Stochastic Control Problems where Small Intervention Costs Have Big Effects

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Abstract. We study an impulse control problem where the cost of interfering in a stochastic system with an impulse of size $\zeta \in \mathbf{R}$ is given by

$$c + \lambda|\zeta|,$$

where c and λ are positive constants. We call λ the *proportional* cost coefficient and c the *intervention cost*. We find the value/cost function V_c for this problem for each $c > 0$ and we show that $\lim_{c \to 0^+} V_c = W$, where W is the value function for the corresponding singular stochastic control problem. Our main result is that

$$\frac{dV_c}{dc} = \infty \qquad \text{at} \quad c = 0.$$

This illustrates that the introduction of an intervention cost $c > 0$, however small, into a system can have a big effect on the value function: the increase in the value function is in no proportion to the increase in c (from $c = 0$).

Key Words. Impulse control, Vanishing intervention cost, Quasi-variational inequalities, Singular stochastic control, Nonrobustness feature.

AMS Classification. 93E20, 60G40, 60J65, 49J40, 35R35.

1.   Introduction

Many mathematical models make stylized assumptions which strictly speaking are not satisfied in real applications. For example, in mathematical finance it is often assumed that the transactions can be performed continuously (i.e., in continuous time) and that there are no transaction costs. In recent years models where the transaction costs are proportional to the size $|\zeta|$ of the transaction have also been studied (see, e.g., [DN] and the references therein). An even more realistic model is obtained by assuming that the interventions occur at discrete times and then with a cost given by

$$c + \lambda|\zeta|,$$

where $c \geq 0, \lambda > 0$ are constants. This leads to the *impulse control* model (see below). A natural question is how the result $V_{c,\lambda}$ (the value function) depends on the parameters c, λ. In this paper we fix $\lambda > 0$ and study how $V_c = V_{c,\lambda}$ depends on c near $c = 0$ in a specific case. It has been shown in [MR1] and [MR2] that (under some conditions) $V_c \to V_0$ as $c \to 0^+$. The purpose of this paper is to show that V_c may not be differentiable at $c = 0$: we give an example where

$$\left[\frac{dV_c}{dc}\right]_{c=0} = \infty \qquad \text{(Theorem 2.6).}$$

We now describe this in more detail.

Suppose that—if there are no interventions—the state $Y_t \in \mathbf{R}^n$ at time t of the system we consider, satisfies an Ito stochastic differential equation of the form

$$Y_t = y + \int_0^t b(Y_r)\,dr + \int_0^t \sigma(Y_r)\,dB_r; \qquad t \geq 0. \tag{1.1}$$

Here $b: \mathbf{R}^n \to \mathbf{R}^n$ and $\sigma: \mathbf{R}^n \to \mathbf{R}^{n \times m}$ are given Lipschitz continuous functions, $B_t = B_t(\omega); t \geq 0, \omega \in \Omega$, is a Brownian motion in $\mathbf{R}^m$ with filtration $\mathcal{F}_t$ and probability law P when starting at $0 \in \mathbf{R}^m$. The point $y \in \mathbf{R}^n$ is the starting value of the system at time 0. (See, e.g., [Ø] for background on stochastic differential equations.)

An *impulse control* for this system is a (possibly finite) sequence

$$v = (\tau_1, \tau_2, \ldots, \tau_k, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_k \ldots)_{k \leq N} \qquad (N \leq \infty), \tag{1.2}$$

where $0 \leq \tau_1 \leq \tau_2 \leq \cdots$ are $\mathcal{F}_t$-stopping times and $\zeta_1, \zeta_2, \ldots$ belong to a given set $Z \subset \mathbf{R}^l$. We interpret $\tau_1, \tau_2, \ldots$ as the *intervention times*, i.e., the times when we decide to intervene. The quantities $\zeta_1, \zeta_2, \ldots$ are the *impulses* we give the system at the times $\tau_1, \tau_2, \ldots$, respectively. If the system is in state $x \in \mathbf{R}^n$ when it gets the impulse $\zeta \in Z$, we assume that it jumps immediately to a new state $x + g(\zeta)$, where $g: Z \to \mathbf{R}^n$ is a given function.

If the impulse control v given by (1.2) is applied to the system $\{Y_t\}$, it gets the form $\{Y_t^v\}$, which is given by

$$Y_t^v = y + \int_0^t b(Y_r^v)\,dr + \int_0^t \sigma(Y_r^v)\,dB_r + \sum_{\tau_k \leq t} g(\zeta_k); \qquad 0 \leq t < T^*. \tag{1.3}$$

We still call $y = Y_{0-}^v$ the *starting point* of Y_t at $t = 0$, although we need not have $Y_0 = y$, because Y_t could possibly jump at $t = 0$. Here $T^* = T^*(\omega)$ is the *explosion time* of the process Y_t^v, defined by

$$T^*(\omega) = \lim_{R \to \infty} (\inf\{t > s; \ |Y_t^v(\omega)| \geq R\}) \leq \infty. \tag{1.4}$$

Let $Q_y = Q_y^v$ denote the law of the stochastic process $\{Y_t^v\}_{t \geq 0}$ starting at $y = Y_{0-}$ at time $t = 0$.

Suppose we are given a family $\mathcal{V}$ of impulse controls, called the *admissible* impulse controls. We assume that all $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$ satisfy

$$\tau_k \to T^* \qquad \text{a.s. } Q_y \quad \text{for all} \quad y \in \mathbf{R}^n. \tag{1.5}$$

Suppose that the *cost rate* when the system is in state x is $f(x)$, where $f \colon \mathbf{R}^n \to \mathbf{R}$ is a given function. Moreover, suppose that the *impulse cost* $K(x, \zeta)$ of giving the system the impulse $\zeta \in Z$ is given by

$$K(x, \zeta) = c + \lambda|\zeta| \qquad \text{(independent of } x\text{)}, \tag{1.6}$$

where $c \geq 0, \lambda > 0$ are constants. Thus the impulse cost consists of a *fixed, minimum cost* c (the *intervention cost*) plus a cost $\lambda|\zeta|$ proportional to the size $|\zeta|$ of the intervention ξ. Then if we apply $v = (\tau_1, \tau_2, \ldots, \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$, we get the *performance* or *total expected cost* $J_c^v(y)$ defined by

$$J_c^v(y) = E_y \left[\int_0^\infty f(Y_t^v) \, dt + \sum_{k=1}^N (c + \lambda|\zeta_k|) e^{-\rho \tau_k} \right], \tag{1.7}$$

where E_y denotes expectation with respect to Q_y. The corresponding impulse control problem is the following:

Problem 1.1 (General Impulse Control Problem).   Find the value function $V_c = V_c(y)$ defined by

$$V_c(y) = \inf_{v \in \mathcal{V}} J_c^v(y), \qquad y \in \mathbf{R}^n, \tag{1.8}$$

and find an *optimal* $v^* \in \mathcal{V}$ such that

$$V_c(y) = J_c^{v^*}(y), \qquad y \in \mathbf{R}^n. \tag{1.9}$$

We refer the reader to [BL] for more information about impulse control and its relation to quasi-variational inequalities (see Theorem 2.1).

A study of impulse control problems for diffusions with jumps can be found in [M].

Note that Problem 1.1 also makes sense if $c = 0$. The infimum is still taken over impulse controls with finitely many jumps in compact time intervals in $[0, T^*)$. However, if $c = 0$ it is conceivable that no optimal impulse control $v^* \in \mathcal{V}$ exists: Since the number of interventions do not matter, only the sizes of them, it is natural to guess that the infimum is obtained by letting the number of interventions go to infinity and at the same time the sizes of the interventions go to zero. This would in the limit bring us to a control outside $\mathcal{V}$. We will show that this is indeed the case in the problem we consider.

