

We use the notation $W_{k,p}$ $(k=1,\ldots;\ 1\le p\le\infty)$ for the class of functions which are absolutely continuous with their derivatives up to order $k-1$ and have kth derivative belonging to L_p. The existence of minimizers for problem (4.3)–(4.4) in the class $W_{m,1}([a, b], \mathbb{R}^n)$ will follow from classical existence results, if we impose that $L(t, x_0, \ldots, x_m)$ is convex with respect to x_m (see [3]). One can put a question of whether every minimizer $x(\cdot) \in W_{m,1}$ has an essentially bounded mth derivative, i.e., belongs to $W_{m,\infty}$. A study of higher-order regularity was done in 1990 by Clarke and Vinter in [11], where they deduced a condition of the Tonelli–Morrey type:

$$\left\| L_{x_i}(t, x_0, \ldots, x_m) \right\| \le \gamma \left(|L(t, x_0, \ldots, x_m)| + \|x_m\| \right) + \eta(t)\, r(x_0, \ldots, x_m),$$

with $i = 0, \ldots, m-1$, γ being a constant, η an integrable function, and r a locally bounded function. Once again, we are able to derive from our main result a condition of a new type for this case also.

Theorem 3. *Provided that for all* $t \in [a, b]$ *and* $x_0, \ldots, x_m \in \mathbb{R}^n$ *the function* $(t, x_0, \ldots, x_m) \to L(t, x_0, \ldots, x_m)$ *is continuously differentiable and the following conditions hold*:

(H1) **coercivity:** *there is a function* $\theta\colon \mathbb{R} \to \mathbb{R}$ *and a constant* ζ *such that*

$$L(t, x_0, \ldots, x_m) \ge \theta(\|x_m\|) > \zeta,$$

and $\lim_{r\to+\infty}(r/\theta(r)) = 0$;

(H2) **growth condition:** *for some constants* γ, β, η, *and* μ *with* $\gamma > 0$, $\beta < 2$, *and* $\mu \ge \max\{\beta - 1, -1\}$,

$$\left(|L_t| + \left\|L_{x_i}\right\|\right) \|x_m\|^{\mu} \le \gamma L^{\beta} + \eta, \qquad i \in \{0, \ldots, m-1\}; \tag{4.5}$$

then all minimizers $\tilde{x}(\cdot) \in W_{m,1}([a, b], \mathbb{R}^n)$ *of the variational problem* (4.3)–(4.4) *belong to the class* $W_{m,\infty}([a, b], \mathbb{R}^n)$.

Remark 6. There are no abnormal extremals for problem (4.3)–(4.4).

Some corollaries can be easily derived.

Corollary 2. *Any minimizer* $\tilde{x}(\cdot)$ *of the functional*

$$\mathcal{J}[x(\cdot)] = \int_a^b L\left(x^{(m)}(t)\right) dt \to \min,$$
$$x(\cdot) \in W_{m,1}\left([a, b], \mathbb{R}^n\right),$$

under the conditions (4.4), *is contained in the space* $W_{m,\infty}([a, b], \mathbb{R}^n)$, *provided* $L(\cdot)$ *is continuously differentiable*; *and, for all* $x_m \in \mathbb{R}^n$ *and some constants* $\xi \in \mathbb{R}$ *and* $\alpha \in]1, +\infty[$, $L(x_m) \ge \|x_m\|^{\alpha} + \xi$.

For $m = 1$ there exists a much stronger result than this latter corollary: if $m = 1$ and L is autonomous, $L = L(x, \dot{x})$, coercive and convex in $\dot{x}$, then all the minimizers of the problem belong to $W_{1,\infty}$ (condition (C1) of the Regularity Results).

The question of whether it can be generalized onto the case of higher-order functionals $L\left(x,\ \dot{x},\ \ldots,\ x^{(m)}\right)$ remained open till recent time. It has been shown in [15] that autonomous higher-order functionals not only may possess minimizers belonging to $W_{m,1} \backslash W_{1,\infty}$ but also exhibit the Lavrentiev phenomenon: their infimum in $W_{m,1}$ can be strictly less than the one in $W_{m,\infty}$.

Acknowledgment

The authors are grateful to A. A. Agrachev for stimulating discussions.

References

1. Ambrosio L, Ascenzi O, Buttazzo G (1989) Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands. J Math Anal Appl 142:301–316
2. Bernstein S (1912) Sur les équations du calcul des variations. Ann Sci École Norm Sup 3:431–485
3. Cesari L (1983) Optimization Theory and Applications. Springer-Verlag, New York
4. Cheng C, Mizel VJ (1996) On the Lavrentiev phenomenon for optimal control problems with second-order dynamics. SIAM J Control Optim 34:2172–2179
5. Clarke FH (1976) The Maximum Principle under minimal hypotheses. SIAM J Control Optim 14:1078–1091
6. Clarke FH (1985) Existence and regularity in the small in the Calculus of Variations. J Differential Equations 59:336–354
7. Clarke FH (1989) Methods of Dynamic and Nonsmooth Optimization. SIAM, Philadelphia, PA
8. Clarke FH, Loewen PD (1989) Variational problems with Lipschitzian minimizers. Ann Inst Poincaré Anal Non Linéaire 6:185–209
9. Clarke FH, Vinter RB (1985) Regularity properties of solutions to the basic problem in the calculus of variations. Trans Amer Math Soc 289:73–98
10. Clarke FH, Vinter RB (1986) Regularity of solutions to variational problems with polynomial Lagrangians. Bull Polish Acad Sci 34:73–81
11. Clarke FH, Vinter RB (1990) A regularity theory for variational problems with higher order derivatives. Trans Amer Math Soc 320:227–251
12. Clarke FH, Vinter RB (1990) Regularity properties of optimal controls. SIAM J Control Optim 28:980–997
13. Gamkrelidze RV (1978) Principles of Optimal Control Theory. Plenum, New York
14. Morrey CB (1966) Multiple Integrals in the Calculus of Variations. Springer-Verlag, Berlin
15. Sarychev AV (1997) First and second-order integral functionals of the calculus of variations which exhibit the Lavrentiev phenomenon. J Dynam Control Systems 3:565–588
16. Sussmann HJ (1994) A strong version of the Maximum Principle under weak hypotheses. Proc 33rd IEEE Conference on Decision and Control, Orlando, FL, pp 1950–1956
17. Tonelli L (1915) Sur une méthode directe du calcul des variations. Rend Circ Mat Palermo 39:233–264
18. Torres DM (1997) Lipschitzian regularity of minimizers in the calculus of variations and optimal control (in Portuguese). MSc thesis, Univ Aveiro, Aveiro

Accepted 15 *March* 1999