From now on we assume that we only allow nonnegative impulses, i.e., that

$$Z = [0, \infty), \tag{1.10}$$

and that the impulse ζ acts linearly on just one of the components of Y_t, say the last one:

$$g(\zeta) = (0, \dots, b\zeta) \in \mathbf{R}^n \qquad \text{for} \quad \zeta \in Z, \tag{1.11}$$

where $b \in \mathbf{R}$ is a constant.

Then if $c = 0$ it is also natural to model the problem above as a *singular stochastic control* problem, as follows:

Let our space Γ of admissible controls consist of all $\mathcal{F}_t$-adapted, t-right continuous processes $\gamma_t(\omega)$ such that the function $t \to \gamma_t(\omega)$ is nondecreasing for a.a. $\omega \in \Omega$. We can associate to $\gamma_t(\omega)$ the measure $d\gamma_t(\omega)$ with the property that

$$\int_{s^+}^t d\gamma_r(\omega) := \int_{(s,t]} d\gamma_r(\omega) = \gamma_t(\omega) - \gamma_s(\omega); \qquad t > s. \tag{1.12}$$

Note that with this notation we have

$$\int_s^t d\gamma_r(\omega) = \Delta\gamma_s(\omega) + \gamma_t(\omega) - \gamma_s(\omega), \tag{1.13}$$

where in general

$$\Delta\gamma_{t_i}(\omega) = \gamma_{t_i}(\omega) - \gamma_{t_i^-}(\omega) \tag{1.14}$$

is the jump of $\gamma_t(\omega)$ at $t = t_i$.

In analogy with (1.3) we now assume that if the admissible control $\gamma_t \in \Gamma$ is applied to the system $\{Y_t\}_{t \geq 0}$, it gets the form $\{Y_t^\gamma\}_{t \geq 0}$ given by the equation (using (1.11))

$$Y_t^\gamma = y + \int_0^t b(Y_r^\gamma)\, dr + \int_0^t \sigma(Y_r^\gamma)\, dB_r + \left(0, \dots, b\int_0^t d\gamma_r\right); \quad t \geq 0. \tag{1.15}$$

Note that this coincides with Y_t^v given by (1.3) if γ_t only increases at $t = \tau_k$ and then performs jumps of size ζ_k, $k = 1, 2, \dots$. Furthermore, in analogy with (1.7) for $c = 0$, we assume that the cost of applying the control $\gamma \in \Gamma$ is given by

$$J^\gamma(y) = E_y\left[\int_0^\infty f(Y_t^\gamma)\, dt + \lambda\int_0^\infty e^{-\rho t}\, d\gamma_t\right]. \tag{1.16}$$

Hence the singular stochastic control problem corresponding to $c = 0$ is the following:

Problem 1.2 (General Singular Stochastic Control Problem).   Find the value function $W(y)$ defined by

$$W(y) = \inf_{\gamma \in \Gamma} J^\gamma(y), \qquad y \in \mathbf{R}^n, \tag{1.17}$$

and find an *optimal* $\gamma^* \in \Gamma$ such that

$$W(y) = J^{\gamma^*}(y), \qquad y \in \mathbf{R}^n. \tag{1.18}$$

Since we can regard the set of impulse controls $v \in \mathcal{V}$ as a subset of the set of singular stochastic controls $\gamma \in \Gamma$ (by identifying an impulse $\zeta > 0$ with the jump $\Delta \gamma = \zeta$), we have

$$W(y) \leq V_c(y) \qquad \text{for all} \quad c \geq 0. \tag{1.19}$$

It was proved in Theorems 2.1 and 2.5 of [MR1] and in [MR2] (if $\lambda = 0$) that

$$W(y) = V_0(y) = \lim_{c \to 0^+} V_c(y). \tag{1.20}$$

In particular, the function $c \to V_c(y)$ is continuous at $c = 0$.

In this paper we study a specific impulse control problem of the form (1.8)–(1.9) in detail (see Problem 2.2) and we investigate how $V_c(y)$ and other quantities of the problem depends on c. Our main result is that $V_c(y)$ is not differentiable at $c = 0$. More precisely, we show that

$$\lim_{c \to 0^+} \frac{d}{dc} V_c(y) = \left[\frac{dV_c}{dc}\right]_{c=0} = \infty \qquad \text{for all } y \tag{1.21}$$

(see Theorem 2.6).

This means that going from $c = 0$ to a small positive c causes an increase in the value function which is in no proportion to the increase in c.

We also solve explicitly the corresponding singular stochastic control problem when $c = 0$ (see Problem 3.1) and we show that the reflecting barrier $\widehat{x}$ in the solution of this problem is the common limit of the impulse barriers $x_0(c)$ and $x_1(c)$ of the impulse problem as $c \to 0$. (See Theorem 3.3.)

This work was motivated by a discussion of a combined stochastic control and impulse control of a problem in [MØ], where a property similar to (1.21) was deduced under certain conditions. (See (4.47) and (4.48) in [MØ].)

This work is also related to [JS], where it is proved that the limit of the impulse barriers as $c \to 0$ (of a different impulse control problem) is equal to the reflecting barrier of a corresponding singular control problem. However, the property (1.21) is not studied there.

It is natural to conjecture that the statement (1.21) holds not just for the case we consider, but more generally for a large class of impulse control problems where the cost of interferring has the form given by (1.6). This will be discussed in a forthcoming paper by Ubøe, Zhang, and the author [ØUZ].

2. The Impulse Control Problem

We begin by stating a verification theorem of quasi-variational type for impulse control problems. The result is a special case of Theorem 3.1 in [BØ2], which again is an elaboration of results in [BL]. First we introduce some notation.

Define the *intervention operator* $\mathcal{N}$ on the space of functions $h: \mathbf{R}^n \to \mathbf{R}$ by

$$\mathcal{N}h(x) = \inf_{\zeta \in Z} \{h(x + g(\zeta)) + c + \lambda|\zeta|\}. \tag{2.1}$$

Suppose that for each $x \in \mathbf{R}^n$ there exists at least one $\hat{\zeta} = \hat{\zeta}(x) \in Z$ such that the infimum in (2.1) is attained and that a *measurable selection* $\zeta = \mathcal{R}_h(x)$ of such minimum points $\hat{\zeta}$ exists. Then we have

$$\mathcal{N}h(x) = h(x + \mathcal{R}_h(s, x)) + c + \lambda|\mathcal{R}_h(s, x)|. \tag{2.2}$$

Note that if we do not have any interventions, then Y_t is an Ito diffusion with a generator A which coincides on $C_0^2(\mathbf{R}^n)$ with the partial differential operator

$$L = \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j}. \tag{2.3}$$

As is customary, we let $C^1(\mathbf{R}^n)$ (resp. $C^2(\mathbf{R}^n)$) denote the set of all once (resp. twice) continuously differentiable functions on $\mathbf{R}^n$, and $C_0^2(\mathbf{R}^n)$ denotes the set of functions in $C^2(\mathbf{R}^n)$ with compact support.

The *Green measure* of Y_t is the "total expected occupation measure" $G = G_Y(\cdot, y)$ defined for each $y \in \mathbf{R}^n$ by

$$G_Y(F, y) = E_y\left[\int_0^\infty \mathcal{X}_F(Y_t)\,dt\right]; \qquad F \subset \mathbf{R}^n \quad \text{(Borel).} \tag{2.4}$$

We say that a continuous function $f\colon \mathbf{R}^n \to \mathbf{R}$ is *stochastically C^2* (with respect to Y) if $Lf(x)$ is defined (pointwise) for a.a. x with respect to the Green measure $G_Y(\cdot, y)$ and the *generalized Dynkin formula* holds for f:

$$E_y[f(Y_\tau)] = f(y) + E_y\left[\int_0^\tau Lf(Y_t)\,dt\right], \qquad y \in \mathbf{R}^n, \tag{2.5}$$

for all bounded stopping times τ which are less than the exit time for Y_t from some compact set. By the classical Dynkin formula all $C^2(\mathbf{R}^n)$ functions are stochastically C^2. More generally, functions which are C^2 except on a "small" set (for Y_t) are stochastically C^2. See [BØ1] for more details.

Theorem 2.1 [BØ2] (Sufficient Quasi-Variational Inequalities for Impulse Control). *Let $c \geq 0$.*

(a) *Suppose we can find a continuous function $\varphi\colon \mathbf{R}^n \to \mathbf{R}$, $\varphi \geq 0$, such that*:

$$\varphi \quad \text{is stochastically } C^2 \text{ with respect to } Y. \tag{2.6}$$

$$L\varphi(y) + f(y) \geq 0 \quad \text{for a.a. } y \text{ with respect to } G_Y(\cdot, z) \quad \text{for all} \quad z \in \mathbf{R}^n. \tag{2.7}$$

$$\varphi(y) \leq \mathcal{N}\varphi(y) \quad \text{for all } y \in \mathbf{R}^n \tag{2.8}$$

$$\lim_{t\to\infty} \varphi(Y_t^v) = 0 \qquad \text{a.s. } Q_y^v \quad \text{for all} \quad v \in \mathcal{V} \quad \text{and all} \quad y \in \mathbf{R}^n. \tag{2.9}$$

The family $\{\varphi(X_\tau^v); \tau \text{ stopping time}\}$ is Q_y-uniformly integrable

$$\text{for all} \quad y \in \mathbf{R}^n \quad \text{and all} \quad v \in \mathcal{V}. \tag{2.10}$$

Then

$$\varphi(y) \leq J_c^v(y) \quad \text{for all } v \in \mathcal{V}, \, y \in \mathbf{R}^n,$$

where $J_c^v(y)$ is defined by (1.7).
   (b) *Define the* continuation region *D by*

$$D = \{y; \varphi(y) < \mathcal{N}\varphi(y)\}. \tag{2.11}$$

Suppose that, in addition to (2.6)–(2.10) above, we have

$$L\varphi(y) + f(y) = 0 \quad \text{for all} \quad y \in D. \tag{2.12}$$

Define the impulse control

$$\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \ldots; \hat{\zeta}_1, \hat{\zeta}_2, \ldots)$$

inductively as follows:
   Put $\hat{\tau}_0 = 0$ and

$$\hat{\tau}_{k+1} = \inf\{t > \hat{\tau}_k; Y_t^{(k)} \notin D\}, \tag{2.13}$$

$$\hat{\zeta}_{k+1} = \mathcal{R}_\varphi \left(Y_{\hat{\tau}_{k+1}^-}^{(k)} \right), \tag{2.14}$$

where $Y_t^{(k)}$ is the result of applying the impulse control $\hat{v} := (\hat{\tau}_1, \ldots, \hat{\tau}_k; \hat{\zeta}_1, \ldots, \hat{\zeta}_k)$, $k = 1, 2, \ldots$, to the process Y_t. Suppose

$$\hat{\tau} \in \mathcal{V} \tag{2.15}$$

Then

$$\varphi(y) = J_c^{\hat{v}}(y) \quad \text{for all } y. \tag{2.16}$$

Hence

$$\varphi(y) = V_c(y) \quad (\text{defined by (1.8)}) \tag{2.17}$$

and therefore

$$v^* = \hat{v} \quad \text{is optimal (i.e., satisfies (1.9)).} \tag{2.18}$$

A Special Case.   We now apply this to a special case. The following impulse control problem is related to the problems studied in [HST] and [V]. When $\lambda = 0$ this problem was solved in Section 1 of [MR1].
   If there are no interventions, we assume that the system Y_t is given by

$$Y_t = (s + t, x + B_t), \quad t \geq 0, \tag{2.19}$$

where B_t is a one-dimensional Brownian motion starting at 0 (so Y_t starts at $y = (s, x)$). If the impulse control $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$ is applied to $\{Y_t\}$, it gets the form

$$Y_t^v = (s + t, X_t^v), \tag{2.20}$$

where

$$X_t^v = x + B_t - \sum_{\tau_k \le t} \zeta_k; \qquad 0 \le t < T^* = \infty. \tag{2.21}$$

We assume that the cost of applying the impulse $\zeta \ge 0$ is given by

$$K(\zeta) = c + \lambda\zeta, \tag{2.22}$$

where $c \ge 0, \lambda \ge 0$ are constants. Assume that the cost rate is

$$f(r, \xi) = e^{-\rho r}\xi^2. \tag{2.23}$$

This leads to a total expected cost $J_c^v(s, x)$ given by

$$J_c^v(s, x) = E_{s,x}\left[\int_0^\infty e^{-\rho(s+t)}(X_t^v)^2\, dt + \sum_k (c + \lambda|\zeta_k|)e^{-\rho(s+\tau_k)}\right]. \tag{2.24}$$

Note that it is not optimal to move X_t downward if X_t is already below 0. Hence it is enough to consider impulse controls $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ such that $\zeta_k \le x + B_{\tau_k} - \sum_{j=1}^{k-1}\zeta$; i.e., $\sum_{j=1}^{k}\zeta_j \le x + B_{\tau_k}$. We let the family of such v satisfying (1.5) be our admissible family $\mathcal{V}$.

Problem 2.2 (Special Impulse Control Problem).   Let $J_c^v(s, x)$ be as in (2.24) and let X_t^v be as in (2.21). For all $c > 0$ find the value function $V_c(s, x)$ and the optimal impulse control $v^* = v_c^* \in \mathcal{V}$ such that

$$V_c(s, x) = \inf_{v \in \mathcal{V}} J_c^v(s, x) = J_c^{v^*}(s, x). \tag{2.25}$$

In order to solve this problem we make some guesses about the value function $V_c(s, x)$ and the continuation region D given by (2.11). Then we verify that this is indeed the solution by applying Theorem 2.1.

First, as a candidate $\varphi(s, x) = \varphi_c(s, x)$ for $V_c(s, x)$ we try to put

$$\varphi(s, x) = e^{-\rho s}\psi(x) \tag{2.26}$$

and as a candidate for the continuation region D we try

$$D = \{(s, x); x < x_1\}, \tag{2.27}$$

where $\psi(x)$ and $x_1 \in \mathbf{R}$ remain to be determined. With this choice of φ we get

$$\begin{aligned}
L\varphi(s, x) + f(s, x) &= \frac{\partial\varphi}{\partial s} + \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2} + e^{-\rho s}x^2 \\
&= e^{-\rho s}\left(\tfrac{1}{2}\psi''(x) - \rho\psi(x) + x^2\right),
\end{aligned}$$

so (2.12) indicates that we choose $\psi(x)$ in $(-\infty, x_1)$ as a solution $h(x)$ of the equation

$$\tfrac{1}{2}h''(x) - \rho h(x) + x^2 = 0. \tag{2.28}$$

The general solution of (2.28) is

$$h(x) = C_1 e^{\sqrt{2\rho} x} + C_2 e^{-\sqrt{2\rho} x} + \frac{1}{\rho} x^2 + \frac{1}{\rho^2}, \tag{2.29}$$

where C_1, C_2 are arbitrary constants.

If we make no interventions, then the expected total cost is

$$J^0(s, x) = E_{s,x} \left[\int_0^\infty e^{-\rho(s+t)} (B_t^x)^2 \, dt \right] = e^{-\rho s} \int_0^\infty e^{-\rho t} (x^2 + t) \, dt$$

$$= e^{-\rho s} \left(\frac{1}{\rho} x^2 + \frac{1}{\rho^2} \right).$$

Hence we must have

$$0 \le \psi(x) \le \frac{1}{\rho} x^2 + \frac{1}{\rho^2} \qquad \text{for all} \quad x \le x_1,$$

and comparing with (2.29) we see that the last inequality is only possible if $C_2 \le 0$. The first inequality rules out that $C_2 < 0$. Hence $C_2 = 0$. Then $C_1 \le 0$ and we put

$$\psi(x) = \frac{1}{\rho} x^2 + \frac{1}{\rho^2} - a e^{\sqrt{2\rho} x} \qquad \text{for} \quad x \le x_1, \tag{2.30}$$

where $a = -C_1 \ge 0$. We guess that $a > 0$.

For $x \ge x_1$ we have $\varphi(x) = \mathcal{N}\varphi(x)$ and hence

$$\psi(x) = \inf_{\zeta > 0} \{ \psi(x - \zeta) + c + \lambda \zeta \}. \tag{2.31}$$

To find the minimum of $h(\zeta) := \psi(x - \zeta) + c + \lambda \zeta$ we look for $\hat{\zeta} = \hat{\zeta}(x)$ such that $h'(\hat{\zeta}) = -\psi'(x - \hat{\zeta}) + \lambda = 0$, i.e.,

$$\psi'(x_0) = \lambda, \tag{2.32}$$

where

$$x_0 = x - \hat{\zeta}(x). \tag{2.33}$$

From (2.31) we get that $\psi(x) = \psi(x - \hat{\zeta}(x)) + c + \lambda \hat{\zeta}(x)$, i.e.,
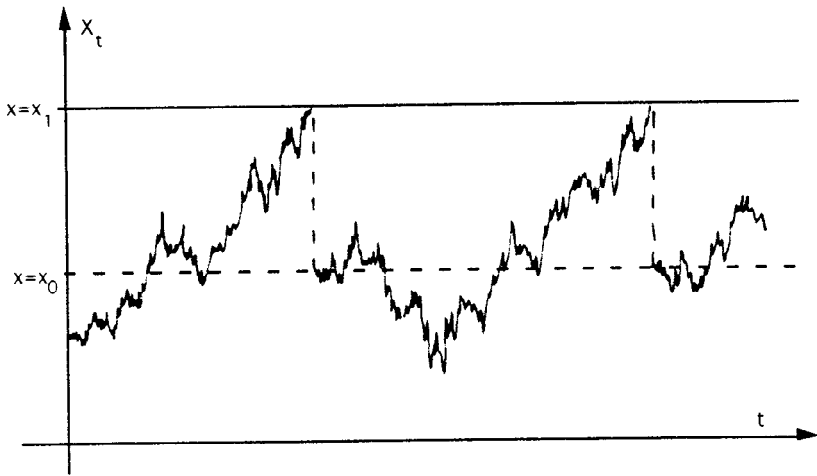
$$\psi(x) = \psi(x_0) + c + \lambda(x - x_0) \qquad \text{for} \quad x \ge x_1. \tag{2.34}$$

In particular,

$$\psi'(x_1) = \lambda \tag{2.35}$$

and

$$\psi(x_1) = \psi(x_0) + c + \lambda(x_1 - x_0). \tag{2.36}$$

To summarize we put

$$\psi(x) = \begin{cases} \dfrac{1}{\rho}x^2 + \dfrac{1}{\rho^2} - ae^{\sqrt{2\rho}\,x} & \text{for} \quad x \le x_1, \\ \psi(x_0) + c + \lambda(x - x_0) & \text{for} \quad x \ge x_1, \end{cases} \tag{2.37}$$

where x_0, x_1, and a are determined by (2.32), (2.35), and (2.36), i.e.,

$$a\sqrt{2\rho}\, e^{\sqrt{2\rho}\,x_0} = \frac{2}{\rho}x_0 - \lambda, \tag{2.38}$$

$$a\sqrt{2\rho}\, e^{\sqrt{2\rho}\,x_1} = \frac{2}{\rho}x_1 - \lambda, \tag{2.39}$$

$$ae^{\sqrt{2\rho}\,x_1} - ae^{\sqrt{2\rho}\,x_0} = \frac{1}{\rho}(x_1^2 - x_0^2) - c - \lambda(x_1 - x_0), \tag{2.40}'$$

where $0 \le x_0 < x_1$ and $a > 0$.

Note that by subtracting (2.38) from (2.39) and rearranging we get

$$(x_1 - x_0)(x_1 + x_0 - 2\hat{x}) = c\rho, \tag{2.40}$$

where

$$\hat{x} := \frac{\rho x}{2} + \frac{1}{\sqrt{2\rho}}. \tag{2.41}$$

To study the solutions of (2.38)–(2.40), we first consider the function

$$g(x) = g_a(x) = a\sqrt{2\rho}\, e^{\sqrt{2\rho}\,x} - \frac{2}{\rho}x + \lambda \qquad \text{for fixed} \quad a > 0. \tag{2.42}$$

Note that

$$g'(x) = 2\rho a e^{\sqrt{2\rho}\,x} - \frac{2}{\rho}$$

so that

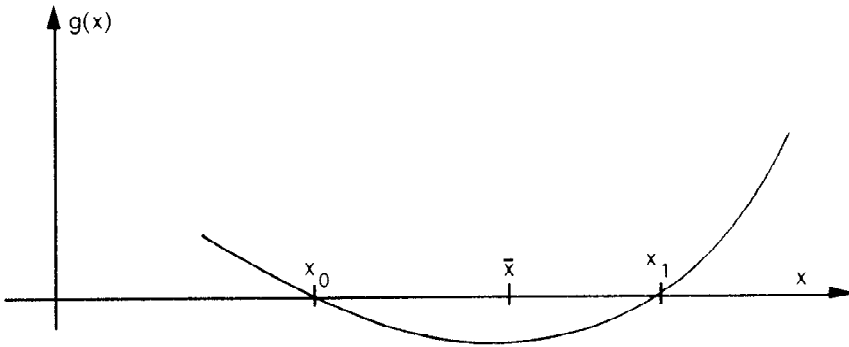$$g'(x) = 0 \quad \Leftrightarrow \quad 2\rho a e^{\sqrt{2\rho}\,x} = \frac{2}{\rho}, \tag{2.43}$$

which has a unique solution $x = \overline{x} = \overline{x}(a)$.

Since

$$g''(x) = \sqrt{2\rho}\, 2\rho a e^{\sqrt{2\rho}\,x} > 0 \qquad \text{for all } x, \tag{2.44}$$

we see that $x = \overline{x}$ is a minimum point for $g(x)$. Then note that from (2.43) we have

$$g(\overline{x}) < 0 \Leftrightarrow a\sqrt{2\rho}\, e^{\sqrt{2\rho}\,\overline{x}} - \frac{2}{\rho}\overline{x} + \lambda < 0$$

$$\Leftrightarrow \frac{2}{\rho\sqrt{2\rho}} - \frac{2}{\rho}\overline{x} + \lambda < 0 \quad \Leftrightarrow \quad \overline{x} > \frac{\rho\lambda}{2} + \frac{1}{\sqrt{2\rho}} = \hat{x}. \tag{2.45}$$

Equivalently, since by (2.43)

$$a = \frac{1}{\rho^2} e^{-\sqrt{2\rho}\,\overline{x}}, \tag{2.46}$$

we get that $g(\overline{x}) < 0$ if and only if $a < A$, where

$$A = \frac{1}{\rho^2} \exp\left(-1 - \frac{\lambda\rho\sqrt{2\rho}}{2}\right) = \frac{1}{\rho^2} \exp(-\sqrt{2\rho}\,\hat{x}). \tag{2.47}$$

From this we conclude that (2.38) and (2.39) have exactly two solutions $x_0 = x_0(a)$ and $x_1 = x_1(a)$ such that

$$0 < x_0(a) < \overline{x}(a) < x_1(a)$$

if and only if $0 < a < A$.

From now on assume that $0 < a < A$. To study how $x_0 = x_0(a)$ depends on a we differentiate (2.38) with respect to a and get

$$\sqrt{2\rho}\, e^{\sqrt{2\rho}\,x_0(a)} + a2\rho e^{\sqrt{2\rho}\,x_0(a)} x_0'(a) = \frac{2}{\rho} x_0'(a)$$

or

$$x_0'(a) = \frac{\sqrt{2\rho}e^{\sqrt{2\rho}\,x_0(a)}}{2/\rho - 2\rho ae^{\sqrt{2\rho}\,x_0(a)}} = \frac{\sqrt{2\rho}\,e^{\sqrt{2\rho}\,x_0(a)}}{-g'(x_0)} > 0. \tag{2.48}$$

Similarly

$$x_1'(a) = \frac{\sqrt{2\rho}\,e^{\sqrt{2\rho}\,x_1(a)}}{2/\rho - 2\rho ae^{\sqrt{2\rho}\,x_1(a)}} = \frac{\sqrt{2\rho}\,e^{\sqrt{2\rho}\,x_1(a)}}{-g'(x_1)} < 0. \tag{2.49}$$

Hence $x_0(a)$ decreases as $a \to 0$ and from (2.38) it follows that

$$\lim_{a\to 0} x_0(a) = \frac{\lambda\rho}{2}. \tag{2.50}$$

On the other hand, $x_1(a)$ *increases* as $a \to 0$ and from (2.39) it follows that

$$\lim_{a\to 0} x_1(a) = \lim_{a\to 0} ae^{\sqrt{2\rho}\,x_1(a)} = \infty. \tag{2.51}$$

If $a \to A$, then $\overline{x}(a) \to \rho\lambda/2 + 1/\sqrt{2\rho}$ by (2.45) and (2.47) and hence

$$g_a(\overline{x}(a)) = \frac{\sqrt{2\rho}}{\rho^2} - \frac{2}{\rho}\overline{x}(a) + \lambda \to 0,$$

while $g_a''(\overline{x}(a))$ is bounded away from 0. Therefore

$$\lim_{a\to A} x_0(a) = \lim_{a\to A} x_1(a) = \lim_{a\to A} \overline{x}(a) = \frac{\rho\lambda}{2} + \frac{1}{\sqrt{2\rho}} = \hat{x}. \tag{2.52}$$

Moreover, we claim that

$$x_0'(a) + x_1'(a) < 0 \qquad \text{for all} \quad a \in (0, A). \tag{2.53}$$

To prove this we use (2.48), (2.49) and rewrite (2.53) as (with $x_0 = x_0(a)$, $x_1 = x_1(a)$)

$$\frac{e^{\sqrt{2\rho}\,x_0}}{2/\rho - 2\rho ae^{\sqrt{2\rho}\,x_0}} < \frac{e^{\sqrt{2\rho}\,x_1}}{2\rho ae^{\sqrt{2\rho}\,x_1} - 2/\rho}.$$

By (2.46) this inequality is equivalent to

$$e^{\sqrt{2\rho}\,x_0}(e^{\sqrt{2\rho}(x_1-\overline{x})} - 1) < e^{\sqrt{2\rho}\,x_1}(1 - e^{\sqrt{2\rho}(x_0-\overline{x})})$$

or, by multiplying with $e^{\sqrt{2\rho}(\overline{x}-x_0)}$,

$$e^{\sqrt{2\rho}\,\overline{x}}(e^{\sqrt{2\rho}(x_1-\overline{x})} - 1) < e^{\sqrt{2\rho}\,x_1}(e^{\sqrt{2\rho}(\overline{x}-x_0)} - 1). \tag{2.54}$$

Since $g'''(x) = 4\rho^2 ae^{\sqrt{2\rho}\,x} > 0$ for all x we must have

$$x_1 - \overline{x} < \overline{x} - x_0$$

and since we also have $\overline{x} < x_1$, we conclude that (2.54)—and hence (2.53)—holds.

We have now proved the following:

Lemma 2.3. *For all $a \in (0, A)$ there exists a unique solution $x_0 = x_0(a) < x_1 = x_1(a)$ of (2.38) and (2.39). Moreover,*

$$x_0'(a) > 0, \quad x_1'(a) < 0, \qquad x_0'(a) + x_1'(a) < 0, \tag{2.55}$$

$$\lim_{a \to 0} x_0(a) = \frac{\lambda\rho}{2}, \qquad \lim_{a \to 0} x_1(a) = \infty, \tag{2.56}$$

$$\lim_{a \to A} x_0(a) = \lim_{a \to A} x_1(a) = \hat{x} = \frac{\lambda\rho}{2} + \frac{1}{\sqrt{2\rho}} \; . \tag{2.57}$$

Accordingly,

$$\frac{\lambda\rho}{2} < x_0(a) < \hat{x} < x_1(a) \tag{2.58}$$

and

$$x_0(a) + x_1(a) > 2\hat{x} \qquad \text{for all} \quad a \in (0, A). \tag{2.59}$$

Using this result we get the existence of a solution a, x_0, x_1 of (2.38)–(2.40):

Lemma 2.4. *For all $c > 0$ there exists $a = a^*(c) \in (0, A)$ such that (2.40) holds. With this choice of $a = a^*$ the triple $a = a^*$, $x_0 = x_0(a^*)$, $x_1 = x_1(a^*)$ is a solution of the system (2.38)–(2.40).*

Proof. Let $L(a) = (x_1(a) - x_0(a))(x_1(a) + x_0(a) - 2\hat{x})$ denote the left-hand side of (2.40). Then by Lemma 2.3 we have

$$\lim_{a \to 0} L(a) = \infty, \qquad \lim_{a \to A} L(a) = 0.$$

Hence, for all $c > 0$ there exists $a = a^*(c)$ such that $L(a) = c\rho$. Now use this value of a in Lemma 2.3. $\qquad\square$

We can summarize this as follows:

Theorem 2.5. *For $c > 0$ let $a^* = a^*(c)$, $x_0 = x_0(a^*)$, and $x_1 = x_1(a^*)$ be as in Lemma 2.4. Define*

$$\varphi_c(s, x) = e^{-\rho s} \psi_c(x) \tag{2.60}$$

with

$$\psi_c(x) = \begin{cases} \dfrac{1}{\rho}x^2 + \dfrac{1}{\rho^2} - a^* e^{\sqrt{2\rho}\,x} & \text{for} \quad x \le x_1(a^*), \\ \psi(x_0(a^*)) + c + \lambda(x - x_0(a^*)) & \text{for} \quad x > x_1(a^*). \end{cases} \tag{2.61}$$

Then φ_c satisfies all the conditions of Theorem 2.1. Hence φ_c solves Problem 2.2, i.e.,

$$\varphi_c(s, x) = V_c(s, x) \qquad \text{defined in (2.25)} \tag{2.62}$$

and the following impulse control $v^* = (\tau_1^*, \tau_2^*, \ldots; \zeta_1^*, \zeta_2^*, \ldots)$ is optimal:
   Set $\tau_0^* = 0$ and define inductively, as in (2.13),

$$\tau_{k+1}^* = \inf\{t > \tau_k^*; X_t^{(k)} \geq x_1(a^*)\} \tag{2.63}$$

and (from (2.33))

$$\zeta_{k+1}^* = \hat{\zeta}(x_1(a^*)) = x_1(a^*) - x_0(a^*); \qquad k = 0, 1, 2, \ldots. \tag{2.64}$$

Remark.   (a) Note that since ζ_{k+1}^* does not depend on k, we clearly have that

$$\tau_k^* \to \infty \qquad \text{as} \quad k \to \infty$$

and therefore $v^* \in \mathcal{V}$.
   (b) It follows from Theorem 2.5 that a^* must be unique (and hence $x_0(a^*)$ and $x_1(a^*)$), because the corresponding $\varphi_c(y)$ is the (unique) value function $V_c(y)$.

   We proceed to study how the solution depends on $c > 0$. Let $a = a(c)$, $x_0 = x_0(c)$, and $x_1 = x_1(c)$ denote the unique solution we have found for (2.38)–(2.40). Differentiating (2.38) and (2.39) with respect to c we get, with $a' = a'(c)$, $x_0' = x_0'(c)$, and $x_1' = x_1'(c)$:

$$a'\sqrt{2\rho}e^{\sqrt{2\rho}\,x_0} + a2\rho e^{\sqrt{2\rho}\,x_0} \cdot x_0' = \frac{2}{\rho}x_0', \tag{2.65}$$

$$a'\sqrt{2\rho}\,e^{\sqrt{2\rho}\,x_1} + a2\rho e^{\sqrt{2\rho}\,x_1} \cdot x_1' = \frac{2}{\rho}x_1'. \tag{2.66}$$

   Differentiating (2.40) with respect to c gives

$$(x_1' - x_0')[x_1 + x_0 - 2\hat{x}] + (x_1 - x_0)[x_1' + x_0'] = \rho$$

or

$$(2x_1 - \lambda\rho)x_1' = (2x_0 - \lambda\rho)x_0' + \frac{2}{\sqrt{2\rho}}(x_1' - x_0') + \rho. \tag{2.67}$$

Now (2.65) and (2.66) can be written, using (2.38) and (2.39),

$$a'\sqrt{2\rho}\,e^{\sqrt{2\rho}\,x_0} + \sqrt{2\rho}\left(\frac{2}{\rho}x_0 - \lambda\right)x_0' = \frac{2}{\rho}x_0', \tag{2.68}$$

$$a'\sqrt{2\rho}\,e^{\sqrt{2\rho}\,x_1} + \sqrt{2\rho}\left(\frac{2}{\rho}x_1 - \lambda\right)x_1' = \frac{2}{\rho}x_1'. \tag{2.69}$$

Subtracting (2.68) from (2.69) and using (2.67) we get

$$a'\sqrt{2\rho}(e^{\sqrt{2\rho}\,x_1} - e^{\sqrt{2\rho}\,x_0}) + \frac{\sqrt{2\rho}}{\rho}\left(\frac{2}{\sqrt{2\rho}}(x_1' - x_0') + \rho\right) = \frac{2}{\rho}(x_1' - x_0')$$

or

$$a'(c) = -(e^{\sqrt{2\rho}\,x_1(c)} - e^{\sqrt{2\rho}\,x_0(c)})^{-1} < 0. \tag{2.70}$$

Combining (2.48) with (2.70) we get

$$x_0'(c) = \frac{dx_0}{dc} = \frac{dx_0}{da} \cdot \frac{da}{dc} < 0 \tag{2.71}$$

and

$$x_1'(c) = \frac{dx_1}{dc} = \frac{dx_1}{da} \cdot \frac{da}{dc} > 0. \tag{2.72}$$

We conclude from (2.70), (2.71), and (2.72) that

$$\hat{a} := \lim_{c \to 0} a(c), \quad \hat{x}_0 := \lim_{c \to 0} x_0(c), \quad \text{and} \quad \hat{x}_1 := \lim_{c \to 0} x_1(c) \tag{2.73}$$

exist. To find these limits, we note that they must solve the system (2.38)–(2.40) with $c = 0$, i.e.,

$$\hat{a}\sqrt{2\rho}\,e^{\sqrt{2\rho}\,\hat{x}_0} = \frac{2}{\rho}\hat{x}_0 - \lambda, \tag{2.74}$$

$$\hat{a}\sqrt{2\rho}\,e^{\sqrt{2\rho}\,\hat{x}_1} = \frac{2}{\rho}\hat{x}_1 - \lambda, \tag{2.75}$$

$$(\hat{x}_1 - \hat{x}_0)(\hat{x}_1 + \hat{x}_0 - 2\hat{x}) = 0. \tag{2.76}$$

Clearly $\hat{a} \leq A$. Assume $\hat{a} < A$. Then by Lemma 2.3 we must have $\hat{x}_0 < \hat{x}_1$ and $\hat{x}_0 + \hat{x}_1 > 2\hat{x}$. However, this contradicts (2.76). Hence $\hat{a} = A$ and therefore

$$\hat{x}_0 = \lim_{a \to A} x_0(a) = \hat{x} \quad \text{and} \quad \hat{x}_1 = \lim_{a \to A} x_1(a) = \hat{x}.$$

From (2.70) it follows that $a'(c) \to -\infty$ as $c > 0^+$. Then by (2.71), (2.71) combined with (2.48), (2.49) we see that $x_0'(c) \to -\infty$ and $x_1'(c) \to \infty$ as $c \to 0^+$.

We summarize this as follows:

Theorem 2.6. *Let $a(c)$, $a_0(c)$, and $x_1(c)$ be the solution given in Theorem 2.5 of the special impulse control problem (Problem 2.2) for $c > 0$. Then*

$$\lim_{c \to 0^+} x_0(c) = \lim_{c \to 0^+} x_1(c) = \frac{\lambda\rho}{2} + \frac{1}{\sqrt{2\rho}} =: \hat{x} \tag{2.77}$$

and

$$\lim_{c \to 0^+} a(c) = \frac{1}{\rho^2} \cdot \exp(-\sqrt{2\rho}\,\hat{x}) =: A. \tag{2.78}$$

Moreover

$$a'(c) = -(e^{\sqrt{2\rho}\,x_1(c)} - e^{\sqrt{2\rho}\,x_0(c)})^{-1} \to -\infty \qquad as \quad c \to 0^+ \tag{2.79}$$

and

$$x_0'(c) \to -\infty, \quad x_1'(c) \to \infty \qquad as \quad c \to 0^+. \tag{2.80}$$

Hence

$$\lim_{c \to 0^+} V_c(s, x) = e^{-\rho s} \psi_0(x), \tag{2.81}$$

where

$$\psi_0(x) = \begin{cases} \dfrac{1}{\rho} x^2 + \dfrac{1}{\rho^2} - A e^{\sqrt{2\rho}\, x} & for \quad x \le \hat{x}, \\[2mm] \psi_0(\hat{x}) + \lambda(x - \hat{x}) & for \quad x > \hat{x}. \end{cases} \tag{2.82}$$

Moreover,

$$\lim_{c \to 0^+} \frac{d}{dc} V_c(s, x) = \infty \qquad for\ all\ s, x. \tag{2.83}$$

3.   The Singular Stochastic Control Problem

We now consider the case $c = 0$ only. As explained in the Introduction (see Problem 1.2) there is a natural singular control problem interpretation of Problem 2.2, as follows:

For $\gamma \in \Gamma$ let the state Y_t^γ of our system be given by

$$Y_t^\gamma = \left(s + t, x + B_t - \int_0^t d\gamma_r \right); \qquad t \ge 0. \tag{3.1}$$

Assume that the cost of applying the control $\gamma \in \Gamma$ is given by

$$J^\gamma(y) = J^\gamma(s, x) = E_{s,x} \left[\int_0^\infty e^{-\rho(s+t)} (X_t^\gamma)^2 \, dt + \lambda \int_0^\infty e^{-\rho(s+t)} \, d\gamma_t \right], \tag{3.2}$$

where

$$X_t^\gamma = x + B_t - \int_0^t d\gamma_r. \qquad \text{(See (2.20)–(2.24).)} \tag{3.3}$$

This leads to the following problem:

Problem 3.1 (Special Singular Stochastic Control Problem when $c = 0$).   Let $J^\gamma(s, x)$ be as in (3.2). Find the value function $W(s, x)$ and the optimal control $\gamma^* \in \Gamma$ such that

$$W(s, x) = \inf_{\gamma \in \Gamma} J^\gamma(s, x) = J^{\gamma^*}(s, x). \tag{3.4}$$

We briefly recall the concept of a *reflected* Ito diffusion:
Let Y_t be an Ito diffusion in $\mathbf{R}^n$ given by

$$dY_t = b(Y_t)\, dt + \sigma(Y_t)\, dB_t; \qquad t \ge 0, \quad Y_0 = y \in \mathbf{R}^n.$$

Let D be a domain in $\mathbf{R}^n$ whose boundary has a tangent at every point. For each $x \in \partial D$ choose a vector $\eta(x) \in \mathbf{R}^n$ pointing into D. Choose $y \in \overline{D}$, the closure of D. Then consider the problem of finding a pair $(\hat{Y}_t, \xi_t)$ of continuous, $\mathcal{F}_t$-adapted stochastic processes with the following properties:

$$\hat{Y}_t \in \overline{D} \qquad \text{for all} \quad t \geq 0, \tag{3.5}$$

$$\xi_t \text{ is a nondecreasing process, increasing only when } t \in \Lambda := \{t; \hat{Y}_t \in \partial D\}, \tag{3.6}$$

$$\Lambda = \Lambda(\omega) \subset [0, \infty) \qquad \text{has Lebesgue measure 0 for a.a. } \omega, \tag{3.7}$$

and

$$d\hat{Y}_t = b(\hat{Y}_t)\, dt + \sigma(\hat{Y}_t)\, dB_t + \eta(\hat{Y}_t)\, d\xi_t, \qquad t \geq 0, \tag{3.8}$$

$$\hat{Y}_0 = y \in \overline{D} \quad \text{and} \quad \xi_0 = 0. \tag{3.9}$$

Equations (3.8)–(3.9) (with the conditions (3.5)–(3.7)) are called a *Skorohod* stochastic *differential equation*. If it has a unique solution, then $\hat{Y}_t$ is called the *reflection* of Y_t at ∂D (in the direction of the vector fields $\eta(x)$, $x \in \partial D$) and ξ_t is called *the local time of Y_t at ∂D*. For more information see [F] and the references therein.

In the special case when $Y_t = Z_t$ is a constant σ times a Brownian motion plus a constant drift in $\mathbf{R}$ and D has the form $D = (-\infty, x^*)$, then there is a simple construction of the reflected process $\widehat{Z}_t$, which in this case is called the *downward reflection* of Z_t at x^*. In fact, $\widehat{Z}_t$ is given by

$$\widehat{Z}_t = Z_t - \xi_t,$$

where

$$\xi_t := \sup_{s \leq t}(Z_s - x^*)^+$$

is the local time of $\{Z_t\}$ at x^*. This is due to Skorohod (see, e.g., p. 222 of [RY]). Here we have used the standard notation $y^+ = \max(y, 0)$.

The pair $(\widehat{Z}_t, \xi_t)$ satisfies all the requirements above, except possibly (3.7). If σ nonzero, it also satisfies (3.7). However, (3.7) is not needed in the result below.

Analogous to the quasi-variational inequality verification result for impulse control (Theorem 2.1) there is a variational inequality verification result for singular stochastic control. The following formulation is sufficient for our purposes.

Note that it is not optimal to move X_t downward if X_t is already below 0. Hence we have

$$W(s, x) := \inf_{\gamma \in \Gamma} J^\gamma(s, x) = \inf_{\gamma \in \Gamma_0} J^\gamma(s, x),$$

where

$$\Gamma_0 = \{\gamma \in \Gamma; \gamma_t \text{ increases only when } X_t \geq 0\}.$$

Theorem 3.2 [LØ] (Sufficient Variational Inequalities for the Singular Stochastic Control Problem).   *Suppose we can find $\varphi \in C^2(\mathbf{R}^2)$, $\varphi \geq 0$, such that*

$$\frac{\partial \varphi}{\partial x} \leq \lambda e^{-\rho t} \qquad everywhere, \tag{3.10}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2} + e^{-\rho t} x^2 \geq 0 \qquad everywhere, \tag{3.11}$$

and

$$\lim_{R \to \infty} E_{s,x}[\varphi(T_R, X_{T_R}^\gamma)] = 0 \qquad for\ all \quad \gamma \in \Gamma_0, \tag{3.12}$$

where

$$T_R = R \wedge \inf\{t > 0;\ |X_t^\gamma| \geq R\}.$$

Then

$$\varphi(t, x) \leq W(t, x). \tag{3.13}$$

Now define the nonintervention region

$$D = \left\{(t, x);\ \frac{\partial \varphi}{\partial x} < \lambda e^{-\rho t}\right\}. \tag{3.14}$$

Assume that

$$D = \{(t, x);\ x < x^*\} \qquad for\ some \quad x^* \in \mathbf{R} \tag{3.15}$$

and that

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2} + e^{-\rho t} x^2 = 0 \qquad in D. \tag{3.16}$$

Let $X_t^0 = x + B_t$ (corresponding to $\gamma = 0$ in (3.3)). Define the following control $\hat{\gamma} \in \Gamma_0$:

$$\hat{\gamma}_t = \begin{cases} (x - x^*)^+ & for \quad t = 0, \\ \xi_t & for \quad t > 0, \end{cases} \tag{3.17}$$

where

$$\xi_t = \sup_{s \leq t}(X_s^0 - x^*)^+. \tag{3.18}$$

Then

$$\varphi(t, x) = W(t, x) \tag{3.19}$$

and

$$\gamma^* = \hat{\gamma} \qquad is\ optimal. \tag{3.20}$$

The proof of Theorem 3.2 is similar to the proof of Theorem 3.3 in [LØ] and is omitted.

Note that the first statement of the following theorem is just a special case of a result in [MR1] (see (1.20)).

Theorem 3.3.   *With $V_c(s, x)$ as in Theorem 2.6 for $c > 0$ and W as in Problem 3.1 we have*

$$\lim_{c \to 0^+} V_c(s, x) = W(s, x).$$

Moreover, the optimal singular stochastic control $\gamma^* \in \Gamma$ for Problem 3.1 coincides with the local time ξ_t at $x = \hat{x}$ where $\widehat{x}$ is as in Theorem 2.6, i.e.,

$$\hat{x} = \frac{\lambda \rho}{2} + \frac{1}{\sqrt{2\rho}} = \lim_{c \to 0^+} x_0(c) = \lim_{c \to 0^+} x_1(c), \tag{3.21}$$

corresponding to an optimal state process $Y_t^{\gamma^*}$ being the downward reflection *of the process $Y_t^0 = (s + t, X_t^0)$ at $\partial D = \{(t, x); x = \hat{x}\}$.*

Proof.   We verify that $\varphi(s, x) = e^{-\rho s} \psi_0(x)$ with ψ_0 given by (2.82) satisfies the requirements of Theorem 3.2. By (2.82) it suffices to verify that $\psi_0(x) \in C^2(\mathbf{R})$ and that the following hold:

$$\psi_0'(x) \leq \lambda \qquad \text{everywhere,} \tag{3.22}$$

$$-\rho \psi_0(x) + \tfrac{1}{2} \psi_0''(x) + x^2 \geq 0 \qquad \text{everywhere,} \tag{3.23}$$

$$D := \{(t, x); \psi_0'(x) < \lambda\} = \{(t, x); x < \hat{x}\}, \tag{3.24}$$

$$-\rho \psi_0(x) + \tfrac{1}{2} \psi_0''(x) + x^2 = 0 \qquad \text{for} \quad x < \hat{x}. \tag{3.25}$$

Moreover, we must check that (3.12) holds, i.e.,

$$\lim_{R \to \infty} E_{s,x}[e^{-\rho T_R} \psi_0(X_{T_R}^\gamma)] = 0 \qquad \text{for all} \quad \gamma \in \Gamma_0. \tag{3.26}$$

To check that $\psi_0 \in C^2(\mathbf{R})$ we first note that

$$\frac{d}{dx} \left(\frac{1}{\rho} x^2 + \frac{1}{\rho^2} - A e^{\sqrt{2\rho}\, x} \right) = \frac{2}{\rho} x - A\sqrt{2\rho}\, e^{\sqrt{2\rho}\, x} = \lambda$$

for $x = \hat{x}$ by (2.74) and (2.77). Since $(d/dx)(\psi_0(\hat{x}) + \lambda(x - \hat{x})) = \lambda$ also, we have

$\psi_0 \in C^1(\mathbf{R})$. Next, since

$$\frac{d^2}{dx^2}\left(\frac{1}{\rho}x^2 + \frac{1}{\rho^2} - Ae^{\sqrt{2\rho}\,x}\right)$$

$$= \frac{2}{\rho} - 2\rho\,Ae^{\sqrt{2\rho}\,x}$$

$$= \frac{2}{\rho} - 2\rho \cdot \frac{1}{\rho^2}\exp(\sqrt{2\rho}(x - \hat{x})) = 0 \qquad \text{for} \quad x = \hat{x}, \tag{3.27}$$

we conclude that $\psi_0 \in C^2(\mathbf{R})$.

For $x \geq \hat{x}$ we have $\psi_0'(x) = \lambda$, while for $x < \hat{x}$ we have

$$\psi_0'(x) = \frac{2}{\rho}x - A\sqrt{2\rho}\,e^{\sqrt{2\rho}\,x} < \lambda$$

because $\psi_0''(x) > 0$ for $x < \hat{x}$ by (3.27). Hence (3.22) holds. Moreover,

$$D = \{x;\ \psi_0'(x) < \lambda\} = \{x;\ x < \hat{x}\}.$$

To verify (3.23) we first show that the function $h(x)$ defined by

$$h(x) := -\rho(\psi_0(\hat{x}) + \lambda(x - \hat{x})) + \tfrac{1}{2}(\psi_0(\hat{x}) + \lambda(x - \hat{x}))'' + x^2$$

$$= -\rho\psi_0(\hat{x}) - \rho\lambda(x - \hat{x}) + x^2$$

satisfies

$$h(x) \geq 0 \qquad \text{for} \quad x \geq \hat{x}. \tag{3.28}$$

Since $h'(x) = -\rho\lambda + 2x > 0$ for $x > \rho\lambda/2$ and $\hat{x} = \rho\lambda/2 + 1/\sqrt{2\rho} > \lambda\rho/2$, it suffices to verify (3.27) for $x = \hat{x}$, i.e., to verify that

$$-\rho\psi_0(\hat{x}) + \hat{x}^2 \geq 0. \tag{3.29}$$

By (2.82) we have

$$-\rho\psi_0(\hat{x}) + \hat{x}^2 = \rho\,Ae^{\sqrt{2\rho}\,\hat{x}} - \frac{1}{\rho} = 0.$$

To verify (3.25), and thereby also completing the verification of (3.23), we note that for $x < \hat{x}$ we have

$$-\rho\psi_0(x) + \tfrac{1}{2}\psi_0''(x) + x^2 = 0$$

by (2.82) and (2.28).

Finally we note that (3.26) clearly holds, because if $\gamma \in \Gamma_0$, then

$$\psi_0(X_t^\gamma) \leq \frac{1}{\rho}B_t^2 + C,$$

for some constant C not depending on t. That completes the proof of Theorem 3.3. $\quad\square$

Summary

We have studied an impulse control problem and found its value function V_c for all positive intervention costs c. Then we have shown that

$$\lim_{c \to 0^+} \frac{d}{dc} V_c = \infty,$$

which implies that increasing the intervention cost from $c = 0$ to a positive c, albeit small, can have a big effect on the value function for the problem.

Acknowledgments

References

[BL] A. Bensoussan and J. L. Lions: Impulse Control and Quasi-Variational Inequalities. Gauthier-Villars, Paris, 1984.

[BØ1] K. A. Brekke and B. Øksendal: The high contact principle as a sufficiency condition for optimal stopping. In D. Lund and B. Øksendal (editors): Stochastic Models and Option Values. North-Holland, Amsterdam, 1991, pp. 187–208.

[BØ2] K. A. Brekke and B. Øksendal: A verification theorem for combined stochastic control and impulse control. In L. Decreusefond, J. Gjerde, B. Øksendal, and S. Ustunel (editors): Stochastic Analysis and Related Topics, Vol. 6. Birkhäuser, Basel, 1998, pp. 211–220.

[DN] M. H. A. Davis and A. Norman: Portfolio selection with transaction costs. Mathematics of Operations Research 15 (1990), 676–713.

[F] M. Freidlin: Functional Integration and Partial Differential Equations. Princeton University Press, Princeton, NJ, 1985.

[HST] J. M. Harrison, T. M. Sellke, and A. J. Taylor: Impulse control of Brownian motion. Mathematics of Operations Research 8 (1983), 454–466.

[JS] M. Jeanblanc-Picqué and A. N. Shiryaev: Optimization of the flow of dividends. Russian Mathematical Surveys 50 (1995), 257–277.

[LØ] E. Lungu and B. Øksendal: Optimal harvesting from a population in a stochastic crowded environment. Preprint, University of Oslo, 1997. Mathematical Biosciences 145 (1997), 47–75.

[M] J. L. Menaldi: Optimal impulse control problems for degenerate diffusions with jumps. Acta Applicandae Mathematicae 8 (1987), 165–198.

[MØ] G. Mundaca and B. Øksendal: Optimal stochastic intervention control with application to the exchange rate. Journal of Mathematical Economics 29 (1998), 225–243.

[MR1] J. L. Menaldi and M. Robin: On some cheap control problems for diffusion processes. Transactions of the AMS, 278 (1983), 771–802.

[MR2] J. L. Menaldi and E. Rofman: On stochastic control problems with impulse cost vanishing. In A. V. Fiacco and K. O. Kortanek (editors): Lecture Notes in Economics and Mathematical Systems, Vol. 215. Springer-Verlag, Berlin, 1983, pp. 218–294.

[Ø] B. Øksendal: Stochastic Differential Equations, fifth edition. Springer-Verlag, New York, 1998.

[ØUZ] B. Øksendal, J. Ubøe, and T. Zhang: Nonrobustness of some impulse control problems with respect to intervention costs (to appear).

[RY] D. Revuz and M. Yor: Continuous Martingales and Brownian Motion. Springer-Verlag, New York, 1991.

[V] H. Varner: Some Impulse Control Problems with Applications to Economics. Cand. scient. thesis, University of Oslo, 1997.